PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a, b) = (z^5 + z^2, z^2 + 1)$

GUO-NIU HAN* AND YINING HU*

ABSTRACT. The continued fraction conjecture claims that the continued fraction defined by the (a, b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a, b) in $\mathbb{F}_2[x] \setminus \mathbb{F}_2$. In this paper we prove the conjecture for $(a, b) = (z^5 + z^2, z^2 + 1)$.

1. INTRODUCTION

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \ldots)$ of elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{\ddots + \cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a, b)-Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \ldots) = (a, b, b, a, b, a, a, b, \ldots)$$

that we shall denote

$$\mathrm{CF}(a, b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When* $(a,b)=(z^5+z^2,z^2+1)$, *the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$, *with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

where, for $\mathrm{CF}(a,b)$,

$$\begin{aligned}
p_0(z)&=z^{17}+z^{15}+z^{14}+z^{13}+z^{12}+z^7+z^5+z^4+z^2+1,\\
p_1(z)&=z^{22}+z^{20}+z^{19}+z^{18}+z^{14}+z^{13}+z^9+z^5+z^4+z^2,\\
p_2(z)&=z^{24}+z^{14}+z^{12}+z^{10}+z^8+z^4,\\
p_3(z)&=z^{22}+z^{20}+z^{19}+z^{18}+z^{14}+z^{13}+z^9+z^5+z^4+z^2,\\
p_4(z)&=z^{20}+z^{17}+z^{16}+z^{15}+z^{13}+z^{10}+z^8+z^7+z^5+z^4+z^2,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$,

$$\begin{aligned}
p_0(z)&=z^{17}+z^{16}+z^{15}+z^{14}+z^{13}+z^8+z^7+z^5+z^2,\\
p_1(z)&=z^{22}+z^{20}+z^{19}+z^{18}+z^{14}+z^{13}+z^9+z^5+z^4+z^2,\\
p_2(z)&=z^{24}+z^{14}+z^{12}+z^{10}+z^8+z^4,\\
p_3(z)&=z^{22}+z^{20}+z^{19}+z^{18}+z^{14}+z^{13}+z^9+z^5+z^4+z^2,\\
p_4(z)&=z^{20}+z^{17}+z^{15}+z^{13}+z^{12}+z^{10}+z^7+z^5+z^2+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x)&=x^{\deg(t_{2^n-1})}\begin{pmatrix} t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x)&=x^{\deg(\bar{t}_{2^n-1})}\begin{pmatrix} \bar{t}_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar{t}_{2^n-2})}\begin{pmatrix} \bar{t}_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(\bar{t}_0)}\begin{pmatrix} \bar{t}_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\geq 0$

$$\begin{aligned}
M_{n+1}(x)&=W_n(x)\cdot M_n(x),\\
W_{n+1}(x)&=M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(175, 175, 175, 175, 175)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z\left(\phi(W^o,0,0)(x,z),\ z^{12}\cdot\phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 630 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 630, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z\left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ to order 770 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 770, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. ∎

Define

$$R^e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned} M^e[:14u] = {} & M^e[:7u] + x^{2u}M^e[:5u] + x^{3u}M^e[:4u] + x^{4u}M^e[:3u] + x^{5u}M^e[:2u] \\ & + x^{6u}M^e[:u] + x^{2u}M^e[:7u] + x^{4u}M^e[:5u] + x^{5u}M^e[:4u] \\ & + x^{6u}M^e[:3u] + x^{7u}M^e[:2u] + x^{8u}M^e[:u] + x^{3u}M^e[:7u] \\ & + x^{5u}M^e[:5u] + x^{6u}M^e[:4u] + x^{7u}M^e[:3u] + x^{8u}M^e[:2u] \\ & + x^{9u}M^e[:u] + x^{5u}M^e[:7u] + x^{7u}M^e[:5u] + x^{8u}M^e[:4u] \\ & + x^{9u}M^e[:3u] + x^{10u}M^e[:2u] + x^{11u}M^e[:u] + x^{6u}M^e[:7u] \end{aligned}$$

$$
\begin{aligned}
&+ x^{8u}M^e[:5u] + x^{9u}M^e[:4u] + x^{10u}M^e[:3u] + x^{11u}M^e[:2u]\\
&+ x^{12u}M^e[:u] + x^{9u}M^e[:5u] + x^{11u}M^e[:3u] + x^{12u}M^e[:2u]\\
&+ x^{13u}M^e[:u] + (x^{7u} + x^{9u} + x^{10u} + x^{12u} + x^{13u})R^e,\\
W^e[:14u] = {}& W^e[:7u] + x^{2u}W^e[:5u] + x^{3u}W^e[:4u] + x^{4u}W^e[:3u] + x^{5u}W^e[:2u]\\
&+ x^{6u}W^e[:u] + x^{2u}W^e[:7u] + x^{4u}W^e[:5u] + x^{5u}W^e[:4u]\\
&+ x^{6u}W^e[:3u] + x^{7u}W^e[:2u] + x^{8u}W^e[:u] + x^{3u}W^e[:7u]\\
&+ x^{5u}W^e[:5u] + x^{6u}W^e[:4u] + x^{7u}W^e[:3u] + x^{8u}W^e[:2u]\\
&+ x^{9u}W^e[:u] + x^{5u}W^e[:7u] + x^{7u}W^e[:5u] + x^{8u}W^e[:4u]\\
&+ x^{9u}W^e[:3u] + x^{10u}W^e[:2u] + x^{11u}W^e[:u] + x^{6u}W^e[:7u]\\
&+ x^{8u}W^e[:5u] + x^{9u}W^e[:4u] + x^{10u}W^e[:3u] + x^{11u}W^e[:2u]\\
&+ x^{12u}W^e[:u] + x^{9u}W^e[:5u] + x^{11u}W^e[:3u] + x^{12u}W^e[:2u]\\
&+ x^{13u}W^e[:u] + (x^{7u} + x^{9u} + x^{10u} + x^{12u} + x^{13u})R^e.
\end{aligned}
$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$
\begin{aligned}
M^o[:14u] = {}& M^o[:7u] + x^{2u}M^o[:5u] + x^{3u}M^o[:4u] + x^{4u}M^o[:3u] + x^{5u}M^o[:2u]\\
&+ x^{6u}M^o[:u] + x^{2u}M^o[:7u] + x^{4u}M^o[:5u] + x^{5u}M^o[:4u]\\
&+ x^{6u}M^o[:3u] + x^{7u}M^o[:2u] + x^{8u}M^o[:u] + x^{3u}M^o[:7u]\\
&+ x^{5u}M^o[:5u] + x^{6u}M^o[:4u] + x^{7u}M^o[:3u] + x^{8u}M^o[:2u]\\
&+ x^{9u}M^o[:u] + x^{5u}M^o[:7u] + x^{7u}M^o[:5u] + x^{8u}M^o[:4u]\\
&+ x^{9u}M^o[:3u] + x^{10u}M^o[:2u] + x^{11u}M^o[:u] + x^{6u}M^o[:7u]\\
&+ x^{8u}M^o[:5u] + x^{9u}M^o[:4u] + x^{10u}M^o[:3u] + x^{11u}M^o[:2u]\\
&+ x^{12u}M^o[:u] + x^{9u}M^o[:5u] + x^{11u}M^o[:3u] + x^{12u}M^o[:2u]\\
&+ x^{13u}M^o[:u] + (x^{7u} + x^{9u} + x^{10u} + x^{12u} + x^{13u})R^o,\\
W^o[:14u] = {}& W^o[:7u] + x^{2u}W^o[:5u] + x^{3u}W^o[:4u] + x^{4u}W^o[:3u] + x^{5u}W^o[:2u]\\
&+ x^{6u}W^o[:u] + x^{2u}W^o[:7u] + x^{4u}W^o[:5u] + x^{5u}W^o[:4u]\\
&+ x^{6u}W^o[:3u] + x^{7u}W^o[:2u] + x^{8u}W^o[:u] + x^{3u}W^o[:7u]\\
&+ x^{5u}W^o[:5u] + x^{6u}W^o[:4u] + x^{7u}W^o[:3u] + x^{8u}W^o[:2u]\\
&+ x^{9u}W^o[:u] + x^{5u}W^o[:7u] + x^{7u}W^o[:5u] + x^{8u}W^o[:4u]\\
&+ x^{9u}W^o[:3u] + x^{10u}W^o[:2u] + x^{11u}W^o[:u] + x^{6u}W^o[:7u]\\
&+ x^{8u}W^o[:5u] + x^{9u}W^o[:4u] + x^{10u}W^o[:3u] + x^{11u}W^o[:2u]\\
&+ x^{12u}W^o[:u] + x^{9u}W^o[:5u] + x^{11u}W^o[:3u] + x^{12u}W^o[:2u]\\
&+ x^{13u}W^o[:u] + (x^{7u} + x^{9u} + x^{10u} + x^{12u} + x^{13u})R^o.
\end{aligned}
$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$T[:14u] = T[:7u] + x^{2u}T[:5u] + x^{3u}T[:4u] + x^{4u}T[:3u] + x^{5u}T[:2u] + x^{6u}T[:u]$$

$$
\begin{aligned}
&+ x^{2u}T[:7u] + x^{4u}T[:5u] + x^{5u}T[:4u] + x^{6u}T[:3u] + x^{7u}T[:2u]\\
&+ x^{8u}T[:u] + x^{3u}T[:7u] + x^{5u}T[:5u] + x^{6u}T[:4u] + x^{7u}T[:3u]\\
&+ x^{8u}T[:2u] + x^{9u}T[:u] + x^{5u}T[:7u] + x^{7u}T[:5u] + x^{8u}T[:4u]\\
&+ x^{9u}T[:3u] + x^{10u}T[:2u] + x^{11u}T[:u] + x^{6u}T[:7u] + x^{8u}T[:5u]\\
&+ x^{9u}T[:4u] + x^{10u}T[:3u] + x^{11u}T[:2u] + x^{12u}T[:u] + x^{9u}T[:5u]\\
&+ x^{11u}T[:3u] + x^{12u}T[:2u] + x^{13u}T[:u].
\end{aligned}
$$

The proof of the other 15 cases are similar. We break down the above identity into 7 parts:

$$
\begin{aligned}
0 &= x^{7u}T[:u] + x^{6u}T[u:2u] + x^{5u}T[2u:3u] + x^{4u}T[3u:4u]\\
&\quad + x^{3u}T[4u:5u] + x^{2u}T[5u:6u] + T[7u:8u],\\
0 &= x^{7u}T[u:2u] + x^{6u}T[2u:3u] + x^{5u}T[3u:4u] + x^{4u}T[4u:5u]\\
&\quad + x^{3u}T[5u:6u] + x^{2u}T[6u:7u] + T[8u:9u],\\
0 &= x^{8u}T[u:2u] + x^{3u}T[6u:7u] + T[9u:10u],\\
0 &= x^{9u}T[u:2u] + x^{7u}T[3u:4u] + x^{6u}T[4u:5u] + x^{5u}T[5u:6u]\\
&\quad + T[10u:11u],\\
0 &= x^{11u}T[:u] + x^{9u}T[2u:3u] + x^{7u}T[4u:5u] + x^{6u}T[5u:6u]\\
&\quad + x^{5u}T[6u:7u] + T[11u:12u],\\
0 &= x^{10u}T[2u:3u] + x^{8u}T[4u:5u] + x^{6u}T[6u:7u] + T[12u:13u],\\
0 &= x^{13u}T[:u] + x^{12u}T[u:2u] + x^{11u}T[2u:3u] + x^{9u}T[4u:5u]\\
&\quad + T[13u:14u],
\end{aligned}
$$

which can be rewritten as

$$
\begin{aligned}
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2]\\
&\quad + T[[101w]_2] + T[[111w]_2], \qquad (2.7)\\
0 &= T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2]\\
&\quad + T[[110w]_2] + T[[1000w]_2], \qquad (2.8)\\
0 &= T[[1w]_2] + T[[110w]_2] + T[[1001w]_2], \qquad (2.9)\\
0 &= T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1010w]_2], \qquad (2.10)\\
0 &= T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[110w]_2]\\
&\quad + T[[1011w]_2], \qquad (2.11)\\
0 &= T[[10w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1100w]_2], \qquad (2.12)\\
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[100w]_2] + T[[1101w]_2], \qquad (2.13)
\end{aligned}
$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$
\begin{aligned}
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2]\\
&\quad + T[[101w]_2] + T[[111w]_2], \qquad (2.14)\\
0 &= T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2]\\
&\quad + T[[110w]_2] + T[[1000w]_2], \qquad (2.15)
\end{aligned}
$$

$$0 = T[[1w]_2] + T[[110w]_2] + T[[1001w]_2], \tag{2.16}$$

$$0 = T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1010w]_2], \tag{2.17}$$

$$\begin{aligned} 0 = {} & T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[110w]_2] \\ & + T[[1011w]_2], \end{aligned} \tag{2.18}$$

$$0 = T[[10w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1100w]_2], \tag{2.19}$$

$$0 = T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[100w]_2] + T[[1101w]_2], \tag{2.20}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 1241 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.20) can be written as

$$\begin{aligned} 0 = {} & \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ & + \tau(A(s,101)) + \tau(A(s,111)), \end{aligned} \tag{2.21}$$

$$\begin{aligned} 0 = {} & \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ & + \tau(A(s,101)) + \tau(A(s,110)) + \tau(A(s,1000)), \end{aligned} \tag{2.22}$$

$$0 = \tau(A(s,1)) + \tau(A(s,110)) + \tau(A(s,1001)), \tag{2.23}$$

$$\begin{aligned} 0 = {} & \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,100)) + \tau(A(s,101)) \\ & + \tau(A(s,1010)), \end{aligned} \tag{2.24}$$

$$\begin{aligned} 0 = {} & \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,101)) \\ & + \tau(A(s,110)) + \tau(A(s,1011)), \end{aligned} \tag{2.25}$$

$$0 = \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,110)) + \tau(A(s,1100)), \tag{2.26}$$

$$\begin{aligned} 0 = {} & \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,100)) \\ & + \tau(A(s,1101)), \end{aligned} \tag{2.27}$$

for all $s \in E_{2k}$, and

$$\begin{aligned} 0 = {} & \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ & + \tau(A(s,101)) + \tau(A(s,111)), \end{aligned} \tag{2.28}$$

$$\begin{aligned} 0 = {} & \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ & + \tau(A(s,101)) + \tau(A(s,110)) + \tau(A(s,1000)), \end{aligned} \tag{2.29}$$

$$0 = \tau(A(s,1)) + \tau(A(s,110)) + \tau(A(s,1001)), \tag{2.30}$$

$$\begin{aligned} 0 = {} & \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,100)) + \tau(A(s,101)) \\ & + \tau(A(s,1010)), \end{aligned} \tag{2.31}$$

$$\begin{aligned} 0 = {} & \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,101)) \\ & + \tau(A(s,110)) + \tau(A(s,1011)), \end{aligned} \tag{2.32}$$

$$0 = \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,110)) + \tau(A(s,1100)), \tag{2.33}$$

$$\begin{aligned} 0 = {} & \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,100)) \\ & + \tau(A(s,1101)), \end{aligned} \tag{2.34}$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{28}), E_{28}) = (A(i, 0^{16}), E_{16})$, so that we only have to verify that identities (2.21) through (2.34) hold for $2 \le k \le 14$, which turns out to be true. $\square$

