PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a,b) = (z^2, z^5 + z^4 + z^2 + z)$

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Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a,b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a,b) in $\mathbb{F}_2[x] \setminus \mathbb{F}_2$. In this paper we prove the conjecture for $(a,b) = (z^2, z^5 + z^4 + z^2 + z)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \ldots)$ of elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{\ddots + \cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a,b)-Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \ldots) = (a, b, b, a, b, a, a, b, \ldots)$$

that we shall denote

$$\mathrm{CF}(a,b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When* $(a, b) = (z^2, z^5 + z^4 + z^2 + z)$*, the two power series* $\mathrm{CF}(a, b)$ *and* $\mathrm{CF}(b, a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4 + p_3(z)y^3 + p_2(z)y^2 + p_1(z)y + p_0(z) = 0$$

where, for $\mathrm{CF}(a, b)$,

$$\begin{aligned}
p_0(z) &= z^{17} + z^{15} + z^{12} + z^{11} + z^6 + z^4 + z + 1,\\
p_1(z) &= z^{22} + z^{19} + z^{18} + z^{17} + z^{14} + z^{13} + z^{10} + z^8 + z^3 + z^2,\\
p_2(z) &= z^{24} + z^{16} + z^{14} + z^8 + z^6 + z^4,\\
p_3(z) &= z^{22} + z^{19} + z^{18} + z^{17} + z^{14} + z^{13} + z^{10} + z^8 + z^3 + z^2,\\
p_4(z) &= z^{20} + z^{17} + z^{15} + z^{14} + z^{12} + z^{11} + z,
\end{aligned}$$

and for $\mathrm{CF}(b, a)$,

$$\begin{aligned}
p_0(z) &= z^{17} + z^{16} + z^{15} + z^{11} + z^6 + z^4 + z,\\
p_1(z) &= z^{22} + z^{19} + z^{18} + z^{17} + z^{14} + z^{13} + z^{10} + z^8 + z^3 + z^2,\\
p_2(z) &= z^{24} + z^{16} + z^{14} + z^8 + z^6 + z^4,\\
p_3(z) &= z^{22} + z^{19} + z^{18} + z^{17} + z^{14} + z^{13} + z^{10} + z^8 + z^3 + z^2,\\
p_4(z) &= z^{20} + z^{17} + z^{16} + z^{15} + z^{14} + z^{11} + z + 1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z) \\ P_{n-1}(z) & Q_{n-1}(z) \end{pmatrix} := \begin{pmatrix} t_n(z) & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t_{n-1}(z) & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} t_0(z) & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a, b) = \frac{P_n(z)}{Q_n(z)} \in \mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a, b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x) = {} & x^{\deg(t_{2^n-1})} \begin{pmatrix} t_{2^n-1}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times x^{\deg(t_{2^n-2})} \begin{pmatrix} t_{2^n-2}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times \\
& \cdots \times x^{\deg(t_0)} \begin{pmatrix} t_0(1/x) & 1 \\ 1 & 0 \end{pmatrix},\\
W_n(x) = {} & x^{\deg(\bar{t}_{2^n-1})} \begin{pmatrix} \bar{t}_{2^n-1}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times x^{\deg(\bar{t}_{2^n-2})} \begin{pmatrix} \bar{t}_{2^n-2}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times \\
& \cdots \times x^{\deg(\bar{t}_0)} \begin{pmatrix} \bar{t}_0(1/x) & 1 \\ 1 & 0 \end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b, a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n \geq 0$

$$\begin{aligned}
M_{n+1}(x) &= W_n(x) \cdot M_n(x),\\
W_{n+1}(x) &= M_n(x) \cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(175, 175, 175, 175, 175)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z\left(\phi(W^o,0,0)(x,z),\ z^{12}\cdot\phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 630 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 630, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z\left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ to order 770 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 770, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. □

Define

$$R^e = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned}
M^e[:14u] = {} & M^e[:7u] + x^u M^e[:6u] + x^{3u}M^e[:4u] + x^{5u}M^e[:2u] + x^u M^e[:7u]\\
&+ x^{2u}M^e[:6u] + x^{4u}M^e[:4u] + x^{6u}M^e[:2u] + x^{2u}M^e[:7u]\\
&+ x^{3u}M^e[:6u] + x^{5u}M^e[:4u] + x^{7u}M^e[:2u] + x^{4u}M^e[:7u]\\
&+ x^{5u}M^e[:6u] + x^{7u}M^e[:4u] + x^{9u}M^e[:2u] + x^{5u}M^e[:7u]\\
&+ x^{6u}M^e[:6u] + x^{8u}M^e[:4u] + x^{10u}M^e[:2u] + x^{7u}M^e[:7u]\\
&+ x^{10u}M^e[:4u] + x^{12u}M^e[:2u] + x^{9u}M^e[:5u] + x^{11u}M^e[:3u]
\end{aligned}$$

$$
\begin{aligned}
&+ x^{13u}M^e[:u] + (x^{7u} + x^{8u} + x^{9u} + x^{11u} + x^{12u})R^e,\\
W^e[:14u] &= W^e[:7u] + x^uW^e[:6u] + x^{3u}W^e[:4u] + x^{5u}W^e[:2u] + x^uW^e[:7u]\\
&+ x^{2u}W^e[:6u] + x^{4u}W^e[:4u] + x^{6u}W^e[:2u] + x^{2u}W^e[:7u]\\
&+ x^{3u}W^e[:6u] + x^{5u}W^e[:4u] + x^{7u}W^e[:2u] + x^{4u}W^e[:7u]\\
&+ x^{5u}W^e[:6u] + x^{7u}W^e[:4u] + x^{9u}W^e[:2u] + x^{5u}W^e[:7u]\\
&+ x^{6u}W^e[:6u] + x^{8u}W^e[:4u] + x^{10u}W^e[:2u] + x^{7u}W^e[:7u]\\
&+ x^{10u}W^e[:4u] + x^{12u}W^e[:2u] + x^{9u}W^e[:5u] + x^{11u}W^e[:3u]\\
&+ x^{13u}W^e[:u] + (x^{7u} + x^{8u} + x^{9u} + x^{11u} + x^{12u})R^e.
\end{aligned}
$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$
\begin{aligned}
M^o[:14u] &= M^o[:7u] + x^uM^o[:6u] + x^{3u}M^o[:4u] + x^{5u}M^o[:2u] + x^uM^o[:7u]\\
&+ x^{2u}M^o[:6u] + x^{4u}M^o[:4u] + x^{6u}M^o[:2u] + x^{2u}M^o[:7u]\\
&+ x^{3u}M^o[:6u] + x^{5u}M^o[:4u] + x^{7u}M^o[:2u] + x^{4u}M^o[:7u]\\
&+ x^{5u}M^o[:6u] + x^{7u}M^o[:4u] + x^{9u}M^o[:2u] + x^{5u}M^o[:7u]\\
&+ x^{6u}M^o[:6u] + x^{8u}M^o[:4u] + x^{10u}M^o[:2u] + x^{7u}M^o[:7u]\\
&+ x^{10u}M^o[:4u] + x^{12u}M^o[:2u] + x^{9u}M^o[:5u] + x^{11u}M^o[:3u]\\
&+ x^{13u}M^o[:u] + (x^{7u} + x^{8u} + x^{9u} + x^{11u} + x^{12u})R^o,\\
W^o[:14u] &= W^o[:7u] + x^uW^o[:6u] + x^{3u}W^o[:4u] + x^{5u}W^o[:2u] + x^uW^o[:7u]\\
&+ x^{2u}W^o[:6u] + x^{4u}W^o[:4u] + x^{6u}W^o[:2u] + x^{2u}W^o[:7u]\\
&+ x^{3u}W^o[:6u] + x^{5u}W^o[:4u] + x^{7u}W^o[:2u] + x^{4u}W^o[:7u]\\
&+ x^{5u}W^o[:6u] + x^{7u}W^o[:4u] + x^{9u}W^o[:2u] + x^{5u}W^o[:7u]\\
&+ x^{6u}W^o[:6u] + x^{8u}W^o[:4u] + x^{10u}W^o[:2u] + x^{7u}W^o[:7u]\\
&+ x^{10u}W^o[:4u] + x^{12u}W^o[:2u] + x^{9u}W^o[:5u] + x^{11u}W^o[:3u]\\
&+ x^{13u}W^o[:u] + (x^{7u} + x^{8u} + x^{9u} + x^{11u} + x^{12u})R^o.
\end{aligned}
$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$
\begin{aligned}
T[:14u] &= T[:7u] + x^uT[:6u] + x^{3u}T[:4u] + x^{5u}T[:2u] + x^uT[:7u] + x^{2u}T[:6u]\\
&+ x^{4u}T[:4u] + x^{6u}T[:2u] + x^{2u}T[:7u] + x^{3u}T[:6u] + x^{5u}T[:4u]\\
&+ x^{7u}T[:2u] + x^{4u}T[:7u] + x^{5u}T[:6u] + x^{7u}T[:4u] + x^{9u}T[:2u]\\
&+ x^{5u}T[:7u] + x^{6u}T[:6u] + x^{8u}T[:4u] + x^{10u}T[:2u] + x^{7u}T[:7u]\\
&+ x^{10u}T[:4u] + x^{12u}T[:2u] + x^{9u}T[:5u] + x^{11u}T[:3u] + x^{13u}T[:u].
\end{aligned}
$$

The proof of the other 15 cases are similar. We break down the above identity into 7 parts:

$$
\begin{aligned}
0 &= x^{7u}T[:u] + x^{5u}T[2u:3u] + x^{3u}T[4u:5u] + x^uT[6u:7u] + T[7u:8u],\\
0 &= x^{8u}T[:u] + x^{7u}T[u:2u] + x^{6u}T[2u:3u] + x^{5u}T[3u:4u]
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{4u}T[4u:5u] + x^{3u}T[5u:6u] + x^{2u}T[6u:7u] + T[8u:9u],\\
0 &= x^{8u}T[u:2u] + x^{6u}T[3u:4u] + x^{4u}T[5u:6u] + T[9u:10u],\\
0 &= x^{8u}T[2u:3u] + x^{6u}T[4u:5u] + x^{4u}T[6u:7u] + T[10u:11u],\\
0 &= x^{11u}T[:u] + x^{9u}T[2u:3u] + x^{8u}T[3u:4u] + x^{7u}T[4u:5u]\\
&\quad + x^{6u}T[5u:6u] + x^{5u}T[6u:7u] + T[11u:12u],\\
0 &= x^{12u}T[:u] + x^{11u}T[u:2u] + x^{10u}T[2u:3u] + x^{9u}T[3u:4u]\\
&\quad + x^{7u}T[5u:6u] + T[12u:13u],\\
0 &= x^{13u}T[:u] + x^{12u}T[u:2u] + x^{11u}T[2u:3u] + x^{10u}T[3u:4u]\\
&\quad + x^{9u}T[4u:5u] + x^{7u}T[6u:7u] + T[13u:14u],
\end{aligned}
$$

which can be rewritten as

$$0 = T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[110w]_2] + T[[111w]_2], \tag{2.7}$$

$$\begin{aligned}0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2]\\ &\quad + T[[101w]_2] + T[[110w]_2] + T[[1000w]_2],\end{aligned} \tag{2.8}$$

$$0 = T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1001w]_2], \tag{2.9}$$

$$0 = T[[10w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1010w]_2], \tag{2.10}$$

$$\begin{aligned}0 &= T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2]\\ &\quad + T[[110w]_2] + T[[1011w]_2],\end{aligned} \tag{2.11}$$

$$\begin{aligned}0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2]\\ &\quad + T[[1100w]_2],\end{aligned} \tag{2.12}$$

$$\begin{aligned}0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2]\\ &\quad + T[[110w]_2] + T[[1101w]_2],\end{aligned} \tag{2.13}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$0 = T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[110w]_2] + T[[111w]_2], \tag{2.14}$$

