PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a,b)=(z^4+z^2,z^3+1)$

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Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a,b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a,b) in $\mathbb{F}_2[x]\setminus\mathbb{F}_2$. In this paper we prove the conjecture for $(a,b)=(z^4+z^2,z^3+1)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a}=(a_0(z),a_1(z),a_2(z),\ldots)$ of elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z))=\cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{a_2(z)+\cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z))=\cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{\ddots+\cfrac{1}{a_n(z)}}}}\in\mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1}+c_{-2}z^{-2}+c_{-3}z^{-3}+\cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a,b)-Thue-Morse sequence

$$\mathbf{t}=(t_0(z),t_1(z),t_2(z),\ldots)=(a,b,b,a,b,a,a,b,\ldots)$$

that we shall denote

$$\mathrm{CF}(a,b):=\mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When* $(a, b) = (z^4 + z^2, z^3 + 1)$*, the two power series* $\mathrm{CF}(a, b)$ *and* $\mathrm{CF}(b, a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4 + p_3(z)y^3 + p_2(z)y^2 + p_1(z)y + p_0(z) = 0$$

where, for $\mathrm{CF}(a, b)$*,*

$$\begin{aligned}
p_0(z) &= z^{17} + z^{15} + z^{14} + z^{12} + z^{10} + z^9 + z^8 + z^7 + z^6 + z^5 + z^4 + z^2 + 1,\\
p_1(z) &= z^{21} + z^{20} + z^{17} + z^{13} + z^8 + z^7 + z^6 + z^2,\\
p_2(z) &= z^{24} + z^{20} + z^{18} + z^{16} + z^{14} + z^4,\\
p_3(z) &= z^{21} + z^{20} + z^{17} + z^{13} + z^8 + z^7 + z^6 + z^2,\\
p_4(z) &= z^{18} + z^{17} + z^{16} + z^{15} + z^{12} + z^{10} + z^9 + z^7 + z^6 + z^5 + z^4 + z^2,
\end{aligned}$$

and for $\mathrm{CF}(b, a)$*,*

$$\begin{aligned}
p_0(z) &= z^{17} + z^{15} + z^{14} + z^{10} + z^9 + z^8 + z^7 + z^6 + z^5 + z^2,\\
p_1(z) &= z^{21} + z^{20} + z^{17} + z^{13} + z^8 + z^7 + z^6 + z^2,\\
p_2(z) &= z^{24} + z^{20} + z^{18} + z^{16} + z^{14} + z^4,\\
p_3(z) &= z^{21} + z^{20} + z^{17} + z^{13} + z^8 + z^7 + z^6 + z^2,\\
p_4(z) &= z^{18} + z^{17} + z^{16} + z^{15} + z^{10} + z^9 + z^7 + z^6 + z^5 + z^2 + 1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a, b) = \frac{P_n(z)}{Q_n(z)} \in \mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a, b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x) &= x^{\deg(t_{2^n-1})}\begin{pmatrix} t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix} \times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad \cdots \times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x) &= x^{\deg(\bar t_{2^n-1})}\begin{pmatrix} \bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix} \times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix} \bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad \cdots \times x^{\deg(\bar t_0)}\begin{pmatrix} \bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b, a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n \geq 0$

$$\begin{aligned}
M_{n+1}(x) &= W_n(x)\cdot M_n(x),\\
W_{n+1}(x) &= M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(175, 175, 175, 175, 175)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \dots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z\left(\phi(W^o,0,0)(x,z),\ z^{12}\cdot\phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 630 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 630, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z\left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ to order 770 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 770, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. □

Define

$$R^e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 2.2. *For all integers $k \ge 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned}
M^e[:14u] = {} & M^e[:7u] + x^{2u}M^e[:5u] + x^{3u}M^e[:4u] + x^{5u}M^e[:2u] + x^{6u}M^e[:u] \\
& + x^{2u}M^e[:7u] + x^{4u}M^e[:5u] + x^{5u}M^e[:4u] + x^{7u}M^e[:2u] \\
& + x^{8u}M^e[:u] + x^{3u}M^e[:7u] + x^{5u}M^e[:5u] + x^{6u}M^e[:4u] \\
& + x^{8u}M^e[:2u] + x^{9u}M^e[:u] + x^{4u}M^e[:7u] + x^{6u}M^e[:5u] \\
& + x^{7u}M^e[:4u] + x^{9u}M^e[:2u] + x^{10u}M^e[:u] + x^{5u}M^e[:7u] \\
& + x^{7u}M^e[:5u] + x^{8u}M^e[:4u] + x^{10u}M^e[:2u] + x^{11u}M^e[:u]
\end{aligned}$$