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \ge 2$, and $u = 2^{2k-1}$,*

$$\begin{aligned} M_{2k} &= M^e[:7u] + x^{2u}M^e[:5u] + x^{3u}M^e[:4u] + x^{4u}M^e[:3u] + x^{5u}M^e[:2u] \\ &\quad + x^{6u}M^e[:u] + x^{7u}R^e, \\ W_{2k} &= W^e[:7u] + x^{2u}W^e[:5u] + x^{3u}W^e[:4u] + x^{4u}W^e[:3u] + x^{5u}W^e[:2u] \\ &\quad + x^{6u}W^e[:u] + x^{7u}R^e. \end{aligned}$$

For all integer $k \ge 2$, and $u = 2^{2k}$,

$$\begin{aligned} M_{2k+1} &= M^o[:7u] + x^{2u}M^o[:5u] + x^{3u}M^o[:4u] + x^{4u}M^o[:3u] + x^{5u}M^o[:2u] \\ &\quad + x^{6u}M^o[:u] + x^{7u}R^o, \\ W_{2k+1} &= W^o[:7u] + x^{2u}W^o[:5u] + x^{3u}W^o[:4u] + x^{4u}W^o[:3u] + x^{5u}W^o[:2u] \\ &\quad + x^{6u}W^o[:u] + x^{7u}R^o. \end{aligned}$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \ge 2$, we claim that

$$\begin{aligned} M_{2k} \wedge W_{2k} &\Rightarrow M_{2k+1} \wedge W_{2k+1}, \\ M_{2k+1} \wedge W_{2k+1} &\Rightarrow M_{2k+2} \wedge W_{2k+2}. \end{aligned}$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$\begin{aligned} W_{2k}M_{2k} = \big(&W^e[:7v] + x^{2v}W^e[:5v] + x^{3v}W^e[:4v] + x^{4v}W^e[:3v] \\ &+ x^{5v}W^e[:2v] + x^{6v}W^e[:v] + x^{7u}R^e\big) \times \big(M^e[:7v] + x^{2v}M^e[:5v] \\ &+ x^{3v}M^e[:4v] + x^{4v}M^e[:3v] + x^{5v}M^e[:2v] + x^{6v}M^e[:v] + x^{7u}R^e \\ &\big); \end{aligned} \tag{2.35}$$

Call this expression *lhs*. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{14v}R^o$. Therefore we only need to prove that their difference is $O(x^{14v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{14v}, to

$$\begin{aligned} &W^e[:14v]M^e[:14v] + x^{4v}W^e[:10v]M^e[:10v] + x^{6v}W^e[:8v]M^e[:8v] \\ &\qquad + x^{8v}W^e[:6v]M^e[:6v] + x^{10v}W^e[:4v]M^e[:4v] + x^{12v}W^e[:2v]M^e[:2v]. \end{aligned}$$

For all $n \le 14$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call *rhs*. Define

$$X := x^v,$$

$$\begin{aligned}a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n,\\ b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n,\\ c &:= R^e.\end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.35) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{14}, b_1, ..., b_{14}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{14})$, which completes the proof. $\square$

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \leq j, k \leq 1$,

$$\begin{aligned}\lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k},\\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k},\\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k},\\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}.\end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a, b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x, y) = \mathrm{Res}_t\left(\phi(M^e, 0, 1)(x, t), y^{12}\phi(M^e, 0, 1)(x, t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x, y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned}q_0(x) &= x^{24} + x^{22} + x^{20} + x^{19} + x^{17} + x^{12} + x^{11} + x^{10} + x^9 + x^7,\\ q_1(x) &= x^{22} + x^{20} + x^{19} + x^{15} + x^{11} + x^{10} + x^6 + x^5 + x^4 + x^2,\\ q_2(x) &= x^{20} + x^{16} + x^{14} + x^{12} + x^{10} + 1,\\ q_3(x) &= x^{22} + x^{20} + x^{19} + x^{15} + x^{11} + x^{10} + x^6 + x^5 + x^4 + x^2,\\ q_4(x) &= x^{22} + x^{20} + x^{19} + x^{17} + x^{16} + x^{14} + x^{11} + x^9 + x^8 + x^7 + x^4.\end{aligned}$$

The polynomial $Q(x, y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x, y)$ of multiplicity m and $R(x, y) := P(x, y)/Q(x, y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 216, and substitute it for y in $R(z, y)$, we get a series with valuation smaller than 216, which proves that $R(z, y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z, y)$ is the minimal polynomial of $\mathrm{CF}(a, b) = f(1/z)$. $\square$

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{120} + x^{118} + x^{108} + x^{106} + x^{104} + x^{100} + x^{98} + x^{96} + x^{92} + x^{90} \\
&\quad + x^{88} + x^{86} + x^{82} + x^{78} + x^{76} + x^{70} + x^{66} + x^{64} + x^{62} + x^{58} + x^{56} \\
&\quad + x^{48} + x^{46} + x^{40} + x^{32} + x^{30} + x^{28} + x^{26} + x^{24}, \\
p_3(x) &= x^{112} + x^{110} + x^{108} + x^{104} + x^{102} + x^{100} + x^{98} + x^{88} + x^{86} + x^{84} \\
&\quad + x^{82} + x^{76} + x^{70} + x^{68} + x^{66} + x^{64} + x^{62} + x^{54} + x^{52} + x^{48} + x^{46} \\
&\quad + x^{42} + x^{38} + x^{36} + x^{20} + x^{16}, \\
p_6(x) &= x^{108} + x^{104} + x^{102} + x^{96} + x^{90} + x^{86} + x^{84} + x^{82} + x^{76} + x^{68} + x^{64} \\
&\quad + x^{62} + x^{58} + x^{54} + x^{48} + x^{44} + x^{38} + x^{36} + x^{30} + x^{28} + x^{20} + x^{18} \\
&\quad + x^{14} + x^{10} + x^{8} + x^{6} + x^{2} + 1, \\
p_9(x) &= x^{116} + x^{114} + x^{110} + x^{108} + x^{106} + x^{98} + x^{96} + x^{90} + x^{88} + x^{86} + x^{84} \\
&\quad + x^{80} + x^{78} + x^{76} + x^{74} + x^{68} + x^{62} + x^{58} + x^{54} + x^{52} + x^{50} + x^{48} \\
&\quad + x^{44} + x^{34} + x^{26} + x^{22} + x^{20} + x^{16} + x^{14} + x^{12} + x^{10} + x^{8} + x^{6} + x^{4}, \\
p_{12}(x) &= x^{116} + x^{114} + x^{112} + x^{84} + x^{82} + x^{80} + x^{52} + x^{50} + x^{48} + x^{4} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0].

For $\phi(M^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{129} + x^{124} + x^{119} + x^{117} + x^{116} + x^{115} + x^{114} + x^{113} + x^{112} + x^{111} \\
&\quad + x^{107} + x^{106} + x^{105} + x^{103} + x^{102} + x^{99} + x^{98} + x^{96} + x^{93} + x^{90} \\
&\quad + x^{88} + x^{86} + x^{85} + x^{83} + x^{82} + x^{81} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} \\
&\quad + x^{74} + x^{73} + x^{71} + x^{69} + x^{67} + x^{65} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} \\
&\quad + x^{52} + x^{47} + x^{45} + x^{43} + x^{42} + x^{40} + x^{39} + x^{37} + x^{36} + x^{35} + x^{33}, \\
p_3(x) &= x^{125} + x^{123} + x^{121} + x^{120} + x^{118} + x^{116} + x^{113} + x^{108} + x^{105} + x^{103} \\
&\quad + x^{100} + x^{98} + x^{93} + x^{91} + x^{89} + x^{88} + x^{86} + x^{85} + x^{84} + x^{83} + x^{80} \\
&\quad + x^{78} + x^{77} + x^{75} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} \\
&\quad + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} \\
&\quad + x^{54} + x^{52} + x^{50} + x^{47} + x^{45} + x^{43} + x^{42} + x^{40} + x^{39} + x^{38} + x^{34} \\
&\quad + x^{33} + x^{28} + x^{25} + x^{23} + x^{20} + x^{18}, \\
p_6(x) &= x^{123} + x^{120} + x^{119} + x^{117} + x^{116} + x^{115} + x^{114} + x^{112} + x^{111} + x^{108} \\
&\quad + x^{103} + x^{100} + x^{97} + x^{95} + x^{94} + x^{93} + x^{92} + x^{90} + x^{89} + x^{86} + x^{85} \\
&\quad + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{76} + x^{73} + x^{70} + x^{67} + x^{65} \\
&\quad + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{58} + x^{57} + x^{55} + x^{54} + x^{53} + x^{52}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{50} + x^{41} + x^{39} + x^{38} + x^{37} + x^{36} + x^{34} + x^{33} + x^{31} + x^{30} + x^{29}\\
&+ x^{28} + x^{27} + x^{26} + x^{24} + x^{21} + x^{19} + x^{18} + x^{16} + x^{15} + x^{12},\\
p_9(x) = {}& x^{121} + x^{119} + x^{116} + x^{114} + x^{113} + x^{109} + x^{108} + x^{107} + x^{104} + x^{103}\\
&+ x^{102} + x^{99} + x^{98} + x^{97} + x^{95} + x^{94} + x^{93} + x^{92} + x^{90} + x^{89} + x^{88}\\
&+ x^{85} + x^{84} + x^{80} + x^{75} + x^{70} + x^{67} + x^{63} + x^{62} + x^{61} + x^{58} + x^{56}\\
&+ x^{53} + x^{51} + x^{48} + x^{47} + x^{46} + x^{45} + x^{42} + x^{40} + x^{37} + x^{33} + x^{32}\\
&+ x^{31} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20}\\
&+ x^{19} + x^{18} + x^{16} + x^{15} + x^{14} + x^{13} + x^{11} + x^{10} + x^{8} + x^{6},\\
p_{12}(x) = {}& x^{119} + x^{116} + x^{111} + x^{108} + x^{107} + x^{104} + x^{103} + x^{100} + x^{99} + x^{96}\\
&+ x^{95} + x^{92} + x^{91} + x^{88} + x^{83} + x^{80} + x^{55} + x^{52} + x^{47} + x^{44} + x^{43}\\
&+ x^{40} + x^{39} + x^{36} + x^{35} + x^{32} + x^{31} + x^{28} + x^{27} + x^{24} + x^{23} + x^{20}\\
&+ x^{19} + x^{16} + x^{15} + x^{12} + x^{11} + x^{8} + x^{3} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 0, 1, 0, 0, 0, 1].

For $\phi(M^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) = {}& x^{129} + x^{124} + x^{119} + x^{117} + x^{116} + x^{115} + x^{114} + x^{113} + x^{112} + x^{111}\\
&+ x^{107} + x^{106} + x^{105} + x^{103} + x^{102} + x^{99} + x^{98} + x^{96} + x^{93} + x^{90}\\
&+ x^{88} + x^{86} + x^{85} + x^{83} + x^{82} + x^{81} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75}\\
&+ x^{74} + x^{73} + x^{71} + x^{69} + x^{67} + x^{65} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54}\\
&+ x^{52} + x^{47} + x^{45} + x^{43} + x^{42} + x^{40} + x^{39} + x^{37} + x^{36} + x^{35} + x^{33},\\
p_3(x) = {}& x^{125} + x^{123} + x^{121} + x^{120} + x^{118} + x^{116} + x^{113} + x^{108} + x^{105} + x^{103}\\
&+ x^{100} + x^{98} + x^{93} + x^{91} + x^{89} + x^{88} + x^{86} + x^{85} + x^{84} + x^{83} + x^{80}\\
&+ x^{78} + x^{77} + x^{75} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66}\\
&+ x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55}\\
&+ x^{54} + x^{52} + x^{50} + x^{47} + x^{45} + x^{43} + x^{42} + x^{40} + x^{39} + x^{38} + x^{34}\\
&+ x^{33} + x^{28} + x^{25} + x^{23} + x^{20} + x^{18},\\
p_6(x) = {}& x^{123} + x^{120} + x^{119} + x^{117} + x^{116} + x^{115} + x^{114} + x^{112} + x^{111} + x^{108}\\
&+ x^{103} + x^{100} + x^{97} + x^{95} + x^{94} + x^{93} + x^{92} + x^{90} + x^{89} + x^{86} + x^{85}\\
&+ x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{76} + x^{73} + x^{70} + x^{67} + x^{65}\\
&+ x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{58} + x^{57} + x^{55} + x^{54} + x^{53} + x^{52}\\
&+ x^{50} + x^{41} + x^{39} + x^{38} + x^{37} + x^{36} + x^{34} + x^{33} + x^{31} + x^{30} + x^{29}\\
&+ x^{28} + x^{27} + x^{26} + x^{24} + x^{21} + x^{19} + x^{18} + x^{16} + x^{15} + x^{12},\\
p_9(x) = {}& x^{121} + x^{119} + x^{116} + x^{114} + x^{113} + x^{109} + x^{108} + x^{107} + x^{104} + x^{103}\\
&+ x^{102} + x^{99} + x^{98} + x^{97} + x^{95} + x^{94} + x^{93} + x^{92} + x^{90} + x^{89} + x^{88}\\
&+ x^{85} + x^{84} + x^{80} + x^{75} + x^{70} + x^{67} + x^{63} + x^{62} + x^{61} + x^{58} + x^{56}\\
&+ x^{53} + x^{51} + x^{48} + x^{47} + x^{46} + x^{45} + x^{42} + x^{40} + x^{37} + x^{33} + x^{32}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{31} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20}\\
&+ x^{19} + x^{18} + x^{16} + x^{15} + x^{14} + x^{13} + x^{11} + x^{10} + x^{8} + x^{6},\\
p_{12}(x) &= x^{119} + x^{116} + x^{111} + x^{108} + x^{107} + x^{104} + x^{103} + x^{100} + x^{99} + x^{96}\\
&+ x^{95} + x^{92} + x^{91} + x^{88} + x^{83} + x^{80} + x^{55} + x^{52} + x^{47} + x^{44} + x^{43}\\
&+ x^{40} + x^{39} + x^{36} + x^{35} + x^{32} + x^{31} + x^{28} + x^{27} + x^{24} + x^{23} + x^{20}\\
&+ x^{19} + x^{16} + x^{15} + x^{12} + x^{11} + x^{8} + x^{3} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 0, 1, 0, 0, 0, 1].