$$\begin{aligned}0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2]\\ &\quad + T[[101w]_2] + T[[110w]_2] + T[[1000w]_2],\end{aligned} \tag{2.15}$$

$$0 = T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1001w]_2], \tag{2.16}$$

$$0 = T[[10w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1010w]_2], \tag{2.17}$$

$$\begin{aligned}0 &= T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2]\\ &\quad + T[[110w]_2] + T[[1011w]_2],\end{aligned} \tag{2.18}$$

$$\begin{aligned}0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2]\\ &\quad + T[[1100w]_2],\end{aligned} \tag{2.19}$$

$$\begin{aligned}0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2]\\ &\quad + T[[110w]_2] + T[[1101w]_2],\end{aligned} \tag{2.20}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 761 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i,w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.20) can be written as

$$\begin{aligned}
&0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,110)) \\
&\quad + \tau(A(s,111)), && (2.21)\\
&0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\
&\quad + \tau(A(s,101)) + \tau(A(s,110)) + \tau(A(s,1000)), && (2.22)\\
&0 = \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,1001)), && (2.23)\\
&0 = \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,110)) + \tau(A(s,1010)), && (2.24)\\
&0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\
&\quad + \tau(A(s,101)) + \tau(A(s,110)) + \tau(A(s,1011)), && (2.25)\\
&0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101)) \\
&\quad + \tau(A(s,1100)), && (2.26)\\
&0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\
&\quad + \tau(A(s,110)) + \tau(A(s,1101)), && (2.27)
\end{aligned}$$

for all $s \in E_{2k}$, and

$$\begin{aligned}
&0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,110)) \\
&\quad + \tau(A(s,111)), && (2.28)\\
&0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\
&\quad + \tau(A(s,101)) + \tau(A(s,110)) + \tau(A(s,1000)), && (2.29)\\
&0 = \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,1001)), && (2.30)\\
&0 = \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,110)) + \tau(A(s,1010)), && (2.31)\\
&0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\
&\quad + \tau(A(s,101)) + \tau(A(s,110)) + \tau(A(s,1011)), && (2.32)\\
&0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101)) \\
&\quad + \tau(A(s,1100)), && (2.33)\\
&0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\
&\quad + \tau(A(s,110)) + \tau(A(s,1101)), && (2.34)
\end{aligned}$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{22}), E_{22}) = (A(i, 0^{16}), E_{16})$, so that we only have to verify that identities (2.21) through (2.34) hold for $2 \leq k \leq 11$, which turns out to be true. ☐

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \geq 2$, and $u = 2^{2k-1}$,*

$$M_{2k} = M^e[:7u] + x^u M^e[:6u] + x^{3u} M^e[:4u] + x^{5u} M^e[:2u] + x^{7u} R^e,$$
$$W_{2k} = W^e[:7u] + x^u W^e[:6u] + x^{3u} W^e[:4u] + x^{5u} W^e[:2u] + x^{7u} R^e.$$

For all integer $k \geq 2$, and $u = 2^{2k}$,

$$M_{2k+1} = M^o[:7u] + x^u M^o[:6u] + x^{3u} M^o[:4u] + x^{5u} M^o[:2u] + x^{7u} R^o,$$
$$W_{2k+1} = W^o[:7u] + x^u W^o[:6u] + x^{3u} W^o[:4u] + x^{5u} W^o[:2u] + x^{7u} R^o.$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \geq 2$, we claim that

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1} \wedge W_{2k+1},$$
$$M_{2k+1} \wedge W_{2k+1} \Rightarrow M_{2k+2} \wedge W_{2k+2}.$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$\begin{aligned} W_{2k} M_{2k} = \big(W^e[:7v] + x^v W^e[:6v] + x^{3v} W^e[:4v] + x^{5v} W^e[:2v] + x^{7u} R^e\big) \times \big(\\ M^e[:7v] + x^v M^e[:6v] + x^{3v} M^e[:4v] + x^{5v} M^e[:2v] + x^{7u} R^e\big); \end{aligned} \tag{2.35}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{14v} R^o$. Therefore we only need to prove that their difference is $O(x^{14v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{14v}, to

$$\begin{aligned} W^e[:14v] M^e[:14v] + x^{2v} W^e[:12v] M^e[:12v] + x^{6v} W^e[:8v] M^e[:8v] \\ + x^{10v} W^e[:4v] M^e[:4v]. \end{aligned}$$

For all $n \leq 14$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v] / X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v] / X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.35) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{14}, b_1, ..., b_{14}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{14})$, which completes the proof. $\square$

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \leq j, k \leq 1$,

$$\begin{aligned} \lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k}, \\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k}, \\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k}, \\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}. \end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a,b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e,0,1)$ and $\phi(M^e,0,0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x,y) = \mathrm{Res}_t\left(\phi(M^e,0,1)(x,t), y^{12}\phi(M^e,0,1)(x,t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x,y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned}
q_0(x) &= x^{24} + x^{23} + x^{20} + x^{18} + x^{13} + x^{12} + x^9 + x^7,\\
q_1(x) &= x^{22} + x^{21} + x^{16} + x^{14} + x^{11} + x^{10} + x^7 + x^6 + x^5 + x^2,\\
q_2(x) &= x^{20} + x^{18} + x^{16} + x^{10} + x^8 + 1,\\
q_3(x) &= x^{22} + x^{21} + x^{16} + x^{14} + x^{11} + x^{10} + x^7 + x^6 + x^5 + x^2,\\
q_4(x) &= x^{23} + x^{13} + x^{12} + x^{10} + x^9 + x^7 + x^4.
\end{aligned}$$

The polynomial $Q(x,y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and $R(x,y) := P(x,y)/Q(x,y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 216, and substitute it for y in $R(z,y)$, we get a series with valuation smaller than 216, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b) = f(1/z)$. □

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e,0,0)$:

$$\begin{aligned}
p_0(x) = {}& x^{138} + x^{118} + x^{116} + x^{112} + x^{110} + x^{106} + x^{100} + x^{98} + x^{94} + x^{86}\\
&+ x^{82} + x^{78} + x^{76} + x^{70} + x^{68} + x^{66} + x^{64} + x^{62} + x^{60} + x^{58} + x^{54}\\
&+ x^{52} + x^{50} + x^{46} + x^{44} + x^{40} + x^{34} + x^{30} + x^{24},\\
p_3(x) = {}& x^{130} + x^{126} + x^{118} + x^{112} + x^{110} + x^{104} + x^{102} + x^{100} + x^{98} + x^{96}\\
&+ x^{92} + x^{82} + x^{78} + x^{76} + x^{74} + x^{68} + x^{66} + x^{62} + x^{58} + x^{56} + x^{50}\\
&+ x^{48} + x^{46} + x^{40} + x^{38} + x^{32} + x^{30} + x^{28} + x^{22} + x^{20} + x^{18} + x^{16},\\
p_6(x) = {}& x^{130} + x^{116} + x^{114} + x^{112} + x^{110} + x^{106} + x^{100} + x^{94} + x^{90} + x^{88}\\
&+ x^{86} + x^{84} + x^{80} + x^{76} + x^{74} + x^{72} + x^{70} + x^{60} + x^{58} + x^{48} + x^{44}\\
&+ x^{42} + x^{38} + x^{34} + x^{24} + x^{22} + x^{20} + x^{18} + x^{16} + x^{10} + x^4 + 1,
\end{aligned}$$

$$\begin{aligned}
p_9(x) &= x^{128}+x^{124}+x^{122}+x^{114}+x^{112}+x^{108}+x^{106}+x^{102}+x^{96}+x^{90}\\
&\quad +x^{84}+x^{80}+x^{78}+x^{72}+x^{68}+x^{64}+x^{62}+x^{60}+x^{52}+x^{50}+x^{46}\\
&\quad +x^{44}+x^{32}+x^{28}+x^{22}+x^{14}+x^{12}+x^{10}+x^{8}+x^{4},\\
p_{12}(x) &= x^{128}+x^{120}+x^{104}+x^{88}+x^{72}+x^{56}+x^{48}+x^{16}+x^{8}+1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0].

For $\phi(M^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{129}+x^{126}+x^{125}+x^{124}+x^{122}+x^{118}+x^{115}+x^{112}+x^{111}+x^{107}\\
&\quad +x^{106}+x^{105}+x^{102}+x^{101}+x^{100}+x^{99}+x^{98}+x^{97}+x^{96}+x^{95}\\
&\quad +x^{94}+x^{93}+x^{92}+x^{91}+x^{89}+x^{88}+x^{87}+x^{86}+x^{85}+x^{81}+x^{78}\\
&\quad +x^{76}+x^{75}+x^{74}+x^{73}+x^{72}+x^{71}+x^{68}+x^{67}+x^{66}+x^{65}+x^{60}\\
&\quad +x^{59}+x^{58}+x^{57}+x^{55}+x^{53}+x^{52}+x^{51}+x^{48}+x^{43}+x^{42}+x^{40}\\
&\quad +x^{36}+x^{35}+x^{34}+x^{33},\\
p_3(x) &= x^{122}+x^{120}+x^{119}+x^{117}+x^{116}+x^{115}+x^{114}+x^{112}+x^{110}+x^{109}\\
&\quad +x^{108}+x^{107}+x^{104}+x^{103}+x^{102}+x^{101}+x^{100}+x^{98}+x^{95}+x^{94}\\
&\quad +x^{88}+x^{85}+x^{84}+x^{78}+x^{75}+x^{74}+x^{66}+x^{64}+x^{63}+x^{61}+x^{60}\\
&\quad +x^{59}+x^{58}+x^{56}+x^{54}+x^{53}+x^{52}+x^{51}+x^{50}+x^{46}+x^{43}+x^{40}\\
&\quad +x^{39}+x^{37}+x^{36}+x^{35}+x^{34}+x^{32}+x^{30}+x^{29}+x^{28}+x^{27}+x^{26}\\
&\quad +x^{22}+x^{19}+x^{18},\\
p_6(x) &= x^{120}+x^{119}+x^{118}+x^{116}+x^{115}+x^{114}+x^{106}+x^{105}+x^{104}+x^{102}\\
&\quad +x^{101}+x^{100}+x^{98}+x^{97}+x^{95}+x^{94}+x^{88}+x^{87}+x^{85}+x^{83}+x^{81}\\
&\quad +x^{79}+x^{77}+x^{75}+x^{73}+x^{72}+x^{70}+x^{69}+x^{67}+x^{65}+x^{64}+x^{62}\\
&\quad +x^{61}+x^{59}+x^{58}+x^{54}+x^{53}+x^{52}+x^{50}+x^{49}+x^{48}+x^{44}+x^{43}\\
&\quad +x^{42}+x^{40}+x^{39}+x^{37}+x^{35}+x^{34}+x^{32}+x^{31}+x^{30}+x^{22}+x^{21}\\
&\quad +x^{20}+x^{18}+x^{17}+x^{15}+x^{13}+x^{12},\\
p_9(x) &= x^{118}+x^{115}+x^{114}+x^{112}+x^{109}+x^{106}+x^{105}+x^{104}+x^{103}+x^{100}\\
&\quad +x^{99}+x^{98}+x^{97}+x^{96}+x^{94}+x^{91}+x^{88}+x^{87}+x^{85}+x^{84}+x^{83}\\
&\quad +x^{82}+x^{80}+x^{77}+x^{76}+x^{74}+x^{72}+x^{71}+x^{69}+x^{67}+x^{65}+x^{64}\\
&\quad +x^{63}+x^{60}+x^{59}+x^{58}+x^{57}+x^{54}+x^{53}+x^{51}+x^{50}+x^{49}+x^{48}\\
&\quad +x^{46}+x^{43}+x^{40}+x^{39}+x^{38}+x^{37}+x^{33}+x^{29}+x^{28}+x^{26}+x^{23}\\
&\quad +x^{22}+x^{18}+x^{16}+x^{15}+x^{14}+x^{13}+x^{12}+x^{10}+x^{7}+x^{6},\\
p_{12}(x) &= x^{116}+x^{115}+x^{113}+x^{111}+x^{109}+x^{108}+x^{104}+x^{103}+x^{101}+x^{99}\\
&\quad +x^{97}+x^{96}+x^{84}+x^{83}+x^{81}+x^{79}+x^{77}+x^{76}+x^{72}+x^{71}+x^{69}\\
&\quad +x^{67}+x^{65}+x^{64}+x^{52}+x^{51}+x^{49}+x^{47}+x^{45}+x^{44}+x^{40}+x^{39}\\
&\quad +x^{37}+x^{36}+x^{32}+x^{31}+x^{29}+x^{28}+x^{24}+x^{23}+x^{21}+x^{20}+x^{16}\\
&\quad +x^{15}+x^{13}+x^{12}+x^{8}+x^{7}+x^{5}+x^{3}+x+1,
\end{aligned}$$

and the initial terms are [0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 0, 1, 1].