$$
\begin{aligned}
&+ x^{6u}M^e[:7u] + x^{8u}M^e[:5u] + x^{9u}M^e[:4u] + x^{11u}M^e[:2u]\\
&+ x^{12u}M^e[:u] + x^{7u}M^e[:7u] + x^{10u}M^e[:4u] + x^{13u}M^e[:u]\\
&+ x^{11u}M^e[:3u] + (x^{7u} + x^{9u} + x^{10u} + x^{11u} + x^{12u} + x^{13u})R^e,\\
W^e[:14u] = {}& W^e[:7u] + x^{2u}W^e[:5u] + x^{3u}W^e[:4u] + x^{5u}W^e[:2u] + x^{6u}W^e[:u]\\
&+ x^{2u}W^e[:7u] + x^{4u}W^e[:5u] + x^{5u}W^e[:4u] + x^{7u}W^e[:2u]\\
&+ x^{8u}W^e[:u] + x^{3u}W^e[:7u] + x^{5u}W^e[:5u] + x^{6u}W^e[:4u]\\
&+ x^{8u}W^e[:2u] + x^{9u}W^e[:u] + x^{4u}W^e[:7u] + x^{6u}W^e[:5u]\\
&+ x^{7u}W^e[:4u] + x^{9u}W^e[:2u] + x^{10u}W^e[:u] + x^{5u}W^e[:7u]\\
&+ x^{7u}W^e[:5u] + x^{8u}W^e[:4u] + x^{10u}W^e[:2u] + x^{11u}W^e[:u]\\
&+ x^{6u}W^e[:7u] + x^{8u}W^e[:5u] + x^{9u}W^e[:4u] + x^{11u}W^e[:2u]\\
&+ x^{12u}W^e[:u] + x^{7u}W^e[:7u] + x^{10u}W^e[:4u] + x^{13u}W^e[:u]\\
&+ x^{11u}W^e[:3u] + (x^{7u} + x^{9u} + x^{10u} + x^{11u} + x^{12u} + x^{13u})R^e.
\end{aligned}
$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$
\begin{aligned}
M^o[:14u] = {}& M^o[:7u] + x^{2u}M^o[:5u] + x^{3u}M^o[:4u] + x^{5u}M^o[:2u] + x^{6u}M^o[:u]\\
&+ x^{2u}M^o[:7u] + x^{4u}M^o[:5u] + x^{5u}M^o[:4u] + x^{7u}M^o[:2u]\\
&+ x^{8u}M^o[:u] + x^{3u}M^o[:7u] + x^{5u}M^o[:5u] + x^{6u}M^o[:4u]\\
&+ x^{8u}M^o[:2u] + x^{9u}M^o[:u] + x^{4u}M^o[:7u] + x^{6u}M^o[:5u]\\
&+ x^{7u}M^o[:4u] + x^{9u}M^o[:2u] + x^{10u}M^o[:u] + x^{5u}M^o[:7u]\\
&+ x^{7u}M^o[:5u] + x^{8u}M^o[:4u] + x^{10u}M^o[:2u] + x^{11u}M^o[:u]\\
&+ x^{6u}M^o[:7u] + x^{8u}M^o[:5u] + x^{9u}M^o[:4u] + x^{11u}M^o[:2u]\\
&+ x^{12u}M^o[:u] + x^{7u}M^o[:7u] + x^{10u}M^o[:4u] + x^{13u}M^o[:u]\\
&+ x^{11u}M^o[:3u] + (x^{7u} + x^{9u} + x^{10u} + x^{11u} + x^{12u} + x^{13u})R^o,\\
W^o[:14u] = {}& W^o[:7u] + x^{2u}W^o[:5u] + x^{3u}W^o[:4u] + x^{5u}W^o[:2u] + x^{6u}W^o[:u]\\
&+ x^{2u}W^o[:7u] + x^{4u}W^o[:5u] + x^{5u}W^o[:4u] + x^{7u}W^o[:2u]\\
&+ x^{8u}W^o[:u] + x^{3u}W^o[:7u] + x^{5u}W^o[:5u] + x^{6u}W^o[:4u]\\
&+ x^{8u}W^o[:2u] + x^{9u}W^o[:u] + x^{4u}W^o[:7u] + x^{6u}W^o[:5u]\\
&+ x^{7u}W^o[:4u] + x^{9u}W^o[:2u] + x^{10u}W^o[:u] + x^{5u}W^o[:7u]\\
&+ x^{7u}W^o[:5u] + x^{8u}W^o[:4u] + x^{10u}W^o[:2u] + x^{11u}W^o[:u]\\
&+ x^{6u}W^o[:7u] + x^{8u}W^o[:5u] + x^{9u}W^o[:4u] + x^{11u}W^o[:2u]\\
&+ x^{12u}W^o[:u] + x^{7u}W^o[:7u] + x^{10u}W^o[:4u] + x^{13u}W^o[:u]\\
&+ x^{11u}W^o[:3u] + (x^{7u} + x^{9u} + x^{10u} + x^{11u} + x^{12u} + x^{13u})R^o.
\end{aligned}
$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$T[:14u] = T[:7u] + x^{2u}T[:5u] + x^{3u}T[:4u] + x^{5u}T[:2u] + x^{6u}T[:u] + x^{2u}T[:7u]$$

$$
\begin{aligned}
&+ x^{4u}T[:5u] + x^{5u}T[:4u] + x^{7u}T[:2u] + x^{8u}T[:u] + x^{3u}T[:7u]\\
&+ x^{5u}T[:5u] + x^{6u}T[:4u] + x^{8u}T[:2u] + x^{9u}T[:u] + x^{4u}T[:7u]\\
&+ x^{6u}T[:5u] + x^{7u}T[:4u] + x^{9u}T[:2u] + x^{10u}T[:u] + x^{5u}T[:7u]\\
&+ x^{7u}T[:5u] + x^{8u}T[:4u] + x^{10u}T[:2u] + x^{11u}T[:u] + x^{6u}T[:7u]\\
&+ x^{8u}T[:5u] + x^{9u}T[:4u] + x^{11u}T[:2u] + x^{12u}T[:u] + x^{7u}T[:7u]\\
&+ x^{10u}T[:4u] + x^{13u}T[:u] + x^{11u}T[:3u].
\end{aligned}
$$

The proof of the other 15 cases are similar. We break down the above identity into 7 parts:

$$
\begin{aligned}
0 &= x^{6u}T[u:2u] + x^{5u}T[2u:3u] + x^{3u}T[4u:5u] + x^{2u}T[5u:6u]\\
&\quad + T[7u:8u],\\
0 &= x^{6u}T[2u:3u] + x^{5u}T[3u:4u] + x^{3u}T[5u:6u] + x^{2u}T[6u:7u]\\
&\quad + T[8u:9u],\\
0 &= x^{9u}T[:u] + x^{8u}T[u:2u] + x^{7u}T[2u:3u] + x^{6u}T[3u:4u]\\
&\quad + x^{4u}T[5u:6u] + x^{3u}T[6u:7u] + T[9u:10u],\\
0 &= x^{10u}T[:u] + x^{7u}T[3u:4u] + x^{5u}T[5u:6u] + x^{4u}T[6u:7u]\\
&\quad + T[10u:11u],\\
0 &= x^{11u}T[:u] + x^{9u}T[2u:3u] + x^{6u}T[5u:6u] + x^{5u}T[6u:7u]\\
&\quad + T[11u:12u],\\
0 &= x^{12u}T[:u] + x^{10u}T[2u:3u] + x^{9u}T[3u:4u] + x^{8u}T[4u:5u]\\
&\quad + x^{7u}T[5u:6u] + x^{6u}T[6u:7u] + T[12u:13u],\\
0 &= x^{13u}T[:u] + x^{11u}T[2u:3u] + x^{10u}T[3u:4u] + x^{7u}T[6u:7u]\\
&\quad + T[13u:14u],
\end{aligned}
$$

which can be rewritten as

$$0 = T[[1w]_2] + T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[111w]_2], \tag{2.7}$$

$$0 = T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2] + T[[1000w]_2], \tag{2.8}$$

$$\begin{aligned} 0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2]\\ &\quad + T[[110w]_2] + T[[1001w]_2], \end{aligned} \tag{2.9}$$

$$0 = T[[w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2] + T[[1010w]_2], \tag{2.10}$$

$$0 = T[[w]_2] + T[[10w]_2] + T[[101w]_2] + T[[110w]_2] + T[[1011w]_2], \tag{2.11}$$

$$\begin{aligned} 0 &= T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2]\\ &\quad + T[[110w]_2] + T[[1100w]_2], \end{aligned} \tag{2.12}$$

$$0 = T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[110w]_2] + T[[1101w]_2], \tag{2.13}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$0 = T[[1w]_2] + T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[111w]_2], \tag{2.14}$$

$$0 = T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2] + T[[1000w]_2], \tag{2.15}$$

$$0 = T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2]$$

$$+T[[110w]_2]+T[[1001w]_2], \tag{2.16}$$

$$0=T[[w]_2]+T[[11w]_2]+T[[101w]_2]+T[[110w]_2]+T[[1010w]_2], \tag{2.17}$$

$$0=T[[w]_2]+T[[10w]_2]+T[[101w]_2]+T[[110w]_2]+T[[1011w]_2], \tag{2.18}$$

$$\begin{aligned}0=&T[[w]_2]+T[[10w]_2]+T[[11w]_2]+T[[100w]_2]+T[[101w]_2]\\&+T[[110w]_2]+T[[1100w]_2],\end{aligned} \tag{2.19}$$

$$0=T[[w]_2]+T[[10w]_2]+T[[11w]_2]+T[[110w]_2]+T[[1101w]_2], \tag{2.20}$$

for $w=0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 1219 states; its transition function and output function can be found in the annex. Let $A(s,w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k}=\{A(i,w):|w|=2k,w\neq 0^{2k}\}.$$

