PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a,b) = (z^5 + z^2, z^2 + z + 1)$

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ABSTRACT. The continued fraction conjecture claims that the continued fraction defined by the (a,b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a,b) in $\mathbb{F}_2[x] \setminus \mathbb{F}_2$. In this paper we prove the conjecture for $(a,b) = (z^5 + z^2, z^2 + z + 1)$.

1. INTRODUCTION

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \ldots)$ of elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{\ddots + \cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a,b)-Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \ldots) = (a, b, b, a, b, a, a, b, \ldots)$$

that we shall denote

$$\mathrm{CF}(a,b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When* $(a, b) = (z^5 + z^2, z^2 + z + 1)$*, the two power series* $\mathrm{CF}(a, b)$ *and* $\mathrm{CF}(b, a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4 + p_3(z)y^3 + p_2(z)y^2 + p_1(z)y + p_0(z) = 0$$

where, for $\mathrm{CF}(a, b)$*,*

$$\begin{aligned}
p_0(z) &= z^{13} + z^{12} + z^9 + z^8 + z^6 + z^4 + z^3 + z^2 + 1,\\
p_1(z) &= z^{18} + z^{17} + z^{13} + z^{12} + z^{10} + z^7 + z^4 + z^2,\\
p_2(z) &= z^{20} + z^{16} + z^{14} + z^4,\\
p_3(z) &= z^{18} + z^{17} + z^{13} + z^{12} + z^{10} + z^7 + z^4 + z^2,\\
p_4(z) &= z^{16} + z^{14} + z^{13} + z^9 + z^8 + z^6 + z^3 + z^2,
\end{aligned}$$

and for $\mathrm{CF}(b, a)$*,*

$$\begin{aligned}
p_0(z) &= z^{13} + z^9 + z^6 + z^4 + z^3 + z^2,\\
p_1(z) &= z^{18} + z^{17} + z^{13} + z^{12} + z^{10} + z^7 + z^4 + z^2,\\
p_2(z) &= z^{20} + z^{16} + z^{14} + z^4,\\
p_3(z) &= z^{18} + z^{17} + z^{13} + z^{12} + z^{10} + z^7 + z^4 + z^2,\\
p_4(z) &= z^{16} + z^{14} + z^{13} + z^{12} + z^9 + z^6 + z^3 + z^2 + 1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a, b) = \frac{P_n(z)}{Q_n(z)} \in \mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a, b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x) = x^{\deg(t_{2^n-1})}&\begin{pmatrix} t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix} \times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix} \times\\
&\cdots \times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x) = x^{\deg(\bar{t}_{2^n-1})}&\begin{pmatrix} \bar{t}_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix} \times x^{\deg(\bar{t}_{2^n-2})}\begin{pmatrix} \bar{t}_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix} \times\\
&\cdots \times x^{\deg(\bar{t}_0)}\begin{pmatrix} \bar{t}_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b, a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n \geq 0$

$$\begin{aligned}
M_{n+1}(x) &= W_n(x)\cdot M_n(x),\\
W_{n+1}(x) &= M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(175, 175, 175, 175, 175)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x, y) = \mathrm{Res}_z \left(\phi(W^o, 0, 0)(x, z),\ z^{12} \cdot \phi(M^o, 0, 0)(x, y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x, y)$. To prove that $\phi_0(x, y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x, y)$ of multiplicity m and that $Q(x, y) := P(x, y)/\phi_0(x, y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 630 and substitute it for y in $Q(x, y)$. We get a series of valuation less than 630, which proves that $Q(x, y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x, y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e, 0, 0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x, y) = \mathrm{Res}_z \left(\phi_0(x, z),\ \phi_1(x, y + z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e, 0, 0)$ is an irreducible factor of $S(x, y)$ of multiplicity μ, and that $Q(x, y) := S(x, y)/\phi(M^e, 0, 0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ to order 770 and substitute it for y in $Q(x, y)$. We get a series of valuation less than 770, and therefore $Q(x, y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e, 0, 0)$, we know that the two series are one and the same. □

Define

$$R^e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 2.2. *For all integers $k \ge 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned} M^e[:14u] = {} & M^e[:7u] + x^u M^e[:6u] + x^{2u} M^e[:5u] + x^{3u} M^e[:4u] + x^u M^e[:7u] \\ & + x^{2u} M^e[:6u] + x^{3u} M^e[:5u] + x^{4u} M^e[:4u] + x^{4u} M^e[:7u] \\ & + x^{5u} M^e[:6u] + x^{6u} M^e[:5u] + x^{7u} M^e[:4u] + x^{5u} M^e[:7u] \\ & + x^{6u} M^e[:6u] + x^{7u} M^e[:5u] + x^{8u} M^e[:4u] + x^{10u} M^e[:4u] \\ & + x^{12u} M^e[:2u] + (x^{7u} + x^{8u} + x^{11u} + x^{12u}) R^e, \end{aligned}$$

$$W^e[:14u] = W^e[:7u] + x^u W^e[:6u] + x^{2u} W^e[:5u] + x^{3u} W^e[:4u] + x^u W^e[:7u]$$

$$+ x^{2u}W^e[:6u] + x^{3u}W^e[:5u] + x^{4u}W^e[:4u] + x^{4u}W^e[:7u]$$
$$+ x^{5u}W^e[:6u] + x^{6u}W^e[:5u] + x^{7u}W^e[:4u] + x^{5u}W^e[:7u]$$
$$+ x^{6u}W^e[:6u] + x^{7u}W^e[:5u] + x^{8u}W^e[:4u] + x^{10u}W^e[:4u]$$
$$+ x^{12u}W^e[:2u] + (x^{7u} + x^{8u} + x^{11u} + x^{12u})R^e.$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$\begin{aligned}
M^o[:14u] = {} & M^o[:7u] + x^u M^o[:6u] + x^{2u}M^o[:5u] + x^{3u}M^o[:4u] + x^u M^o[:7u] \\
& + x^{2u}M^o[:6u] + x^{3u}M^o[:5u] + x^{4u}M^o[:4u] + x^{4u}M^o[:7u] \\
& + x^{5u}M^o[:6u] + x^{6u}M^o[:5u] + x^{7u}M^o[:4u] + x^{5u}M^o[:7u] \\
& + x^{6u}M^o[:6u] + x^{7u}M^o[:5u] + x^{8u}M^o[:4u] + x^{10u}M^o[:4u] \\
& + x^{12u}M^o[:2u] + (x^{7u} + x^{8u} + x^{11u} + x^{12u})R^o, \\
W^o[:14u] = {} & W^o[:7u] + x^u W^o[:6u] + x^{2u}W^o[:5u] + x^{3u}W^o[:4u] + x^u W^o[:7u] \\
& + x^{2u}W^o[:6u] + x^{3u}W^o[:5u] + x^{4u}W^o[:4u] + x^{4u}W^o[:7u] \\
& + x^{5u}W^o[:6u] + x^{6u}W^o[:5u] + x^{7u}W^o[:4u] + x^{5u}W^o[:7u] \\
& + x^{6u}W^o[:6u] + x^{7u}W^o[:5u] + x^{8u}W^o[:4u] + x^{10u}W^o[:4u] \\
& + x^{12u}W^o[:2u] + (x^{7u} + x^{8u} + x^{11u} + x^{12u})R^o.
\end{aligned}$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$\begin{aligned}
T[:14u] = {} & T[:7u] + x^u T[:6u] + x^{2u}T[:5u] + x^{3u}T[:4u] + x^u T[:7u] + x^{2u}T[:6u] \\
& + x^{3u}T[:5u] + x^{4u}T[:4u] + x^{4u}T[:7u] + x^{5u}T[:6u] + x^{6u}T[:5u] \\
& + x^{7u}T[:4u] + x^{5u}T[:7u] + x^{6u}T[:6u] + x^{7u}T[:5u] + x^{8u}T[:4u] \\
& + x^{10u}T[:4u] + x^{12u}T[:2u].
\end{aligned}$$

The proof of the other 15 cases are similar. We break down the above identity into 7 parts:

$$\begin{aligned}
0 &= x^{3u}T[4u:5u] + x^{2u}T[5u:6u] + x^u T[6u:7u] + T[7u:8u], \\
0 &= x^{8u}T[:u] + x^{4u}T[4u:5u] + T[8u:9u], \\
0 &= x^{8u}T[u:2u] + x^{4u}T[5u:6u] + T[9u:10u], \\
0 &= x^{10u}T[:u] + x^{8u}T[2u:3u] + x^{4u}T[6u:7u] + T[10u:11u], \\
0 &= x^{10u}T[u:2u] + x^{8u}T[3u:4u] + x^{7u}T[4u:5u] + x^{6u}T[5u:6u] \\
&\quad + x^{5u}T[6u:7u] + T[11u:12u], \\
0 &= x^{12u}T[:u] + x^{10u}T[2u:3u] + T[12u:13u], \\
0 &= x^{12u}T[u:2u] + x^{10u}T[3u:4u] + T[13u:14u],
\end{aligned}$$

which can be rewritten as

$$0 = T[[100w]_2] + T[[101w]_2] + T[[110w]_2] + T[[111w]_2], \tag{2.7}$$

$$0 = T[[w]_2] + T[[100w]_2] + T[[1000w]_2], \tag{2.8}$$

$$0 = T[[1w]_2] + T[[101w]_2] + T[[1001w]_2], \tag{2.9}$$
$$0 = T[[w]_2] + T[[10w]_2] + T[[110w]_2] + T[[1010w]_2], \tag{2.10}$$
$$\begin{aligned} 0 = {} & T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[110w]_2] \\ & + T[[1011w]_2], \end{aligned} \tag{2.11}$$
$$0 = T[[w]_2] + T[[10w]_2] + T[[1100w]_2], \tag{2.12}$$
$$0 = T[[1w]_2] + T[[11w]_2] + T[[1101w]_2], \tag{2.13}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$0 = T[[100w]_2] + T[[101w]_2] + T[[110w]_2] + T[[111w]_2], \tag{2.14}$$
$$0 = T[[w]_2] + T[[100w]_2] + T[[1000w]_2], \tag{2.15}$$
$$0 = T[[1w]_2] + T[[101w]_2] + T[[1001w]_2], \tag{2.16}$$
$$0 = T[[w]_2] + T[[10w]_2] + T[[110w]_2] + T[[1010w]_2], \tag{2.17}$$
$$\begin{aligned} 0 = {} & T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[110w]_2] \\ & + T[[1011w]_2], \end{aligned} \tag{2.18}$$
$$0 = T[[w]_2] + T[[10w]_2] + T[[1100w]_2], \tag{2.19}$$
$$0 = T[[1w]_2] + T[[11w]_2] + T[[1101w]_2], \tag{2.20}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 769 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.20) can be written as