For $\phi(M^e, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{129} + x^{126} + x^{125} + x^{124} + x^{122} + x^{118} + x^{115} + x^{112} + x^{111} + x^{107} \\
&\quad + x^{106} + x^{105} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} \\
&\quad + x^{94} + x^{93} + x^{92} + x^{91} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{81} + x^{78} \\
&\quad + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{68} + x^{67} + x^{66} + x^{65} + x^{60} \\
&\quad + x^{59} + x^{58} + x^{57} + x^{55} + x^{53} + x^{52} + x^{51} + x^{48} + x^{43} + x^{42} + x^{40} \\
&\quad + x^{36} + x^{35} + x^{34} + x^{33}, \\
p_3(x) &= x^{122} + x^{120} + x^{119} + x^{117} + x^{116} + x^{115} + x^{114} + x^{112} + x^{110} + x^{109} \\
&\quad + x^{108} + x^{107} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{98} + x^{95} + x^{94} \\
&\quad + x^{88} + x^{85} + x^{84} + x^{78} + x^{75} + x^{74} + x^{66} + x^{64} + x^{63} + x^{61} + x^{60} \\
&\quad + x^{59} + x^{58} + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{46} + x^{43} + x^{40} \\
&\quad + x^{39} + x^{37} + x^{36} + x^{35} + x^{34} + x^{32} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} \\
&\quad + x^{22} + x^{19} + x^{18}, \\
p_6(x) &= x^{120} + x^{119} + x^{118} + x^{116} + x^{115} + x^{114} + x^{106} + x^{105} + x^{104} + x^{102} \\
&\quad + x^{101} + x^{100} + x^{98} + x^{97} + x^{95} + x^{94} + x^{88} + x^{87} + x^{85} + x^{83} + x^{81} \\
&\quad + x^{79} + x^{77} + x^{75} + x^{73} + x^{72} + x^{70} + x^{69} + x^{67} + x^{65} + x^{64} + x^{62} \\
&\quad + x^{61} + x^{59} + x^{58} + x^{54} + x^{53} + x^{52} + x^{50} + x^{49} + x^{48} + x^{44} + x^{43} \\
&\quad + x^{42} + x^{40} + x^{39} + x^{37} + x^{35} + x^{34} + x^{32} + x^{31} + x^{30} + x^{22} + x^{21} \\
&\quad + x^{20} + x^{18} + x^{17} + x^{15} + x^{13} + x^{12}, \\
p_9(x) &= x^{118} + x^{115} + x^{114} + x^{112} + x^{109} + x^{106} + x^{105} + x^{104} + x^{103} + x^{100} \\
&\quad + x^{99} + x^{98} + x^{97} + x^{96} + x^{94} + x^{91} + x^{88} + x^{87} + x^{85} + x^{84} + x^{83} \\
&\quad + x^{82} + x^{80} + x^{77} + x^{76} + x^{74} + x^{72} + x^{71} + x^{69} + x^{67} + x^{65} + x^{64} \\
&\quad + x^{63} + x^{60} + x^{59} + x^{58} + x^{57} + x^{54} + x^{53} + x^{51} + x^{50} + x^{49} + x^{48} \\
&\quad + x^{46} + x^{43} + x^{40} + x^{39} + x^{38} + x^{37} + x^{33} + x^{29} + x^{28} + x^{26} + x^{23} \\
&\quad + x^{22} + x^{18} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{10} + x^{7} + x^{6}, \\
p_{12}(x) &= x^{116} + x^{115} + x^{113} + x^{111} + x^{109} + x^{108} + x^{104} + x^{103} + x^{101} + x^{99} \\
&\quad + x^{97} + x^{96} + x^{84} + x^{83} + x^{81} + x^{79} + x^{77} + x^{76} + x^{72} + x^{71} + x^{69} \\
&\quad + x^{67} + x^{65} + x^{64} + x^{52} + x^{51} + x^{49} + x^{47} + x^{45} + x^{44} + x^{40} + x^{39} \\
&\quad + x^{37} + x^{36} + x^{32} + x^{31} + x^{29} + x^{28} + x^{24} + x^{23} + x^{21} + x^{20} + x^{16} \\
&\quad + x^{15} + x^{13} + x^{12} + x^{8} + x^{7} + x^{5} + x^{3} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 0, 1, 1].

For $\phi(M^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{120} + x^{114} + x^{112} + x^{110} + x^{106} + x^{100} + x^{88} + x^{82} + x^{78} + x^{76} \\
&\quad + x^{74} + x^{68} + x^{58} + x^{56} + x^{54} + x^{50} + x^{46} + x^{44} + x^{42},
\end{aligned}$$

$$\begin{aligned}
p_3(x) &= x^{108} + x^{106} + x^{104} + x^{96} + x^{92} + x^{90} + x^{88} + x^{86} + x^{78} + x^{72} + x^{66} \\
&\quad + x^{60} + x^{52} + x^{48} + x^{46} + x^{42} + x^{40} + x^{38} + x^{34} + x^{30} + x^{28} + x^{26} \\
&\quad + x^{24} + x^{16} + x^{14} + x^{12}, \\
p_6(x) &= x^{108} + x^{106} + x^{100} + x^{98} + x^{92} + x^{90} + x^{82} + x^{80} + x^{72} + x^{68} + x^{66} \\
&\quad + x^{56} + x^{52} + x^{50} + x^{48} + x^{44} + x^{36} + x^{34} + x^{26} + x^{18} + x^{8} + x^{4} + x^{2} \\
&\quad + 1, \\
p_9(x) &= x^{104} + x^{102} + x^{100} + x^{98} + x^{96} + x^{92} + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} \\
&\quad + x^{76} + x^{74} + x^{70} + x^{68} + x^{66} + x^{60} + x^{58} + x^{52} + x^{48} + x^{46} + x^{40} \\
&\quad + x^{38} + x^{36} + x^{32} + x^{30} + x^{28} + x^{26} + x^{24} + x^{18} + x^{10} + x^{8} + x^{6} + x^{4}, \\
p_{12}(x) &= x^{104} + x^{102} + x^{98} + x^{96} + x^{72} + x^{70} + x^{66} + x^{64} + x^{40} + x^{38} + x^{34} \\
&\quad + x^{32} + x^{24} + x^{22} + x^{18} + x^{16} + x^{8} + x^{6} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0].

For $\phi(W^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{120} + x^{114} + x^{98} + x^{94} + x^{92} + x^{90} + x^{88} + x^{86} + x^{82} + x^{80} + x^{78} \\
&\quad + x^{74} + x^{72} + x^{64} + x^{60} + x^{58} + x^{56} + x^{54} + x^{50} + x^{46} + x^{42} + x^{40} \\
&\quad + x^{34} + x^{26} + x^{24}, \\
p_3(x) &= x^{108} + x^{106} + x^{104} + x^{100} + x^{94} + x^{86} + x^{78} + x^{76} + x^{72} + x^{70} + x^{66} \\
&\quad + x^{62} + x^{56} + x^{50} + x^{44} + x^{42} + x^{38} + x^{36} + x^{20} + x^{16}, \\
p_6(x) &= x^{108} + x^{106} + x^{98} + x^{92} + x^{86} + x^{84} + x^{82} + x^{80} + x^{76} + x^{68} + x^{60} \\
&\quad + x^{58} + x^{56} + x^{52} + x^{50} + x^{48} + x^{46} + x^{44} + x^{42} + x^{36} + x^{34} + x^{32} \\
&\quad + x^{30} + x^{28} + x^{22} + x^{18} + x^{16} + x^{12} + x^{10} + x^{4} + x^{2} + 1, \\
p_9(x) &= x^{104} + x^{102} + x^{100} + x^{98} + x^{96} + x^{92} + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} \\
&\quad + x^{76} + x^{74} + x^{70} + x^{68} + x^{66} + x^{60} + x^{58} + x^{52} + x^{48} + x^{46} + x^{40} \\
&\quad + x^{38} + x^{36} + x^{32} + x^{30} + x^{28} + x^{26} + x^{24} + x^{18} + x^{10} + x^{8} + x^{6} + x^{4}, \\
p_{12}(x) &= x^{104} + x^{102} + x^{98} + x^{96} + x^{72} + x^{70} + x^{66} + x^{64} + x^{40} + x^{38} + x^{34} \\
&\quad + x^{32} + x^{24} + x^{22} + x^{18} + x^{16} + x^{8} + x^{6} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0].

For $\phi(W^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{129} + x^{124} + x^{121} + x^{117} + x^{114} + x^{101} + x^{98} + x^{94} + x^{93} + x^{90} \\
&\quad + x^{89} + x^{88} + x^{87} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{75} + x^{74} \\
&\quad + x^{67} + x^{65} + x^{64} + x^{62} + x^{56} + x^{54} + x^{51} + x^{49} + x^{48} + x^{47} + x^{46} \\
&\quad + x^{43} + x^{42} + x^{41} + x^{37} + x^{36} + x^{35} + x^{33}, \\
p_3(x) &= x^{122} + x^{120} + x^{119} + x^{117} + x^{116} + x^{115} + x^{114} + x^{112} + x^{110} + x^{109} \\
&\quad + x^{108} + x^{107} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{98} + x^{95} + x^{94} \\
&\quad + x^{88} + x^{85} + x^{84} + x^{78} + x^{75} + x^{74} + x^{66} + x^{64} + x^{63} + x^{61} + x^{60}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{59} + x^{58} + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{46} + x^{43} + x^{40}\\
&+ x^{39} + x^{37} + x^{36} + x^{35} + x^{34} + x^{32} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26}\\
&+ x^{22} + x^{19} + x^{18},\\
p_6(x) &= x^{120} + x^{119} + x^{118} + x^{116} + x^{115} + x^{114} + x^{106} + x^{105} + x^{104} + x^{102}\\
&+ x^{101} + x^{100} + x^{98} + x^{97} + x^{95} + x^{94} + x^{88} + x^{87} + x^{85} + x^{83} + x^{81}\\
&+ x^{79} + x^{77} + x^{75} + x^{73} + x^{72} + x^{70} + x^{69} + x^{67} + x^{65} + x^{64} + x^{62}\\
&+ x^{61} + x^{59} + x^{58} + x^{54} + x^{53} + x^{52} + x^{50} + x^{49} + x^{48} + x^{44} + x^{43}\\
&+ x^{42} + x^{40} + x^{39} + x^{37} + x^{35} + x^{34} + x^{32} + x^{31} + x^{30} + x^{22} + x^{21}\\
&+ x^{20} + x^{18} + x^{17} + x^{15} + x^{13} + x^{12},\\
p_9(x) &= x^{118} + x^{115} + x^{114} + x^{112} + x^{109} + x^{106} + x^{105} + x^{104} + x^{103} + x^{100}\\
&+ x^{99} + x^{98} + x^{97} + x^{96} + x^{94} + x^{91} + x^{88} + x^{87} + x^{85} + x^{84} + x^{83}\\
&+ x^{82} + x^{80} + x^{77} + x^{76} + x^{74} + x^{72} + x^{71} + x^{69} + x^{67} + x^{65} + x^{64}\\
&+ x^{63} + x^{60} + x^{59} + x^{58} + x^{57} + x^{54} + x^{53} + x^{51} + x^{50} + x^{49} + x^{48}\\
&+ x^{46} + x^{43} + x^{40} + x^{39} + x^{38} + x^{37} + x^{33} + x^{29} + x^{28} + x^{26} + x^{23}\\
&+ x^{22} + x^{18} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{10} + x^{7} + x^{6},\\
p_{12}(x) &= x^{116} + x^{115} + x^{113} + x^{111} + x^{109} + x^{108} + x^{104} + x^{103} + x^{101} + x^{99}\\
&+ x^{97} + x^{96} + x^{84} + x^{83} + x^{81} + x^{79} + x^{77} + x^{76} + x^{72} + x^{71} + x^{69}\\
&+ x^{67} + x^{65} + x^{64} + x^{52} + x^{51} + x^{49} + x^{47} + x^{45} + x^{44} + x^{40} + x^{39}\\
&+ x^{37} + x^{36} + x^{32} + x^{31} + x^{29} + x^{28} + x^{24} + x^{23} + x^{21} + x^{20} + x^{16}\\
&+ x^{15} + x^{13} + x^{12} + x^{8} + x^{7} + x^{5} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 1, 1, 1, 0, 1, 1, 0, 0, 0, 0, 0].