Identities (2.7) through (2.20) can be written as

$$\begin{aligned}0=&\tau(A(s,1))+\tau(A(s,10))+\tau(A(s,100))+\tau(A(s,101))\\&+\tau(A(s,111)),\end{aligned} \tag{2.21}$$

$$\begin{aligned}0=&\tau(A(s,10))+\tau(A(s,11))+\tau(A(s,101))+\tau(A(s,110))\\&+\tau(A(s,1000)),\end{aligned} \tag{2.22}$$

$$\begin{aligned}0=&\tau(A(s,\epsilon))+\tau(A(s,1))+\tau(A(s,10))+\tau(A(s,11))+\tau(A(s,101))\\&+\tau(A(s,110))+\tau(A(s,1001)),\end{aligned} \tag{2.23}$$

$$\begin{aligned}0=&\tau(A(s,\epsilon))+\tau(A(s,11))+\tau(A(s,101))+\tau(A(s,110))\\&+\tau(A(s,1010)),\end{aligned} \tag{2.24}$$

$$\begin{aligned}0=&\tau(A(s,\epsilon))+\tau(A(s,10))+\tau(A(s,101))+\tau(A(s,110))\\&+\tau(A(s,1011)),\end{aligned} \tag{2.25}$$

$$\begin{aligned}0=&\tau(A(s,\epsilon))+\tau(A(s,10))+\tau(A(s,11))+\tau(A(s,100))\\&+\tau(A(s,101))+\tau(A(s,110))+\tau(A(s,1100)),\end{aligned} \tag{2.26}$$

$$\begin{aligned}0=&\tau(A(s,\epsilon))+\tau(A(s,10))+\tau(A(s,11))+\tau(A(s,110))\\&+\tau(A(s,1101)),\end{aligned} \tag{2.27}$$

for all $s\in E_{2k}$, and

$$\begin{aligned}0=&\tau(A(s,1))+\tau(A(s,10))+\tau(A(s,100))+\tau(A(s,101))\\&+\tau(A(s,111)),\end{aligned} \tag{2.28}$$

$$\begin{aligned}0=&\tau(A(s,10))+\tau(A(s,11))+\tau(A(s,101))+\tau(A(s,110))\\&+\tau(A(s,1000)),\end{aligned} \tag{2.29}$$

$$\begin{aligned}0=&\tau(A(s,\epsilon))+\tau(A(s,1))+\tau(A(s,10))+\tau(A(s,11))+\tau(A(s,101))\\&+\tau(A(s,110))+\tau(A(s,1001)),\end{aligned} \tag{2.30}$$

$$\begin{aligned}0=&\tau(A(s,\epsilon))+\tau(A(s,11))+\tau(A(s,101))+\tau(A(s,110))\\&+\tau(A(s,1010)),\end{aligned} \tag{2.31}$$

$$\begin{aligned}0=&\tau(A(s,\epsilon))+\tau(A(s,10))+\tau(A(s,101))+\tau(A(s,110))\\&+\tau(A(s,1011)),\end{aligned} \tag{2.32}$$

$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ &\quad + \tau(A(s,101)) + \tau(A(s,110)) + \tau(A(s,1100)),\end{aligned} \tag{2.33}$$

$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,110)) \\ &\quad + \tau(A(s,1101)),\end{aligned} \tag{2.34}$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{20}), E_{20}) = (A(i, 0^{18}), E_{18})$, so that we only have to verify that identities (2.21) through (2.34) hold for $2 \le k \le 10$, which turns out to be true. $\square$

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \ge 2$, and $u = 2^{2k-1}$,*

$$\begin{aligned}M_{2k} &= M^e[:7u] + x^{2u}M^e[:5u] + x^{3u}M^e[:4u] + x^{5u}M^e[:2u] + x^{6u}M^e[:u] \\ &\quad + x^{7u}R^e, \\ W_{2k} &= W^e[:7u] + x^{2u}W^e[:5u] + x^{3u}W^e[:4u] + x^{5u}W^e[:2u] + x^{6u}W^e[:u] \\ &\quad + x^{7u}R^e.\end{aligned}$$

For all integer $k \ge 2$, and $u = 2^{2k}$,

$$\begin{aligned}M_{2k+1} &= M^o[:7u] + x^{2u}M^o[:5u] + x^{3u}M^o[:4u] + x^{5u}M^o[:2u] + x^{6u}M^o[:u] \\ &\quad + x^{7u}R^o, \\ W_{2k+1} &= W^o[:7u] + x^{2u}W^o[:5u] + x^{3u}W^o[:4u] + x^{5u}W^o[:2u] + x^{6u}W^o[:u] \\ &\quad + x^{7u}R^o.\end{aligned}$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \ge 2$, we claim that

$$\begin{aligned}M_{2k} \wedge W_{2k} &\Rightarrow M_{2k+1} \wedge W_{2k+1}, \\ M_{2k+1} \wedge W_{2k+1} &\Rightarrow M_{2k+2} \wedge W_{2k+2}.\end{aligned}$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$\begin{aligned}W_{2k}M_{2k} &= \big(W^e[:7v] + x^{2v}W^e[:5v] + x^{3v}W^e[:4v] + x^{5v}W^e[:2v] + x^{6v}W^e[:v] \\ &\quad + x^{7u}R^e\big) \times \big(M^e[:7v] + x^{2v}M^e[:5v] + x^{3v}M^e[:4v] + x^{5v}M^e[:2v] \\ &\quad + x^{6v}M^e[:v] + x^{7u}R^e\big);\end{aligned} \tag{2.35}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{14v}R^o$. Therefore we only need to prove that their difference is $O(x^{14v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{14v}, to

$$\begin{aligned}&W^e[:14v]M^e[:14v] + x^{4v}W^e[:10v]M^e[:10v] + x^{6v}W^e[:8v]M^e[:8v] \\ &\qquad + x^{10v}W^e[:4v]M^e[:4v] + x^{12v}W^e[:2v]M^e[:2v].\end{aligned}$$

For all $n \leq 14$, replace the occurrences of $W^e[: n \cdot v]$ and $M^e[: n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.35) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{14}, b_1, ..., b_{14}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{14})$, which completes the proof. □

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \leq j, k \leq 1$,

$$\begin{aligned} \lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k}, \\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k}, \\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k}, \\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}. \end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a, b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x, y) = \mathrm{Res}_t\left(\phi(M^e, 0, 1)(x, t), y^{12}\phi(M^e, 0, 1)(x, t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x, y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned} q_0(x) &= x^{24} + x^{22} + x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{12} + x^{10} + x^9 \\ &\quad + x^7, \\ q_1(x) &= x^{22} + x^{18} + x^{17} + x^{16} + x^{11} + x^7 + x^4 + x^3, \\ q_2(x) &= x^{20} + x^{10} + x^8 + x^6 + x^4 + 1, \\ q_3(x) &= x^{22} + x^{18} + x^{17} + x^{16} + x^{11} + x^7 + x^4 + x^3, \\ q_4(x) &= x^{22} + x^{20} + x^{19} + x^{18} + x^{17} + x^{15} + x^{14} + x^{12} + x^9 + x^8 + x^7 + x^6. \end{aligned}$$