$$0 = \tau(A(s, 100)) + \tau(A(s, 101)) + \tau(A(s, 110)) + \tau(A(s, 111)), \tag{2.21}$$
$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 100)) + \tau(A(s, 1000)), \tag{2.22}$$
$$0 = \tau(A(s, 1)) + \tau(A(s, 101)) + \tau(A(s, 1001)), \tag{2.23}$$
$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 10)) + \tau(A(s, 110)) + \tau(A(s, 1010)), \tag{2.24}$$
$$\begin{aligned} 0 = {} & \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 101)) \\ & + \tau(A(s, 110)) + \tau(A(s, 1011)), \end{aligned} \tag{2.25}$$
$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 10)) + \tau(A(s, 1100)), \tag{2.26}$$
$$0 = \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 1101)), \tag{2.27}$$

for all $s \in E_{2k}$, and

$$0 = \tau(A(s, 100)) + \tau(A(s, 101)) + \tau(A(s, 110)) + \tau(A(s, 111)), \tag{2.28}$$
$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 100)) + \tau(A(s, 1000)), \tag{2.29}$$
$$0 = \tau(A(s, 1)) + \tau(A(s, 101)) + \tau(A(s, 1001)), \tag{2.30}$$
$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 10)) + \tau(A(s, 110)) + \tau(A(s, 1010)), \tag{2.31}$$
$$\begin{aligned} 0 = {} & \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 101)) \\ & + \tau(A(s, 110)) + \tau(A(s, 1011)), \end{aligned} \tag{2.32}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,1100)), \tag{2.33}$$
$$0 = \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,1101)), \tag{2.34}$$

for $s = A(i, 0^{2k})$. We find that $(A(i,0^{42}), E_{42}) = (A(i,0^{14}), E_{14})$, so that we only have to verify that identities (2.21) through (2.34) hold for $2 \le k \le 21$, which turns out to be true. $\square$

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \ge 2$, and $u = 2^{2k-1}$,*

$$M_{2k} = M^e[:7u] + x^u M^e[:6u] + x^{2u} M^e[:5u] + x^{3u} M^e[:4u] + x^{7u} R^e,$$
$$W_{2k} = W^e[:7u] + x^u W^e[:6u] + x^{2u} W^e[:5u] + x^{3u} W^e[:4u] + x^{7u} R^e.$$

For all integer $k \ge 2$, and $u = 2^{2k}$,

$$M_{2k+1} = M^o[:7u] + x^u M^o[:6u] + x^{2u} M^o[:5u] + x^{3u} M^o[:4u] + x^{7u} R^o,$$
$$W_{2k+1} = W^o[:7u] + x^u W^o[:6u] + x^{2u} W^o[:5u] + x^{3u} W^o[:4u] + x^{7u} R^o.$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \ge 2$, we claim that

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1} \wedge W_{2k+1},$$
$$M_{2k+1} \wedge W_{2k+1} \Rightarrow M_{2k+2} \wedge W_{2k+2}.$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$W_{2k} M_{2k} = \big(W^e[:7v] + x^v W^e[:6v] + x^{2v} W^e[:5v] + x^{3v} W^e[:4v] + x^{7u} R^e\big) \times \big($$
$$M^e[:7v] + x^v M^e[:6v] + x^{2v} M^e[:5v] + x^{3v} M^e[:4v] + x^{7u} R^e\big); \tag{2.35}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{14v} R^o$. Therefore we only need to prove that their difference is $O(x^{14v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{14v}, to

$$W^e[:14v]M^e[:14v] + x^{2v} W^e[:12v]M^e[:12v] + x^{4v} W^e[:10v]M^e[:10v]$$
$$+ x^{6v} W^e[:8v]M^e[:8v].$$

For all $n \le 14$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$X := x^v,$$
$$a_n := W^e[n \cdot v : (n+1) \cdot v]/X^n,$$
$$b_n := M^e[n \cdot v : (n+1) \cdot v]/X^n,$$
$$c := R^e.$$

Using the notation introduced above, we can represent the expressions lhs (2.35) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{14}, b_1, ..., b_{14}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{14})$, which completes the proof. $\square$

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \leq j, k \leq 1$,

$$\lim_{n\to\infty} M_{2n,j,k} = M^e_{j,k},$$
$$\lim_{n\to\infty} M_{2n+1,j,k} = M^o_{j,k},$$
$$\lim_{n\to\infty} W_{2n,j,k} = W^e_{j,k},$$
$$\lim_{n\to\infty} W_{2n+1,j,k} = W^o_{j,k}.$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a, b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x, y) = \mathrm{Res}_t \left(\phi(M^e, 0, 1)(x, t), y^{12}\phi(M^e, 0, 1)(x, t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x, y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned}
q_0(x) &= x^{20} + x^{18} + x^{17} + x^{16} + x^{14} + x^{12} + x^{11} + x^8 + x^7,\\
q_1(x) &= x^{18} + x^{16} + x^{13} + x^{10} + x^8 + x^7 + x^3 + x^2,\\
q_2(x) &= x^{16} + x^6 + x^4 + 1,\\
q_3(x) &= x^{18} + x^{16} + x^{13} + x^{10} + x^8 + x^7 + x^3 + x^2,\\
q_4(x) &= x^{18} + x^{17} + x^{14} + x^{12} + x^{11} + x^7 + x^6 + x^4.
\end{aligned}$$

The polynomial $Q(x, y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x, y)$ of multiplicity m and $R(x, y) := P(x, y)/Q(x, y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 216, and substitute it for y in $R(z, y)$, we get a series with valuation smaller than 216, which proves that $R(z, y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z, y)$ is the minimal polynomial of $\mathrm{CF}(a, b) = f(1/z)$. $\square$

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{102} + x^{94} + x^{92} + x^{88} + x^{84} + x^{60} + x^{58} + x^{52} + x^{48} + x^{44} + x^{42} \\
&\quad + x^{40} + x^{36} + x^{32} + x^{28} + x^{26} + x^{24}, \\
p_3(x) &= x^{96} + x^{94} + x^{92} + x^{90} + x^{84} + x^{82} + x^{80} + x^{74} + x^{72} + x^{70} + x^{62} \\
&\quad + x^{60} + x^{58} + x^{56} + x^{54} + x^{52} + x^{40} + x^{38} + x^{34} + x^{32} + x^{28} + x^{26} \\
&\quad + x^{22} + x^{16}, \\
p_6(x) &= x^{96} + x^{92} + x^{86} + x^{84} + x^{82} + x^{80} + x^{76} + x^{68} + x^{64} + x^{60} + x^{56} \\
&\quad + x^{54} + x^{52} + x^{46} + x^{40} + x^{38} + x^{36} + x^{34} + x^{30} + x^{28} + x^{22} + x^{20} \\
&\quad + x^{14} + x^{12} + x^{8} + x^{4} + x^{2} + 1, \\
p_9(x) &= x^{98} + x^{96} + x^{94} + x^{92} + x^{90} + x^{88} + x^{86} + x^{80} + x^{76} + x^{66} + x^{60} \\
&\quad + x^{56} + x^{54} + x^{50} + x^{38} + x^{30} + x^{28} + x^{26} + x^{24} + x^{22} + x^{18} + x^{16} \\
&\quad + x^{10} + x^{4}, \\
p_{12}(x) &= x^{98} + x^{96} + x^{90} + x^{88} + x^{82} + x^{80} + x^{50} + x^{48} + x^{42} + x^{40} + x^{34} \\
&\quad + x^{32} + x^{18} + x^{16} + x^{10} + x^{8} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0].

For $\phi(M^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{111} + x^{109} + x^{108} + x^{107} + x^{106} + x^{103} + x^{101} + x^{99} + x^{98} + x^{97} \\
&\quad + x^{96} + x^{95} + x^{92} + x^{91} + x^{89} + x^{86} + x^{84} + x^{83} + x^{82} + x^{81} + x^{78} \\
&\quad + x^{77} + x^{75} + x^{71} + x^{69} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} \\
&\quad + x^{60} + x^{56} + x^{55} + x^{54} + x^{52} + x^{51} + x^{50} + x^{48} + x^{47} + x^{46} + x^{45} \\
&\quad + x^{43} + x^{41} + x^{39} + x^{36} + x^{35} + x^{33}, \\
p_3(x) &= x^{107} + x^{106} + x^{105} + x^{104} + x^{103} + x^{100} + x^{98} + x^{97} + x^{96} + x^{92} \\
&\quad + x^{91} + x^{90} + x^{86} + x^{81} + x^{80} + x^{76} + x^{73} + x^{72} + x^{71} + x^{70} + x^{68} \\
&\quad + x^{66} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{52} + x^{51} + x^{47} + x^{43} + x^{41} \\
&\quad + x^{40} + x^{38} + x^{36} + x^{31} + x^{30} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{20} \\
&\quad + x^{18}, \\
p_6(x) &= x^{105} + x^{104} + x^{97} + x^{96} + x^{95} + x^{94} + x^{91} + x^{90} + x^{89} + x^{88} + x^{85} \\
&\quad + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{67} + x^{66} + x^{65} + x^{64} \\
&\quad + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} \\
&\quad + x^{52} + x^{51} + x^{50} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} \\
&\quad + x^{33} + x^{32} + x^{29} + x^{28} + x^{25} + x^{24} + x^{23} + x^{22} + x^{15} + x^{14} + x^{13} \\
&\quad + x^{12}, \\
p_9(x) &= x^{103} + x^{102} + x^{101} + x^{100} + x^{98} + x^{97} + x^{95} + x^{94} + x^{92} + x^{91} + x^{87} \\
&\quad + x^{82} + x^{81} + x^{80} + x^{77} + x^{75} + x^{74} + x^{72} + x^{70} + x^{69} + x^{68} + x^{67}
\end{aligned}$$

$$\begin{aligned}
&+x^{66}+x^{65}+x^{63}+x^{61}+x^{58}+x^{56}+x^{55}+x^{54}+x^{51}+x^{49}+x^{48}\\
&+x^{46}+x^{45}+x^{40}+x^{35}+x^{34}+x^{33}+x^{32}+x^{30}+x^{29}+x^{27}+x^{26}\\
&+x^{25}+x^{23}+x^{20}+x^{18}+x^{15}+x^{14}+x^{11}+x^{6},\\
p_{12}(x)&=x^{101}+x^{100}+x^{97}+x^{96}+x^{93}+x^{92}+x^{89}+x^{88}+x^{85}+x^{84}+x^{81}\\
&+x^{80}+x^{53}+x^{52}+x^{49}+x^{48}+x^{45}+x^{44}+x^{41}+x^{40}+x^{37}+x^{36}\\
&+x^{33}+x^{32}+x^{21}+x^{20}+x^{17}+x^{16}+x^{13}+x^{12}+x^{9}+x^{8}+x^{5}+x^{4}\\
&+x+1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 1].