For $\phi(W^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{129} + x^{124} + x^{121} + x^{117} + x^{114} + x^{101} + x^{98} + x^{94} + x^{93} + x^{90}\\
&+ x^{89} + x^{88} + x^{87} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{75} + x^{74}\\
&+ x^{67} + x^{65} + x^{64} + x^{62} + x^{56} + x^{54} + x^{51} + x^{49} + x^{48} + x^{47} + x^{46}\\
&+ x^{43} + x^{42} + x^{41} + x^{37} + x^{36} + x^{35} + x^{33},\\
p_3(x) &= x^{122} + x^{120} + x^{119} + x^{117} + x^{116} + x^{115} + x^{114} + x^{112} + x^{110} + x^{109}\\
&+ x^{108} + x^{107} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{98} + x^{95} + x^{94}\\
&+ x^{88} + x^{85} + x^{84} + x^{78} + x^{75} + x^{74} + x^{66} + x^{64} + x^{63} + x^{61} + x^{60}\\
&+ x^{59} + x^{58} + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{46} + x^{43} + x^{40}\\
&+ x^{39} + x^{37} + x^{36} + x^{35} + x^{34} + x^{32} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26}\\
&+ x^{22} + x^{19} + x^{18},\\
p_6(x) &= x^{120} + x^{119} + x^{118} + x^{116} + x^{115} + x^{114} + x^{106} + x^{105} + x^{104} + x^{102}\\
&+ x^{101} + x^{100} + x^{98} + x^{97} + x^{95} + x^{94} + x^{88} + x^{87} + x^{85} + x^{83} + x^{81}\\
&+ x^{79} + x^{77} + x^{75} + x^{73} + x^{72} + x^{70} + x^{69} + x^{67} + x^{65} + x^{64} + x^{62}
\end{aligned}
$$

$$+ x^{61} + x^{59} + x^{58} + x^{54} + x^{53} + x^{52} + x^{50} + x^{49} + x^{48} + x^{44} + x^{43} + x^{42} + x^{40} + x^{39} + x^{37} + x^{35} + x^{34} + x^{32} + x^{31} + x^{30} + x^{22} + x^{21} + x^{20} + x^{18} + x^{17} + x^{15} + x^{13} + x^{12},$$

$$p_9(x) = x^{118} + x^{115} + x^{114} + x^{112} + x^{109} + x^{106} + x^{105} + x^{104} + x^{103} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{94} + x^{91} + x^{88} + x^{87} + x^{85} + x^{84} + x^{83} + x^{82} + x^{80} + x^{77} + x^{76} + x^{74} + x^{72} + x^{71} + x^{69} + x^{67} + x^{65} + x^{64} + x^{63} + x^{60} + x^{59} + x^{58} + x^{57} + x^{54} + x^{53} + x^{51} + x^{50} + x^{49} + x^{48} + x^{46} + x^{43} + x^{40} + x^{39} + x^{38} + x^{37} + x^{33} + x^{29} + x^{28} + x^{26} + x^{23} + x^{22} + x^{18} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{10} + x^{7} + x^{6},$$

$$p_{12}(x) = x^{116} + x^{115} + x^{113} + x^{111} + x^{109} + x^{108} + x^{104} + x^{103} + x^{101} + x^{99} + x^{97} + x^{96} + x^{84} + x^{83} + x^{81} + x^{79} + x^{77} + x^{76} + x^{72} + x^{71} + x^{69} + x^{67} + x^{65} + x^{64} + x^{52} + x^{51} + x^{49} + x^{47} + x^{45} + x^{44} + x^{40} + x^{39} + x^{37} + x^{36} + x^{32} + x^{31} + x^{29} + x^{28} + x^{24} + x^{23} + x^{21} + x^{20} + x^{16} + x^{15} + x^{13} + x^{12} + x^{8} + x^{7} + x^{5} + x^{3} + x + 1,$$

and the initial terms are [0, 0, 0, 0, 0, 1, 1, 1, 0, 1, 1, 0, 0, 0, 0, 0].

For $\phi(W^e, 1, 1)$:

$$p_0(x) = x^{138} + x^{132} + x^{128} + x^{126} + x^{120} + x^{116} + x^{112} + x^{110} + x^{98} + x^{96} + x^{94} + x^{92} + x^{90} + x^{86} + x^{82} + x^{76} + x^{74} + x^{70} + x^{68} + x^{66} + x^{64} + x^{62} + x^{58} + x^{52} + x^{48} + x^{44} + x^{42},$$

$$p_3(x) = x^{130} + x^{126} + x^{124} + x^{122} + x^{118} + x^{112} + x^{106} + x^{102} + x^{88} + x^{86} + x^{82} + x^{78} + x^{74} + x^{70} + x^{66} + x^{64} + x^{62} + x^{58} + x^{54} + x^{52} + x^{48} + x^{46} + x^{42} + x^{40} + x^{38} + x^{36} + x^{34} + x^{32} + x^{28} + x^{26} + x^{24} + x^{20} + x^{16} + x^{12},$$

$$p_6(x) = x^{130} + x^{124} + x^{122} + x^{120} + x^{114} + x^{108} + x^{104} + x^{98} + x^{96} + x^{94} + x^{90} + x^{88} + x^{76} + x^{74} + x^{72} + x^{58} + x^{56} + x^{54} + x^{50} + x^{48} + x^{44} + x^{38} + x^{30} + x^{26} + x^{24} + x^{12} + x^{10} + x^{8} + x^{4} + 1,$$

$$p_9(x) = x^{128} + x^{124} + x^{122} + x^{114} + x^{112} + x^{108} + x^{106} + x^{102} + x^{96} + x^{90} + x^{84} + x^{80} + x^{78} + x^{72} + x^{68} + x^{64} + x^{62} + x^{60} + x^{52} + x^{50} + x^{46} + x^{44} + x^{32} + x^{28} + x^{22} + x^{14} + x^{12} + x^{10} + x^{8} + x^{4},$$

$$p_{12}(x) = x^{128} + x^{120} + x^{104} + x^{88} + x^{72} + x^{56} + x^{48} + x^{16} + x^{8} + 1,$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0].

For $\phi(M^o, 0, 0)$:

$$p_0(x) = x^{129} + x^{126} + x^{119} + x^{116} + x^{115} + x^{114} + x^{113} + x^{110} + x^{107} + x^{106} + x^{102} + x^{101} + x^{100} + x^{97} + x^{96} + x^{92} + x^{88} + x^{85} + x^{83} + x^{82} + x^{80} + x^{79} + x^{77} + x^{75} + x^{74} + x^{71} + x^{70} + x^{67} + x^{65} + x^{64} + x^{63} + x^{62}$$

$$
\begin{aligned}
&+ x^{57} + x^{54} + x^{53} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43}\\
&+ x^{42} + x^{41} + x^{32} + x^{30} + x^{28} + x^{27} + x^{25} + x^{24},\\
p_3(x) &= x^{124} + x^{118} + x^{113} + x^{112} + x^{110} + x^{108} + x^{106} + x^{105} + x^{104} + x^{103}\\
&+ x^{102} + x^{100} + x^{97} + x^{96} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{85}\\
&+ x^{83} + x^{82} + x^{80} + x^{77} + x^{76} + x^{74} + x^{73} + x^{71} + x^{69} + x^{64} + x^{63}\\
&+ x^{62} + x^{61} + x^{60} + x^{58} + x^{56} + x^{55} + x^{54} + x^{53} + x^{51} + x^{50} + x^{48}\\
&+ x^{46} + x^{43} + x^{42} + x^{38} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{29} + x^{28}\\
&+ x^{24} + x^{22} + x^{20} + x^{19} + x^{17} + x^{16},\\
p_6(x) &= x^{124} + x^{118} + x^{115} + x^{111} + x^{109} + x^{104} + x^{99} + x^{97} + x^{96} + x^{95}\\
&+ x^{94} + x^{93} + x^{92} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83}\\
&+ x^{82} + x^{81} + x^{78} + x^{76} + x^{75} + x^{74} + x^{73} + x^{71} + x^{70} + x^{68} + x^{67}\\
&+ x^{66} + x^{65} + x^{58} + x^{55} + x^{54} + x^{52} + x^{51} + x^{49} + x^{44} + x^{41} + x^{40}\\
&+ x^{37} + x^{35} + x^{34} + x^{33} + x^{29} + x^{24} + x^{22} + x^{21} + x^{19} + x^{17} + x^{16}\\
&+ x^{15} + x^{14} + x^{13} + x^{11} + x^{9} + x^{7} + x^{5} + x^{3} + x + 1,\\
p_9(x) &= x^{124} + x^{123} + x^{121} + x^{120} + x^{112} + x^{111} + x^{109} + x^{107} + x^{105} + x^{104}\\
&+ x^{92} + x^{91} + x^{89} + x^{88} + x^{80} + x^{79} + x^{77} + x^{75} + x^{73} + x^{72} + x^{60}\\
&+ x^{59} + x^{57} + x^{56} + x^{48} + x^{47} + x^{45} + x^{44} + x^{32} + x^{31} + x^{29} + x^{28}\\
&+ x^{16} + x^{15} + x^{13} + x^{11} + x^{9} + x^{8},\\
p_{12}(x) &= x^{124} + x^{123} + x^{121} + x^{120} + x^{116} + x^{115} + x^{113} + x^{112} + x^{108} + x^{107}\\
&+ x^{105} + x^{103} + x^{101} + x^{99} + x^{97} + x^{96} + x^{92} + x^{91} + x^{89} + x^{88} + x^{84}\\
&+ x^{83} + x^{81} + x^{80} + x^{76} + x^{75} + x^{73} + x^{71} + x^{69} + x^{67} + x^{65} + x^{64}\\
&+ x^{60} + x^{59} + x^{57} + x^{56} + x^{52} + x^{51} + x^{49} + x^{48} + x^{40} + x^{39} + x^{37}\\
&+ x^{36} + x^{24} + x^{23} + x^{21} + x^{20} + x^{12} + x^{11} + x^{9} + x^{7} + x^{5} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 1, 1, 1, 1, 1, 0, 0, 1, 1].