The polynomial $Q(x, y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x, y)$ of multiplicity m and $R(x, y) := P(x, y)/Q(x, y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 216, and substitute it for y in $R(z,y)$, we get a series with valuation smaller than 216, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b)=f(1/z)$. $\square$

3. Annexe

All 16 polynomials are of the form

$$p_0(x)+p_3(x)y^3+p_6(x)y^6+p_9(x)y^9+p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e,0,0)$:

$$\begin{aligned}
p_0(x) &= x^{126}+x^{124}+x^{122}+x^{104}+x^{100}+x^{98}+x^{96}+x^{92}+x^{90}+x^{86}+x^{82}\\
&\quad +x^{76}+x^{72}+x^{66}+x^{62}+x^{58}+x^{54}+x^{50}+x^{44}+x^{40}+x^{36},\\
p_3(x) &= x^{116}+x^{112}+x^{108}+x^{106}+x^{104}+x^{102}+x^{100}+x^{98}+x^{96}+x^{84}\\
&\quad +x^{82}+x^{78}+x^{76}+x^{74}+x^{66}+x^{62}+x^{58}+x^{56}+x^{52}+x^{50}+x^{46}\\
&\quad +x^{44}+x^{42}+x^{32}+x^{24}+x^{20},\\
p_6(x) &= x^{118}+x^{114}+x^{112}+x^{108}+x^{104}+x^{102}+x^{100}+x^{94}+x^{92}+x^{84}\\
&\quad +x^{82}+x^{80}+x^{78}+x^{74}+x^{68}+x^{62}+x^{60}+x^{58}+x^{54}+x^{52}+x^{50}\\
&\quad +x^{48}+x^{46}+x^{32}+x^{28}+x^{20}+x^{18}+x^{12}+x^{10}+x^{6}+x^{2}+1,\\
p_9(x) &= x^{122}+x^{120}+x^{118}+x^{116}+x^{114}+x^{112}+x^{108}+x^{104}+x^{100}+x^{90}\\
&\quad +x^{84}+x^{82}+x^{80}+x^{68}+x^{64}+x^{56}+x^{46}+x^{38}+x^{36}+x^{32}+x^{30}\\
&\quad +x^{28}+x^{26}+x^{24}+x^{14}+x^{12}+x^{10}+x^{6},\\
p_{12}(x) &= x^{122}+x^{120}+x^{106}+x^{104}+x^{98}+x^{96}+x^{90}+x^{88}+x^{82}+x^{80}+x^{66}\\
&\quad +x^{64}+x^{42}+x^{40}+x^{26}+x^{24}+x^{18}+x^{16}+x^{2}+1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0].

For $\phi(M^e,0,1)$:

$$\begin{aligned}
p_0(x) &= x^{129}+x^{125}+x^{124}+x^{123}+x^{121}+x^{120}+x^{118}+x^{117}+x^{115}+x^{111}\\
&\quad +x^{109}+x^{108}+x^{107}+x^{106}+x^{103}+x^{102}+x^{98}+x^{96}+x^{95}+x^{94}\\
&\quad +x^{91}+x^{90}+x^{88}+x^{85}+x^{84}+x^{83}+x^{82}+x^{80}+x^{79}+x^{75}+x^{73}\\
&\quad +x^{71}+x^{69}+x^{64}+x^{61}+x^{60}+x^{59}+x^{55}+x^{53}+x^{50}+x^{49}+x^{44}\\
&\quad +x^{43}+x^{42}+x^{41}+x^{40}+x^{39},\\
p_3(x) &= x^{125}+x^{122}+x^{120}+x^{119}+x^{115}+x^{113}+x^{112}+x^{111}+x^{107}+x^{104}\\
&\quad +x^{103}+x^{102}+x^{100}+x^{99}+x^{93}+x^{90}+x^{88}+x^{87}+x^{83}+x^{81}+x^{80}\\
&\quad +x^{79}+x^{75}+x^{72}+x^{71}+x^{70}+x^{69}+x^{68}+x^{67}+x^{66}+x^{64}+x^{63}\\
&\quad +x^{61}+x^{59}+x^{58}+x^{57}+x^{53}+x^{50}+x^{49}+x^{48}+x^{47}+x^{46}+x^{44}\\
&\quad +x^{43}+x^{41}+x^{38}+x^{36}+x^{32}+x^{31}+x^{30}+x^{28}+x^{27},\\
p_6(x) &= x^{123}+x^{120}+x^{115}+x^{113}+x^{112}+x^{110}+x^{107}+x^{104}+x^{97}+x^{95}
\end{aligned}$$

$$\begin{aligned}
&+ x^{94} + x^{92} + x^{89} + x^{86} + x^{85} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} \\
&+ x^{77} + x^{76} + x^{74} + x^{71} + x^{68} + x^{67} + x^{64} + x^{61} + x^{59} + x^{58} + x^{57} \\
&+ x^{56} + x^{54} + x^{53} + x^{50} + x^{45} + x^{42} + x^{37} + x^{35} + x^{34} + x^{32} + x^{29} \\
&+ x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{18}, \\
p_9(x) &= x^{121} + x^{118} + x^{116} + x^{115} + x^{107} + x^{105} + x^{104} + x^{103} + x^{89} + x^{87} \\
&+ x^{86} + x^{82} + x^{81} + x^{80} + x^{78} + x^{77} + x^{75} + x^{72} + x^{71} + x^{70} + x^{69} \\
&+ x^{68} + x^{67} + x^{66} + x^{64} + x^{60} + x^{58} + x^{55} + x^{54} + x^{53} + x^{49} + x^{47} \\
&+ x^{46} + x^{45} + x^{41} + x^{39} + x^{38} + x^{37} + x^{35} + x^{32} + x^{30} + x^{29} + x^{23} \\
&+ x^{21} + x^{20} + x^{17} + x^{16} + x^{15} + x^{13} + x^{11} + x^{10} + x^{9}, \\
p_{12}(x) &= x^{119} + x^{116} + x^{111} + x^{108} + x^{107} + x^{104} + x^{103} + x^{100} + x^{99} + x^{96} \\
&+ x^{95} + x^{92} + x^{91} + x^{88} + x^{87} + x^{84} + x^{83} + x^{80} + x^{79} + x^{76} + x^{75} \\
&+ x^{72} + x^{67} + x^{64} + x^{39} + x^{36} + x^{31} + x^{28} + x^{27} + x^{24} + x^{23} + x^{20} \\
&+ x^{19} + x^{16} + x^{15} + x^{12} + x^{11} + x^{8} + x^{3} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 1].