For $\phi(M^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{138} + x^{136} + x^{134} + x^{132} + x^{130} + x^{126} + x^{124} + x^{120} + x^{110} + x^{106}\\
&+ x^{104} + x^{96} + x^{94} + x^{88} + x^{84} + x^{80} + x^{78} + x^{76} + x^{74} + x^{72} + x^{70}\\
&+ x^{68} + x^{66} + x^{64} + x^{60} + x^{56} + x^{46} + x^{44} + x^{42},\\
p_3(x) &= x^{126} + x^{124} + x^{122} + x^{116} + x^{114} + x^{110} + x^{108} + x^{104} + x^{98} + x^{94}\\
&+ x^{90} + x^{88} + x^{76} + x^{72} + x^{68} + x^{54} + x^{52} + x^{48} + x^{44} + x^{38} + x^{34}\\
&+ x^{26} + x^{14} + x^{12},\\
p_6(x) &= x^{126} + x^{124} + x^{120} + x^{118} + x^{114} + x^{112} + x^{102} + x^{96} + x^{74} + x^{70}\\
&+ x^{66} + x^{64} + x^{62} + x^{58} + x^{56} + x^{54} + x^{48} + x^{42} + x^{38} + x^{36} + x^{32}\\
&+ x^{28} + x^{26} + x^{22} + x^{20} + x^{18} + x^{16} + x^{14} + x^{12} + x^{8} + x^{6} + x^{4} + x^{2}\\
&+ 1,\\
p_9(x) &= x^{122} + x^{120} + x^{118} + x^{116} + x^{112} + x^{108} + x^{102} + x^{100} + x^{98} + x^{96}\\
&+ x^{90} + x^{88} + x^{86} + x^{78} + x^{76} + x^{72} + x^{62} + x^{58} + x^{56} + x^{52} + x^{50}\\
&+ x^{42} + x^{40} + x^{32} + x^{30} + x^{26} + x^{24} + x^{22} + x^{20} + x^{16} + x^{12} + x^{10}\\
&+ x^{6} + x^{4},\\
p_{12}(x) &= x^{122} + x^{120} + x^{106} + x^{104} + x^{98} + x^{96} + x^{82} + x^{80} + x^{58} + x^{56} + x^{42}\\
&+ x^{40} + x^{34} + x^{32} + x^{26} + x^{24} + x^{18} + x^{16} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{138} + x^{136} + x^{134} + x^{132} + x^{128} + x^{126} + x^{124} + x^{122} + x^{112} + x^{110}\\
&+ x^{98} + x^{96} + x^{92} + x^{90} + x^{86} + x^{84} + x^{82} + x^{80} + x^{76} + x^{72} + x^{70}\\
&+ x^{66} + x^{64} + x^{62} + x^{58} + x^{56} + x^{50} + x^{48} + x^{44} + x^{38} + x^{30} + x^{26}\\
&+ x^{24},\\
p_3(x) &= x^{126} + x^{124} + x^{122} + x^{118} + x^{108} + x^{100} + x^{98} + x^{94} + x^{92} + x^{90}\\
&+ x^{88} + x^{82} + x^{76} + x^{68} + x^{60} + x^{58} + x^{54} + x^{52} + x^{50} + x^{46} + x^{42}\\
&+ x^{40} + x^{38} + x^{34} + x^{32} + x^{28} + x^{20} + x^{16},\\
p_6(x) &= x^{126} + x^{124} + x^{120} + x^{116} + x^{114} + x^{110} + x^{106} + x^{96} + x^{92} + x^{90}\\
&+ x^{88} + x^{86} + x^{84} + x^{80} + x^{78} + x^{76} + x^{74} + x^{72} + x^{68} + x^{56} + x^{54}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{48} + x^{46} + x^{44} + x^{40} + x^{38} + x^{36} + x^{34} + x^{32} + x^{26} + x^{24} + x^{20}\\
&+ x^{16} + x^{10} + x^{6} + x^{4} + x^{2} + 1,\\
p_9(x) &= x^{122} + x^{120} + x^{118} + x^{116} + x^{112} + x^{108} + x^{102} + x^{100} + x^{98} + x^{96}\\
&+ x^{90} + x^{88} + x^{86} + x^{78} + x^{76} + x^{72} + x^{62} + x^{58} + x^{56} + x^{52} + x^{50}\\
&+ x^{42} + x^{40} + x^{32} + x^{30} + x^{26} + x^{24} + x^{22} + x^{20} + x^{16} + x^{12} + x^{10}\\
&+ x^{6} + x^{4},\\
p_{12}(x) &= x^{122} + x^{120} + x^{106} + x^{104} + x^{98} + x^{96} + x^{82} + x^{80} + x^{58} + x^{56} + x^{42}\\
&+ x^{40} + x^{34} + x^{32} + x^{26} + x^{24} + x^{18} + x^{16} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{129} + x^{127} + x^{125} + x^{120} + x^{117} + x^{116} + x^{114} + x^{109} + x^{104} + x^{98}\\
&+ x^{96} + x^{95} + x^{94} + x^{92} + x^{91} + x^{89} + x^{88} + x^{87} + x^{82} + x^{81} + x^{78}\\
&+ x^{72} + x^{71} + x^{70} + x^{69} + x^{66} + x^{65} + x^{62} + x^{61} + x^{59} + x^{58} + x^{57}\\
&+ x^{54} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{45} + x^{44} + x^{40} + x^{38} + x^{37}\\
&+ x^{34} + x^{33},\\
p_3(x) &= x^{125} + x^{123} + x^{121} + x^{120} + x^{118} + x^{116} + x^{113} + x^{108} + x^{105} + x^{103}\\
&+ x^{100} + x^{98} + x^{93} + x^{91} + x^{89} + x^{88} + x^{86} + x^{85} + x^{84} + x^{83} + x^{80}\\
&+ x^{78} + x^{77} + x^{75} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66}\\
&+ x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55}\\
&+ x^{54} + x^{52} + x^{50} + x^{47} + x^{45} + x^{43} + x^{42} + x^{40} + x^{39} + x^{38} + x^{34}\\
&+ x^{33} + x^{28} + x^{25} + x^{23} + x^{20} + x^{18},\\
p_6(x) &= x^{123} + x^{120} + x^{119} + x^{117} + x^{116} + x^{115} + x^{114} + x^{112} + x^{111} + x^{108}\\
&+ x^{103} + x^{100} + x^{97} + x^{95} + x^{94} + x^{93} + x^{92} + x^{90} + x^{89} + x^{86} + x^{85}\\
&+ x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{76} + x^{73} + x^{70} + x^{67} + x^{65}\\
&+ x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{58} + x^{57} + x^{55} + x^{54} + x^{53} + x^{52}\\
&+ x^{50} + x^{41} + x^{39} + x^{38} + x^{37} + x^{36} + x^{34} + x^{33} + x^{31} + x^{30} + x^{29}\\
&+ x^{28} + x^{27} + x^{26} + x^{24} + x^{21} + x^{19} + x^{18} + x^{16} + x^{15} + x^{12},\\
p_9(x) &= x^{121} + x^{119} + x^{116} + x^{114} + x^{113} + x^{109} + x^{108} + x^{107} + x^{104} + x^{103}\\
&+ x^{102} + x^{99} + x^{98} + x^{97} + x^{95} + x^{94} + x^{93} + x^{92} + x^{90} + x^{89} + x^{88}\\
&+ x^{85} + x^{84} + x^{80} + x^{75} + x^{70} + x^{67} + x^{63} + x^{62} + x^{61} + x^{58} + x^{56}\\
&+ x^{53} + x^{51} + x^{48} + x^{47} + x^{46} + x^{45} + x^{42} + x^{40} + x^{37} + x^{33} + x^{32}\\
&+ x^{31} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20}\\
&+ x^{19} + x^{18} + x^{16} + x^{15} + x^{14} + x^{13} + x^{11} + x^{10} + x^{8} + x^{6},\\
p_{12}(x) &= x^{119} + x^{116} + x^{111} + x^{108} + x^{107} + x^{104} + x^{103} + x^{100} + x^{99} + x^{96}\\
&+ x^{95} + x^{92} + x^{91} + x^{88} + x^{83} + x^{80} + x^{55} + x^{52} + x^{47} + x^{44} + x^{43}
\end{aligned}
$$

$$+ x^{40} + x^{39} + x^{36} + x^{35} + x^{32} + x^{31} + x^{28} + x^{27} + x^{24} + x^{23} + x^{20} + x^{19} + x^{16} + x^{15} + x^{12} + x^{11} + x^{8} + x^{3} + 1,$$

and the initial terms are [0, 0, 1, 0, 1, 0, 1, 0, 1, 1, 0, 0, 0, 1, 1, 1].

For $\phi(W^e, 1, 0)$:

$$\begin{aligned}
p_0(x) = {}& x^{129} + x^{127} + x^{125} + x^{120} + x^{117} + x^{116} + x^{114} + x^{109} + x^{104} + x^{98}\\
&+ x^{96} + x^{95} + x^{94} + x^{92} + x^{91} + x^{89} + x^{88} + x^{87} + x^{82} + x^{81} + x^{78}\\
&+ x^{72} + x^{71} + x^{70} + x^{69} + x^{66} + x^{65} + x^{62} + x^{61} + x^{59} + x^{58} + x^{57}\\
&+ x^{54} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{45} + x^{44} + x^{40} + x^{38} + x^{37}\\
&+ x^{34} + x^{33},\\
p_3(x) = {}& x^{125} + x^{123} + x^{121} + x^{120} + x^{118} + x^{116} + x^{113} + x^{108} + x^{105} + x^{103}\\
&+ x^{100} + x^{98} + x^{93} + x^{91} + x^{89} + x^{88} + x^{86} + x^{85} + x^{84} + x^{83} + x^{80}\\
&+ x^{78} + x^{77} + x^{75} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66}\\
&+ x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55}\\
&+ x^{54} + x^{52} + x^{50} + x^{47} + x^{45} + x^{43} + x^{42} + x^{40} + x^{39} + x^{38} + x^{34}\\
&+ x^{33} + x^{28} + x^{25} + x^{23} + x^{20} + x^{18},\\
p_6(x) = {}& x^{123} + x^{120} + x^{119} + x^{117} + x^{116} + x^{115} + x^{114} + x^{112} + x^{111} + x^{108}\\
&+ x^{103} + x^{100} + x^{97} + x^{95} + x^{94} + x^{93} + x^{92} + x^{90} + x^{89} + x^{86} + x^{85}\\
&+ x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{76} + x^{73} + x^{70} + x^{67} + x^{65}\\
&+ x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{58} + x^{57} + x^{55} + x^{54} + x^{53} + x^{52}\\
&+ x^{50} + x^{41} + x^{39} + x^{38} + x^{37} + x^{36} + x^{34} + x^{33} + x^{31} + x^{30} + x^{29}\\
&+ x^{28} + x^{27} + x^{26} + x^{24} + x^{21} + x^{19} + x^{18} + x^{16} + x^{15} + x^{12},\\
p_9(x) = {}& x^{121} + x^{119} + x^{116} + x^{114} + x^{113} + x^{109} + x^{108} + x^{107} + x^{104} + x^{103}\\
&+ x^{102} + x^{99} + x^{98} + x^{97} + x^{95} + x^{94} + x^{93} + x^{92} + x^{90} + x^{89} + x^{88}\\
&+ x^{85} + x^{84} + x^{80} + x^{75} + x^{70} + x^{67} + x^{63} + x^{62} + x^{61} + x^{58} + x^{56}\\
&+ x^{53} + x^{51} + x^{48} + x^{47} + x^{46} + x^{45} + x^{42} + x^{40} + x^{37} + x^{33} + x^{32}\\
&+ x^{31} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20}\\
&+ x^{19} + x^{18} + x^{16} + x^{15} + x^{14} + x^{13} + x^{11} + x^{10} + x^{8} + x^{6},\\
p_{12}(x) = {}& x^{119} + x^{116} + x^{111} + x^{108} + x^{107} + x^{104} + x^{103} + x^{100} + x^{99} + x^{96}\\
&+ x^{95} + x^{92} + x^{91} + x^{88} + x^{83} + x^{80} + x^{55} + x^{52} + x^{47} + x^{44} + x^{43}\\
&+ x^{40} + x^{39} + x^{36} + x^{35} + x^{32} + x^{31} + x^{28} + x^{27} + x^{24} + x^{23} + x^{20}\\
&+ x^{19} + x^{16} + x^{15} + x^{12} + x^{11} + x^{8} + x^{3} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 1, 0, 1, 0, 1, 0, 1, 1, 0, 0, 0, 1, 1, 1].