For $\phi(M^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{128} + x^{126} + x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{112} + x^{108} + x^{105}\\
&+ x^{103} + x^{101} + x^{99} + x^{98} + x^{95} + x^{94} + x^{92} + x^{91} + x^{89} + x^{83} + x^{82}\\
&+ x^{81} + x^{79} + x^{76} + x^{72} + x^{71} + x^{67} + x^{64} + x^{60} + x^{59} + x^{56} + x^{55}\\
&+ x^{54} + x^{51} + x^{49} + x^{48} + x^{47} + x^{46} + x^{40} + x^{38} + x^{37} + x^{36} + x^{35}\\
&+ x^{33},\\
p_3(x) &= x^{114} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} + x^{106} + x^{105} + x^{103}\\
&+ x^{101} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{88} + x^{86} + x^{84} + x^{81}\\
&+ x^{79} + x^{77} + x^{74} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{50}\\
&+ x^{49} + x^{47} + x^{45} + x^{42} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28}\\
&+ x^{26} + x^{25} + x^{23} + x^{21} + x^{18},
\end{aligned}
$$

$$\begin{aligned}
p_6(x) &= x^{116} + x^{114} + x^{102} + x^{100} + x^{94} + x^{84} + x^{80} + x^{76} + x^{72} + x^{66} + x^{62} \\
&\quad + x^{60} + x^{56} + x^{54} + x^{52} + x^{46} + x^{44} + x^{42} + x^{36} + x^{32} + x^{30} + x^{18} \\
&\quad + x^{16} + x^{12}, \\
p_9(x) &= x^{118} + x^{117} + x^{114} + x^{113} + x^{111} + x^{109} + x^{108} + x^{107} + x^{105} + x^{102} \\
&\quad + x^{86} + x^{85} + x^{82} + x^{81} + x^{79} + x^{77} + x^{76} + x^{75} + x^{73} + x^{70} + x^{54} \\
&\quad + x^{53} + x^{50} + x^{49} + x^{47} + x^{45} + x^{44} + x^{43} + x^{41} + x^{37} + x^{34} + x^{33} \\
&\quad + x^{31} + x^{29} + x^{28} + x^{27} + x^{25} + x^{21} + x^{18} + x^{17} + x^{15} + x^{13} + x^{12} \\
&\quad + x^{11} + x^{9} + x^{6}, \\
p_{12}(x) &= x^{120} + x^{112} + x^{104} + x^{100} + x^{96} + x^{88} + x^{80} + x^{72} + x^{68} + x^{64} + x^{56} \\
&\quad + x^{48} + x^{36} + x^{20} + x^{8} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0].

For $\phi(M^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{138} + x^{135} + x^{133} + x^{132} + x^{129} + x^{128} + x^{125} + x^{124} + x^{123} + x^{120} \\
&\quad + x^{119} + x^{117} + x^{114} + x^{110} + x^{109} + x^{106} + x^{104} + x^{102} + x^{98} + x^{97} \\
&\quad + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{89} + x^{84} + x^{83} + x^{81} + x^{76} \\
&\quad + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{67} + x^{64} + x^{63} + x^{61} \\
&\quad + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} \\
&\quad + x^{44} + x^{43} + x^{42} + x^{37} + x^{34} + x^{33}, \\
p_3(x) &= x^{130} + x^{129} + x^{128} + x^{127} + x^{123} + x^{120} + x^{119} + x^{117} + x^{116} + x^{112} \\
&\quad + x^{110} + x^{108} + x^{105} + x^{103} + x^{102} + x^{100} + x^{99} + x^{97} + x^{96} + x^{94} \\
&\quad + x^{93} + x^{91} + x^{89} + x^{85} + x^{84} + x^{83} + x^{79} + x^{78} + x^{77} + x^{73} + x^{72} \\
&\quad + x^{71} + x^{67} + x^{64} + x^{63} + x^{61} + x^{60} + x^{58} + x^{57} + x^{55} + x^{54} + x^{52} \\
&\quad + x^{51} + x^{49} + x^{48} + x^{46} + x^{45} + x^{43} + x^{42} + x^{40} + x^{39} + x^{37} + x^{36} \\
&\quad + x^{34} + x^{33} + x^{31} + x^{30} + x^{28} + x^{27} + x^{26} + x^{23} + x^{22} + x^{21} + x^{18}, \\
p_6(x) &= x^{132} + x^{130} + x^{128} + x^{126} + x^{120} + x^{112} + x^{110} + x^{104} + x^{102} + x^{98} \\
&\quad + x^{94} + x^{88} + x^{84} + x^{82} + x^{76} + x^{74} + x^{72} + x^{66} + x^{60} + x^{58} + x^{56} \\
&\quad + x^{50} + x^{48} + x^{46} + x^{40} + x^{24} + x^{22} + x^{20} + x^{18} + x^{12}, \\
p_9(x) &= x^{134} + x^{133} + x^{127} + x^{126} + x^{124} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} \\
&\quad + x^{117} + x^{115} + x^{112} + x^{108} + x^{107} + x^{106} + x^{105} + x^{101} + x^{95} + x^{94} \\
&\quad + x^{92} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{83} + x^{80} + x^{76} + x^{75} \\
&\quad + x^{74} + x^{73} + x^{69} + x^{63} + x^{62} + x^{60} + x^{58} + x^{57} + x^{56} + x^{55} + x^{51} \\
&\quad + x^{48} + x^{47} + x^{46} + x^{43} + x^{40} + x^{39} + x^{38} + x^{35} + x^{32} + x^{31} + x^{30} \\
&\quad + x^{27} + x^{24} + x^{23} + x^{21} + x^{19} + x^{16} + x^{12} + x^{11} + x^{10} + x^{9} + x^{6}, \\
p_{12}(x) &= x^{136} + x^{132} + x^{128} + x^{116} + x^{112} + x^{104} + x^{100} + x^{84} + x^{80} + x^{72} \\
&\quad + x^{68} + x^{56} + x^{40} + x^{32} + x^{20} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 1, 1, 1, 0, 1, 1, 0, 1, 1, 0, 0].

For $\phi(M^o, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{129} + x^{125} + x^{120} + x^{115} + x^{113} + x^{111} + x^{108} + x^{103} + x^{102} + x^{101} \\
&\quad + x^{94} + x^{93} + x^{92} + x^{90} + x^{86} + x^{84} + x^{83} + x^{82} + x^{75} + x^{74} + x^{73} \\
&\quad + x^{71} + x^{70} + x^{69} + x^{68} + x^{65} + x^{64} + x^{62} + x^{60} + x^{59} + x^{58} + x^{55} \\
&\quad + x^{54} + x^{52} + x^{50} + x^{47} + x^{42}, \\
p_3(x) &= x^{124} + x^{120} + x^{119} + x^{114} + x^{109} + x^{104} + x^{99} + x^{98} + x^{97} + x^{95} \\
&\quad + x^{93} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{80} + x^{77} + x^{75} + x^{74} \\
&\quad + x^{72} + x^{71} + x^{69} + x^{61} + x^{60} + x^{57} + x^{55} + x^{52} + x^{51} + x^{49} + x^{46} \\
&\quad + x^{44} + x^{42} + x^{41} + x^{39} + x^{36} + x^{35} + x^{30} + x^{26} + x^{25} + x^{22}, \\
p_6(x) &= x^{124} + x^{120} + x^{119} + x^{115} + x^{114} + x^{113} + x^{107} + x^{106} + x^{103} + x^{102} \\
&\quad + x^{100} + x^{98} + x^{97} + x^{93} + x^{91} + x^{90} + x^{87} + x^{86} + x^{85} + x^{82} + x^{79} \\
&\quad + x^{78} + x^{77} + x^{76} + x^{74} + x^{72} + x^{71} + x^{69} + x^{67} + x^{66} + x^{65} + x^{62} \\
&\quad + x^{61} + x^{59} + x^{55} + x^{53} + x^{52} + x^{50} + x^{49} + x^{43} + x^{42} + x^{41} + x^{39} \\
&\quad + x^{36} + x^{35} + x^{32} + x^{30} + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{18} \\
&\quad + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{8} + x^{7} + x^{5} + x^{3} + x + 1, \\
p_9(x) &= x^{124} + x^{123} + x^{121} + x^{120} + x^{112} + x^{111} + x^{109} + x^{107} + x^{105} + x^{104} \\
&\quad + x^{92} + x^{91} + x^{89} + x^{88} + x^{80} + x^{79} + x^{77} + x^{75} + x^{73} + x^{72} + x^{60} \\
&\quad + x^{59} + x^{57} + x^{56} + x^{48} + x^{47} + x^{45} + x^{44} + x^{32} + x^{31} + x^{29} + x^{28} \\
&\quad + x^{16} + x^{15} + x^{13} + x^{11} + x^{9} + x^{8}, \\
p_{12}(x) &= x^{124} + x^{123} + x^{121} + x^{120} + x^{116} + x^{115} + x^{113} + x^{112} + x^{108} + x^{107} \\
&\quad + x^{105} + x^{103} + x^{101} + x^{99} + x^{97} + x^{96} + x^{92} + x^{91} + x^{89} + x^{88} + x^{84} \\
&\quad + x^{83} + x^{81} + x^{80} + x^{76} + x^{75} + x^{73} + x^{71} + x^{69} + x^{67} + x^{65} + x^{64} \\
&\quad + x^{60} + x^{59} + x^{57} + x^{56} + x^{52} + x^{51} + x^{49} + x^{48} + x^{40} + x^{39} + x^{37} \\
&\quad + x^{36} + x^{24} + x^{23} + x^{21} + x^{20} + x^{12} + x^{11} + x^{9} + x^{7} + x^{5} + x^{3} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 1, 1, 0, 0, 1].

For $\phi(W^o, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{129} + x^{126} + x^{119} + x^{116} + x^{115} + x^{114} + x^{113} + x^{110} + x^{107} + x^{106} \\
&\quad + x^{102} + x^{101} + x^{100} + x^{97} + x^{96} + x^{92} + x^{88} + x^{85} + x^{83} + x^{82} + x^{80} \\
&\quad + x^{79} + x^{77} + x^{75} + x^{74} + x^{71} + x^{70} + x^{67} + x^{65} + x^{64} + x^{63} + x^{62} \\
&\quad + x^{57} + x^{54} + x^{53} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} \\
&\quad + x^{42} + x^{41} + x^{32} + x^{30} + x^{28} + x^{27} + x^{25} + x^{24}, \\
p_3(x) &= x^{124} + x^{118} + x^{113} + x^{112} + x^{110} + x^{108} + x^{106} + x^{105} + x^{104} + x^{103} \\
&\quad + x^{102} + x^{100} + x^{97} + x^{96} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{85} \\
&\quad + x^{83} + x^{82} + x^{80} + x^{77} + x^{76} + x^{74} + x^{73} + x^{71} + x^{69} + x^{64} + x^{63}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{62} + x^{61} + x^{60} + x^{58} + x^{56} + x^{55} + x^{54} + x^{53} + x^{51} + x^{50} + x^{48} \\
&+ x^{46} + x^{43} + x^{42} + x^{38} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{29} + x^{28} \\
&+ x^{24} + x^{22} + x^{20} + x^{19} + x^{17} + x^{16}, \\
p_6(x) &= x^{124} + x^{118} + x^{115} + x^{111} + x^{109} + x^{104} + x^{99} + x^{97} + x^{96} + x^{95} \\
&+ x^{94} + x^{93} + x^{92} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} \\
&+ x^{82} + x^{81} + x^{78} + x^{76} + x^{75} + x^{74} + x^{73} + x^{71} + x^{70} + x^{68} + x^{67} \\
&+ x^{66} + x^{65} + x^{58} + x^{55} + x^{54} + x^{52} + x^{51} + x^{49} + x^{44} + x^{41} + x^{40} \\
&+ x^{37} + x^{35} + x^{34} + x^{33} + x^{29} + x^{24} + x^{22} + x^{21} + x^{19} + x^{17} + x^{16} \\
&+ x^{15} + x^{14} + x^{13} + x^{11} + x^{9} + x^{7} + x^{5} + x^{3} + x + 1, \\
p_9(x) &= x^{124} + x^{123} + x^{121} + x^{120} + x^{112} + x^{111} + x^{109} + x^{107} + x^{105} + x^{104} \\
&+ x^{92} + x^{91} + x^{89} + x^{88} + x^{80} + x^{79} + x^{77} + x^{75} + x^{73} + x^{72} + x^{60} \\
&+ x^{59} + x^{57} + x^{56} + x^{48} + x^{47} + x^{45} + x^{44} + x^{32} + x^{31} + x^{29} + x^{28} \\
&+ x^{16} + x^{15} + x^{13} + x^{11} + x^{9} + x^{8}, \\
p_{12}(x) &= x^{124} + x^{123} + x^{121} + x^{120} + x^{116} + x^{115} + x^{113} + x^{112} + x^{108} + x^{107} \\
&+ x^{105} + x^{103} + x^{101} + x^{99} + x^{97} + x^{96} + x^{92} + x^{91} + x^{89} + x^{88} + x^{84} \\
&+ x^{83} + x^{81} + x^{80} + x^{76} + x^{75} + x^{73} + x^{71} + x^{69} + x^{67} + x^{65} + x^{64} \\
&+ x^{60} + x^{59} + x^{57} + x^{56} + x^{52} + x^{51} + x^{49} + x^{48} + x^{40} + x^{39} + x^{37} \\
&+ x^{36} + x^{24} + x^{23} + x^{21} + x^{20} + x^{12} + x^{11} + x^{9} + x^{7} + x^{5} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 1, 1, 1, 1, 1, 0, 0, 1, 1].