For $\phi(M^e, 1, 0)$:

$$\begin{aligned}
p_0(x)&=x^{111}+x^{109}+x^{108}+x^{107}+x^{106}+x^{103}+x^{101}+x^{99}+x^{98}+x^{97}\\
&+x^{96}+x^{95}+x^{92}+x^{91}+x^{89}+x^{86}+x^{84}+x^{83}+x^{82}+x^{81}+x^{78}\\
&+x^{77}+x^{75}+x^{71}+x^{69}+x^{68}+x^{67}+x^{66}+x^{65}+x^{64}+x^{63}+x^{62}\\
&+x^{60}+x^{56}+x^{55}+x^{54}+x^{52}+x^{51}+x^{50}+x^{48}+x^{47}+x^{46}+x^{45}\\
&+x^{43}+x^{41}+x^{39}+x^{36}+x^{35}+x^{33},\\
p_3(x)&=x^{107}+x^{106}+x^{105}+x^{104}+x^{103}+x^{100}+x^{98}+x^{97}+x^{96}+x^{92}\\
&+x^{91}+x^{90}+x^{86}+x^{81}+x^{80}+x^{76}+x^{73}+x^{72}+x^{71}+x^{70}+x^{68}\\
&+x^{66}+x^{59}+x^{58}+x^{57}+x^{56}+x^{55}+x^{52}+x^{51}+x^{47}+x^{43}+x^{41}\\
&+x^{40}+x^{38}+x^{36}+x^{31}+x^{30}+x^{26}+x^{25}+x^{24}+x^{23}+x^{22}+x^{20}\\
&+x^{18},\\
p_6(x)&=x^{105}+x^{104}+x^{97}+x^{96}+x^{95}+x^{94}+x^{91}+x^{90}+x^{89}+x^{88}+x^{85}\\
&+x^{84}+x^{83}+x^{82}+x^{81}+x^{80}+x^{79}+x^{78}+x^{67}+x^{66}+x^{65}+x^{64}\\
&+x^{63}+x^{62}+x^{61}+x^{60}+x^{59}+x^{58}+x^{57}+x^{56}+x^{55}+x^{54}+x^{53}\\
&+x^{52}+x^{51}+x^{50}+x^{47}+x^{46}+x^{45}+x^{44}+x^{43}+x^{42}+x^{41}+x^{40}\\
&+x^{33}+x^{32}+x^{29}+x^{28}+x^{25}+x^{24}+x^{23}+x^{22}+x^{15}+x^{14}+x^{13}\\
&+x^{12},\\
p_9(x)&=x^{103}+x^{102}+x^{101}+x^{100}+x^{98}+x^{97}+x^{95}+x^{94}+x^{92}+x^{91}+x^{87}\\
&+x^{82}+x^{81}+x^{80}+x^{77}+x^{75}+x^{74}+x^{72}+x^{70}+x^{69}+x^{68}+x^{67}\\
&+x^{66}+x^{65}+x^{63}+x^{61}+x^{58}+x^{56}+x^{55}+x^{54}+x^{51}+x^{49}+x^{48}\\
&+x^{46}+x^{45}+x^{40}+x^{35}+x^{34}+x^{33}+x^{32}+x^{30}+x^{29}+x^{27}+x^{26}\\
&+x^{25}+x^{23}+x^{20}+x^{18}+x^{15}+x^{14}+x^{11}+x^{6},\\
p_{12}(x)&=x^{101}+x^{100}+x^{97}+x^{96}+x^{93}+x^{92}+x^{89}+x^{88}+x^{85}+x^{84}+x^{81}\\
&+x^{80}+x^{53}+x^{52}+x^{49}+x^{48}+x^{45}+x^{44}+x^{41}+x^{40}+x^{37}+x^{36}\\
&+x^{33}+x^{32}+x^{21}+x^{20}+x^{17}+x^{16}+x^{13}+x^{12}+x^{9}+x^{8}+x^{5}+x^{4}\\
&+x+1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 1].

For $\phi(M^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{120} + x^{116} + x^{114} + x^{108} + x^{106} + x^{100} + x^{98} + x^{96} + x^{92} + x^{78} \\
&\quad + x^{68} + x^{64} + x^{60} + x^{58} + x^{54} + x^{52} + x^{50} + x^{48} + x^{46} + x^{44} + x^{42}, \\
p_3(x) &= x^{108} + x^{104} + x^{102} + x^{100} + x^{96} + x^{94} + x^{92} + x^{90} + x^{84} + x^{80} + x^{78} \\
&\quad + x^{76} + x^{74} + x^{64} + x^{56} + x^{48} + x^{46} + x^{42} + x^{38} + x^{36} + x^{34} + x^{32} \\
&\quad + x^{28} + x^{26} + x^{22} + x^{20} + x^{14} + x^{12}, \\
p_6(x) &= x^{108} + x^{102} + x^{98} + x^{94} + x^{92} + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{74} \\
&\quad + x^{70} + x^{68} + x^{64} + x^{60} + x^{58} + x^{56} + x^{52} + x^{50} + x^{48} + x^{40} + x^{38} \\
&\quad + x^{36} + x^{34} + x^{32} + x^{28} + x^{26} + x^{22} + x^{18} + x^{6} + x^{4} + 1, \\
p_9(x) &= x^{104} + x^{100} + x^{90} + x^{84} + x^{78} + x^{76} + x^{72} + x^{70} + x^{68} + x^{64} + x^{60} \\
&\quad + x^{56} + x^{52} + x^{50} + x^{44} + x^{42} + x^{40} + x^{38} + x^{36} + x^{32} + x^{26} + x^{24} \\
&\quad + x^{22} + x^{14} + x^{12} + x^{8} + x^{6} + x^{4}, \\
p_{12}(x) &= x^{104} + x^{80} + x^{56} + x^{32} + x^{24} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{120} + x^{116} + x^{114} + x^{112} + x^{108} + x^{106} + x^{104} + x^{100} + x^{96} + x^{86} \\
&\quad + x^{82} + x^{80} + x^{76} + x^{72} + x^{70} + x^{68} + x^{64} + x^{62} + x^{60} + x^{56} + x^{54} \\
&\quad + x^{52} + x^{48} + x^{44} + x^{42} + x^{40} + x^{30} + x^{28} + x^{24}, \\
p_3(x) &= x^{108} + x^{104} + x^{102} + x^{94} + x^{88} + x^{82} + x^{74} + x^{64} + x^{60} + x^{50} + x^{48} \\
&\quad + x^{46} + x^{42} + x^{32} + x^{30} + x^{28} + x^{24} + x^{22} + x^{18} + x^{16}, \\
p_6(x) &= x^{108} + x^{102} + x^{100} + x^{98} + x^{94} + x^{88} + x^{86} + x^{80} + x^{78} + x^{74} + x^{72} \\
&\quad + x^{70} + x^{68} + x^{64} + x^{58} + x^{56} + x^{52} + x^{48} + x^{36} + x^{32} + x^{30} + x^{24} \\
&\quad + x^{22} + x^{20} + x^{16} + x^{14} + x^{8} + x^{6} + x^{4} + 1, \\
p_9(x) &= x^{104} + x^{100} + x^{90} + x^{84} + x^{78} + x^{76} + x^{72} + x^{70} + x^{68} + x^{64} + x^{60} \\
&\quad + x^{56} + x^{52} + x^{50} + x^{44} + x^{42} + x^{40} + x^{38} + x^{36} + x^{32} + x^{26} + x^{24} \\
&\quad + x^{22} + x^{14} + x^{12} + x^{8} + x^{6} + x^{4}, \\
p_{12}(x) &= x^{104} + x^{80} + x^{56} + x^{32} + x^{24} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0].

For $\phi(W^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{111} + x^{107} + x^{104} + x^{103} + x^{101} + x^{100} + x^{99} + x^{98} + x^{95} + x^{91} \\
&\quad + x^{90} + x^{89} + x^{87} + x^{85} + x^{84} + x^{82} + x^{81} + x^{79} + x^{78} + x^{77} + x^{75} \\
&\quad + x^{73} + x^{72} + x^{71} + x^{70} + x^{68} + x^{65} + x^{63} + x^{62} + x^{61} + x^{59} + x^{57} \\
&\quad + x^{53} + x^{48} + x^{47} + x^{46} + x^{45} + x^{43} + x^{42} + x^{41} + x^{40} + x^{37} + x^{35} \\
&\quad + x^{34} + x^{33}, \\
p_3(x) &= x^{107} + x^{106} + x^{105} + x^{104} + x^{103} + x^{100} + x^{98} + x^{97} + x^{96} + x^{92}
\end{aligned}$$

$$
\begin{aligned}
&+x^{91}+x^{90}+x^{86}+x^{81}+x^{80}+x^{76}+x^{73}+x^{72}+x^{71}+x^{70}+x^{68}\\
&+x^{66}+x^{59}+x^{58}+x^{57}+x^{56}+x^{55}+x^{52}+x^{51}+x^{47}+x^{43}+x^{41}\\
&+x^{40}+x^{38}+x^{36}+x^{31}+x^{30}+x^{26}+x^{25}+x^{24}+x^{23}+x^{22}+x^{20}\\
&+x^{18},\\
p_6(x)&=x^{105}+x^{104}+x^{97}+x^{96}+x^{95}+x^{94}+x^{91}+x^{90}+x^{89}+x^{88}+x^{85}\\
&+x^{84}+x^{83}+x^{82}+x^{81}+x^{80}+x^{79}+x^{78}+x^{67}+x^{66}+x^{65}+x^{64}\\
&+x^{63}+x^{62}+x^{61}+x^{60}+x^{59}+x^{58}+x^{57}+x^{56}+x^{55}+x^{54}+x^{53}\\
&+x^{52}+x^{51}+x^{50}+x^{47}+x^{46}+x^{45}+x^{44}+x^{43}+x^{42}+x^{41}+x^{40}\\
&+x^{33}+x^{32}+x^{29}+x^{28}+x^{25}+x^{24}+x^{23}+x^{22}+x^{15}+x^{14}+x^{13}\\
&+x^{12},\\
p_9(x)&=x^{103}+x^{102}+x^{101}+x^{100}+x^{98}+x^{97}+x^{95}+x^{94}+x^{92}+x^{91}+x^{87}\\
&+x^{82}+x^{81}+x^{80}+x^{77}+x^{75}+x^{74}+x^{72}+x^{70}+x^{69}+x^{68}+x^{67}\\
&+x^{66}+x^{65}+x^{63}+x^{61}+x^{58}+x^{56}+x^{55}+x^{54}+x^{51}+x^{49}+x^{48}\\
&+x^{46}+x^{45}+x^{40}+x^{35}+x^{34}+x^{33}+x^{32}+x^{30}+x^{29}+x^{27}+x^{26}\\
&+x^{25}+x^{23}+x^{20}+x^{18}+x^{15}+x^{14}+x^{11}+x^{6},\\
p_{12}(x)&=x^{101}+x^{100}+x^{97}+x^{96}+x^{93}+x^{92}+x^{89}+x^{88}+x^{85}+x^{84}+x^{81}\\
&+x^{80}+x^{53}+x^{52}+x^{49}+x^{48}+x^{45}+x^{44}+x^{41}+x^{40}+x^{37}+x^{36}\\
&+x^{33}+x^{32}+x^{21}+x^{20}+x^{17}+x^{16}+x^{13}+x^{12}+x^{9}+x^{8}+x^{5}+x^{4}\\
&+x+1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 1, 0, 1, 1, 0, 1, 0, 1, 1, 1, 0, 0, 1].