For $\phi(M^e, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{129} + x^{125} + x^{124} + x^{123} + x^{121} + x^{120} + x^{118} + x^{117} + x^{115} + x^{111} \\
&+ x^{109} + x^{108} + x^{107} + x^{106} + x^{103} + x^{102} + x^{98} + x^{96} + x^{95} + x^{94} \\
&+ x^{91} + x^{90} + x^{88} + x^{85} + x^{84} + x^{83} + x^{82} + x^{80} + x^{79} + x^{75} + x^{73} \\
&+ x^{71} + x^{69} + x^{64} + x^{61} + x^{60} + x^{59} + x^{55} + x^{53} + x^{50} + x^{49} + x^{44} \\
&+ x^{43} + x^{42} + x^{41} + x^{40} + x^{39}, \\
p_3(x) &= x^{125} + x^{122} + x^{120} + x^{119} + x^{115} + x^{113} + x^{112} + x^{111} + x^{107} + x^{104} \\
&+ x^{103} + x^{102} + x^{100} + x^{99} + x^{93} + x^{90} + x^{88} + x^{87} + x^{83} + x^{81} + x^{80} \\
&+ x^{79} + x^{75} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{64} + x^{63} \\
&+ x^{61} + x^{59} + x^{58} + x^{57} + x^{53} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{44} \\
&+ x^{43} + x^{41} + x^{38} + x^{36} + x^{32} + x^{31} + x^{30} + x^{28} + x^{27}, \\
p_6(x) &= x^{123} + x^{120} + x^{115} + x^{113} + x^{112} + x^{110} + x^{107} + x^{104} + x^{97} + x^{95} \\
&+ x^{94} + x^{92} + x^{89} + x^{86} + x^{85} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} \\
&+ x^{77} + x^{76} + x^{74} + x^{71} + x^{68} + x^{67} + x^{64} + x^{61} + x^{59} + x^{58} + x^{57} \\
&+ x^{56} + x^{54} + x^{53} + x^{50} + x^{45} + x^{42} + x^{37} + x^{35} + x^{34} + x^{32} + x^{29} \\
&+ x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{18}, \\
p_9(x) &= x^{121} + x^{118} + x^{116} + x^{115} + x^{107} + x^{105} + x^{104} + x^{103} + x^{89} + x^{87} \\
&+ x^{86} + x^{82} + x^{81} + x^{80} + x^{78} + x^{77} + x^{75} + x^{72} + x^{71} + x^{70} + x^{69} \\
&+ x^{68} + x^{67} + x^{66} + x^{64} + x^{60} + x^{58} + x^{55} + x^{54} + x^{53} + x^{49} + x^{47} \\
&+ x^{46} + x^{45} + x^{41} + x^{39} + x^{38} + x^{37} + x^{35} + x^{32} + x^{30} + x^{29} + x^{23} \\
&+ x^{21} + x^{20} + x^{17} + x^{16} + x^{15} + x^{13} + x^{11} + x^{10} + x^{9},
\end{aligned}$$

$$\begin{aligned}p_{12}(x) &= x^{119}+x^{116}+x^{111}+x^{108}+x^{107}+x^{104}+x^{103}+x^{100}+x^{99}+x^{96}\\&+x^{95}+x^{92}+x^{91}+x^{88}+x^{87}+x^{84}+x^{83}+x^{80}+x^{79}+x^{76}+x^{75}\\&+x^{72}+x^{67}+x^{64}+x^{39}+x^{36}+x^{31}+x^{28}+x^{27}+x^{24}+x^{23}+x^{20}\\&+x^{19}+x^{16}+x^{15}+x^{12}+x^{11}+x^{8}+x^{3}+1,\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 1].

For $\phi(M^e, 1, 1)$:

$$\begin{aligned}p_0(x) &= x^{132}+x^{130}+x^{126}+x^{120}+x^{118}+x^{116}+x^{114}+x^{110}+x^{108}+x^{106}\\&+x^{102}+x^{100}+x^{98}+x^{96}+x^{94}+x^{90}+x^{88}+x^{86}+x^{84}+x^{78}+x^{76}\\&+x^{74}+x^{72}+x^{70}+x^{66}+x^{64}+x^{62}+x^{58}+x^{56}+x^{44}+x^{42},\\p_3(x) &= x^{120}+x^{118}+x^{116}+x^{94}+x^{92}+x^{90}+x^{88}+x^{86}+x^{82}+x^{74}+x^{72}\\&+x^{66}+x^{60}+x^{58}+x^{54}+x^{52}+x^{48}+x^{46}+x^{42}+x^{38}+x^{36}+x^{32}\\&+x^{30}+x^{26}+x^{24}+x^{22}+x^{20}+x^{18},\\p_6(x) &= x^{120}+x^{118}+x^{114}+x^{110}+x^{108}+x^{106}+x^{102}+x^{100}+x^{86}+x^{80}\\&+x^{78}+x^{74}+x^{70}+x^{64}+x^{62}+x^{60}+x^{58}+x^{54}+x^{48}+x^{44}+x^{42}\\&+x^{40}+x^{38}+x^{30}+x^{26}+x^{18}+x^{14}+x^{12}+x^{6}+x^{4}+x^{2}+1,\\p_9(x) &= x^{116}+x^{114}+x^{110}+x^{106}+x^{104}+x^{102}+x^{100}+x^{98}+x^{96}+x^{92}\\&+x^{90}+x^{86}+x^{78}+x^{66}+x^{64}+x^{58}+x^{54}+x^{50}+x^{44}+x^{38}+x^{32}\\&+x^{30}+x^{26}+x^{24}+x^{22}+x^{16}+x^{14}+x^{6},\\p_{12}(x) &= x^{116}+x^{114}+x^{112}+x^{68}+x^{66}+x^{64}+x^{36}+x^{34}+x^{32}+x^{4}+x^{2}+1,\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0].

For $\phi(W^e, 0, 0)$:

$$\begin{aligned}p_0(x) &= x^{132}+x^{130}+x^{126}+x^{124}+x^{122}+x^{118}+x^{116}+x^{114}+x^{112}+x^{104}\\&+x^{102}+x^{100}+x^{98}+x^{96}+x^{94}+x^{92}+x^{90}+x^{88}+x^{84}+x^{82}+x^{80}\\&+x^{76}+x^{62}+x^{58}+x^{56}+x^{50}+x^{44}+x^{42}+x^{36},\\p_3(x) &= x^{120}+x^{118}+x^{116}+x^{112}+x^{110}+x^{108}+x^{104}+x^{100}+x^{96}+x^{94}\\&+x^{92}+x^{84}+x^{82}+x^{78}+x^{76}+x^{72}+x^{70}+x^{68}+x^{64}+x^{58}+x^{56}\\&+x^{54}+x^{46}+x^{44}+x^{42}+x^{36}+x^{32}+x^{30}+x^{26}+x^{20},\\p_6(x) &= x^{120}+x^{118}+x^{114}+x^{112}+x^{108}+x^{102}+x^{96}+x^{90}+x^{86}+x^{84}\\&+x^{82}+x^{74}+x^{70}+x^{68}+x^{60}+x^{58}+x^{56}+x^{54}+x^{52}+x^{42}+x^{40}\\&+x^{36}+x^{34}+x^{32}+x^{30}+x^{26}+x^{24}+x^{22}+x^{20}+x^{18}+x^{6}+x^{4}+x^{2}\\&+1,\\p_9(x) &= x^{116}+x^{114}+x^{110}+x^{106}+x^{104}+x^{102}+x^{100}+x^{98}+x^{96}+x^{92}\\&+x^{90}+x^{86}+x^{78}+x^{66}+x^{64}+x^{58}+x^{54}+x^{50}+x^{44}+x^{38}+x^{32}\\&+x^{30}+x^{26}+x^{24}+x^{22}+x^{16}+x^{14}+x^{6},\\p_{12}(x) &= x^{116}+x^{114}+x^{112}+x^{68}+x^{66}+x^{64}+x^{36}+x^{34}+x^{32}+x^{4}+x^{2}+1,\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{129} + x^{127} + x^{121} + x^{120} + x^{119} + x^{117} + x^{113} + x^{109} + x^{108} + x^{107} \\
&\quad + x^{105} + x^{104} + x^{102} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{92} + x^{91} + x^{88} \\
&\quad + x^{85} + x^{81} + x^{78} + x^{73} + x^{71} + x^{68} + x^{66} + x^{65} + x^{64} + x^{62} + x^{61} \\
&\quad + x^{60} + x^{59} + x^{56} + x^{52} + x^{50} + x^{47} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39}, \\
p_3(x) &= x^{125} + x^{122} + x^{120} + x^{119} + x^{115} + x^{113} + x^{112} + x^{111} + x^{107} + x^{104} \\
&\quad + x^{103} + x^{102} + x^{100} + x^{99} + x^{93} + x^{90} + x^{88} + x^{87} + x^{83} + x^{81} + x^{80} \\
&\quad + x^{79} + x^{75} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{64} + x^{63} \\
&\quad + x^{61} + x^{59} + x^{58} + x^{57} + x^{53} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{44} \\
&\quad + x^{43} + x^{41} + x^{38} + x^{36} + x^{32} + x^{31} + x^{30} + x^{28} + x^{27}, \\
p_6(x) &= x^{123} + x^{120} + x^{115} + x^{113} + x^{112} + x^{110} + x^{107} + x^{104} + x^{97} + x^{95} \\
&\quad + x^{94} + x^{92} + x^{89} + x^{86} + x^{85} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} \\
&\quad + x^{77} + x^{76} + x^{74} + x^{71} + x^{68} + x^{67} + x^{64} + x^{61} + x^{59} + x^{58} + x^{57} \\
&\quad + x^{56} + x^{54} + x^{53} + x^{50} + x^{45} + x^{42} + x^{37} + x^{35} + x^{34} + x^{32} + x^{29} \\
&\quad + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{18}, \\
p_9(x) &= x^{121} + x^{118} + x^{116} + x^{115} + x^{107} + x^{105} + x^{104} + x^{103} + x^{89} + x^{87} \\
&\quad + x^{86} + x^{82} + x^{81} + x^{80} + x^{78} + x^{77} + x^{75} + x^{72} + x^{71} + x^{70} + x^{69} \\
&\quad + x^{68} + x^{67} + x^{66} + x^{64} + x^{60} + x^{58} + x^{55} + x^{54} + x^{53} + x^{49} + x^{47} \\
&\quad + x^{46} + x^{45} + x^{41} + x^{39} + x^{38} + x^{37} + x^{35} + x^{32} + x^{30} + x^{29} + x^{23} \\
&\quad + x^{21} + x^{20} + x^{17} + x^{16} + x^{15} + x^{13} + x^{11} + x^{10} + x^{9}, \\
p_{12}(x) &= x^{119} + x^{116} + x^{111} + x^{108} + x^{107} + x^{104} + x^{103} + x^{100} + x^{99} + x^{96} \\
&\quad + x^{95} + x^{92} + x^{91} + x^{88} + x^{87} + x^{84} + x^{83} + x^{80} + x^{79} + x^{76} + x^{75} \\
&\quad + x^{72} + x^{67} + x^{64} + x^{39} + x^{36} + x^{31} + x^{28} + x^{27} + x^{24} + x^{23} + x^{20} \\
&\quad + x^{19} + x^{16} + x^{15} + x^{12} + x^{11} + x^{8} + x^{3} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1].

For $\phi(W^e, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{129} + x^{127} + x^{121} + x^{120} + x^{119} + x^{117} + x^{113} + x^{109} + x^{108} + x^{107} \\
&\quad + x^{105} + x^{104} + x^{102} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{92} + x^{91} + x^{88} \\
&\quad + x^{85} + x^{81} + x^{78} + x^{73} + x^{71} + x^{68} + x^{66} + x^{65} + x^{64} + x^{62} + x^{61} \\
&\quad + x^{60} + x^{59} + x^{56} + x^{52} + x^{50} + x^{47} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39}, \\
p_3(x) &= x^{125} + x^{122} + x^{120} + x^{119} + x^{115} + x^{113} + x^{112} + x^{111} + x^{107} + x^{104} \\
&\quad + x^{103} + x^{102} + x^{100} + x^{99} + x^{93} + x^{90} + x^{88} + x^{87} + x^{83} + x^{81} + x^{80} \\
&\quad + x^{79} + x^{75} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{64} + x^{63} \\
&\quad + x^{61} + x^{59} + x^{58} + x^{57} + x^{53} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{44}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{43} + x^{41} + x^{38} + x^{36} + x^{32} + x^{31} + x^{30} + x^{28} + x^{27},\\
p_6(x) &= x^{123} + x^{120} + x^{115} + x^{113} + x^{112} + x^{110} + x^{107} + x^{104} + x^{97} + x^{95}\\
&+ x^{94} + x^{92} + x^{89} + x^{86} + x^{85} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78}\\
&+ x^{77} + x^{76} + x^{74} + x^{71} + x^{68} + x^{67} + x^{64} + x^{61} + x^{59} + x^{58} + x^{57}\\
&+ x^{56} + x^{54} + x^{53} + x^{50} + x^{45} + x^{42} + x^{37} + x^{35} + x^{34} + x^{32} + x^{29}\\
&+ x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{18},\\
p_9(x) &= x^{121} + x^{118} + x^{116} + x^{115} + x^{107} + x^{105} + x^{104} + x^{103} + x^{89} + x^{87}\\
&+ x^{86} + x^{82} + x^{81} + x^{80} + x^{78} + x^{77} + x^{75} + x^{72} + x^{71} + x^{70} + x^{69}\\
&+ x^{68} + x^{67} + x^{66} + x^{64} + x^{60} + x^{58} + x^{55} + x^{54} + x^{53} + x^{49} + x^{47}\\
&+ x^{46} + x^{45} + x^{41} + x^{39} + x^{38} + x^{37} + x^{35} + x^{32} + x^{30} + x^{29} + x^{23}\\
&+ x^{21} + x^{20} + x^{17} + x^{16} + x^{15} + x^{13} + x^{11} + x^{10} + x^{9},\\
p_{12}(x) &= x^{119} + x^{116} + x^{111} + x^{108} + x^{107} + x^{104} + x^{103} + x^{100} + x^{99} + x^{96}\\
&+ x^{95} + x^{92} + x^{91} + x^{88} + x^{87} + x^{84} + x^{83} + x^{80} + x^{79} + x^{76} + x^{75}\\
&+ x^{72} + x^{67} + x^{64} + x^{39} + x^{36} + x^{31} + x^{28} + x^{27} + x^{24} + x^{23} + x^{20}\\
&+ x^{19} + x^{16} + x^{15} + x^{12} + x^{11} + x^{8} + x^{3} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1].