For $\phi(W^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{138} + x^{135} + x^{133} + x^{132} + x^{129} + x^{128} + x^{125} + x^{124} + x^{123} + x^{120} \\
&+ x^{119} + x^{117} + x^{114} + x^{110} + x^{109} + x^{106} + x^{104} + x^{102} + x^{98} + x^{97} \\
&+ x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{89} + x^{84} + x^{83} + x^{81} + x^{76} \\
&+ x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{67} + x^{64} + x^{63} + x^{61} \\
&+ x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} \\
&+ x^{44} + x^{43} + x^{42} + x^{37} + x^{34} + x^{33}, \\
p_3(x) &= x^{130} + x^{129} + x^{128} + x^{127} + x^{123} + x^{120} + x^{119} + x^{117} + x^{116} + x^{112} \\
&+ x^{110} + x^{108} + x^{105} + x^{103} + x^{102} + x^{100} + x^{99} + x^{97} + x^{96} + x^{94} \\
&+ x^{93} + x^{91} + x^{89} + x^{85} + x^{84} + x^{83} + x^{79} + x^{78} + x^{77} + x^{73} + x^{72} \\
&+ x^{71} + x^{67} + x^{64} + x^{63} + x^{61} + x^{60} + x^{58} + x^{57} + x^{55} + x^{54} + x^{52} \\
&+ x^{51} + x^{49} + x^{48} + x^{46} + x^{45} + x^{43} + x^{42} + x^{40} + x^{39} + x^{37} + x^{36} \\
&+ x^{34} + x^{33} + x^{31} + x^{30} + x^{28} + x^{27} + x^{26} + x^{23} + x^{22} + x^{21} + x^{18}, \\
p_6(x) &= x^{132} + x^{130} + x^{128} + x^{126} + x^{120} + x^{112} + x^{110} + x^{104} + x^{102} + x^{98} \\
&+ x^{94} + x^{88} + x^{84} + x^{82} + x^{76} + x^{74} + x^{72} + x^{66} + x^{60} + x^{58} + x^{56} \\
&+ x^{50} + x^{48} + x^{46} + x^{40} + x^{24} + x^{22} + x^{20} + x^{18} + x^{12},
\end{aligned}
$$

$$\begin{aligned}
p_9(x) &= x^{134} + x^{133} + x^{127} + x^{126} + x^{124} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} \\
&+ x^{117} + x^{115} + x^{112} + x^{108} + x^{107} + x^{106} + x^{105} + x^{101} + x^{95} + x^{94} \\
&+ x^{92} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{83} + x^{80} + x^{76} + x^{75} \\
&+ x^{74} + x^{73} + x^{69} + x^{63} + x^{62} + x^{60} + x^{58} + x^{57} + x^{56} + x^{55} + x^{51} \\
&+ x^{48} + x^{47} + x^{46} + x^{43} + x^{40} + x^{39} + x^{38} + x^{35} + x^{32} + x^{31} + x^{30} \\
&+ x^{27} + x^{24} + x^{23} + x^{21} + x^{19} + x^{16} + x^{12} + x^{11} + x^{10} + x^{9} + x^{6}, \\
p_{12}(x) &= x^{136} + x^{132} + x^{128} + x^{116} + x^{112} + x^{104} + x^{100} + x^{84} + x^{80} + x^{72} \\
&+ x^{68} + x^{56} + x^{40} + x^{32} + x^{20} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 1, 1, 1, 0, 1, 1, 0, 1, 1, 0, 0].

For $\phi(W^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{128} + x^{126} + x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{112} + x^{108} + x^{105} \\
&+ x^{103} + x^{101} + x^{99} + x^{98} + x^{95} + x^{94} + x^{92} + x^{91} + x^{89} + x^{83} + x^{82} \\
&+ x^{81} + x^{79} + x^{76} + x^{72} + x^{71} + x^{67} + x^{64} + x^{60} + x^{59} + x^{56} + x^{55} \\
&+ x^{54} + x^{51} + x^{49} + x^{48} + x^{47} + x^{46} + x^{40} + x^{38} + x^{37} + x^{36} + x^{35} \\
&+ x^{33}, \\
p_3(x) &= x^{114} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} + x^{106} + x^{105} + x^{103} \\
&+ x^{101} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{88} + x^{86} + x^{84} + x^{81} \\
&+ x^{79} + x^{77} + x^{74} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{50} \\
&+ x^{49} + x^{47} + x^{45} + x^{42} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} \\
&+ x^{26} + x^{25} + x^{23} + x^{21} + x^{18}, \\
p_6(x) &= x^{116} + x^{114} + x^{102} + x^{100} + x^{94} + x^{84} + x^{80} + x^{76} + x^{72} + x^{66} + x^{62} \\
&+ x^{60} + x^{56} + x^{54} + x^{52} + x^{46} + x^{44} + x^{42} + x^{36} + x^{32} + x^{30} + x^{18} \\
&+ x^{16} + x^{12}, \\
p_9(x) &= x^{118} + x^{117} + x^{114} + x^{113} + x^{111} + x^{109} + x^{108} + x^{107} + x^{105} + x^{102} \\
&+ x^{86} + x^{85} + x^{82} + x^{81} + x^{79} + x^{77} + x^{76} + x^{75} + x^{73} + x^{70} + x^{54} \\
&+ x^{53} + x^{50} + x^{49} + x^{47} + x^{45} + x^{44} + x^{43} + x^{41} + x^{37} + x^{34} + x^{33} \\
&+ x^{31} + x^{29} + x^{28} + x^{27} + x^{25} + x^{21} + x^{18} + x^{17} + x^{15} + x^{13} + x^{12} \\
&+ x^{11} + x^{9} + x^{6}, \\
p_{12}(x) &= x^{120} + x^{112} + x^{104} + x^{100} + x^{96} + x^{88} + x^{80} + x^{72} + x^{68} + x^{64} + x^{56} \\
&+ x^{48} + x^{36} + x^{20} + x^{8} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0].

For $\phi(W^o, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{129} + x^{125} + x^{120} + x^{115} + x^{113} + x^{111} + x^{108} + x^{103} + x^{102} + x^{101} \\
&+ x^{94} + x^{93} + x^{92} + x^{90} + x^{86} + x^{84} + x^{83} + x^{82} + x^{75} + x^{74} + x^{73} \\
&+ x^{71} + x^{70} + x^{69} + x^{68} + x^{65} + x^{64} + x^{62} + x^{60} + x^{59} + x^{58} + x^{55}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{54} + x^{52} + x^{50} + x^{47} + x^{42},\\
p_3(x) &= x^{124} + x^{120} + x^{119} + x^{114} + x^{109} + x^{104} + x^{99} + x^{98} + x^{97} + x^{95}\\
&+ x^{93} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{80} + x^{77} + x^{75} + x^{74}\\
&+ x^{72} + x^{71} + x^{69} + x^{61} + x^{60} + x^{57} + x^{55} + x^{52} + x^{51} + x^{49} + x^{46}\\
&+ x^{44} + x^{42} + x^{41} + x^{39} + x^{36} + x^{35} + x^{30} + x^{26} + x^{25} + x^{22},\\
p_6(x) &= x^{124} + x^{120} + x^{119} + x^{115} + x^{114} + x^{113} + x^{107} + x^{106} + x^{103} + x^{102}\\
&+ x^{100} + x^{98} + x^{97} + x^{93} + x^{91} + x^{90} + x^{87} + x^{86} + x^{85} + x^{82} + x^{79}\\
&+ x^{78} + x^{77} + x^{76} + x^{74} + x^{72} + x^{71} + x^{69} + x^{67} + x^{66} + x^{65} + x^{62}\\
&+ x^{61} + x^{59} + x^{55} + x^{53} + x^{52} + x^{50} + x^{49} + x^{43} + x^{42} + x^{41} + x^{39}\\
&+ x^{36} + x^{35} + x^{32} + x^{30} + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{18}\\
&+ x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{8} + x^{7} + x^{5} + x^{3} + x + 1,\\
p_9(x) &= x^{124} + x^{123} + x^{121} + x^{120} + x^{112} + x^{111} + x^{109} + x^{107} + x^{105} + x^{104}\\
&+ x^{92} + x^{91} + x^{89} + x^{88} + x^{80} + x^{79} + x^{77} + x^{75} + x^{73} + x^{72} + x^{60}\\
&+ x^{59} + x^{57} + x^{56} + x^{48} + x^{47} + x^{45} + x^{44} + x^{32} + x^{31} + x^{29} + x^{28}\\
&+ x^{16} + x^{15} + x^{13} + x^{11} + x^{9} + x^{8},\\
p_{12}(x) &= x^{124} + x^{123} + x^{121} + x^{120} + x^{116} + x^{115} + x^{113} + x^{112} + x^{108} + x^{107}\\
&+ x^{105} + x^{103} + x^{101} + x^{99} + x^{97} + x^{96} + x^{92} + x^{91} + x^{89} + x^{88} + x^{84}\\
&+ x^{83} + x^{81} + x^{80} + x^{76} + x^{75} + x^{73} + x^{71} + x^{69} + x^{67} + x^{65} + x^{64}\\
&+ x^{60} + x^{59} + x^{57} + x^{56} + x^{52} + x^{51} + x^{49} + x^{48} + x^{40} + x^{39} + x^{37}\\
&+ x^{36} + x^{24} + x^{23} + x^{21} + x^{20} + x^{12} + x^{11} + x^{9} + x^{7} + x^{5} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 1, 1, 0, 0, 1].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	153	[267, 268]	306	[190, 453]	459	[227, 610]	612	[455, 669]
1	[3, 4]	154	[269, 270]	307	[423, 395]	460	[611, 612]	613	[140, 48]
2	[5, 6]	155	[4, 271]	308	[204, 454]	461	[613, 259]	614	[475, 219]
3	[7, 8]	156	[272, 273]	309	[36, 411]	462	[281, 98]	615	[56, 197]
4	[9, 10]	157	[274, 275]	310	[455, 456]	463	[614, 109]	616	[686, 440]
5	[11, 12]	158	[276, 277]	311	[457, 219]	464	[28, 615]	617	[392, 460]
6	[13, 14]	159	[278, 279]	312	[128, 205]	465	[616, 313]	618	[656, 478]
7	[15, 16]	160	[280, 281]	313	[458, 459]	466	[617, 618]	619	[687, 34]
8	[17, 18]	161	[282, 283]	314	[440, 460]	467	[619, 316]	620	[457, 218]
9	[19, 20]	162	[284, 285]	315	[461, 148]	468	[620, 621]	621	[204, 676]
10	[21, 22]	163	[100, 286]	316	[49, 424]	469	[622, 623]	622	[458, 661]
11	[23, 24]	164	[287, 288]	317	[130, 385]	470	[624, 625]	623	[180, 146]
12	[25, 26]	165	[289, 290]	318	[462, 153]	471	[518, 77]	624	[391, 399]