For $\phi(W^e,1,0)$:

$$
\begin{aligned}
p_0(x)&=x^{111}+x^{107}+x^{104}+x^{103}+x^{101}+x^{100}+x^{99}+x^{98}+x^{95}+x^{91}\\
&+x^{90}+x^{89}+x^{87}+x^{85}+x^{84}+x^{82}+x^{81}+x^{79}+x^{78}+x^{77}+x^{75}\\
&+x^{73}+x^{72}+x^{71}+x^{70}+x^{68}+x^{65}+x^{63}+x^{62}+x^{61}+x^{59}+x^{57}\\
&+x^{53}+x^{48}+x^{47}+x^{46}+x^{45}+x^{43}+x^{42}+x^{41}+x^{40}+x^{37}+x^{35}\\
&+x^{34}+x^{33},\\
p_3(x)&=x^{107}+x^{106}+x^{105}+x^{104}+x^{103}+x^{100}+x^{98}+x^{97}+x^{96}+x^{92}\\
&+x^{91}+x^{90}+x^{86}+x^{81}+x^{80}+x^{76}+x^{73}+x^{72}+x^{71}+x^{70}+x^{68}\\
&+x^{66}+x^{59}+x^{58}+x^{57}+x^{56}+x^{55}+x^{52}+x^{51}+x^{47}+x^{43}+x^{41}\\
&+x^{40}+x^{38}+x^{36}+x^{31}+x^{30}+x^{26}+x^{25}+x^{24}+x^{23}+x^{22}+x^{20}\\
&+x^{18},\\
p_6(x)&=x^{105}+x^{104}+x^{97}+x^{96}+x^{95}+x^{94}+x^{91}+x^{90}+x^{89}+x^{88}+x^{85}\\
&+x^{84}+x^{83}+x^{82}+x^{81}+x^{80}+x^{79}+x^{78}+x^{67}+x^{66}+x^{65}+x^{64}\\
&+x^{63}+x^{62}+x^{61}+x^{60}+x^{59}+x^{58}+x^{57}+x^{56}+x^{55}+x^{54}+x^{53}\\
&+x^{52}+x^{51}+x^{50}+x^{47}+x^{46}+x^{45}+x^{44}+x^{43}+x^{42}+x^{41}+x^{40}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{33} + x^{32} + x^{29} + x^{28} + x^{25} + x^{24} + x^{23} + x^{22} + x^{15} + x^{14} + x^{13} \\
&+ x^{12}, \\
p_9(x) = {} & x^{103} + x^{102} + x^{101} + x^{100} + x^{98} + x^{97} + x^{95} + x^{94} + x^{92} + x^{91} + x^{87} \\
&+ x^{82} + x^{81} + x^{80} + x^{77} + x^{75} + x^{74} + x^{72} + x^{70} + x^{69} + x^{68} + x^{67} \\
&+ x^{66} + x^{65} + x^{63} + x^{61} + x^{58} + x^{56} + x^{55} + x^{54} + x^{51} + x^{49} + x^{48} \\
&+ x^{46} + x^{45} + x^{40} + x^{35} + x^{34} + x^{33} + x^{32} + x^{30} + x^{29} + x^{27} + x^{26} \\
&+ x^{25} + x^{23} + x^{20} + x^{18} + x^{15} + x^{14} + x^{11} + x^{6}, \\
p_{12}(x) = {} & x^{101} + x^{100} + x^{97} + x^{96} + x^{93} + x^{92} + x^{89} + x^{88} + x^{85} + x^{84} + x^{81} \\
&+ x^{80} + x^{53} + x^{52} + x^{49} + x^{48} + x^{45} + x^{44} + x^{41} + x^{40} + x^{37} + x^{36} \\
&+ x^{33} + x^{32} + x^{21} + x^{20} + x^{17} + x^{16} + x^{13} + x^{12} + x^{9} + x^{8} + x^{5} + x^{4} \\
&+ x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 1, 0, 1, 1, 0, 1, 0, 1, 1, 1, 0, 0, 1].

For $\phi(W^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) = {} & x^{102} + x^{98} + x^{96} + x^{94} + x^{82} + x^{80} + x^{78} + x^{74} + x^{72} + x^{70} + x^{68} \\
&+ x^{64} + x^{56} + x^{44} + x^{42}, \\
p_3(x) = {} & x^{96} + x^{88} + x^{80} + x^{78} + x^{76} + x^{72} + x^{66} + x^{60} + x^{56} + x^{54} + x^{48} \\
&+ x^{42} + x^{38} + x^{32} + x^{28} + x^{26} + x^{24} + x^{20} + x^{16} + x^{12}, \\
p_6(x) = {} & x^{96} + x^{94} + x^{86} + x^{80} + x^{78} + x^{74} + x^{52} + x^{48} + x^{44} + x^{42} + x^{34} \\
&+ x^{32} + x^{30} + x^{28} + x^{26} + x^{22} + x^{20} + x^{18} + x^{16} + x^{12} + x^{10} + x^{4} \\
&+ x^{2} + 1, \\
p_9(x) = {} & x^{98} + x^{96} + x^{94} + x^{92} + x^{90} + x^{88} + x^{86} + x^{80} + x^{76} + x^{66} + x^{60} \\
&+ x^{56} + x^{54} + x^{50} + x^{38} + x^{30} + x^{28} + x^{26} + x^{24} + x^{22} + x^{18} + x^{16} \\
&+ x^{10} + x^{4}, \\
p_{12}(x) = {} & x^{98} + x^{96} + x^{90} + x^{88} + x^{82} + x^{80} + x^{50} + x^{48} + x^{42} + x^{40} + x^{34} \\
&+ x^{32} + x^{18} + x^{16} + x^{10} + x^{8} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0].

For $\phi(M^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) = {} & x^{111} + x^{109} + x^{108} + x^{106} + x^{103} + x^{102} + x^{101} + x^{98} + x^{96} + x^{92} \\
&+ x^{91} + x^{90} + x^{89} + x^{86} + x^{85} + x^{84} + x^{79} + x^{76} + x^{73} + x^{67} + x^{66} \\
&+ x^{62} + x^{61} + x^{59} + x^{53} + x^{51} + x^{40} + x^{37} + x^{34} + x^{32} + x^{31} + x^{29} \\
&+ x^{28} + x^{25} + x^{24}, \\
p_3(x) = {} & x^{109} + x^{108} + x^{107} + x^{105} + x^{103} + x^{101} + x^{100} + x^{97} + x^{93} + x^{92} \\
&+ x^{91} + x^{89} + x^{88} + x^{87} + x^{86} + x^{83} + x^{82} + x^{80} + x^{73} + x^{71} + x^{69} \\
&+ x^{68} + x^{67} + x^{66} + x^{63} + x^{62} + x^{58} + x^{57} + x^{55} + x^{51} + x^{49} + x^{48} \\
&+ x^{46} + x^{43} + x^{41} + x^{37} + x^{36} + x^{33} + x^{32} + x^{31} + x^{29} + x^{28} + x^{27}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{26} + x^{24} + x^{23} + x^{17} + x^{16},\\
p_6(x) &= x^{109} + x^{108} + x^{107} + x^{105} + x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96}\\
&+ x^{95} + x^{86} + x^{83} + x^{82} + x^{81} + x^{79} + x^{78} + x^{76} + x^{74} + x^{73} + x^{72}\\
&+ x^{71} + x^{70} + x^{68} + x^{67} + x^{65} + x^{63} + x^{60} + x^{59} + x^{57} + x^{54} + x^{53}\\
&+ x^{51} + x^{50} + x^{45} + x^{42} + x^{41} + x^{40} + x^{37} + x^{36} + x^{35} + x^{32} + x^{31}\\
&+ x^{30} + x^{29} + x^{22} + x^{21} + x^{18} + x^{17} + x^{15} + x^{5} + x^{4} + x + 1,\\
p_9(x) &= x^{109} + x^{108} + x^{105} + x^{104} + x^{89} + x^{88} + x^{61} + x^{60} + x^{57} + x^{56} + x^{41}\\
&+ x^{40} + x^{29} + x^{28} + x^{25} + x^{24} + x^{9} + x^{8},\\
p_{12}(x) &= x^{109} + x^{108} + x^{105} + x^{104} + x^{101} + x^{100} + x^{97} + x^{96} + x^{93} + x^{92}\\
&+ x^{85} + x^{84} + x^{81} + x^{80} + x^{61} + x^{60} + x^{57} + x^{56} + x^{53} + x^{52} + x^{49}\\
&+ x^{48} + x^{45} + x^{44} + x^{37} + x^{36} + x^{33} + x^{32} + x^{29} + x^{28} + x^{25} + x^{24}\\
&+ x^{21} + x^{20} + x^{17} + x^{16} + x^{13} + x^{12} + x^{5} + x^{4} + x + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 1, 1, 1, 0, 0, 1, 0, 0, 1, 0].