For $\phi(W^e, 1, 1)$:

$$\begin{aligned}
p_0(x) = {}& x^{120} + x^{118} + x^{116} + x^{114} + x^{110} + x^{104} + x^{102} + x^{100} + x^{98} + x^{96}\\
&+ x^{94} + x^{92} + x^{90} + x^{88} + x^{84} + x^{82} + x^{78} + x^{76} + x^{70} + x^{66} + x^{64}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{62} + x^{60} + x^{58} + x^{52} + x^{48} + x^{42},\\
p_3(x) &= x^{108} + x^{104} + x^{100} + x^{96} + x^{94} + x^{92} + x^{84} + x^{82} + x^{80} + x^{72} + x^{70}\\
&+ x^{66} + x^{64} + x^{60} + x^{54} + x^{50} + x^{48} + x^{46} + x^{36} + x^{32} + x^{26} + x^{22}\\
&+ x^{20} + x^{16} + x^{14} + x^{12},\\
p_6(x) &= x^{112} + x^{110} + x^{106} + x^{100} + x^{92} + x^{72} + x^{64} + x^{56} + x^{54} + x^{52} + x^{50}\\
&+ x^{46} + x^{42} + x^{40} + x^{38} + x^{36} + x^{34} + x^{30} + x^{28} + x^{26} + x^{24} + x^{20}\\
&+ x^{16} + x^{6} + x^{2} + 1,\\
p_9(x) &= x^{116} + x^{114} + x^{110} + x^{108} + x^{106} + x^{98} + x^{96} + x^{90} + x^{88} + x^{86} + x^{84}\\
&+ x^{80} + x^{78} + x^{76} + x^{74} + x^{68} + x^{62} + x^{58} + x^{54} + x^{52} + x^{50} + x^{48}\\
&+ x^{44} + x^{34} + x^{26} + x^{22} + x^{20} + x^{16} + x^{14} + x^{12} + x^{10} + x^{8} + x^{6} + x^{4},\\
p_{12}(x) &= x^{116} + x^{114} + x^{112} + x^{84} + x^{82} + x^{80} + x^{52} + x^{50} + x^{48} + x^{4} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0].

For $\phi(M^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{129} + x^{125} + x^{124} + x^{123} + x^{121} + x^{119} + x^{117} + x^{116} + x^{114} + x^{112}\\
&+ x^{110} + x^{108} + x^{107} + x^{106} + x^{105} + x^{104} + x^{101} + x^{100} + x^{99} + x^{97}\\
&+ x^{96} + x^{89} + x^{87} + x^{84} + x^{83} + x^{82} + x^{80} + x^{76} + x^{75} + x^{74} + x^{73}\\
&+ x^{72} + x^{71} + x^{70} + x^{69} + x^{67} + x^{65} + x^{57} + x^{56} + x^{52} + x^{51} + x^{50}\\
&+ x^{49} + x^{42} + x^{41} + x^{30} + x^{28} + x^{27} + x^{24},\\
p_3(x) &= x^{127} + x^{125} + x^{124} + x^{121} + x^{120} + x^{117} + x^{115} + x^{114} + x^{113} + x^{112}\\
&+ x^{102} + x^{101} + x^{100} + x^{98} + x^{95} + x^{92} + x^{90} + x^{89} + x^{88} + x^{83} + x^{82}\\
&+ x^{81} + x^{78} + x^{75} + x^{74} + x^{73} + x^{71} + x^{69} + x^{64} + x^{63} + x^{62} + x^{61}\\
&+ x^{56} + x^{55} + x^{54} + x^{51} + x^{48} + x^{45} + x^{44} + x^{40} + x^{39} + x^{37} + x^{34}\\
&+ x^{30} + x^{27} + x^{24} + x^{22} + x^{20} + x^{19} + x^{16},\\
p_6(x) &= x^{127} + x^{125} + x^{124} + x^{121} + x^{120} + x^{115} + x^{114} + x^{107} + x^{106} + x^{105}\\
&+ x^{103} + x^{102} + x^{101} + x^{100} + x^{95} + x^{88} + x^{86} + x^{85} + x^{81} + x^{78}\\
&+ x^{72} + x^{69} + x^{67} + x^{66} + x^{65} + x^{63} + x^{62} + x^{59} + x^{57} + x^{55} + x^{54}\\
&+ x^{53} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{42} + x^{41} + x^{38} + x^{37} + x^{36}\\
&+ x^{35} + x^{34} + x^{31} + x^{29} + x^{27} + x^{22} + x^{15} + x^{14} + x^{3} + 1,\\
p_9(x) &= x^{127} + x^{124} + x^{111} + x^{108} + x^{107} + x^{104} + x^{91} + x^{88} + x^{63} + x^{60}\\
&+ x^{47} + x^{44} + x^{43} + x^{40} + x^{31} + x^{28} + x^{27} + x^{24} + x^{11} + x^{8},\\
p_{12}(x) &= x^{127} + x^{124} + x^{119} + x^{116} + x^{103} + x^{100} + x^{99} + x^{96} + x^{95} + x^{92}\\
&+ x^{83} + x^{80} + x^{63} + x^{60} + x^{55} + x^{52} + x^{39} + x^{36} + x^{35} + x^{32} + x^{23}\\
&+ x^{20} + x^{19} + x^{16} + x^{15} + x^{12} + x^{3} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1, 1, 0].

For $\phi(M^o, 0, 1)$:

$$\begin{aligned}
p_0(x) = {} & x^{138} + x^{136} + x^{132} + x^{129} + x^{128} + x^{125} + x^{123} + x^{122} + x^{120} + x^{118} \\
& + x^{111} + x^{109} + x^{106} + x^{104} + x^{101} + x^{99} + x^{98} + x^{97} + x^{95} + x^{93} \\
& + x^{82} + x^{79} + x^{77} + x^{75} + x^{74} + x^{72} + x^{71} + x^{70} + x^{69} + x^{66} + x^{65} \\
& + x^{62} + x^{61} + x^{58} + x^{57} + x^{56} + x^{55} + x^{49} + x^{45} + x^{43} + x^{38} + x^{37} \\
& + x^{33}, \\
p_3(x) = {} & x^{124} + x^{121} + x^{119} + x^{118} + x^{116} + x^{115} + x^{113} + x^{112} + x^{110} + x^{109} \\
& + x^{107} + x^{106} + x^{104} + x^{103} + x^{101} + x^{98} + x^{92} + x^{89} + x^{87} + x^{86} \\
& + x^{83} + x^{80} + x^{79} + x^{77} + x^{74} + x^{52} + x^{49} + x^{47} + x^{46} + x^{44} + x^{43} \\
& + x^{41} + x^{40} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{30} + x^{29} \\
& + x^{27} + x^{26} + x^{24} + x^{23} + x^{21} + x^{18}, \\
p_6(x) = {} & x^{126} + x^{124} + x^{122} + x^{118} + x^{114} + x^{102} + x^{94} + x^{90} + x^{86} + x^{74} \\
& + x^{70} + x^{68} + x^{66} + x^{64} + x^{62} + x^{56} + x^{50} + x^{44} + x^{38} + x^{32} + x^{30} \\
& + x^{28} + x^{26} + x^{24} + x^{22} + x^{16} + x^{14} + x^{12}, \\
p_9(x) = {} & x^{128} + x^{125} + x^{124} + x^{123} + x^{121} + x^{119} + x^{118} + x^{117} + x^{116} + x^{114} \\
& + x^{113} + x^{111} + x^{108} + x^{106} + x^{103} + x^{101} + x^{100} + x^{98} + x^{97} + x^{96} \\
& + x^{95} + x^{93} + x^{91} + x^{90} + x^{89} + x^{86} + x^{64} + x^{61} + x^{60} + x^{59} + x^{57} \\
& + x^{55} + x^{54} + x^{53} + x^{52} + x^{50} + x^{49} + x^{47} + x^{44} + x^{42} + x^{39} + x^{37} \\
& + x^{36} + x^{34} + x^{33} + x^{31} + x^{28} + x^{26} + x^{23} + x^{21} + x^{20} + x^{18} + x^{17} \\
& + x^{16} + x^{15} + x^{13} + x^{11} + x^{10} + x^{9} + x^{6}, \\
p_{12}(x) = {} & x^{130} + x^{128} + x^{118} + x^{116} + x^{106} + x^{104} + x^{102} + x^{100} + x^{90} + x^{88} \\
& + x^{82} + x^{80} + x^{66} + x^{64} + x^{54} + x^{52} + x^{42} + x^{40} + x^{38} + x^{36} + x^{34} \\
& + x^{32} + x^{26} + x^{24} + x^{22} + x^{20} + x^{10} + x^{8} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 0, 1, 1, 1, 1, 0].

For $\phi(M^o, 1, 0)$:

$$\begin{aligned}
p_0(x) = {} & x^{124} + x^{122} + x^{119} + x^{118} + x^{117} + x^{116} + x^{114} + x^{113} + x^{112} + x^{111} \\
& + x^{110} + x^{108} + x^{107} + x^{106} + x^{104} + x^{102} + x^{101} + x^{97} + x^{96} + x^{91} \\
& + x^{90} + x^{85} + x^{83} + x^{81} + x^{80} + x^{79} + x^{75} + x^{72} + x^{68} + x^{67} + x^{64} \\
& + x^{61} + x^{60} + x^{59} + x^{56} + x^{54} + x^{53} + x^{52} + x^{48} + x^{44} + x^{42} + x^{41} \\
& + x^{39} + x^{38} + x^{36} + x^{34} + x^{33}, \\
p_3(x) = {} & x^{122} + x^{120} + x^{119} + x^{116} + x^{115} + x^{114} + x^{113} + x^{111} + x^{109} + x^{107} \\
& + x^{106} + x^{105} + x^{104} + x^{101} + x^{100} + x^{98} + x^{90} + x^{88} + x^{87} + x^{84} \\
& + x^{83} + x^{81} + x^{80} + x^{77} + x^{76} + x^{74} + x^{50} + x^{48} + x^{47} + x^{44} + x^{43} \\
& + x^{42} + x^{41} + x^{39} + x^{37} + x^{35} + x^{33} + x^{31} + x^{29} + x^{27} + x^{26} + x^{25} \\
& + x^{24} + x^{21} + x^{20} + x^{18},
\end{aligned}$$

$$\begin{aligned}
p_6(x) &= x^{124} + x^{118} + x^{114} + x^{108} + x^{104} + x^{100} + x^{98} + x^{96} + x^{92} + x^{84} \\
&\quad + x^{78} + x^{76} + x^{74} + x^{70} + x^{68} + x^{66} + x^{60} + x^{56} + x^{52} + x^{50} + x^{42} \\
&\quad + x^{40} + x^{36} + x^{32} + x^{26} + x^{24} + x^{22} + x^{20} + x^{18} + x^{12}, \\
p_9(x) &= x^{126} + x^{124} + x^{123} + x^{122} + x^{120} + x^{118} + x^{115} + x^{113} + x^{107} + x^{105} \\
&\quad + x^{99} + x^{97} + x^{94} + x^{92} + x^{90} + x^{89} + x^{88} + x^{86} + x^{62} + x^{60} + x^{59} \\
&\quad + x^{58} + x^{56} + x^{54} + x^{51} + x^{49} + x^{43} + x^{41} + x^{35} + x^{33} + x^{27} + x^{25} \\
&\quad + x^{19} + x^{17} + x^{14} + x^{12} + x^{10} + x^{9} + x^{8} + x^{6}, \\
p_{12}(x) &= x^{128} + x^{124} + x^{120} + x^{116} + x^{104} + x^{92} + x^{84} + x^{80} + x^{64} + x^{60} + x^{56} \\
&\quad + x^{52} + x^{40} + x^{32} + x^{24} + x^{12} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0].

For $\phi(M^o, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{129} + x^{127} + x^{119} + x^{117} + x^{116} + x^{113} + x^{111} + x^{109} + x^{107} + x^{104} \\
&\quad + x^{101} + x^{100} + x^{97} + x^{96} + x^{95} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} \\
&\quad + x^{85} + x^{84} + x^{82} + x^{81} + x^{80} + x^{74} + x^{73} + x^{70} + x^{68} + x^{62} + x^{60} \\
&\quad + x^{58} + x^{56} + x^{55} + x^{51} + x^{50} + x^{48} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42}, \\
p_3(x) &= x^{127} + x^{125} + x^{124} + x^{123} + x^{121} + x^{115} + x^{113} + x^{112} + x^{111} + x^{109} \\
&\quad + x^{108} + x^{106} + x^{103} + x^{102} + x^{101} + x^{99} + x^{98} + x^{96} + x^{95} + x^{92} \\
&\quad + x^{91} + x^{90} + x^{89} + x^{85} + x^{81} + x^{80} + x^{79} + x^{78} + x^{76} + x^{74} + x^{73} \\
&\quad + x^{72} + x^{71} + x^{69} + x^{64} + x^{63} + x^{62} + x^{61} + x^{56} + x^{55} + x^{54} + x^{44} \\
&\quad + x^{42} + x^{40} + x^{36} + x^{35} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} \\
&\quad + x^{24} + x^{23} + x^{22}, \\
p_6(x) &= x^{127} + x^{125} + x^{124} + x^{123} + x^{121} + x^{117} + x^{112} + x^{111} + x^{108} + x^{106} \\
&\quad + x^{105} + x^{104} + x^{102} + x^{101} + x^{99} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} \\
&\quad + x^{91} + x^{90} + x^{89} + x^{83} + x^{82} + x^{80} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} \\
&\quad + x^{68} + x^{67} + x^{66} + x^{61} + x^{60} + x^{58} + x^{56} + x^{52} + x^{49} + x^{44} + x^{42} \\
&\quad + x^{36} + x^{33} + x^{32} + x^{30} + x^{28} + x^{27} + x^{26} + x^{22} + x^{21} + x^{18} + x^{14} \\
&\quad + x^{12} + x^{11} + x^{8} + x^{3} + 1, \\
p_9(x) &= x^{127} + x^{124} + x^{111} + x^{108} + x^{107} + x^{104} + x^{91} + x^{88} + x^{63} + x^{60} \\
&\quad + x^{47} + x^{44} + x^{43} + x^{40} + x^{31} + x^{28} + x^{27} + x^{24} + x^{11} + x^{8}, \\
p_{12}(x) &= x^{127} + x^{124} + x^{119} + x^{116} + x^{103} + x^{100} + x^{99} + x^{96} + x^{95} + x^{92} \\
&\quad + x^{83} + x^{80} + x^{63} + x^{60} + x^{55} + x^{52} + x^{39} + x^{36} + x^{35} + x^{32} + x^{23} \\
&\quad + x^{20} + x^{19} + x^{16} + x^{15} + x^{12} + x^{3} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 0, 1, 1, 1, 0].