For $\phi(W^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{124} + x^{120} + x^{118} + x^{116} + x^{114} + x^{92} + x^{90} + x^{84} + x^{72}\\
&+ x^{70} + x^{66} + x^{64} + x^{58} + x^{54} + x^{50} + x^{46} + x^{44} + x^{42},\\
p_3(x) &= x^{118} + x^{112} + x^{110} + x^{108} + x^{106} + x^{98} + x^{94} + x^{92} + x^{90} + x^{88} + x^{86}\\
&+ x^{80} + x^{78} + x^{76} + x^{72} + x^{70} + x^{62} + x^{60} + x^{54} + x^{52} + x^{50} + x^{44}\\
&+ x^{40} + x^{28} + x^{26} + x^{24} + x^{20} + x^{18},\\
p_6(x) &= x^{116} + x^{110} + x^{108} + x^{98} + x^{94} + x^{92} + x^{90} + x^{88} + x^{86} + x^{78} + x^{74}\\
&+ x^{70} + x^{68} + x^{64} + x^{62} + x^{56} + x^{54} + x^{50} + x^{48} + x^{46} + x^{44} + x^{38}\\
&+ x^{36} + x^{30} + x^{28} + x^{26} + x^{20} + x^{18} + x^{16} + x^{14} + x^{10} + x^{6} + x^{2} + 1,\\
p_9(x) &= x^{122} + x^{120} + x^{118} + x^{116} + x^{114} + x^{112} + x^{108} + x^{104} + x^{100} + x^{90}\\
&+ x^{84} + x^{82} + x^{80} + x^{68} + x^{64} + x^{56} + x^{46} + x^{38} + x^{36} + x^{32} + x^{30}\\
&+ x^{28} + x^{26} + x^{24} + x^{14} + x^{12} + x^{10} + x^{6},\\
p_{12}(x) &= x^{122} + x^{120} + x^{106} + x^{104} + x^{98} + x^{96} + x^{90} + x^{88} + x^{82} + x^{80} + x^{66}\\
&+ x^{64} + x^{42} + x^{40} + x^{26} + x^{24} + x^{18} + x^{16} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0].

For $\phi(M^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{129} + x^{124} + x^{120} + x^{119} + x^{118} + x^{116} + x^{114} + x^{111} + x^{110} + x^{108}\\
&+ x^{107} + x^{104} + x^{102} + x^{101} + x^{96} + x^{94} + x^{93} + x^{91} + x^{88} + x^{86}\\
&+ x^{85} + x^{84} + x^{83} + x^{81} + x^{73} + x^{71} + x^{68} + x^{67} + x^{64} + x^{56} + x^{53}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{50} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39} + x^{38} + x^{36},\\
p_3(x) &= x^{127} + x^{125} + x^{124} + x^{120} + x^{115} + x^{113} + x^{111} + x^{110} + x^{109} + x^{108}\\
&+ x^{107} + x^{106} + x^{105} + x^{103} + x^{95} + x^{92} + x^{90} + x^{88} + x^{87} + x^{84}\\
&+ x^{80} + x^{75} + x^{71} + x^{70} + x^{69} + x^{65} + x^{64} + x^{63} + x^{62} + x^{60} + x^{58}\\
&+ x^{57} + x^{54} + x^{53} + x^{48} + x^{47} + x^{46} + x^{45} + x^{43} + x^{42} + x^{41} + x^{40}\\
&+ x^{39} + x^{37} + x^{35} + x^{34} + x^{31} + x^{30} + x^{29} + x^{27} + x^{26} + x^{24},\\
p_6(x) &= x^{127} + x^{125} + x^{124} + x^{120} + x^{117} + x^{115} + x^{113} + x^{112} + x^{110} + x^{109}\\
&+ x^{107} + x^{106} + x^{104} + x^{102} + x^{100} + x^{99} + x^{98} + x^{96} + x^{93} + x^{90}\\
&+ x^{89} + x^{88} + x^{86} + x^{85} + x^{84} + x^{82} + x^{81} + x^{80} + x^{79} + x^{76} + x^{74}\\
&+ x^{73} + x^{72} + x^{71} + x^{69} + x^{67} + x^{65} + x^{62} + x^{60} + x^{58} + x^{57} + x^{49}\\
&+ x^{43} + x^{42} + x^{41} + x^{38} + x^{36} + x^{34} + x^{32} + x^{31} + x^{30} + x^{28} + x^{24}\\
&+ x^{22} + x^{21} + x^{19} + x^{17} + x^{14} + x^{11} + x^{8} + x^{3} + 1,\\
p_9(x) &= x^{127} + x^{124} + x^{119} + x^{116} + x^{115} + x^{112} + x^{103} + x^{100} + x^{99} + x^{96}\\
&+ x^{87} + x^{84} + x^{83} + x^{80} + x^{79} + x^{76} + x^{47} + x^{44} + x^{39} + x^{36} + x^{35}\\
&+ x^{32} + x^{23} + x^{20} + x^{19} + x^{16} + x^{15} + x^{12},\\
p_{12}(x) &= x^{127} + x^{124} + x^{115} + x^{112} + x^{111} + x^{108} + x^{107} + x^{104} + x^{95} + x^{92}\\
&+ x^{91} + x^{88} + x^{75} + x^{72} + x^{67} + x^{64} + x^{47} + x^{44} + x^{35} + x^{32} + x^{31}\\
&+ x^{28} + x^{27} + x^{24} + x^{11} + x^{8} + x^{3} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0].