For $\phi(M^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{120} + x^{113} + x^{112} + x^{109} + x^{108} + x^{107} + x^{106} + x^{103} + x^{102} + x^{101}\\
&+ x^{100} + x^{93} + x^{91} + x^{86} + x^{84} + x^{82} + x^{81} + x^{79} + x^{77} + x^{76} + x^{75}\\
&+ x^{72} + x^{71} + x^{69} + x^{68} + x^{66} + x^{63} + x^{61} + x^{60} + x^{58} + x^{56} + x^{55}\\
&+ x^{54} + x^{53} + x^{52} + x^{49} + x^{45} + x^{39} + x^{38} + x^{37} + x^{36} + x^{34} + x^{33},\\
p_3(x) &= x^{106} + x^{104} + x^{101} + x^{98} + x^{95} + x^{92} + x^{90} + x^{89} + x^{88} + x^{86} + x^{85}\\
&+ x^{83} + x^{79} + x^{76} + x^{73} + x^{70} + x^{67} + x^{66} + x^{58} + x^{56} + x^{53} + x^{48}\\
&+ x^{47} + x^{45} + x^{44} + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{33}\\
&+ x^{31} + x^{30} + x^{28} + x^{27} + x^{26} + x^{25} + x^{22} + x^{19} + x^{18},\\
p_6(x) &= x^{108} + x^{104} + x^{100} + x^{98} + x^{96} + x^{92} + x^{90} + x^{86} + x^{80} + x^{78} + x^{70}\\
&+ x^{68} + x^{52} + x^{48} + x^{42} + x^{40} + x^{36} + x^{26} + x^{24} + x^{22} + x^{18} + x^{16}\\
&+ x^{14} + x^{12},\\
p_9(x) &= x^{110} + x^{108} + x^{106} + x^{105} + x^{104} + x^{102} + x^{101} + x^{100} + x^{99} + x^{96}\\
&+ x^{94} + x^{93} + x^{92} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{62} + x^{60} + x^{58}\\
&+ x^{57} + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} + x^{48} + x^{46} + x^{45} + x^{44} + x^{42}\\
&+ x^{41} + x^{40} + x^{39} + x^{38} + x^{30} + x^{28} + x^{26} + x^{25} + x^{24} + x^{22} + x^{21}\\
&+ x^{20} + x^{19} + x^{16} + x^{14} + x^{13} + x^{12} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6},\\
p_{12}(x) &= x^{112} + x^{92} + x^{88} + x^{80} + x^{64} + x^{44} + x^{40} + x^{12} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 0, 1].

For $\phi(M^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{104} + x^{101} + x^{100} + x^{98} + x^{96} + x^{88} + x^{86} + x^{84} + x^{79} + x^{77} + x^{75}\\
&+ x^{74} + x^{72} + x^{71} + x^{69} + x^{66} + x^{63} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{54} + x^{53} + x^{52} + x^{51} + x^{49} + x^{47} + x^{46} + x^{42} + x^{41} + x^{40} + x^{37} \\
&+ x^{36} + x^{35} + x^{33}, \\
p_3(x) &= x^{102} + x^{100} + x^{98} + x^{97} + x^{96} + x^{93} + x^{92} + x^{91} + x^{89} + x^{87} + x^{86} \\
&+ x^{84} + x^{83} + x^{81} + x^{80} + x^{77} + x^{76} + x^{75} + x^{73} + x^{71} + x^{67} + x^{66} \\
&+ x^{54} + x^{52} + x^{50} + x^{49} + x^{48} + x^{46} + x^{45} + x^{43} + x^{42} + x^{40} + x^{39} \\
&+ x^{38} + x^{37} + x^{32} + x^{31} + x^{29} + x^{28} + x^{26} + x^{25} + x^{23} + x^{19} + x^{18}, \\
p_6(x) &= x^{104} + x^{96} + x^{94} + x^{90} + x^{88} + x^{84} + x^{82} + x^{80} + x^{78} + x^{66} + x^{64} \\
&+ x^{62} + x^{60} + x^{58} + x^{56} + x^{54} + x^{52} + x^{50} + x^{46} + x^{44} + x^{42} + x^{40} \\
&+ x^{32} + x^{28} + x^{24} + x^{22} + x^{14} + x^{12}, \\
p_9(x) &= x^{106} + x^{104} + x^{101} + x^{98} + x^{96} + x^{95} + x^{94} + x^{91} + x^{89} + x^{88} + x^{87} \\
&+ x^{86} + x^{58} + x^{56} + x^{53} + x^{50} + x^{48} + x^{47} + x^{46} + x^{43} + x^{41} + x^{40} \\
&+ x^{39} + x^{38} + x^{26} + x^{24} + x^{21} + x^{18} + x^{16} + x^{15} + x^{14} + x^{11} + x^{9} \\
&+ x^{8} + x^{7} + x^{6}, \\
p_{12}(x) &= x^{108} + x^{104} + x^{100} + x^{96} + x^{92} + x^{84} + x^{80} + x^{60} + x^{56} + x^{52} + x^{48} \\
&+ x^{44} + x^{36} + x^{32} + x^{28} + x^{24} + x^{20} + x^{16} + x^{12} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 1, 0, 1, 0, 1, 0, 1, 1, 0, 1, 1, 1, 0].

For $\phi(M^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{111} + x^{105} + x^{103} + x^{100} + x^{97} + x^{95} + x^{93} + x^{91} + x^{87} + x^{85} + x^{84} \\
&+ x^{83} + x^{81} + x^{79} + x^{78} + x^{75} + x^{73} + x^{71} + x^{69} + x^{66} + x^{65} + x^{64} \\
&+ x^{60} + x^{59} + x^{57} + x^{54} + x^{53} + x^{51} + x^{42}, \\
p_3(x) &= x^{109} + x^{108} + x^{107} + x^{104} + x^{103} + x^{96} + x^{95} + x^{94} + x^{90} + x^{85} + x^{84} \\
&+ x^{83} + x^{82} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{70} \\
&+ x^{69} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{58} + x^{56} + x^{55} \\
&+ x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{42} + x^{41} + x^{37} + x^{36} + x^{35} \\
&+ x^{34} + x^{33} + x^{32} + x^{31} + x^{25} + x^{22}, \\
p_6(x) &= x^{109} + x^{108} + x^{107} + x^{104} + x^{103} + x^{99} + x^{97} + x^{96} + x^{94} + x^{91} + x^{90} \\
&+ x^{89} + x^{88} + x^{86} + x^{85} + x^{84} + x^{80} + x^{77} + x^{74} + x^{73} + x^{72} + x^{71} \\
&+ x^{70} + x^{69} + x^{66} + x^{65} + x^{62} + x^{61} + x^{58} + x^{56} + x^{55} + x^{52} + x^{47} \\
&+ x^{46} + x^{45} + x^{43} + x^{41} + x^{40} + x^{35} + x^{33} + x^{31} + x^{30} + x^{29} + x^{25} \\
&+ x^{24} + x^{23} + x^{20} + x^{18} + x^{16} + x^{14} + x^{13} + x^{12} + x^{9} + x^{8} + x^{5} + x^{4} \\
&+ x + 1, \\
p_9(x) &= x^{109} + x^{108} + x^{105} + x^{104} + x^{89} + x^{88} + x^{61} + x^{60} + x^{57} + x^{56} + x^{41} \\
&+ x^{40} + x^{29} + x^{28} + x^{25} + x^{24} + x^{9} + x^{8}, \\
p_{12}(x) &= x^{109} + x^{108} + x^{105} + x^{104} + x^{101} + x^{100} + x^{97} + x^{96} + x^{93} + x^{92} \\
&+ x^{85} + x^{84} + x^{81} + x^{80} + x^{61} + x^{60} + x^{57} + x^{56} + x^{53} + x^{52} + x^{49}
\end{aligned}
$$

$$+ x^{48} + x^{45} + x^{44} + x^{37} + x^{36} + x^{33} + x^{32} + x^{29} + x^{28} + x^{25} + x^{24} + x^{21} + x^{20} + x^{17} + x^{16} + x^{13} + x^{12} + x^{5} + x^{4} + x + 1,$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0].

For $\phi(W^o, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{111} + x^{109} + x^{108} + x^{106} + x^{103} + x^{102} + x^{101} + x^{98} + x^{96} + x^{92} \\
&\quad + x^{91} + x^{90} + x^{89} + x^{86} + x^{85} + x^{84} + x^{79} + x^{76} + x^{73} + x^{67} + x^{66} \\
&\quad + x^{62} + x^{61} + x^{59} + x^{53} + x^{51} + x^{40} + x^{37} + x^{34} + x^{32} + x^{31} + x^{29} \\
&\quad + x^{28} + x^{25} + x^{24}, \\
p_3(x) &= x^{109} + x^{108} + x^{107} + x^{105} + x^{103} + x^{101} + x^{100} + x^{97} + x^{93} + x^{92} \\
&\quad + x^{91} + x^{89} + x^{88} + x^{87} + x^{86} + x^{83} + x^{82} + x^{80} + x^{73} + x^{71} + x^{69} \\
&\quad + x^{68} + x^{67} + x^{66} + x^{63} + x^{62} + x^{58} + x^{57} + x^{55} + x^{51} + x^{49} + x^{48} \\
&\quad + x^{46} + x^{43} + x^{41} + x^{37} + x^{36} + x^{33} + x^{32} + x^{31} + x^{29} + x^{28} + x^{27} \\
&\quad + x^{26} + x^{24} + x^{23} + x^{17} + x^{16}, \\
p_6(x) &= x^{109} + x^{108} + x^{107} + x^{105} + x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} \\
&\quad + x^{95} + x^{86} + x^{83} + x^{82} + x^{81} + x^{79} + x^{78} + x^{76} + x^{74} + x^{73} + x^{72} \\
&\quad + x^{71} + x^{70} + x^{68} + x^{67} + x^{65} + x^{63} + x^{60} + x^{59} + x^{57} + x^{54} + x^{53} \\
&\quad + x^{51} + x^{50} + x^{45} + x^{42} + x^{41} + x^{40} + x^{37} + x^{36} + x^{35} + x^{32} + x^{31} \\
&\quad + x^{30} + x^{29} + x^{22} + x^{21} + x^{18} + x^{17} + x^{15} + x^{5} + x^{4} + x + 1, \\
p_9(x) &= x^{109} + x^{108} + x^{105} + x^{104} + x^{89} + x^{88} + x^{61} + x^{60} + x^{57} + x^{56} + x^{41} \\
&\quad + x^{40} + x^{29} + x^{28} + x^{25} + x^{24} + x^{9} + x^{8}, \\
p_{12}(x) &= x^{109} + x^{108} + x^{105} + x^{104} + x^{101} + x^{100} + x^{97} + x^{96} + x^{93} + x^{92} \\
&\quad + x^{85} + x^{84} + x^{81} + x^{80} + x^{61} + x^{60} + x^{57} + x^{56} + x^{53} + x^{52} + x^{49} \\
&\quad + x^{48} + x^{45} + x^{44} + x^{37} + x^{36} + x^{33} + x^{32} + x^{29} + x^{28} + x^{25} + x^{24} \\
&\quad + x^{21} + x^{20} + x^{17} + x^{16} + x^{13} + x^{12} + x^{5} + x^{4} + x + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 1, 0, 1, 1, 1, 0, 0, 1, 0, 0, 1, 0].