13	[27, 28]	166	[291, 292]	319	[463, 410]	472	[626, 625]	625	[144, 664]
14	[29, 30]	167	[293, 294]	320	[12, 442]	473	[627, 628]	626	[41, 486]
15	[31, 32]	168	[295, 296]	321	[14, 464]	474	[629, 630]	627	[688, 189]
16	[33, 34]	169	[297, 298]	322	[194, 167]	475	[631, 632]	628	[463, 183]
17	[35, 36]	170	[299, 300]	323	[465, 418]	476	[633, 634]	629	[448, 681]
18	[37, 38]	171	[301, 302]	324	[389, 466]	477	[277, 635]	630	[420, 662]
19	[39, 40]	172	[303, 304]	325	[210, 396]	478	[636, 64]	631	[689, 177]
20	[41, 42]	173	[305, 306]	326	[467, 387]	479	[637, 512]	632	[136, 172]
21	[43, 44]	174	[307, 308]	327	[49, 455]	480	[638, 558]	633	[59, 417]
22	[45, 46]	175	[18, 309]	328	[206, 58]	481	[639, 640]	634	[135, 57]
23	[47, 48]	176	[66, 310]	329	[401, 468]	482	[641, 547]	635	[419, 428]
24	[49, 50]	177	[311, 312]	330	[215, 469]	483	[642, 643]	636	[199, 141]
25	[51, 52]	178	[313, 314]	331	[33, 470]	484	[68, 644]	637	[690, 189]
26	[53, 54]	179	[315, 75]	332	[210, 220]	485	[645, 646]	638	[691, 478]
27	[55, 56]	180	[316, 317]	333	[201, 410]	486	[647, 648]	639	[665, 683]
28	[57, 58]	181	[318, 319]	334	[197, 459]	487	[649, 650]	640	[422, 692]
29	[59, 60]	182	[320, 321]	335	[33, 204]	488	[651, 652]	641	[461, 407]
30	[61, 62]	183	[322, 323]	336	[385, 471]	489	[653, 115]	642	[138, 434]
31	[63, 64]	184	[324, 325]	337	[472, 208]	490	[654, 655]	643	[404, 393]
32	[65, 66]	185	[326, 299]	338	[213, 387]	491	[32, 127]	644	[135, 692]
33	[67, 68]	186	[327, 225]	339	[39, 60]	492	[43, 421]	645	[57, 207]
34	[69, 70]	187	[319, 328]	340	[437, 169]	493	[656, 145]	646	[173, 399]
35	[71, 72]	188	[329, 330]	341	[473, 60]	494	[32, 657]	647	[9, 169]
36	[73, 74]	189	[331, 332]	342	[474, 438]	495	[34, 484]	648	[668, 195]
37	[75, 76]	190	[333, 334]	343	[458, 198]	496	[658, 429]	649	[693, 429]
38	[77, 78]	191	[335, 336]	344	[422, 206]	497	[136, 436]	650	[479, 182]
39	[79, 80]	192	[337, 338]	345	[475, 210]	498	[420, 394]	651	[131, 471]
40	[81, 82]	193	[339, 340]	346	[423, 424]	499	[8, 177]	652	[219, 694]
41	[83, 84]	194	[341, 342]	347	[36, 153]	500	[446, 486]	653	[695, 148]
42	[85, 86]	195	[343, 344]	348	[385, 386]	501	[185, 429]	654	[401, 411]
43	[87, 88]	196	[345, 346]	349	[424, 476]	502	[659, 421]	655	[14, 696]
44	[89, 90]	197	[112, 347]	350	[477, 478]	503	[465, 660]	656	[697, 343]
45	[91, 92]	198	[348, 349]	351	[479, 188]	504	[197, 661]	657	[64, 261]
46	[93, 94]	199	[350, 351]	352	[128, 130]	505	[429, 197]	658	[698, 274]
47	[95, 96]	200	[352, 228]	353	[480, 424]	506	[659, 662]	659	[699, 700]
48	[97, 98]	201	[353, 267]	354	[141, 481]	507	[14, 663]	660	[346, 223]
49	[99, 100]	202	[229, 354]	355	[482, 177]	508	[183, 153]	661	[565, 701]
50	[101, 102]	203	[355, 356]	356	[479, 483]	509	[656, 664]	662	[702, 703]
51	[103, 104]	204	[357, 358]	357	[12, 460]	510	[665, 666]	663	[251, 704]
52	[105, 106]	205	[110, 222]	358	[465, 177]	511	[180, 667]	664	[705, 651]
53	[107, 108]	206	[334, 359]	359	[410, 153]	512	[154, 173]	665	[706, 707]
54	[109, 110]	207	[336, 360]	360	[204, 484]	513	[37, 147]	666	[708, 709]
55	[111, 112]	208	[361, 362]	361	[199, 426]	514	[433, 149]	667	[710, 711]
56	[113, 114]	209	[233, 363]	362	[455, 425]	515	[668, 464]	668	[712, 713]

57	[115, 116]	210	[364, 365]	363	[129, 150]	516	[424, 669]	669	[650, 714]
58	[117, 118]	211	[366, 367]	364	[391, 189]	517	[389, 141]	670	[715, 91]
59	[119, 120]	212	[368, 245]	365	[53, 186]	518	[385, 129]	671	[513, 716]
60	[121, 122]	213	[369, 370]	366	[171, 137]	519	[168, 413]	672	[717, 382]
61	[123, 124]	214	[371, 372]	367	[418, 485]	520	[53, 429]	673	[718, 719]
62	[125, 126]	215	[373, 374]	368	[166, 486]	521	[11, 61]	674	[720, 28]
63	[127, 128]	216	[375, 376]	369	[487, 424]	522	[146, 38]	675	[292, 70]
64	[129, 130]	217	[377, 378]	370	[457, 414]	523	[446, 42]	676	[721, 722]
65	[32, 131]	218	[379, 380]	371	[199, 466]	524	[443, 170]	677	[88, 102]
66	[132, 133]	219	[381, 382]	372	[34, 454]	525	[665, 159]	678	[96, 723]
67	[134, 135]	220	[383, 384]	373	[416, 40]	526	[440, 214]	679	[84, 247]
68	[136, 137]	221	[200, 132]	374	[173, 189]	527	[206, 441]	680	[724, 725]
69	[138, 139]	222	[129, 32]	375	[37, 488]	528	[135, 206]	681	[726, 727]
70	[53, 56]	223	[200, 200]	376	[56, 430]	529	[670, 204]	682	[728, 380]
71	[140, 141]	224	[385, 128]	377	[489, 422]	530	[446, 167]	683	[729, 730]
72	[33, 142]	225	[131, 129]	378	[181, 483]	531	[656, 180]	684	[120, 645]
73	[51, 143]	226	[131, 386]	379	[138, 208]	532	[455, 476]	685	[114, 731]
74	[144, 145]	227	[200, 133]	380	[144, 478]	533	[48, 481]	686	[732, 611]
75	[146, 147]	228	[131, 128]	381	[401, 153]	534	[659, 394]	687	[733, 734]
76	[148, 149]	229	[130, 127]	382	[465, 400]	535	[402, 453]	688	[104, 541]
77	[128, 150]	230	[154, 387]	383	[416, 490]	536	[141, 415]	689	[735, 504]
78	[48, 151]	231	[175, 388]	384	[56, 458]	537	[177, 419]	690	[283, 365]
79	[152, 153]	232	[389, 48]	385	[317, 348]	538	[161, 438]	691	[338, 710]
80	[154, 155]	233	[32, 385]	386	[262, 491]	539	[668, 663]	692	[231, 736]
81	[138, 149]	234	[390, 62]	387	[492, 493]	540	[12, 62]	693	[737, 608]
82	[156, 157]	235	[213, 391]	388	[494, 495]	541	[402, 422]	694	[378, 738]
83	[158, 159]	236	[392, 214]	389	[496, 497]	542	[56, 671]	695	[739, 361]
84	[160, 10]	237	[45, 393]	390	[498, 499]	543	[672, 399]	696	[635, 740]
85	[161, 162]	238	[43, 394]	391	[500, 501]	544	[432, 410]	697	[741, 173]
86	[163, 164]	239	[185, 54]	392	[502, 503]	545	[391, 174]	698	[742, 61]
87	[158, 165]	240	[49, 395]	393	[504, 505]	546	[404, 673]	699	[444, 436]
88	[166, 167]	241	[219, 396]	394	[506, 507]	547	[419, 388]	700	[474, 143]
89	[168, 169]	242	[397, 56]	395	[508, 509]	548	[407, 149]	701	[210, 694]
90	[156, 170]	243	[171, 398]	396	[510, 511]	549	[473, 40]	702	[194, 447]
91	[171, 172]	244	[155, 399]	397	[512, 513]	550	[445, 662]	703	[215, 449]
92	[173, 174]	245	[8, 400]	398	[514, 515]	551	[51, 162]	704	[418, 178]
93	[175, 176]	246	[401, 184]	399	[370, 516]	552	[404, 46]	705	[217, 219]
94	[177, 178]	247	[402, 191]	400	[517, 518]	553	[423, 455]	706	[743, 460]
95	[179, 180]	248	[168, 403]	401	[519, 520]	554	[674, 180]	707	[463, 401]
96	[181, 182]	249	[404, 405]	402	[521, 522]	555	[171, 436]	708	[155, 744]
97	[183, 184]	250	[406, 407]	403	[523, 524]	556	[392, 62]	709	[668, 696]
98	[185, 186]	251	[37, 408]	404	[525, 526]	557	[185, 56]	710	[197, 685]
99	[187, 56]	252	[203, 34]	405	[527, 528]	558	[136, 398]	711	[387, 399]
100	[181, 188]	253	[132, 209]	406	[529, 16]	559	[478, 667]	712	[479, 681]