For $\phi(W^o, 0, 0)$:

$$p_0(x) = x^{129} + x^{125} + x^{124} + x^{123} + x^{121} + x^{119} + x^{117} + x^{116} + x^{114} + x^{112}$$

$$
\begin{aligned}
&+ x^{110} + x^{108} + x^{107} + x^{106} + x^{105} + x^{104} + x^{101} + x^{100} + x^{99} + x^{97}\\
&+ x^{96} + x^{89} + x^{87} + x^{84} + x^{83} + x^{82} + x^{80} + x^{76} + x^{75} + x^{74} + x^{73}\\
&+ x^{72} + x^{71} + x^{70} + x^{69} + x^{67} + x^{65} + x^{57} + x^{56} + x^{52} + x^{51} + x^{50}\\
&+ x^{49} + x^{42} + x^{41} + x^{30} + x^{28} + x^{27} + x^{24},\\
p_3(x) &= x^{127} + x^{125} + x^{124} + x^{121} + x^{120} + x^{117} + x^{115} + x^{114} + x^{113} + x^{112}\\
&+ x^{102} + x^{101} + x^{100} + x^{98} + x^{95} + x^{92} + x^{90} + x^{89} + x^{88} + x^{83} + x^{82}\\
&+ x^{81} + x^{78} + x^{75} + x^{74} + x^{73} + x^{71} + x^{69} + x^{64} + x^{63} + x^{62} + x^{61}\\
&+ x^{56} + x^{55} + x^{54} + x^{51} + x^{48} + x^{45} + x^{44} + x^{40} + x^{39} + x^{37} + x^{34}\\
&+ x^{30} + x^{27} + x^{24} + x^{22} + x^{20} + x^{19} + x^{16},\\
p_6(x) &= x^{127} + x^{125} + x^{124} + x^{121} + x^{120} + x^{115} + x^{114} + x^{107} + x^{106} + x^{105}\\
&+ x^{103} + x^{102} + x^{101} + x^{100} + x^{95} + x^{88} + x^{86} + x^{85} + x^{81} + x^{78}\\
&+ x^{72} + x^{69} + x^{67} + x^{66} + x^{65} + x^{63} + x^{62} + x^{59} + x^{57} + x^{55} + x^{54}\\
&+ x^{53} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{42} + x^{41} + x^{38} + x^{37} + x^{36}\\
&+ x^{35} + x^{34} + x^{31} + x^{29} + x^{27} + x^{22} + x^{15} + x^{14} + x^{3} + 1,\\
p_9(x) &= x^{127} + x^{124} + x^{111} + x^{108} + x^{107} + x^{104} + x^{91} + x^{88} + x^{63} + x^{60}\\
&+ x^{47} + x^{44} + x^{43} + x^{40} + x^{31} + x^{28} + x^{27} + x^{24} + x^{11} + x^{8},\\
p_{12}(x) &= x^{127} + x^{124} + x^{119} + x^{116} + x^{103} + x^{100} + x^{99} + x^{96} + x^{95} + x^{92}\\
&+ x^{83} + x^{80} + x^{63} + x^{60} + x^{55} + x^{52} + x^{39} + x^{36} + x^{35} + x^{32} + x^{23}\\
&+ x^{20} + x^{19} + x^{16} + x^{15} + x^{12} + x^{3} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1, 1, 0].

For $\phi(W^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{124} + x^{122} + x^{119} + x^{118} + x^{117} + x^{116} + x^{114} + x^{113} + x^{112} + x^{111}\\
&+ x^{110} + x^{108} + x^{107} + x^{106} + x^{104} + x^{102} + x^{101} + x^{97} + x^{96} + x^{91}\\
&+ x^{90} + x^{85} + x^{83} + x^{81} + x^{80} + x^{79} + x^{75} + x^{72} + x^{68} + x^{67} + x^{64}\\
&+ x^{61} + x^{60} + x^{59} + x^{56} + x^{54} + x^{53} + x^{52} + x^{48} + x^{44} + x^{42} + x^{41}\\
&+ x^{39} + x^{38} + x^{36} + x^{34} + x^{33},\\
p_3(x) &= x^{122} + x^{120} + x^{119} + x^{116} + x^{115} + x^{114} + x^{113} + x^{111} + x^{109} + x^{107}\\
&+ x^{106} + x^{105} + x^{104} + x^{101} + x^{100} + x^{98} + x^{90} + x^{88} + x^{87} + x^{84}\\
&+ x^{83} + x^{81} + x^{80} + x^{77} + x^{76} + x^{74} + x^{50} + x^{48} + x^{47} + x^{44} + x^{43}\\
&+ x^{42} + x^{41} + x^{39} + x^{37} + x^{35} + x^{33} + x^{31} + x^{29} + x^{27} + x^{26} + x^{25}\\
&+ x^{24} + x^{21} + x^{20} + x^{18},\\
p_6(x) &= x^{124} + x^{118} + x^{114} + x^{108} + x^{104} + x^{100} + x^{98} + x^{96} + x^{92} + x^{84}\\
&+ x^{78} + x^{76} + x^{74} + x^{70} + x^{68} + x^{66} + x^{60} + x^{56} + x^{52} + x^{50} + x^{42}\\
&+ x^{40} + x^{36} + x^{32} + x^{26} + x^{24} + x^{22} + x^{20} + x^{18} + x^{12},\\
p_9(x) &= x^{126} + x^{124} + x^{123} + x^{122} + x^{120} + x^{118} + x^{115} + x^{113} + x^{107} + x^{105}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{99} + x^{97} + x^{94} + x^{92} + x^{90} + x^{89} + x^{88} + x^{86} + x^{62} + x^{60} + x^{59}\\
&+ x^{58} + x^{56} + x^{54} + x^{51} + x^{49} + x^{43} + x^{41} + x^{35} + x^{33} + x^{27} + x^{25}\\
&+ x^{19} + x^{17} + x^{14} + x^{12} + x^{10} + x^{9} + x^{8} + x^{6},\\
p_{12}(x) &= x^{128} + x^{124} + x^{120} + x^{116} + x^{104} + x^{92} + x^{84} + x^{80} + x^{64} + x^{60} + x^{56}\\
&+ x^{52} + x^{40} + x^{32} + x^{24} + x^{12} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0].

For $\phi(W^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{138} + x^{136} + x^{132} + x^{129} + x^{128} + x^{125} + x^{123} + x^{122} + x^{120} + x^{118}\\
&+ x^{111} + x^{109} + x^{106} + x^{104} + x^{101} + x^{99} + x^{98} + x^{97} + x^{95} + x^{93}\\
&+ x^{82} + x^{79} + x^{77} + x^{75} + x^{74} + x^{72} + x^{71} + x^{70} + x^{69} + x^{66} + x^{65}\\
&+ x^{62} + x^{61} + x^{58} + x^{57} + x^{56} + x^{55} + x^{49} + x^{45} + x^{43} + x^{38} + x^{37}\\
&+ x^{33},\\
p_3(x) &= x^{124} + x^{121} + x^{119} + x^{118} + x^{116} + x^{115} + x^{113} + x^{112} + x^{110} + x^{109}\\
&+ x^{107} + x^{106} + x^{104} + x^{103} + x^{101} + x^{98} + x^{92} + x^{89} + x^{87} + x^{86}\\
&+ x^{83} + x^{80} + x^{79} + x^{77} + x^{74} + x^{52} + x^{49} + x^{47} + x^{46} + x^{44} + x^{43}\\
&+ x^{41} + x^{40} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{30} + x^{29}\\
&+ x^{27} + x^{26} + x^{24} + x^{23} + x^{21} + x^{18},\\
p_6(x) &= x^{126} + x^{124} + x^{122} + x^{118} + x^{114} + x^{102} + x^{94} + x^{90} + x^{86} + x^{74}\\
&+ x^{70} + x^{68} + x^{66} + x^{64} + x^{62} + x^{56} + x^{50} + x^{44} + x^{38} + x^{32} + x^{30}\\
&+ x^{28} + x^{26} + x^{24} + x^{22} + x^{16} + x^{14} + x^{12},\\
p_9(x) &= x^{128} + x^{125} + x^{124} + x^{123} + x^{121} + x^{119} + x^{118} + x^{117} + x^{116} + x^{114}\\
&+ x^{113} + x^{111} + x^{108} + x^{106} + x^{103} + x^{101} + x^{100} + x^{98} + x^{97} + x^{96}\\
&+ x^{95} + x^{93} + x^{91} + x^{90} + x^{89} + x^{86} + x^{64} + x^{61} + x^{60} + x^{59} + x^{57}\\
&+ x^{55} + x^{54} + x^{53} + x^{52} + x^{50} + x^{49} + x^{47} + x^{44} + x^{42} + x^{39} + x^{37}\\
&+ x^{36} + x^{34} + x^{33} + x^{31} + x^{28} + x^{26} + x^{23} + x^{21} + x^{20} + x^{18} + x^{17}\\
&+ x^{16} + x^{15} + x^{13} + x^{11} + x^{10} + x^{9} + x^{6},\\
p_{12}(x) &= x^{130} + x^{128} + x^{118} + x^{116} + x^{106} + x^{104} + x^{102} + x^{100} + x^{90} + x^{88}\\
&+ x^{82} + x^{80} + x^{66} + x^{64} + x^{54} + x^{52} + x^{42} + x^{40} + x^{38} + x^{36} + x^{34}\\
&+ x^{32} + x^{26} + x^{24} + x^{22} + x^{20} + x^{10} + x^{8} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 0, 1, 1, 1, 1, 0].

For $\phi(W^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{129} + x^{127} + x^{119} + x^{117} + x^{116} + x^{113} + x^{111} + x^{109} + x^{107} + x^{104}\\
&+ x^{101} + x^{100} + x^{97} + x^{96} + x^{95} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86}\\
&+ x^{85} + x^{84} + x^{82} + x^{81} + x^{80} + x^{74} + x^{73} + x^{70} + x^{68} + x^{62} + x^{60}\\
&+ x^{58} + x^{56} + x^{55} + x^{51} + x^{50} + x^{48} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42},
\end{aligned}
$$

$$\begin{aligned}
p_3(x) &= x^{127} + x^{125} + x^{124} + x^{123} + x^{121} + x^{115} + x^{113} + x^{112} + x^{111} + x^{109} \\
&\quad + x^{108} + x^{106} + x^{103} + x^{102} + x^{101} + x^{99} + x^{98} + x^{96} + x^{95} + x^{92} \\
&\quad + x^{91} + x^{90} + x^{89} + x^{85} + x^{81} + x^{80} + x^{79} + x^{78} + x^{76} + x^{74} + x^{73} \\
&\quad + x^{72} + x^{71} + x^{69} + x^{64} + x^{63} + x^{62} + x^{61} + x^{56} + x^{55} + x^{54} + x^{44} \\
&\quad + x^{42} + x^{40} + x^{36} + x^{35} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} \\
&\quad + x^{24} + x^{23} + x^{22}, \\
p_6(x) &= x^{127} + x^{125} + x^{124} + x^{123} + x^{121} + x^{117} + x^{112} + x^{111} + x^{108} + x^{106} \\
&\quad + x^{105} + x^{104} + x^{102} + x^{101} + x^{99} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} \\
&\quad + x^{91} + x^{90} + x^{89} + x^{83} + x^{82} + x^{80} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} \\
&\quad + x^{68} + x^{67} + x^{66} + x^{61} + x^{60} + x^{58} + x^{56} + x^{52} + x^{49} + x^{44} + x^{42} \\
&\quad + x^{36} + x^{33} + x^{32} + x^{30} + x^{28} + x^{27} + x^{26} + x^{22} + x^{21} + x^{18} + x^{14} \\
&\quad + x^{12} + x^{11} + x^{8} + x^{3} + 1, \\
p_9(x) &= x^{127} + x^{124} + x^{111} + x^{108} + x^{107} + x^{104} + x^{91} + x^{88} + x^{63} + x^{60} \\
&\quad + x^{47} + x^{44} + x^{43} + x^{40} + x^{31} + x^{28} + x^{27} + x^{24} + x^{11} + x^{8}, \\
p_{12}(x) &= x^{127} + x^{124} + x^{119} + x^{116} + x^{103} + x^{100} + x^{99} + x^{96} + x^{95} + x^{92} \\
&\quad + x^{83} + x^{80} + x^{63} + x^{60} + x^{55} + x^{52} + x^{39} + x^{36} + x^{35} + x^{32} + x^{23} \\
&\quad + x^{20} + x^{19} + x^{16} + x^{15} + x^{12} + x^{3} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 0, 1, 1, 1, 0].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	311	[516, 517]	622	[180, 182]	933	[1066, 1067]
1	[3, 4]	312	[518, 519]	623	[217, 533]	934	[1068, 1069]
2	[5, 6]	313	[520, 421]	624	[217, 217]	935	[1070, 751]
3	[7, 8]	314	[521, 522]	625	[458, 535]	936	[98, 1071]
4	[9, 10]	315	[523, 421]	626	[925, 153]	937	[1072, 1073]
5	[11, 12]	316	[120, 524]	627	[576, 547]	938	[1074, 838]
6	[13, 14]	317	[399, 188]	628	[912, 157]	939	[1075, 1076]
7	[15, 16]	318	[457, 525]	629	[400, 189]	940	[66, 1077]
8	[17, 18]	319	[515, 526]	630	[198, 420]	941	[1078, 1079]
9	[19, 20]	320	[527, 120]	631	[214, 573]	942	[1080, 1081]
10	[21, 22]	321	[528, 428]	632	[417, 48]	943	[1082, 278]
11	[23, 24]	322	[529, 530]	633	[926, 449]	944	[655, 725]
12	[25, 26]	323	[154, 146]	634	[927, 549]	945	[1083, 1084]
13	[27, 28]	324	[410, 144]	635	[548, 928]	946	[1085, 784]
14	[29, 30]	325	[531, 503]	636	[549, 467]	947	[277, 1086]
15	[31, 32]	326	[532, 448]	637	[142, 475]	948	[1016, 17]
16	[33, 34]	327	[533, 534]	638	[416, 929]	949	[1087, 1088]