For $\phi(W^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{104} + x^{101} + x^{100} + x^{98} + x^{96} + x^{88} + x^{86} + x^{84} + x^{79} + x^{77} + x^{75} \\
&\quad + x^{74} + x^{72} + x^{71} + x^{69} + x^{66} + x^{63} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} \\
&\quad + x^{54} + x^{53} + x^{52} + x^{51} + x^{49} + x^{47} + x^{46} + x^{42} + x^{41} + x^{40} + x^{37} \\
&\quad + x^{36} + x^{35} + x^{33}, \\
p_3(x) &= x^{102} + x^{100} + x^{98} + x^{97} + x^{96} + x^{93} + x^{92} + x^{91} + x^{89} + x^{87} + x^{86} \\
&\quad + x^{84} + x^{83} + x^{81} + x^{80} + x^{77} + x^{76} + x^{75} + x^{73} + x^{71} + x^{67} + x^{66} \\
&\quad + x^{54} + x^{52} + x^{50} + x^{49} + x^{48} + x^{46} + x^{45} + x^{43} + x^{42} + x^{40} + x^{39} \\
&\quad + x^{38} + x^{37} + x^{32} + x^{31} + x^{29} + x^{28} + x^{26} + x^{25} + x^{23} + x^{19} + x^{18}, \\
p_6(x) &= x^{104} + x^{96} + x^{94} + x^{90} + x^{88} + x^{84} + x^{82} + x^{80} + x^{78} + x^{66} + x^{64}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{62} + x^{60} + x^{58} + x^{56} + x^{54} + x^{52} + x^{50} + x^{46} + x^{44} + x^{42} + x^{40}\\
&+ x^{32} + x^{28} + x^{24} + x^{22} + x^{14} + x^{12},\\
p_9(x) &= x^{106} + x^{104} + x^{101} + x^{98} + x^{96} + x^{95} + x^{94} + x^{91} + x^{89} + x^{88} + x^{87}\\
&+ x^{86} + x^{58} + x^{56} + x^{53} + x^{50} + x^{48} + x^{47} + x^{46} + x^{43} + x^{41} + x^{40}\\
&+ x^{39} + x^{38} + x^{26} + x^{24} + x^{21} + x^{18} + x^{16} + x^{15} + x^{14} + x^{11} + x^{9}\\
&+ x^{8} + x^{7} + x^{6},\\
p_{12}(x) &= x^{108} + x^{104} + x^{100} + x^{96} + x^{92} + x^{84} + x^{80} + x^{60} + x^{56} + x^{52} + x^{48}\\
&+ x^{44} + x^{36} + x^{32} + x^{28} + x^{24} + x^{20} + x^{16} + x^{12} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 1, 0, 1, 0, 1, 0, 1, 1, 0, 1, 1, 1, 0].

For $\phi(W^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{120} + x^{113} + x^{112} + x^{109} + x^{108} + x^{107} + x^{106} + x^{103} + x^{102} + x^{101}\\
&+ x^{100} + x^{93} + x^{91} + x^{86} + x^{84} + x^{82} + x^{81} + x^{79} + x^{77} + x^{76} + x^{75}\\
&+ x^{72} + x^{71} + x^{69} + x^{68} + x^{66} + x^{63} + x^{61} + x^{60} + x^{58} + x^{56} + x^{55}\\
&+ x^{54} + x^{53} + x^{52} + x^{49} + x^{45} + x^{39} + x^{38} + x^{37} + x^{36} + x^{34} + x^{33},\\
p_3(x) &= x^{106} + x^{104} + x^{101} + x^{98} + x^{95} + x^{92} + x^{90} + x^{89} + x^{88} + x^{86} + x^{85}\\
&+ x^{83} + x^{79} + x^{76} + x^{73} + x^{70} + x^{67} + x^{66} + x^{58} + x^{56} + x^{53} + x^{48}\\
&+ x^{47} + x^{45} + x^{44} + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{33}\\
&+ x^{31} + x^{30} + x^{28} + x^{27} + x^{26} + x^{25} + x^{22} + x^{19} + x^{18},\\
p_6(x) &= x^{108} + x^{104} + x^{100} + x^{98} + x^{96} + x^{92} + x^{90} + x^{86} + x^{80} + x^{78} + x^{70}\\
&+ x^{68} + x^{52} + x^{48} + x^{42} + x^{40} + x^{36} + x^{26} + x^{24} + x^{22} + x^{18} + x^{16}\\
&+ x^{14} + x^{12},\\
p_9(x) &= x^{110} + x^{108} + x^{106} + x^{105} + x^{104} + x^{102} + x^{101} + x^{100} + x^{99} + x^{96}\\
&+ x^{94} + x^{93} + x^{92} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{62} + x^{60} + x^{58}\\
&+ x^{57} + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} + x^{48} + x^{46} + x^{45} + x^{44} + x^{42}\\
&+ x^{41} + x^{40} + x^{39} + x^{38} + x^{30} + x^{28} + x^{26} + x^{25} + x^{24} + x^{22} + x^{21}\\
&+ x^{20} + x^{19} + x^{16} + x^{14} + x^{13} + x^{12} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6},\\
p_{12}(x) &= x^{112} + x^{92} + x^{88} + x^{80} + x^{64} + x^{44} + x^{40} + x^{12} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 0, 1].

For $\phi(W^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{111} + x^{105} + x^{103} + x^{100} + x^{97} + x^{95} + x^{93} + x^{91} + x^{87} + x^{85} + x^{84}\\
&+ x^{83} + x^{81} + x^{79} + x^{78} + x^{75} + x^{73} + x^{71} + x^{69} + x^{66} + x^{65} + x^{64}\\
&+ x^{60} + x^{59} + x^{57} + x^{54} + x^{53} + x^{51} + x^{42},\\
p_3(x) &= x^{109} + x^{108} + x^{107} + x^{104} + x^{103} + x^{96} + x^{95} + x^{94} + x^{90} + x^{85} + x^{84}\\
&+ x^{83} + x^{82} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{70}\\
&+ x^{69} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{58} + x^{56} + x^{55}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{42} + x^{41} + x^{37} + x^{36} + x^{35} \\
&+ x^{34} + x^{33} + x^{32} + x^{31} + x^{25} + x^{22}, \\
p_6(x) = {} & x^{109} + x^{108} + x^{107} + x^{104} + x^{103} + x^{99} + x^{97} + x^{96} + x^{94} + x^{91} + x^{90} \\
&+ x^{89} + x^{88} + x^{86} + x^{85} + x^{84} + x^{80} + x^{77} + x^{74} + x^{73} + x^{72} + x^{71} \\
&+ x^{70} + x^{69} + x^{66} + x^{65} + x^{62} + x^{61} + x^{58} + x^{56} + x^{55} + x^{52} + x^{47} \\
&+ x^{46} + x^{45} + x^{43} + x^{41} + x^{40} + x^{35} + x^{33} + x^{31} + x^{30} + x^{29} + x^{25} \\
&+ x^{24} + x^{23} + x^{20} + x^{18} + x^{16} + x^{14} + x^{13} + x^{12} + x^{9} + x^{8} + x^{5} + x^{4} \\
&+ x + 1, \\
p_9(x) = {} & x^{109} + x^{108} + x^{105} + x^{104} + x^{89} + x^{88} + x^{61} + x^{60} + x^{57} + x^{56} + x^{41} \\
&+ x^{40} + x^{29} + x^{28} + x^{25} + x^{24} + x^{9} + x^{8}, \\
p_{12}(x) = {} & x^{109} + x^{108} + x^{105} + x^{104} + x^{101} + x^{100} + x^{97} + x^{96} + x^{93} + x^{92} \\
&+ x^{85} + x^{84} + x^{81} + x^{80} + x^{61} + x^{60} + x^{57} + x^{56} + x^{53} + x^{52} + x^{49} \\
&+ x^{48} + x^{45} + x^{44} + x^{37} + x^{36} + x^{33} + x^{32} + x^{29} + x^{28} + x^{25} + x^{24} \\
&+ x^{21} + x^{20} + x^{17} + x^{16} + x^{13} + x^{12} + x^{5} + x^{4} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	154	[276, 277]	308	[209, 150]	462	[575, 576]	616	[694, 327]
1	[3, 4]	155	[119, 246]	309	[442, 130]	463	[577, 495]	617	[15, 268]
2	[5, 6]	156	[117, 278]	310	[443, 381]	464	[65, 267]	618	[702, 587]
3	[7, 8]	157	[279, 231]	311	[444, 38]	465	[550, 98]	619	[587, 703]
4	[9, 10]	158	[280, 271]	312	[445, 368]	466	[578, 579]	620	[6, 704]
5	[11, 12]	159	[281, 282]	313	[161, 46]	467	[110, 119]	621	[495, 110]
6	[13, 14]	160	[77, 91]	314	[129, 58]	468	[580, 581]	622	[705, 706]
7	[15, 16]	161	[283, 29]	315	[446, 52]	469	[514, 582]	623	[273, 521]
8	[17, 18]	162	[18, 284]	316	[447, 448]	470	[92, 347]	624	[530, 559]
9	[19, 20]	163	[285, 286]	317	[59, 449]	471	[484, 222]	625	[578, 703]
10	[21, 22]	164	[287, 116]	318	[48, 450]	472	[583, 584]	626	[100, 329]
11	[23, 24]	165	[288, 6]	319	[451, 165]	473	[268, 285]	627	[707, 513]
12	[25, 26]	166	[71, 289]	320	[452, 453]	474	[522, 585]	628	[708, 709]
13	[27, 28]	167	[3, 290]	321	[120, 56]	475	[586, 587]	629	[548, 710]
14	[29, 30]	168	[291, 292]	322	[454, 455]	476	[588, 282]	630	[535, 114]
15	[31, 32]	169	[293, 294]	323	[46, 14]	477	[589, 505]	631	[338, 711]
16	[33, 34]	170	[16, 285]	324	[456, 457]	478	[76, 249]	632	[271, 712]
17	[35, 36]	171	[263, 23]	325	[456, 439]	479	[537, 100]	633	[327, 265]
18	[37, 38]	172	[266, 111]	326	[458, 438]	480	[504, 590]	634	[223, 713]
19	[39, 40]	173	[295, 296]	327	[459, 205]	481	[591, 255]	635	[574, 484]
20	[41, 42]	174	[4, 297]	328	[460, 461]	482	[592, 593]	636	[262, 228]