101	[155, 189]	254	[175, 409]	407	[530, 531]	560	[433, 139]	713	[387, 189]
102	[190, 191]	255	[410, 411]	408	[491, 532]	561	[675, 139]	714	[429, 458]
103	[192, 159]	256	[34, 412]	409	[226, 533]	562	[154, 391]	715	[745, 422]
104	[193, 143]	257	[9, 413]	410	[534, 535]	563	[219, 211]	716	[173, 744]
105	[194, 42]	258	[190, 135]	411	[72, 536]	564	[217, 210]	717	[160, 413]
106	[14, 195]	259	[217, 414]	412	[356, 537]	565	[130, 657]	718	[146, 488]
107	[196, 148]	260	[48, 415]	413	[538, 539]	566	[419, 409]	719	[478, 746]
108	[197, 198]	261	[128, 32]	414	[540, 541]	567	[440, 442]	720	[747, 440]
109	[199, 48]	262	[130, 131]	415	[243, 542]	568	[437, 413]	721	[427, 666]
110	[132, 200]	263	[416, 417]	416	[543, 544]	569	[670, 34]	722	[429, 671]
111	[201, 36]	264	[418, 419]	417	[545, 546]	570	[161, 143]	723	[180, 37]
112	[57, 202]	265	[420, 421]	418	[16, 494]	571	[8, 660]	724	[748, 184]
113	[203, 204]	266	[190, 422]	419	[547, 548]	572	[203, 470]	725	[432, 183]
114	[129, 205]	267	[423, 50]	420	[549, 550]	573	[34, 676]	726	[391, 744]
115	[206, 207]	268	[424, 425]	421	[551, 552]	574	[677, 460]	727	[156, 452]
116	[148, 208]	269	[47, 141]	422	[553, 352]	575	[155, 174]	728	[445, 394]
117	[200, 209]	270	[389, 426]	423	[554, 555]	576	[163, 469]	729	[183, 468]
118	[210, 211]	271	[8, 418]	424	[556, 557]	577	[192, 165]	730	[45, 673]
119	[212, 184]	272	[427, 165]	425	[558, 559]	578	[659, 44]	731	[204, 412]
120	[213, 173]	273	[387, 174]	426	[560, 266]	579	[215, 164]	732	[487, 455]
121	[12, 214]	274	[175, 428]	427	[561, 562]	580	[177, 175]	733	[749, 418]
122	[215, 216]	275	[429, 430]	428	[312, 563]	581	[194, 486]	734	[181, 681]
123	[168, 10]	276	[431, 62]	429	[564, 565]	582	[402, 135]	735	[747, 61]
124	[144, 180]	277	[432, 36]	430	[566, 567]	583	[424, 456]	736	[387, 744]
125	[217, 218]	278	[433, 434]	431	[568, 509]	584	[48, 678]	737	[695, 407]
126	[219, 220]	279	[163, 216]	432	[569, 564]	585	[61, 214]	738	[422, 57]
127	[221, 65]	280	[435, 436]	433	[570, 571]	586	[679, 139]	739	[480, 455]
128	[222, 221]	281	[437, 10]	434	[572, 573]	587	[463, 36]	740	[180, 746]
129	[223, 224]	282	[435, 172]	435	[574, 230]	588	[680, 188]	741	[750, 252]
130	[225, 226]	283	[41, 167]	436	[575, 576]	589	[160, 169]	742	[751, 232]
131	[224, 227]	284	[51, 438]	437	[577, 265]	590	[680, 681]	743	[700, 557]
132	[132, 132]	285	[156, 439]	438	[578, 579]	591	[166, 42]	744	[270, 752]
133	[228, 229]	286	[440, 62]	439	[108, 580]	592	[43, 662]	745	[753, 375]
134	[230, 231]	287	[57, 441]	440	[581, 582]	593	[443, 439]	746	[754, 755]
135	[232, 233]	288	[418, 175]	441	[363, 583]	594	[682, 399]	747	[756, 601]
136	[234, 235]	289	[392, 442]	442	[24, 584]	595	[213, 155]	748	[757, 618]
137	[236, 237]	290	[443, 157]	443	[497, 585]	596	[419, 176]	749	[544, 718]
138	[238, 239]	291	[444, 172]	444	[586, 587]	597	[422, 450]	750	[758, 219]
139	[240, 241]	292	[445, 421]	445	[588, 589]	598	[61, 442]	751	[687, 204]
140	[242, 243]	293	[446, 447]	446	[590, 591]	599	[193, 52]	752	[141, 678]
141	[244, 245]	294	[45, 405]	447	[592, 593]	600	[478, 146]	753	[742, 440]
142	[246, 247]	295	[448, 188]	448	[594, 595]	601	[457, 210]	754	[206, 202]
143	[248, 249]	296	[193, 438]	449	[596, 597]	602	[59, 490]	755	[61, 460]
144	[250, 251]	297	[9, 403]	450	[522, 598]	603	[478, 37]	756	[758, 210]

145	[252, 253]	298	[163, 449]	451	[599, 358]	604	[433, 208]	757	[474, 52]
146	[254, 255]	299	[146, 408]	452	[328, 600]	605	[427, 683]	758	[759, 712]
147	[253, 256]	300	[135, 450]	453	[601, 117]	606	[177, 485]	759	[760, 135]
148	[257, 258]	301	[451, 208]	454	[602, 603]	607	[684, 418]	760	[587, 622]
149	[259, 260]	302	[432, 401]	455	[604, 499]	608	[458, 685]		
150	[261, 262]	303	[183, 411]	456	[605, 606]	609	[407, 208]		
151	[263, 264]	304	[443, 452]	457	[607, 366]	610	[141, 151]		
152	[265, 266]	305	[161, 52]	458	[608, 609]	611	[203, 142]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	109	1	218	1	327	1	436	0	545	0	654	0
1	0	110	0	219	0	328	0	437	1	546	0	655	1
2	0	111	0	220	1	329	0	438	1	547	1	656	1
3	0	112	0	221	0	330	1	439	0	548	0	657	0
4	0	113	0	222	0	331	0	440	0	549	0	658	1
5	0	114	0	223	0	332	0	441	0	550	0	659	1
6	0	115	0	224	1	333	0	442	1	551	1	660	1
7	0	116	0	225	1	334	0	443	0	552	0	661	1
8	0	117	0	226	1	335	0	444	1	553	1	662	0
9	0	118	0	227	0	336	1	445	0	554	1	663	1
10	1	119	1	228	1	337	1	446	0	555	0	664	0
11	0	120	0	229	1	338	0	447	1	556	1	665	0
12	1	121	1	230	0	339	0	448	0	557	1	666	0
13	0	122	1	231	1	340	1	449	1	558	0	667	0
14	1	123	0	232	1	341	0	450	1	559	1	668	1
15	0	124	1	233	1	342	0	451	0	560	1	669	1
16	0	125	0	234	0	343	0	452	0	561	0	670	1
17	0	126	0	235	0	344	1	453	0	562	0	671	1
18	1	127	0	236	1	345	1	454	1	563	0	672	1
19	0	128	0	237	0	346	1	455	1	564	0	673	1
20	1	129	0	238	1	347	1	456	0	565	1	674	1
21	1	130	1	239	1	348	1	457	0	566	1	675	0
22	0	131	1	240	1	349	1	458	0	567	0	676	0
23	0	132	0	241	0	350	1	459	0	568	1	677	1
24	1	133	1	242	0	351	1	460	0	569	1	678	1
25	1	134	0	243	0	352	0	461	0	570	1	679	1
26	1	135	1	244	0	353	0	462	1	571	0	680	0
27	0	136	0	245	0	354	0	463	1	572	0	681	0
28	0	137	1	246	0	355	0	464	0	573	1	682	0
29	1	138	1	247	0	356	1	465	0	574	1	683	0
30	0	139	1	248	0	357	1	466	1	575	0	684	0
31	0	140	0	249	0	358	0	467	1	576	1	685	0
32	1	141	0	250	1	359	1	468	0	577	1	686	0

33	0	142	0	251	1	360	1	469	0	578	1	687	1
34	1	143	0	252	0	361	1	470	0	579	1	688	0
35	0	144	1	253	0	362	1	471	1	580	0	689	1
36	1	145	0	254	1	363	0	472	1	581	0	690	1
37	1	146	1	255	1	364	0	473	0	582	0	691	0
38	0	147	0	256	1	365	1	474	0	583	1	692	1
39	0	148	0	257	0	366	0	475	1	584	0	693	0
40	1	149	0	258	0	367	0	476	1	585	0	694	1
41	1	150	0	259	0	368	1	477	1	586	1	695	0
42	1	151	1	260	0	369	0	478	1	587	1	696	1
43	1	152	0	261	0	370	0	479	1	588	0	697	1
44	0	153	1	262	1	371	1	480	0	589	1	698	1
45	0	154	0	263	1	372	1	481	0	590	0	699	1
46	1	155	0	264	0	373	1	482	0	591	1	700	0
47	0	156	0	265	0	374	1	483	1	592	1	701	0
48	0	157	1	266	0	375	1	484	0	593	0	702	0
49	1	158	1	267	1	376	0	485	0	594	0	703	1
50	0	159	1	268	1	377	0	486	0	595	0	704	0
51	1	160	1	269	0	378	1	487	0	596	1	705	0
52	0	161	1	270	1	379	1	488	1	597	1	706	0
53	1	162	1	271	0	380	1	489	0	598	0	707	1
54	1	163	1	272	0	381	0	490	0	599	0	708	0
55	0	164	0	273	1	382	0	491	1	600	1	709	1
56	0	165	1	274	1	383	1	492	1	601	0	710	0
57	0	166	1	275	0	384	0	493	1	602	1	711	1
58	0	167	0	276	1	385	1	494	1	603	1	712	1
59	1	168	0	277	1	386	1	495	1	604	1	713	1
60	1	169	0	278	1	387	1	496	1	605	0	714	0
61	0	170	1	279	1	388	1	497	0	606	0	715	1
62	0	171	0	280	1	389	1	498	0	607	0	716	1
63	0	172	0	281	1	390	0	499	0	608	0	717	1
64	0	173	1	282	1	391	0	500	0	609	0	718	1
65	1	174	1	283	1	392	1	501	1	610	0	719	1
66	0	175	1	284	1	393	0	502	1	611	0	720	1
67	0	176	0	285	0	394	1	503	0	612	1	721	0
68	0	177	0	286	0	395	0	504	0	613	0	722	0
69	1	178	0	287	0	396	0	505	0	614	1	723	1
70	1	179	0	288	0	397	0	506	1	615	0	724	0
71	0	180	1	289	1	398	1	507	1	616	0	725	1
72	0	181	1	290	0	399	0	508	0	617	1	726	0
73	1	182	1	291	1	400	1	509	1	618	1	727	0
74	1	183	0	292	0	401	0	510	0	619	1	728	0
75	1	184	1	293	0	402	0	511	1	620	0	729	0
76	0	185	1	294	0	403	0	512	0	621	1	730	0

77	0	186	1	295	0	404	0	513	1	622	0	731	1
78	0	187	1	296	0	405	0	514	1	623	1	732	0
79	0	188	0	297	0	406	1	515	1	624	0	733	1
80	0	189	0	298	1	407	0	516	1	625	1	734	1
81	1	190	0	299	1	408	1	517	1	626	1	735	1
82	0	191	0	300	1	409	1	518	1	627	0	736	1
83	1	192	1	301	0	410	1	519	0	628	1	737	0
84	1	193	0	302	1	411	0	520	1	629	0	738	1
85	1	194	0	303	0	412	1	521	0	630	0	739	0
86	1	195	0	304	0	413	1	522	1	631	1	740	1
87	1	196	1	305	1	414	1	523	0	632	0	741	1
88	1	197	0	306	0	415	0	524	0	633	1	742	1
89	0	198	1	307	1	416	1	525	0	634	1	743	0
90	0	199	1	308	1	417	0	526	0	635	1	744	1
91	0	200	0	309	1	418	0	527	0	636	1	745	1
92	1	201	0	310	1	419	1	528	1	637	1	746	0
93	1	202	1	311	0	420	0	529	1	638	0	747	1
94	0	203	0	312	0	421	1	530	0	639	0	748	0
95	0	204	1	313	0	422	1	531	1	640	1	749	1
96	1	205	0	314	0	423	1	532	1	641	0	750	1
97	0	206	0	315	0	424	1	533	0	642	1	751	1
98	1	207	1	316	1	425	0	534	1	643	0	752	0
99	1	208	1	317	1	426	1	535	0	644	1	753	1
100	1	209	1	318	1	427	0	536	0	645	0	754	0
101	0	210	0	319	1	428	0	537	0	646	1	755	0
102	0	211	0	320	1	429	0	538	1	647	0	756	1
103	1	212	1	321	1	430	1	539	1	648	1	757	0
104	0	213	0	322	0	431	1	540	1	649	0	758	1
105	0	214	1	323	0	432	1	541	0	650	1	759	1
106	1	215	1	324	1	433	1	542	0	651	1	760	1
107	1	216	1	325	0	434	0	543	1	652	0		
108	0	217	0	326	1	435	1	544	1	653	0		

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