17	[35, 36]	328	[459, 535]	639	[472, 471]	950	[1089, 1090]
18	[37, 35]	329	[536, 459]	640	[202, 488]	951	[1091, 1092]
19	[38, 39]	330	[534, 217]	641	[543, 214]	952	[1093, 749]
20	[40, 41]	331	[537, 535]	642	[930, 931]	953	[682, 1094]
21	[42, 43]	332	[538, 536]	643	[142, 142]	954	[1095, 1096]
22	[44, 45]	333	[534, 534]	644	[38, 932]	955	[1097, 1098]
23	[46, 47]	334	[479, 539]	645	[933, 549]	956	[1099, 1100]
24	[48, 49]	335	[208, 540]	646	[934, 922]	957	[1101, 1102]
25	[50, 51]	336	[541, 481]	647	[545, 903]	958	[297, 663]
26	[52, 53]	337	[542, 543]	648	[485, 935]	959	[1103, 1104]
27	[54, 55]	338	[59, 544]	649	[575, 157]	960	[1069, 1105]
28	[56, 57]	339	[545, 546]	650	[9, 932]	961	[1106, 1107]
29	[58, 59]	340	[151, 547]	651	[399, 146]	962	[1108, 1109]
30	[60, 49]	341	[499, 548]	652	[158, 462]	963	[1110, 1111]
31	[61, 62]	342	[430, 549]	653	[936, 435]	964	[706, 1112]
32	[63, 64]	343	[445, 424]	654	[35, 937]	965	[1113, 1114]
33	[65, 66]	344	[464, 205]	655	[938, 167]	966	[1115, 634]
34	[67, 68]	345	[550, 551]	656	[939, 505]	967	[1116, 1117]
35	[69, 70]	346	[164, 490]	657	[170, 940]	968	[1118, 845]
36	[30, 71]	347	[552, 157]	658	[941, 901]	969	[1119, 587]
37	[72, 73]	348	[502, 552]	659	[558, 130]	970	[1120, 1121]
38	[74, 75]	349	[190, 553]	660	[179, 458]	971	[666, 357]
39	[76, 77]	350	[479, 509]	661	[910, 153]	972	[1122, 1123]
40	[78, 79]	351	[418, 447]	662	[396, 443]	973	[1124, 1125]
41	[80, 81]	352	[554, 555]	663	[493, 402]	974	[1126, 1127]
42	[82, 83]	353	[556, 557]	664	[502, 156]	975	[1128, 1119]
43	[84, 85]	354	[409, 126]	665	[438, 942]	976	[1129, 1130]
44	[86, 87]	355	[558, 559]	666	[943, 53]	977	[1131, 1132]
45	[88, 89]	356	[481, 135]	667	[900, 560]	978	[1133, 1134]
46	[90, 91]	357	[451, 560]	668	[944, 901]	979	[1135, 1136]
47	[92, 93]	358	[561, 210]	669	[542, 924]	980	[1086, 1137]
48	[94, 95]	359	[179, 459]	670	[140, 929]	981	[1138, 1139]
49	[96, 94]	360	[441, 562]	671	[487, 906]	982	[1140, 735]
50	[97, 98]	361	[164, 563]	672	[918, 36]	983	[1141, 1142]
51	[99, 100]	362	[564, 565]	673	[489, 563]	984	[1143, 1144]
52	[101, 102]	363	[566, 493]	674	[458, 182]	985	[1145, 1146]
53	[103, 104]	364	[567, 568]	675	[215, 536]	986	[1147, 1148]
54	[105, 106]	365	[214, 193]	676	[180, 215]	987	[798, 1149]
55	[107, 108]	366	[194, 555]	677	[180, 535]	988	[1150, 1151]
56	[109, 110]	367	[200, 539]	678	[176, 945]	989	[1152, 310]
57	[111, 112]	368	[527, 569]	679	[946, 134]	990	[70, 1122]
58	[113, 114]	369	[127, 526]	680	[947, 175]	991	[1153, 1154]
59	[115, 116]	370	[570, 409]	681	[560, 948]	992	[758, 628]
60	[117, 118]	371	[56, 571]	682	[576, 185]	993	[1155, 887]

61	[119, 120]	372	[572, 573]	683	[938, 576]	994	[1156, 1157]
62	[121, 122]	373	[47, 574]	684	[493, 501]	995	[1158, 1159]
63	[123, 124]	374	[575, 491]	685	[55, 509]	996	[1160, 1161]
64	[125, 126]	375	[499, 564]	686	[541, 540]	997	[1162, 788]
65	[127, 128]	376	[548, 187]	687	[941, 120]	998	[1163, 1164]
66	[129, 130]	377	[166, 576]	688	[949, 409]	999	[1165, 1166]
67	[131, 132]	378	[202, 398]	689	[410, 147]	1000	[1167, 1168]
68	[133, 134]	379	[572, 577]	690	[581, 461]	1001	[1169, 1170]
69	[135, 136]	380	[578, 480]	691	[950, 159]	1002	[1171, 1172]
70	[52, 137]	381	[208, 579]	692	[951, 163]	1003	[1173, 220]
71	[59, 138]	382	[528, 520]	693	[450, 191]	1004	[795, 1078]
72	[139, 140]	383	[580, 497]	694	[440, 404]	1005	[1174, 96]
73	[141, 142]	384	[581, 502]	695	[538, 584]	1006	[1175, 1176]
74	[143, 144]	385	[580, 495]	696	[536, 180]	1007	[1177, 1178]
75	[145, 146]	386	[531, 45]	697	[535, 536]	1008	[1179, 351]
76	[143, 147]	387	[436, 582]	698	[952, 489]	1009	[1180, 1181]
77	[148, 149]	388	[583, 197]	699	[927, 492]	1010	[1182, 1183]
78	[150, 151]	389	[524, 568]	700	[38, 10]	1011	[910, 987]
79	[152, 153]	390	[421, 404]	701	[953, 954]	1012	[471, 563]
80	[154, 155]	391	[534, 533]	702	[396, 36]	1013	[912, 1184]
81	[156, 157]	392	[535, 584]	703	[135, 955]	1014	[160, 564]
82	[158, 159]	393	[459, 215]	704	[457, 404]	1015	[14, 35]
83	[40, 160]	394	[584, 459]	705	[421, 956]	1016	[550, 184]
84	[161, 162]	395	[538, 181]	706	[941, 957]	1017	[552, 491]
85	[8, 163]	396	[585, 586]	707	[515, 958]	1018	[58, 1006]
86	[164, 165]	397	[587, 588]	708	[959, 54]	1019	[127, 958]
87	[166, 167]	398	[589, 590]	709	[922, 960]	1020	[975, 924]
88	[168, 169]	399	[591, 592]	710	[468, 144]	1021	[902, 977]
89	[170, 171]	400	[593, 374]	711	[496, 549]	1022	[440, 956]
90	[172, 173]	401	[594, 595]	712	[204, 961]	1023	[999, 520]
91	[174, 175]	402	[596, 597]	713	[472, 489]	1024	[189, 1185]
92	[176, 177]	403	[598, 599]	714	[31, 34]	1025	[566, 467]
93	[178, 52]	404	[600, 601]	715	[962, 963]	1026	[1186, 510]
94	[179, 180]	405	[602, 603]	716	[476, 899]	1027	[908, 48]
95	[181, 180]	406	[604, 605]	717	[964, 965]	1028	[406, 526]
96	[182, 181]	407	[291, 606]	718	[60, 447]	1029	[578, 452]
97	[183, 184]	408	[607, 608]	719	[426, 958]	1030	[959, 133]
98	[167, 185]	409	[609, 103]	720	[453, 517]	1031	[152, 546]
99	[186, 187]	410	[610, 611]	721	[966, 456]	1032	[918, 443]
100	[188, 189]	411	[4, 612]	722	[54, 134]	1033	[930, 996]
101	[190, 191]	412	[613, 614]	723	[217, 216]	1034	[49, 197]
102	[192, 193]	413	[615, 616]	724	[216, 533]	1035	[172, 481]
103	[142, 141]	414	[110, 617]	725	[967, 442]	1036	[468, 147]
104	[194, 195]	415	[618, 619]	726	[167, 547]	1037	[919, 954]

105	[196, 194]	416	[620, 621]	727	[564, 187]	1038	[917, 973]
106	[197, 48]	417	[622, 623]	728	[207, 552]	1039	[465, 184]
107	[198, 199]	418	[624, 625]	729	[524, 907]	1040	[1187, 1188]
108	[200, 201]	419	[626, 627]	730	[126, 137]	1041	[447, 908]
109	[202, 203]	420	[628, 629]	731	[450, 553]	1042	[55, 201]
110	[204, 205]	421	[630, 631]	732	[943, 137]	1043	[12, 961]
111	[189, 206]	422	[632, 633]	733	[460, 207]	1044	[166, 151]
112	[207, 156]	423	[634, 635]	734	[968, 502]	1045	[1189, 169]
113	[208, 173]	424	[343, 636]	735	[133, 200]	1046	[506, 963]
114	[209, 210]	425	[637, 638]	736	[419, 57]	1047	[473, 965]
115	[211, 212]	426	[639, 640]	737	[508, 955]	1048	[513, 908]
116	[213, 214]	427	[641, 630]	738	[969, 193]	1049	[898, 477]
117	[215, 179]	428	[642, 643]	739	[426, 128]	1050	[429, 497]
118	[216, 217]	429	[644, 645]	740	[970, 483]	1051	[498, 160]
119	[218, 219]	430	[646, 647]	741	[971, 184]	1052	[1190, 931]
120	[220, 221]	431	[648, 649]	742	[971, 972]	1053	[197, 418]
121	[222, 223]	432	[650, 651]	743	[973, 36]	1054	[943, 1191]
122	[224, 225]	433	[652, 653]	744	[126, 53]	1055	[455, 1192]
123	[226, 227]	434	[654, 655]	745	[974, 452]	1056	[924, 969]
124	[228, 229]	435	[656, 657]	746	[975, 543]	1057	[952, 977]
125	[230, 231]	436	[658, 659]	747	[948, 57]	1058	[549, 493]
126	[232, 233]	437	[660, 260]	748	[478, 53]	1059	[452, 419]
127	[87, 97]	438	[661, 662]	749	[447, 197]	1060	[527, 901]
128	[234, 235]	439	[663, 664]	750	[47, 929]	1061	[989, 517]
129	[236, 65]	440	[665, 666]	751	[976, 424]	1062	[966, 1192]
130	[237, 238]	441	[246, 667]	752	[41, 548]	1063	[409, 52]
131	[239, 240]	442	[668, 669]	753	[904, 41]	1064	[38, 530]
132	[241, 242]	443	[73, 670]	754	[486, 928]	1065	[936, 470]
133	[243, 244]	444	[671, 672]	755	[971, 466]	1066	[970, 151]
134	[245, 246]	445	[592, 673]	756	[977, 490]	1067	[487, 398]
135	[247, 248]	446	[674, 675]	757	[42, 495]	1068	[512, 1006]
136	[249, 250]	447	[676, 677]	758	[145, 400]	1069	[475, 142]
137	[251, 252]	448	[678, 679]	759	[9, 530]	1070	[498, 41]
138	[114, 253]	449	[680, 681]	760	[463, 503]	1071	[37, 918]
139	[254, 255]	450	[378, 682]	761	[978, 973]	1072	[908, 446]
140	[256, 257]	451	[683, 626]	762	[979, 442]	1073	[474, 555]
141	[258, 259]	452	[83, 684]	763	[494, 43]	1074	[46, 140]
142	[260, 258]	453	[685, 686]	764	[469, 435]	1075	[974, 480]
143	[261, 262]	454	[687, 688]	765	[11, 922]	1076	[129, 559]
144	[263, 264]	455	[689, 690]	766	[905, 488]	1077	[128, 50]
145	[265, 266]	456	[691, 692]	767	[976, 206]	1078	[921, 12]
146	[267, 268]	457	[693, 694]	768	[484, 980]	1079	[550, 466]
147	[269, 270]	458	[695, 696]	769	[973, 443]	1080	[444, 1185]
148	[271, 272]	459	[391, 697]	770	[41, 564]	1081	[423, 548]