21	[43, 44]	175	[298, 231]	329	[156, 155]	483	[594, 356]	637	[575, 510]
22	[45, 46]	176	[244, 299]	330	[195, 49]	484	[595, 444]	638	[249, 502]
23	[47, 48]	177	[300, 301]	331	[370, 418]	485	[478, 136]	639	[555, 531]
24	[49, 50]	178	[302, 292]	332	[462, 459]	486	[596, 197]	640	[547, 714]
25	[51, 52]	179	[22, 303]	333	[463, 399]	487	[479, 137]	641	[279, 257]
26	[12, 53]	180	[300, 304]	334	[464, 147]	488	[597, 378]	642	[715, 63]
27	[54, 55]	181	[305, 306]	335	[444, 465]	489	[427, 188]	643	[716, 280]
28	[56, 57]	182	[307, 308]	336	[466, 467]	490	[598, 456]	644	[317, 65]
29	[58, 59]	183	[309, 310]	337	[52, 185]	491	[599, 435]	645	[502, 243]
30	[60, 61]	184	[222, 311]	338	[468, 469]	492	[206, 31]	646	[717, 495]
31	[62, 63]	185	[312, 313]	339	[470, 471]	493	[144, 395]	647	[705, 230]
32	[64, 65]	186	[314, 315]	340	[472, 473]	494	[425, 171]	648	[718, 716]
33	[66, 67]	187	[316, 317]	341	[129, 167]	495	[483, 217]	649	[541, 288]
34	[68, 69]	188	[318, 319]	342	[474, 475]	496	[600, 601]	650	[717, 109]
35	[70, 71]	189	[320, 321]	343	[363, 129]	497	[602, 388]	651	[719, 720]
36	[72, 73]	190	[94, 259]	344	[186, 169]	498	[603, 604]	652	[721, 712]
37	[74, 75]	191	[322, 323]	345	[476, 218]	499	[605, 606]	653	[81, 4]
38	[76, 77]	192	[324, 325]	346	[121, 57]	500	[204, 607]	654	[89, 350]
39	[78, 79]	193	[326, 327]	347	[477, 182]	501	[462, 204]	655	[722, 723]
40	[80, 81]	194	[328, 329]	348	[478, 479]	502	[608, 477]	656	[592, 238]
41	[82, 83]	195	[302, 330]	349	[480, 481]	503	[609, 610]	657	[716, 720]
42	[84, 85]	196	[331, 77]	350	[168, 449]	504	[135, 611]	658	[724, 486]
43	[86, 87]	197	[332, 333]	351	[131, 190]	505	[612, 203]	659	[519, 725]
44	[88, 89]	198	[306, 219]	352	[197, 191]	506	[613, 372]	660	[82, 345]
45	[67, 90]	199	[334, 331]	353	[54, 211]	507	[614, 444]	661	[726, 726]
46	[91, 92]	200	[335, 331]	354	[482, 483]	508	[615, 362]	662	[727, 227]
47	[93, 94]	201	[282, 100]	355	[167, 59]	509	[213, 616]	663	[715, 542]
48	[95, 96]	202	[336, 337]	356	[453, 471]	510	[130, 48]	664	[718, 719]
49	[97, 98]	203	[338, 339]	357	[484, 342]	511	[442, 47]	665	[728, 729]
50	[99, 75]	204	[340, 341]	358	[259, 485]	512	[413, 617]	666	[591, 730]
51	[100, 101]	205	[342, 311]	359	[486, 284]	513	[618, 619]	667	[699, 71]
52	[102, 103]	206	[286, 343]	360	[301, 487]	514	[620, 621]	668	[99, 731]
53	[24, 104]	207	[319, 339]	361	[488, 489]	515	[622, 420]	669	[22, 732]
54	[105, 106]	208	[344, 345]	362	[490, 289]	516	[622, 2]	670	[250, 333]
55	[107, 108]	209	[63, 346]	363	[80, 3]	517	[623, 40]	671	[544, 733]
56	[109, 110]	210	[228, 23]	364	[240, 115]	518	[624, 625]	672	[78, 90]
57	[111, 112]	211	[243, 347]	365	[491, 22]	519	[41, 626]	673	[571, 734]
58	[113, 114]	212	[348, 304]	366	[306, 266]	520	[627, 628]	674	[19, 709]
59	[115, 74]	213	[349, 350]	367	[89, 492]	521	[629, 630]	675	[586, 578]
60	[116, 117]	214	[351, 321]	368	[493, 494]	522	[631, 632]	676	[282, 328]
61	[118, 119]	215	[5, 25]	369	[495, 118]	523	[633, 481]	677	[549, 735]
62	[120, 121]	216	[352, 353]	370	[496, 78]	524	[634, 612]	678	[96, 355]
63	[122, 123]	217	[354, 355]	371	[107, 497]	525	[635, 470]	679	[96, 736]
64	[124, 31]	218	[356, 253]	372	[498, 89]	526	[633, 636]	680	[85, 737]

65	[125, 126]	219	[357, 204]	373	[499, 24]	527	[458, 456]	681	[225, 317]
66	[60, 127]	220	[358, 359]	374	[500, 501]	528	[470, 637]	682	[696, 708]
67	[128, 129]	221	[43, 179]	375	[502, 92]	529	[638, 469]	683	[720, 721]
68	[130, 131]	222	[360, 194]	376	[503, 85]	530	[639, 640]	684	[286, 308]
69	[132, 133]	223	[361, 54]	377	[504, 505]	531	[641, 144]	685	[562, 221]
70	[134, 135]	224	[362, 213]	378	[506, 259]	532	[612, 642]	686	[268, 572]
71	[136, 137]	225	[52, 125]	379	[486, 507]	533	[643, 644]	687	[256, 96]
72	[138, 139]	226	[363, 364]	380	[508, 223]	534	[645, 210]	688	[738, 582]
73	[140, 141]	227	[365, 53]	381	[509, 510]	535	[168, 646]	689	[29, 739]
74	[142, 143]	228	[187, 366]	382	[296, 511]	536	[466, 647]	690	[208, 149]
75	[144, 142]	229	[367, 160]	383	[512, 513]	537	[648, 152]	691	[447, 461]
76	[145, 146]	230	[185, 126]	384	[514, 515]	538	[185, 140]	692	[358, 424]
77	[147, 148]	231	[368, 61]	385	[516, 517]	539	[435, 476]	693	[740, 429]
78	[149, 150]	232	[369, 368]	386	[85, 313]	540	[367, 34]	694	[479, 741]
79	[151, 152]	233	[362, 370]	387	[518, 252]	541	[161, 13]	695	[689, 35]
80	[8, 153]	234	[196, 50]	388	[519, 520]	542	[649, 35]	696	[742, 42]
81	[154, 155]	235	[371, 175]	389	[521, 30]	543	[650, 120]	697	[743, 744]
82	[156, 157]	236	[372, 373]	390	[522, 494]	544	[443, 122]	698	[658, 745]
83	[158, 159]	237	[374, 375]	391	[290, 523]	545	[464, 197]	699	[463, 746]
84	[160, 161]	238	[218, 201]	392	[524, 325]	546	[651, 652]	700	[747, 748]
85	[162, 163]	239	[376, 377]	393	[327, 306]	547	[653, 654]	701	[645, 54]
86	[132, 164]	240	[378, 142]	394	[525, 319]	548	[655, 656]	702	[749, 750]
87	[165, 166]	241	[379, 188]	395	[526, 81]	549	[602, 657]	703	[750, 751]
88	[167, 168]	242	[35, 165]	396	[304, 527]	550	[614, 37]	704	[385, 662]
89	[169, 170]	243	[380, 381]	397	[309, 270]	551	[658, 383]	705	[140, 752]
90	[127, 156]	244	[382, 383]	398	[528, 351]	552	[454, 178]	706	[472, 748]
91	[146, 171]	245	[384, 159]	399	[296, 294]	553	[659, 660]	707	[683, 753]
92	[148, 172]	246	[216, 12]	400	[241, 529]	554	[661, 662]	708	[753, 652]
93	[173, 174]	247	[385, 386]	401	[530, 117]	555	[663, 664]	709	[619, 619]
94	[175, 176]	248	[207, 32]	402	[227, 95]	556	[400, 406]	710	[686, 656]
95	[177, 178]	249	[387, 388]	403	[531, 99]	557	[127, 154]	711	[650, 392]
96	[53, 179]	250	[389, 390]	404	[532, 94]	558	[609, 441]	712	[683, 651]
97	[36, 180]	251	[134, 175]	405	[525, 338]	559	[665, 193]	713	[673, 640]
98	[181, 182]	252	[391, 392]	406	[533, 5]	560	[666, 146]	714	[628, 754]
99	[183, 132]	253	[376, 393]	407	[94, 235]	561	[56, 667]	715	[740, 755]
100	[184, 185]	254	[381, 394]	408	[488, 28]	562	[216, 365]	716	[756, 681]
101	[186, 8]	255	[395, 143]	409	[277, 224]	563	[599, 483]	717	[476, 200]
102	[181, 187]	256	[364, 167]	410	[534, 240]	564	[668, 463]	718	[623, 757]
103	[188, 189]	257	[396, 397]	411	[535, 536]	565	[669, 212]	719	[754, 751]
104	[48, 190]	258	[398, 163]	412	[66, 78]	566	[635, 453]	720	[758, 683]
105	[191, 192]	259	[174, 399]	413	[281, 537]	567	[182, 670]	721	[751, 628]
106	[193, 194]	260	[400, 389]	414	[314, 323]	568	[671, 372]	722	[218, 759]
107	[195, 196]	261	[389, 401]	415	[538, 341]	569	[151, 672]	723	[430, 10]
108	[197, 148]	262	[402, 47]	416	[539, 540]	570	[160, 45]	724	[638, 760]