149	[16, 273]	460	[698, 286]	771	[581, 935]	1082	[1193, 942]
150	[274, 275]	461	[699, 288]	772	[500, 402]	1083	[993, 39]
151	[276, 277]	462	[700, 701]	773	[981, 982]	1084	[1194, 954]
152	[278, 279]	463	[702, 604]	774	[962, 507]	1085	[997, 448]
153	[280, 281]	464	[703, 704]	775	[405, 579]	1086	[541, 173]
154	[282, 283]	465	[705, 706]	776	[121, 210]	1087	[133, 479]
155	[284, 285]	466	[707, 708]	777	[983, 132]	1088	[438, 51]
156	[286, 287]	467	[647, 709]	778	[520, 457]	1089	[468, 413]
157	[288, 289]	468	[710, 711]	779	[984, 985]	1090	[916, 41]
158	[290, 291]	469	[712, 713]	780	[986, 134]	1091	[167, 1195]
159	[292, 293]	470	[714, 715]	781	[128, 438]	1092	[938, 151]
160	[294, 295]	471	[716, 222]	782	[925, 987]	1093	[139, 47]
161	[296, 297]	472	[717, 718]	783	[204, 960]	1094	[909, 922]
162	[298, 299]	473	[719, 720]	784	[583, 908]	1095	[1196, 982]
163	[300, 301]	474	[721, 722]	785	[448, 511]	1096	[32, 940]
164	[302, 303]	475	[723, 724]	786	[923, 988]	1097	[493, 1009]
165	[304, 305]	476	[725, 726]	787	[428, 421]	1098	[431, 467]
166	[306, 307]	477	[727, 728]	788	[989, 454]	1099	[418, 513]
167	[308, 309]	478	[729, 730]	789	[481, 450]	1100	[582, 574]
168	[310, 311]	479	[731, 732]	790	[900, 452]	1101	[933, 566]
169	[312, 313]	480	[733, 111]	791	[990, 559]	1102	[444, 424]
170	[314, 315]	481	[734, 347]	792	[216, 216]	1103	[913, 943]
171	[62, 316]	482	[735, 736]	793	[182, 536]	1104	[190, 136]
172	[18, 247]	483	[737, 738]	794	[397, 203]	1105	[1006, 544]
173	[317, 249]	484	[108, 739]	795	[471, 490]	1106	[1005, 446]
174	[318, 319]	485	[740, 741]	796	[37, 396]	1107	[510, 449]
175	[320, 321]	486	[742, 743]	797	[478, 991]	1108	[983, 1197]
176	[322, 323]	487	[744, 618]	798	[974, 560]	1109	[543, 969]
177	[324, 325]	488	[745, 746]	799	[451, 992]	1110	[1198, 1199]
178	[326, 251]	489	[747, 748]	800	[427, 520]	1111	[120, 476]
179	[327, 328]	490	[749, 750]	801	[993, 10]	1112	[174, 1199]
180	[329, 330]	491	[751, 752]	802	[993, 932]	1113	[1200, 177]
181	[331, 332]	492	[753, 754]	803	[434, 470]	1114	[213, 969]
182	[333, 329]	493	[755, 756]	804	[532, 926]	1115	[993, 530]
183	[334, 335]	494	[757, 758]	805	[529, 10]	1116	[126, 1191]
184	[336, 337]	495	[759, 760]	806	[950, 433]	1117	[527, 957]
185	[307, 338]	496	[761, 762]	807	[994, 45]	1118	[934, 464]
186	[339, 340]	497	[763, 764]	808	[914, 554]	1119	[1193, 439]
187	[295, 341]	498	[765, 766]	809	[437, 475]	1120	[1186, 1188]
188	[342, 343]	499	[690, 767]	810	[931, 544]	1121	[48, 447]
189	[266, 344]	500	[768, 769]	811	[145, 188]	1122	[944, 957]
190	[345, 346]	501	[754, 770]	812	[995, 471]	1123	[1201, 409]
191	[347, 348]	502	[771, 772]	813	[925, 903]	1124	[50, 1202]
192	[349, 350]	503	[773, 774]	814	[523, 440]	1125	[457, 422]

193	[351, 352]	504	[775, 776]	815	[42, 915]	1126	[995, 164]
194	[353, 354]	505	[777, 778]	816	[158, 162]	1127	[967, 911]
195	[355, 356]	506	[779, 780]	817	[953, 920]	1128	[543, 192]
196	[357, 358]	507	[311, 781]	818	[12, 205]	1129	[979, 980]
197	[118, 359]	508	[782, 783]	819	[14, 918]	1130	[483, 547]
198	[360, 361]	509	[784, 785]	820	[537, 538]	1131	[514, 1202]
199	[362, 363]	510	[786, 787]	821	[535, 181]	1132	[969, 573]
200	[364, 365]	511	[788, 789]	822	[533, 217]	1133	[964, 474]
201	[106, 366]	512	[790, 791]	823	[459, 538]	1134	[1005, 48]
202	[367, 368]	513	[792, 793]	824	[536, 537]	1135	[192, 998]
203	[369, 370]	514	[794, 795]	825	[537, 215]	1136	[172, 540]
204	[371, 372]	515	[796, 661]	826	[446, 513]	1137	[1201, 125]
205	[24, 373]	516	[797, 798]	827	[996, 138]	1138	[541, 579]
206	[374, 375]	517	[799, 800]	828	[916, 423]	1139	[1198, 175]
207	[323, 376]	518	[801, 753]	829	[564, 928]	1140	[123, 519]
208	[377, 378]	519	[802, 803]	830	[183, 466]	1141	[950, 162]
209	[379, 380]	520	[804, 632]	831	[997, 926]	1142	[485, 502]
210	[381, 382]	521	[805, 284]	832	[437, 141]	1143	[580, 43]
211	[383, 384]	522	[806, 807]	833	[452, 948]	1144	[412, 400]
212	[385, 386]	523	[808, 600]	834	[214, 998]	1145	[529, 932]
213	[387, 388]	524	[219, 702]	835	[941, 569]	1146	[994, 503]
214	[389, 390]	525	[809, 810]	836	[119, 957]	1147	[1190, 996]
215	[328, 391]	526	[811, 99]	837	[999, 428]	1148	[418, 49]
216	[392, 393]	527	[812, 813]	838	[908, 418]	1149	[570, 125]
217	[394, 395]	528	[814, 693]	839	[140, 574]	1150	[529, 39]
218	[14, 396]	529	[815, 734]	840	[922, 205]	1151	[951, 149]
219	[397, 398]	530	[816, 817]	841	[430, 492]	1152	[969, 577]
220	[399, 400]	531	[818, 819]	842	[406, 128]	1153	[197, 60]
221	[401, 402]	532	[667, 680]	843	[578, 560]	1154	[416, 574]
222	[403, 404]	533	[533, 533]	844	[1000, 520]	1155	[158, 433]
223	[405, 173]	534	[820, 394]	845	[979, 911]	1156	[576, 1195]
224	[406, 407]	535	[332, 821]	846	[464, 960]	1157	[995, 489]
225	[408, 409]	536	[697, 822]	847	[186, 565]	1158	[1187, 510]
226	[410, 411]	537	[823, 331]	848	[423, 564]	1159	[141, 141]
227	[412, 146]	538	[824, 825]	849	[211, 1001]	1160	[1200, 945]
228	[410, 413]	539	[826, 827]	850	[1002, 421]	1161	[1002, 457]
229	[414, 149]	540	[828, 829]	851	[560, 419]	1162	[208, 481]
230	[415, 416]	541	[272, 830]	852	[950, 462]	1163	[49, 1005]
231	[417, 418]	542	[315, 389]	853	[154, 400]	1164	[582, 929]
232	[419, 420]	543	[831, 832]	854	[1003, 144]	1165	[986, 200]
233	[421, 422]	544	[791, 833]	855	[1004, 470]	1166	[524, 477]
234	[40, 423]	545	[834, 835]	856	[403, 525]	1167	[125, 943]
235	[189, 424]	546	[836, 837]	857	[172, 579]	1168	[1193, 51]
236	[134, 425]	547	[838, 839]	858	[949, 125]	1169	[432, 159]

237	[426, 407]	548	[766, 840]	859	[496, 566]	1170	[1194, 920]
238	[427, 428]	549	[841, 596]	860	[186, 928]	1171	[415, 582]
239	[429, 43]	550	[631, 842]	861	[944, 569]	1172	[446, 49]
240	[430, 431]	551	[843, 844]	862	[1000, 428]	1173	[161, 433]
241	[432, 433]	552	[845, 846]	863	[1005, 418]	1174	[182, 179]
242	[434, 435]	553	[847, 848]	864	[1006, 138]	1175	[556, 988]
243	[436, 416]	554	[849, 850]	865	[484, 911]	1176	[54, 200]
244	[437, 142]	555	[358, 851]	866	[977, 563]	1177	[500, 1203]
245	[438, 439]	556	[852, 853]	867	[445, 1007]	1178	[155, 445]
246	[440, 422]	557	[854, 855]	868	[461, 156]	1179	[196, 554]
247	[441, 442]	558	[856, 843]	869	[445, 206]	1180	[976, 1007]
248	[35, 443]	559	[857, 858]	870	[183, 551]	1181	[146, 445]
249	[444, 206]	560	[859, 860]	871	[400, 445]	1182	[432, 462]
250	[188, 445]	561	[303, 861]	872	[968, 207]	1183	[927, 431]
251	[446, 447]	562	[319, 862]	873	[1008, 192]	1184	[1204, 1205]
252	[448, 449]	563	[863, 864]	874	[401, 1009]	1185	[1206, 1207]
253	[173, 450]	564	[865, 866]	875	[160, 548]	1186	[1117, 1208]
254	[451, 452]	565	[867, 868]	876	[419, 199]	1187	[1209, 1210]
255	[453, 454]	566	[227, 869]	877	[926, 511]	1188	[1211, 1212]
256	[455, 456]	567	[870, 818]	878	[173, 135]	1189	[1213, 1110]
257	[428, 457]	568	[221, 871]	879	[465, 972]	1190	[1028, 1214]
258	[458, 215]	569	[872, 727]	880	[483, 185]	1191	[1215, 1216]
259	[181, 459]	570	[873, 101]	881	[948, 571]	1192	[1217, 1218]
260	[215, 181]	571	[874, 875]	882	[403, 422]	1193	[1219, 1220]
261	[9, 39]	572	[876, 744]	883	[417, 60]	1194	[1221, 1126]
262	[460, 461]	573	[832, 877]	884	[931, 138]	1195	[1222, 1223]
263	[161, 462]	574	[91, 878]	885	[921, 204]	1196	[380, 1224]
264	[463, 45]	575	[879, 880]	886	[397, 488]	1197	[1225, 1226]
265	[11, 464]	576	[881, 882]	887	[933, 431]	1198	[738, 1227]
266	[465, 466]	577	[883, 884]	888	[401, 501]	1199	[1228, 1229]
267	[412, 155]	578	[885, 886]	889	[143, 413]	1200	[1230, 1070]
268	[467, 402]	579	[887, 888]	890	[968, 461]	1201	[313, 232]
269	[468, 411]	580	[889, 890]	891	[978, 918]	1202	[1231, 1232]
270	[469, 470]	581	[891, 892]	892	[545, 153]	1203	[1233, 1234]
271	[471, 165]	582	[893, 894]	893	[1010, 1001]	1204	[486, 1235]
272	[472, 164]	583	[895, 624]	894	[946, 200]	1205	[146, 189]
273	[32, 171]	584	[896, 897]	895	[181, 458]	1206	[548, 1235]
274	[473, 474]	585	[898, 899]	896	[584, 537]	1207	[461, 552]
275	[475, 141]	586	[572, 193]	897	[584, 180]	1208	[209, 122]
276	[476, 477]	587	[192, 573]	898	[1011, 1012]	1209	[406, 958]
277	[478, 137]	588	[900, 480]	899	[1013, 1014]	1210	[990, 130]
278	[479, 201]	589	[119, 901]	900	[1015, 1016]	1211	[1010, 212]
279	[451, 480]	590	[482, 54]	901	[611, 1017]	1212	[178, 126]
280	[405, 481]	591	[902, 489]	902	[1018, 863]	1213	[944, 120]

281	[482, 133]	592	[152, 903]	903	[1019, 1020]	1214	[947, 1199]
282	[150, 483]	593	[904, 499]	904	[1021, 865]	1215	[583, 1005]
283	[484, 442]	594	[905, 906]	905	[1022, 90]	1216	[996, 544]
284	[485, 461]	595	[151, 185]	906	[223, 1023]	1217	[429, 495]
285	[486, 187]	596	[467, 501]	907	[1024, 1025]	1218	[1004, 435]
286	[487, 488]	597	[492, 467]	908	[95, 676]	1219	[967, 562]
287	[489, 490]	598	[567, 907]	909	[1026, 1027]	1220	[12, 960]
288	[156, 491]	599	[200, 509]	910	[233, 1028]	1221	[396, 937]
289	[492, 493]	600	[49, 908]	911	[1029, 1030]	1222	[141, 437]
290	[494, 495]	601	[554, 195]	912	[1031, 1032]	1223	[474, 195]
291	[496, 431]	602	[909, 204]	913	[1033, 1034]	1224	[516, 454]
292	[42, 497]	603	[910, 903]	914	[1035, 355]	1225	[580, 915]
293	[148, 163]	604	[909, 12]	915	[1036, 1037]	1226	[414, 163]
294	[498, 499]	605	[441, 911]	916	[1038, 1039]	1227	[119, 569]
295	[500, 501]	606	[912, 491]	917	[1040, 1041]	1228	[900, 992]
296	[161, 159]	607	[913, 126]	918	[28, 1042]	1229	[408, 125]
297	[460, 502]	608	[567, 477]	919	[1043, 1044]	1230	[1003, 147]
298	[143, 411]	609	[914, 194]	920	[1045, 1046]	1231	[552, 1184]
299	[44, 503]	610	[494, 915]	921	[1047, 1048]	1232	[431, 493]
300	[504, 505]	611	[916, 160]	922	[1049, 685]	1233	[156, 1236]
301	[506, 507]	612	[8, 149]	923	[1050, 1051]	1234	[935, 552]
302	[508, 136]	613	[917, 918]	924	[1052, 1053]	1235	[1237, 1238]
303	[134, 509]	614	[905, 398]	925	[1054, 113]	1236	[1239, 1240]
304	[475, 437]	615	[494, 497]	926	[1055, 1056]	1237	[575, 1236]
305	[510, 511]	616	[919, 920]	927	[1057, 755]	1238	[155, 189]
306	[512, 59]	617	[921, 922]	928	[635, 1058]	1239	[467, 1203]
307	[513, 197]	618	[426, 526]	929	[255, 1059]	1240	[935, 156]
308	[514, 51]	619	[561, 122]	930	[1060, 1061]		
309	[134, 201]	620	[923, 557]	931	[1062, 1063]		
310	[515, 128]	621	[924, 214]	932	[1064, 1065]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	207	1	414	0	621	1	828	1	1035	0
1	0	208	0	415	1	622	1	829	0	1036	1
2	0	209	1	416	1	623	1	830	0	1037	0
3	0	210	0	417	1	624	1	831	0	1038	1
4	1	211	1	418	1	625	1	832	0	1039	1
5	0	212	1	419	1	626	1	833	1	1040	1
6	0	213	1	420	0	627	0	834	0	1041	1
7	0	214	0	421	1	628	0	835	1	1042	1
8	1	215	0	422	1	629	0	836	0	1043	0
9	1	216	1	423	0	630	1	837	1	1044	0
10	0	217	1	424	0	631	0	838	0	1045	0

11	0	218	0	425	0	632	1	839	0	1046	1
12	0	219	0	426	1	633	0	840	1	1047	1
13	0	220	0	427	0	634	0	841	0	1048	1
14	0	221	1	428	1	635	1	842	1	1049	1
15	0	222	0	429	1	636	0	843	1	1050	1
16	0	223	1	430	0	637	0	844	1	1051	0
17	1	224	1	431	1	638	1	845	0	1052	1
18	0	225	1	432	1	639	1	846	1	1053	1
19	1	226	0	433	0	640	0	847	0	1054	1
20	1	227	1	434	1	641	0	848	0	1055	0
21	0	228	0	435	1	642	1	849	1	1056	1
22	1	229	0	436	1	643	0	850	1	1057	0
23	0	230	1	437	0	644	1	851	1	1058	0
24	0	231	1	438	1	645	1	852	1	1059	1
25	0	232	1	439	1	646	0	853	1	1060	1
26	0	233	1	440	1	647	0	854	0	1061	1
27	0	234	1	441	1	648	1	855	1	1062	0
28	0	235	1	442	0	649	1	856	0	1063	0
29	0	236	1	443	1	650	1	857	0	1064	1
30	0	237	1	444	1	651	0	858	1	1065	0
31	0	238	0	445	0	652	0	859	1	1066	1
32	0	239	1	446	1	653	0	860	0	1067	1
33	0	240	0	447	1	654	1	861	0	1068	0
34	1	241	1	448	0	655	0	862	1	1069	1
35	1	242	1	449	0	656	1	863	0	1070	0
36	0	243	1	450	0	657	0	864	1	1071	0
37	0	244	0	451	0	658	1	865	0	1072	0
38	1	245	1	452	1	659	0	866	0	1073	0
39	1	246	1	453	1	660	0	867	0	1074	0
40	1	247	1	454	1	661	1	868	0	1075	1
41	1	248	1	455	0	662	1	869	0	1076	1
42	0	249	1	456	1	663	1	870	0	1077	1
43	0	250	0	457	0	664	1	871	0	1078	1
44	1	251	1	458	1	665	1	872	0	1079	0
45	0	252	0	459	0	666	1	873	0	1080	1
46	0	253	0	460	0	667	0	874	1	1081	0
47	0	254	0	461	0	668	0	875	0	1082	1
48	0	255	1	462	1	669	0	876	1	1083	0
49	0	256	0	463	1	670	0	877	0	1084	1
50	0	257	1	464	1	671	1	878	0	1085	0
51	0	258	1	465	1	672	0	879	1	1086	1
52	0	259	0	466	0	673	0	880	1	1087	1
53	0	260	0	467	0	674	1	881	0	1088	1
54	0	261	1	468	1	675	0	882	0	1089	1

55	1	262	0	469	0	676	1	883	1	1090	1
56	0	263	0	470	0	677	1	884	0	1091	0
57	1	264	1	471	1	678	0	885	1	1092	0
58	0	265	0	472	1	679	0	886	0	1093	0
59	1	266	1	473	1	680	0	887	1	1094	1
60	0	267	1	474	0	681	1	888	1	1095	1
61	0	268	0	475	1	682	0	889	1	1096	0
62	0	269	1	476	1	683	0	890	0	1097	1
63	0	270	0	477	0	684	1	891	1	1098	1
64	1	271	1	478	0	685	1	892	0	1099	1
65	0	272	1	479	0	686	1	893	1	1100	1
66	1	273	0	480	0	687	1	894	0	1101	1
67	1	274	1	481	0	688	1	895	0	1102	1
68	1	275	1	482	1	689	0	896	1	1103	1
69	1	276	1	483	1	690	1	897	1	1104	0
70	0	277	0	484	0	691	1	898	1	1105	1
71	1	278	0	485	1	692	0	899	0	1106	0
72	0	279	0	486	1	693	0	900	0	1107	1
73	1	280	1	487	1	694	1	901	1	1108	1
74	1	281	1	488	1	695	1	902	0	1109	0
75	0	282	1	489	0	696	1	903	0	1110	1
76	1	283	0	490	1	697	1	904	0	1111	0
77	1	284	1	491	0	698	0	905	1	1112	0
78	1	285	1	492	0	699	0	906	1	1113	0
79	0	286	1	493	1	700	1	907	1	1114	1
80	1	287	0	494	0	701	0	908	0	1115	0
81	1	288	1	495	1	702	1	909	1	1116	1
82	0	289	0	496	1	703	1	910	1	1117	1
83	1	290	0	497	0	704	0	911	1	1118	0
84	0	291	1	498	0	705	1	912	0	1119	1
85	1	292	0	499	1	706	1	913	1	1120	1
86	1	293	1	500	0	707	0	914	0	1121	0
87	0	294	0	501	1	708	1	915	1	1122	0
88	0	295	0	502	1	709	1	916	1	1123	0
89	0	296	0	503	1	710	1	917	1	1124	0
90	0	297	0	504	1	711	1	918	0	1125	0
91	0	298	1	505	1	712	0	919	0	1126	1
92	0	299	1	506	1	713	1	920	0	1127	1
93	0	300	1	507	0	714	0	921	1	1128	0
94	0	301	1	508	1	715	1	922	1	1129	0
95	0	302	1	509	0	716	1	923	1	1130	1
96	0	303	1	510	1	717	1	924	1	1131	0
97	0	304	1	511	1	718	0	925	1	1132	1
98	0	305	1	512	0	719	1	926	0	1133	1

99	0	306	0	513	1	720	1	927	0	1134	0
100	0	307	1	514	0	721	0	928	1	1135	0
101	0	308	0	515	0	722	0	929	1	1136	0
102	0	309	1	516	0	723	1	930	1	1137	0
103	0	310	0	517	0	724	1	931	0	1138	1
104	1	311	0	518	0	725	1	932	1	1139	1
105	0	312	0	519	0	726	0	933	1	1140	0
106	1	313	0	520	0	727	0	934	0	1141	1
107	1	314	0	521	0	728	1	935	0	1142	1
108	0	315	0	522	1	729	0	936	0	1143	1
109	0	316	0	523	0	730	1	937	0	1144	1
110	0	317	0	524	0	731	0	938	0	1145	0
111	1	318	0	525	0	732	1	939	1	1146	0
112	1	319	0	526	0	733	0	940	1	1147	1
113	0	320	1	527	1	734	0	941	1	1148	1
114	1	321	0	528	0	735	1	942	1	1149	0
115	1	322	0	529	0	736	1	943	1	1150	0
116	1	323	1	530	0	737	1	944	0	1151	0
117	0	324	0	531	0	738	1	945	0	1152	1
118	1	325	0	532	0	739	1	946	0	1153	1
119	0	326	0	533	0	740	1	947	0	1154	1
120	0	327	0	534	0	741	1	948	0	1155	0
121	0	328	0	535	1	742	1	949	1	1156	0
122	1	329	1	536	1	743	0	950	1	1157	1
123	0	330	0	537	0	744	1	951	0	1158	1
124	0	331	0	538	1	745	1	952	0	1159	1
125	1	332	1	539	1	746	0	953	0	1160	0
126	1	333	0	540	1	747	0	954	1	1161	1
127	0	334	0	541	1	748	0	955	1	1162	0
128	1	335	0	542	0	749	1	956	1	1163	0
129	1	336	1	543	0	750	0	957	1	1164	1
130	1	337	0	544	0	751	0	958	0	1165	1
131	1	338	1	545	0	752	1	959	1	1166	0
132	1	339	0	546	0	753	0	960	1	1167	1
133	1	340	1	547	0	754	1	961	0	1168	1
134	1	341	1	548	1	755	1	962	1	1169	1
135	1	342	0	549	0	756	0	963	1	1170	1
136	1	343	0	550	0	757	0	964	1	1171	1
137	1	344	1	551	1	758	0	965	0	1172	1
138	1	345	0	552	0	759	1	966	0	1173	0
139	0	346	1	553	0	760	1	967	1	1174	0
140	0	347	0	554	1	761	1	968	0	1175	1
141	1	348	1	555	1	762	0	969	1	1176	0
142	0	349	0	556	1	763	0	970	1	1177	0

143	1	350	0	557	0	764	0	971	1	1178	1
144	0	351	1	558	0	765	0	972	0	1179	0
145	0	352	1	559	0	766	1	973	0	1180	0
146	1	353	1	560	1	767	0	974	1	1181	1
147	1	354	0	561	1	768	0	975	0	1182	1
148	1	355	0	562	0	769	0	976	0	1183	0
149	0	356	0	563	0	770	1	977	0	1184	1
150	1	357	0	564	0	771	1	978	1	1185	1
151	1	358	1	565	0	772	0	979	0	1186	1
152	0	359	0	566	1	773	1	980	1	1187	1
153	1	360	1	567	0	774	1	981	1	1188	1
154	1	361	1	568	1	775	1	982	0	1189	0
155	1	362	0	569	0	776	0	983	1	1190	1
156	1	363	1	570	0	777	1	984	1	1191	0
157	1	364	0	571	1	778	0	985	0	1192	1
158	0	365	0	572	1	779	1	986	1	1193	1
159	0	366	1	573	0	780	1	987	1	1194	1
160	0	367	0	574	0	781	1	988	0	1195	1
161	0	368	1	575	1	782	1	989	1	1196	1
162	1	369	0	576	0	783	0	990	0	1197	1
163	1	370	0	577	1	784	0	991	1	1198	1
164	1	371	0	578	1	785	0	992	0	1199	0
165	1	372	1	579	1	786	1	993	0	1200	0
166	0	373	0	580	1	787	1	994	0	1201	0
167	0	374	1	581	1	788	1	995	1	1202	0
168	0	375	1	582	1	789	0	996	0	1203	1
169	0	376	1	583	0	790	0	997	0	1204	1
170	0	377	0	584	1	791	0	998	0	1205	1
171	0	378	0	585	1	792	1	999	1	1206	1
172	0	379	1	586	1	793	0	1000	1	1207	0
173	0	380	1	587	0	794	0	1001	1	1208	1
174	0	381	0	588	0	795	1	1002	1	1209	1
175	1	382	0	589	0	796	0	1003	0	1210	0
176	0	383	1	590	1	797	0	1004	1	1211	1
177	0	384	1	591	0	798	1	1005	0	1212	0
178	0	385	1	592	0	799	0	1006	1	1213	0
179	0	386	0	593	0	800	0	1007	0	1214	0
180	1	387	1	594	1	801	0	1008	0	1215	0
181	0	388	0	595	1	802	0	1009	0	1216	0
182	0	389	0	596	0	803	1	1010	1	1217	1
183	0	390	1	597	0	804	0	1011	1	1218	1
184	1	391	0	598	0	805	0	1012	1	1219	1
185	1	392	1	599	0	806	1	1013	0	1220	0
186	0	393	0	600	0	807	0	1014	0	1221	1

187	0	394	1	601	1	808	0	1015	0	1222	1
188	0	395	1	602	1	809	0	1016	0	1223	0
189	1	396	1	603	1	810	0	1017	0	1224	0
190	0	397	0	604	1	811	0	1018	0	1225	1
191	0	398	0	605	1	812	1	1019	0	1226	0
192	0	399	0	606	0	813	1	1020	0	1227	0
193	1	400	0	607	1	814	0	1021	0	1228	0
194	1	401	1	608	0	815	0	1022	1	1229	1
195	0	402	0	609	0	816	0	1023	1	1230	0
196	0	403	0	610	0	817	0	1024	1	1231	0
197	1	404	0	611	1	818	0	1025	1	1232	1
198	1	405	1	612	1	819	0	1026	1	1233	1
199	0	406	1	613	1	820	0	1027	0	1234	0
200	0	407	1	614	1	821	1	1028	1	1235	1
201	1	408	1	615	0	822	0	1029	1	1236	0
202	0	409	0	616	0	823	0	1030	1	1237	1
203	0	410	0	617	1	824	1	1031	0	1238	1
204	0	411	1	618	1	825	0	1032	0	1239	0
205	0	412	1	619	1	826	1	1033	1	1240	0
206	1	413	0	620	1	827	0	1034	0		

Université de Strasbourg, CNRS, IRMA UMR 7501, F-67000 Strasbourg, France
Email address: guoniu.han@unistra.fr

School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan, PR China
Email address: huyining@hust.edu.cn