109	[198, 199]	263	[403, 49]	417	[295, 293]	571	[673, 674]	725	[653, 412]
110	[200, 201]	264	[395, 404]	418	[304, 487]	572	[675, 647]	726	[761, 685]
111	[202, 203]	265	[405, 202]	419	[110, 245]	573	[634, 202]	727	[209, 762]
112	[204, 205]	266	[406, 401]	420	[539, 73]	574	[676, 360]	728	[12, 43]
113	[206, 207]	267	[407, 408]	421	[541, 5]	575	[666, 425]	729	[374, 408]
114	[208, 209]	268	[126, 141]	422	[542, 241]	576	[677, 606]	730	[422, 132]
115	[210, 211]	269	[409, 410]	423	[532, 506]	577	[158, 678]	731	[631, 664]
116	[212, 213]	270	[191, 172]	424	[543, 351]	578	[679, 414]	732	[138, 199]
117	[214, 133]	271	[411, 412]	425	[346, 242]	579	[680, 681]	733	[624, 475]
118	[215, 216]	272	[413, 414]	426	[544, 106]	580	[677, 682]	734	[615, 212]
119	[217, 218]	273	[398, 415]	427	[543, 320]	581	[676, 193]	735	[597, 144]
120	[219, 111]	274	[416, 11]	428	[342, 545]	582	[652, 683]	736	[603, 630]
121	[220, 221]	275	[170, 162]	429	[516, 546]	583	[680, 684]	737	[643, 660]
122	[87, 222]	276	[196, 417]	430	[547, 548]	584	[620, 685]	738	[743, 743]
123	[223, 224]	277	[194, 393]	431	[6, 26]	585	[434, 477]	739	[600, 397]
124	[225, 64]	278	[213, 418]	432	[549, 353]	586	[686, 687]	740	[698, 262]
125	[226, 227]	279	[170, 398]	433	[550, 551]	587	[688, 651]	741	[580, 69]
126	[103, 228]	280	[419, 420]	434	[538, 552]	588	[446, 184]	742	[317, 700]
127	[229, 230]	281	[46, 421]	435	[347, 299]	589	[671, 436]	743	[763, 546]
128	[231, 232]	282	[61, 156]	436	[553, 303]	590	[173, 463]	744	[744, 744]
129	[233, 234]	283	[369, 60]	437	[287, 530]	591	[689, 451]	745	[548, 764]
130	[235, 236]	284	[36, 166]	438	[554, 256]	592	[207, 208]	746	[724, 18]
131	[237, 238]	285	[32, 209]	439	[555, 115]	593	[416, 216]	747	[696, 19]
132	[239, 240]	286	[34, 45]	440	[111, 556]	594	[193, 376]	748	[722, 275]
133	[241, 242]	287	[422, 214]	441	[508, 277]	595	[334, 76]	749	[765, 714]
134	[243, 244]	288	[423, 424]	442	[557, 558]	596	[690, 691]	750	[766, 697]
135	[245, 246]	289	[135, 176]	443	[559, 560]	597	[119, 692]	751	[714, 764]
136	[247, 248]	290	[153, 162]	444	[561, 293]	598	[320, 693]	752	[557, 570]
137	[249, 91]	291	[425, 426]	445	[562, 563]	599	[256, 354]	753	[767, 520]
138	[250, 251]	292	[182, 366]	446	[84, 312]	600	[694, 305]	754	[513, 708]
139	[252, 253]	293	[392, 56]	447	[266, 564]	601	[542, 346]	755	[726, 744]
140	[254, 255]	294	[409, 189]	448	[565, 104]	602	[244, 695]	756	[727, 554]
141	[227, 256]	295	[427, 409]	449	[290, 297]	603	[512, 696]	757	[231, 258]
142	[257, 258]	296	[428, 429]	450	[500, 261]	604	[81, 290]	758	[768, 719]
143	[259, 236]	297	[8, 170]	451	[564, 112]	605	[499, 565]	759	[728, 248]
144	[260, 261]	298	[430, 431]	452	[565, 566]	606	[583, 697]	760	[280, 721]
145	[262, 263]	299	[381, 123]	453	[567, 108]	607	[698, 103]	761	[21, 553]
146	[264, 116]	300	[372, 432]	454	[71, 568]	608	[285, 307]	762	[690, 337]
147	[265, 266]	301	[433, 136]	455	[531, 74]	609	[699, 490]	763	[652, 688]
148	[267, 268]	302	[434, 181]	456	[569, 570]	610	[559, 278]	764	[749, 754]
149	[269, 270]	303	[435, 217]	457	[571, 234]	611	[518, 356]	765	[627, 749]
150	[65, 15]	304	[436, 373]	458	[534, 555]	612	[700, 15]	766	[750, 627]
151	[271, 272]	305	[437, 406]	459	[572, 286]	613	[263, 499]	767	[619, 618]
152	[273, 29]	306	[438, 439]	460	[324, 573]	614	[252, 701]	768	[651, 758]

153	[274, 275]	307	[440, 441]	461	[574, 87]	615	[88, 349]

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	110	0	220	1	330	1	440	1	550	1	660	1
1	0	111	1	221	0	331	0	441	0	551	0	661	0
2	0	112	1	222	1	332	1	442	1	552	1	662	0
3	0	113	1	223	1	333	0	443	1	553	1	663	0
4	1	114	1	224	1	334	0	444	0	554	0	664	0
5	0	115	0	225	0	335	0	445	1	555	0	665	1
6	1	116	0	226	0	336	1	446	0	556	1	666	1
7	0	117	1	227	0	337	0	447	0	557	1	667	0
8	0	118	0	228	0	338	1	448	1	558	0	668	1
9	1	119	1	229	1	339	0	449	1	559	1	669	1
10	0	120	0	230	1	340	1	450	1	560	1	670	1
11	0	121	1	231	1	341	1	451	1	561	0	671	1
12	1	122	0	232	0	342	1	452	1	562	1	672	1
13	1	123	1	233	1	343	0	453	1	563	0	673	1
14	1	124	0	234	0	344	1	454	1	564	1	674	1
15	0	125	0	235	1	345	0	455	0	565	1	675	0
16	0	126	0	236	0	346	1	456	1	566	0	676	0
17	0	127	1	237	1	347	1	457	1	567	1	677	1
18	1	128	1	238	1	348	0	458	0	568	1	678	1
19	1	129	1	239	0	349	1	459	0	569	1	679	1
20	1	130	1	240	0	350	1	460	1	570	0	680	1
21	0	131	1	241	1	351	1	461	0	571	1	681	0
22	1	132	0	242	0	352	1	462	1	572	0	682	0
23	0	133	1	243	0	353	1	463	0	573	1	683	0
24	1	134	0	244	1	354	1	464	0	574	0	684	1
25	1	135	1	245	1	355	0	465	1	575	1	685	1
26	1	136	1	246	1	356	1	466	1	576	1	686	0
27	1	137	0	247	1	357	0	467	0	577	0	687	0
28	0	138	1	248	1	358	1	468	1	578	1	688	1
29	1	139	1	249	0	359	1	469	1	579	1	689	1
30	0	140	1	250	1	360	1	470	0	580	1	690	1
31	0	141	0	251	0	361	1	471	0	581	0	691	0
32	0	142	1	252	1	362	1	472	1	582	1	692	1
33	0	143	1	253	0	363	0	473	0	583	1	693	0
34	1	144	1	254	1	364	0	474	1	584	1	694	0
35	0	145	0	255	0	365	0	475	0	585	1	695	1
36	1	146	0	256	0	366	0	476	0	586	0	696	0
37	1	147	0	257	1	367	1	477	1	587	1	697	1
38	0	148	0	258	0	368	1	478	0	588	0	698	0
39	1	149	1	259	1	369	0	479	0	589	1	699	0

40	0	150	0	260	1	370	0	480	1	590	0	700	0
41	1	151	1	261	1	371	1	481	1	591	1	701	0
42	0	152	0	262	0	372	0	482	1	592	1	702	0
43	0	153	1	263	0	373	0	483	0	593	1	703	1
44	0	154	0	264	0	374	1	484	0	594	0	704	1
45	1	155	1	265	0	375	0	485	0	595	0	705	1
46	0	156	1	266	0	376	0	486	1	596	1	706	1
47	0	157	0	267	0	377	1	487	0	597	1	707	0
48	0	158	0	268	0	378	0	488	1	598	1	708	1
49	1	159	0	269	1	379	1	489	0	599	0	709	1
50	1	160	0	270	1	380	0	490	1	600	0	710	0
51	1	161	0	271	1	381	1	491	0	601	0	711	0
52	0	162	1	272	0	382	1	492	1	602	1	712	0
53	1	163	0	273	0	383	0	493	1	603	0	713	1
54	1	164	0	274	1	384	1	494	1	604	0	714	1
55	1	165	0	275	0	385	1	495	0	605	0	715	0
56	0	166	1	276	0	386	1	496	0	606	1	716	0
57	1	167	0	277	1	387	0	497	1	607	0	717	0
58	1	168	1	278	1	388	1	498	0	608	0	718	0
59	0	169	1	279	0	389	1	499	0	609	0	719	0
60	0	170	0	280	0	390	1	500	1	610	1	720	0
61	0	171	0	281	0	391	1	501	1	611	0	721	1
62	0	172	0	282	0	392	1	502	0	612	0	722	1
63	0	173	0	283	0	393	0	503	0	613	0	723	0
64	0	174	1	284	1	394	0	504	1	614	1	724	0
65	0	175	0	285	0	395	0	505	0	615	0	725	0
66	0	176	1	286	1	396	1	506	0	616	0	726	0
67	1	177	0	287	0	397	1	507	1	617	0	727	0
68	1	178	1	288	0	398	0	508	0	618	0	728	1
69	0	179	1	289	1	399	1	509	1	619	1	729	1
70	0	180	0	290	1	400	1	510	1	620	1	730	0
71	1	181	0	291	1	401	0	511	1	621	0	731	1
72	1	182	1	292	1	402	0	512	0	622	1	732	1
73	1	183	1	293	1	403	0	513	0	623	0	733	0
74	1	184	1	294	1	404	0	514	1	624	0	734	0
75	1	185	1	295	0	405	0	515	1	625	1	735	1
76	0	186	1	296	1	406	0	516	1	626	1	736	0
77	0	187	0	297	0	407	0	517	0	627	0	737	0
78	1	188	0	298	0	408	1	518	0	628	1	738	1
79	1	189	1	299	1	409	1	519	1	629	1	739	0
80	0	190	0	300	0	410	0	520	0	630	1	740	0
81	0	191	1	301	1	411	1	521	1	631	1	741	1
82	1	192	1	302	1	412	0	522	1	632	1	742	0
83	0	193	0	303	1	413	0	523	0	633	0	743	1

84	0	194	1	304	1	414	1	524	1	634	1	744	0
85	1	195	1	305	0	415	1	525	0	635	0	745	1
86	0	196	0	306	0	416	1	526	0	636	0	746	0
87	0	197	1	307	1	417	0	527	0	637	1	747	0
88	0	198	0	308	0	418	1	528	0	638	0	748	1
89	1	199	0	309	1	419	0	529	0	639	0	749	0
90	1	200	0	310	1	420	1	530	0	640	0	750	1
91	0	201	0	311	0	421	0	531	0	641	0	751	1
92	0	202	1	312	1	422	0	532	0	642	0	752	1
93	0	203	1	313	0	423	0	533	0	643	0	753	1
94	0	204	1	314	1	424	0	534	0	644	0	754	0
95	0	205	1	315	0	425	1	535	1	645	0	755	0
96	1	206	1	316	0	426	1	536	1	646	0	756	0
97	1	207	1	317	0	427	0	537	0	647	1	757	1
98	0	208	1	318	0	428	1	538	1	648	0	758	0
99	1	209	0	319	1	429	1	539	1	649	0	759	1
100	1	210	0	320	1	430	0	540	1	650	0	760	0
101	1	211	0	321	0	431	1	541	0	651	0	761	0
102	0	212	0	322	1	432	1	542	0	652	1	762	1
103	0	213	1	323	0	433	1	543	0	653	0	763	1
104	0	214	1	324	1	434	1	544	1	654	1	764	0
105	1	215	0	325	1	435	1	545	0	655	1	765	0
106	0	216	1	326	0	436	1	546	0	656	1	766	1
107	1	217	1	327	0	437	0	547	0	657	0	767	1
108	1	218	1	328	1	438	0	548	1	658	0	768	0
109	0	219	0	329	1	439	0	549	1	659	1		

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