For $\phi(M^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{136} + x^{132} + x^{130} + x^{128} + x^{127} + x^{125} + x^{124} + x^{123} + x^{120} + x^{118}\\
&+ x^{114} + x^{113} + x^{106} + x^{105} + x^{104} + x^{100} + x^{95} + x^{94} + x^{90} + x^{89}\\
&+ x^{88} + x^{87} + x^{86} + x^{85} + x^{83} + x^{80} + x^{79} + x^{76} + x^{74} + x^{72} + x^{71}\\
&+ x^{70} + x^{69} + x^{65} + x^{64} + x^{61} + x^{59} + x^{57} + x^{54} + x^{53} + x^{52} + x^{50}\\
&+ x^{49} + x^{48} + x^{44} + x^{43} + x^{41} + x^{40} + x^{39},\\
p_3(x) &= x^{122} + x^{117} + x^{116} + x^{113} + x^{111} + x^{109} + x^{107} + x^{106} + x^{105} + x^{103}\\
&+ x^{100} + x^{99} + x^{90} + x^{85} + x^{84} + x^{81} + x^{79} + x^{77} + x^{75} + x^{74} + x^{73}\\
&+ x^{71} + x^{68} + x^{67} + x^{66} + x^{61} + x^{60} + x^{58} + x^{57} + x^{55} + x^{52} + x^{51}\\
&+ x^{42} + x^{37} + x^{36} + x^{34} + x^{33} + x^{31} + x^{28} + x^{27},\\
p_6(x) &= x^{124} + x^{120} + x^{116} + x^{114} + x^{112} + x^{110} + x^{108} + x^{104} + x^{98} + x^{96}\\
&+ x^{94} + x^{92} + x^{90} + x^{84} + x^{76} + x^{74} + x^{72} + x^{64} + x^{62} + x^{60} + x^{56}\\
&+ x^{50} + x^{46} + x^{42} + x^{38} + x^{36} + x^{34} + x^{32} + x^{30} + x^{28} + x^{20} + x^{18},\\
p_9(x) &= x^{126} + x^{121} + x^{120} + x^{118} + x^{117} + x^{116} + x^{115} + x^{114} + x^{112} + x^{110}\\
&+ x^{109} + x^{108} + x^{107} + x^{106} + x^{104} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98}\\
&+ x^{96} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{88} + x^{86} + x^{85} + x^{84} + x^{83}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{82} + x^{80} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{46} + x^{41} + x^{40} + x^{38}\\
&+ x^{37} + x^{36} + x^{35} + x^{34} + x^{32} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{24}\\
&+ x^{22} + x^{21} + x^{20} + x^{19} + x^{18} + x^{16} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9},\\
p_{12}(x) &= x^{128} + x^{124} + x^{116} + x^{104} + x^{96} + x^{88} + x^{76} + x^{72} + x^{68} + x^{64} + x^{48}\\
&+ x^{44} + x^{36} + x^{24} + x^{12} + x^{8} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0].
For $\phi(M^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{121} + x^{120} + x^{118} + x^{115} + x^{114} + x^{113} + x^{112} + x^{111} + x^{110}\\
&+ x^{109} + x^{108} + x^{106} + x^{105} + x^{103} + x^{101} + x^{100} + x^{99} + x^{95} + x^{94}\\
&+ x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{87} + x^{85} + x^{79} + x^{76} + x^{70} + x^{69}\\
&+ x^{68} + x^{62} + x^{59} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{49}\\
&+ x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{40} + x^{39},\\
p_3(x) &= x^{124} + x^{122} + x^{120} + x^{119} + x^{117} + x^{116} + x^{114} + x^{113} + x^{111} + x^{110}\\
&+ x^{108} + x^{107} + x^{105} + x^{104} + x^{102} + x^{101} + x^{100} + x^{99} + x^{92} + x^{90}\\
&+ x^{88} + x^{87} + x^{85} + x^{84} + x^{82} + x^{81} + x^{79} + x^{78} + x^{76} + x^{75} + x^{73}\\
&+ x^{72} + x^{70} + x^{69} + x^{67} + x^{66} + x^{64} + x^{63} + x^{61} + x^{57} + x^{56} + x^{54}\\
&+ x^{53} + x^{52} + x^{51} + x^{44} + x^{42} + x^{40} + x^{39} + x^{37} + x^{33} + x^{32} + x^{30}\\
&+ x^{29} + x^{28} + x^{27},\\
p_6(x) &= x^{126} + x^{124} + x^{116} + x^{112} + x^{110} + x^{106} + x^{104} + x^{100} + x^{96} + x^{90}\\
&+ x^{88} + x^{84} + x^{82} + x^{72} + x^{70} + x^{68} + x^{66} + x^{58} + x^{54} + x^{52} + x^{50}\\
&+ x^{48} + x^{46} + x^{38} + x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{18},\\
p_9(x) &= x^{128} + x^{126} + x^{124} + x^{123} + x^{121} + x^{118} + x^{117} + x^{116} + x^{114} + x^{111}\\
&+ x^{110} + x^{109} + x^{108} + x^{107} + x^{106} + x^{102} + x^{101} + x^{100} + x^{98} + x^{95}\\
&+ x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{86} + x^{85} + x^{84} + x^{82} + x^{80} + x^{79}\\
&+ x^{77} + x^{74} + x^{73} + x^{48} + x^{46} + x^{44} + x^{43} + x^{41} + x^{38} + x^{37} + x^{36}\\
&+ x^{34} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{22} + x^{21} + x^{20} + x^{18}\\
&+ x^{16} + x^{15} + x^{13} + x^{10} + x^{9},\\
p_{12}(x) &= x^{130} + x^{128} + x^{122} + x^{120} + x^{110} + x^{108} + x^{106} + x^{104} + x^{98} + x^{96}\\
&+ x^{94} + x^{92} + x^{90} + x^{88} + x^{78} + x^{76} + x^{66} + x^{64} + x^{50} + x^{48} + x^{42}\\
&+ x^{40} + x^{30} + x^{28} + x^{26} + x^{24} + x^{14} + x^{12} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 1, 0].
For $\phi(M^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{129} + x^{127} + x^{125} + x^{121} + x^{119} + x^{116} + x^{113} + x^{111} + x^{109} + x^{108}\\
&+ x^{107} + x^{103} + x^{101} + x^{99} + x^{97} + x^{95} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89}\\
&+ x^{88} + x^{87} + x^{86} + x^{82} + x^{79} + x^{76} + x^{75} + x^{74} + x^{72} + x^{70} + x^{69}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{68} + x^{67} + x^{62} + x^{60} + x^{58} + x^{56} + x^{51} + x^{50} + x^{48} + x^{47} + x^{46} \\
&+ x^{45} + x^{42}, \\
p_3(x) &= x^{127} + x^{125} + x^{124} + x^{123} + x^{119} + x^{116} + x^{115} + x^{109} + x^{107} + x^{106} \\
&+ x^{103} + x^{102} + x^{99} + x^{96} + x^{95} + x^{92} + x^{91} + x^{90} + x^{81} + x^{80} + x^{79} \\
&+ x^{78} + x^{76} + x^{75} + x^{73} + x^{71} + x^{69} + x^{65} + x^{64} + x^{62} + x^{59} + x^{58} \\
&+ x^{56} + x^{53} + x^{51} + x^{47} + x^{46} + x^{45} + x^{42} + x^{41} + x^{37} + x^{36} + x^{34} \\
&+ x^{33} + x^{32} + x^{31} + x^{29} + x^{26}, \\
p_6(x) &= x^{127} + x^{125} + x^{124} + x^{123} + x^{119} + x^{117} + x^{116} + x^{109} + x^{107} + x^{106} \\
&+ x^{104} + x^{103} + x^{102} + x^{99} + x^{98} + x^{97} + x^{96} + x^{94} + x^{93} + x^{90} + x^{88} \\
&+ x^{87} + x^{83} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{73} + x^{71} + x^{69} + x^{67} \\
&+ x^{65} + x^{63} + x^{62} + x^{61} + x^{55} + x^{54} + x^{53} + x^{52} + x^{50} + x^{49} + x^{47} \\
&+ x^{45} + x^{44} + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{32} + x^{31} + x^{30} + x^{29} \\
&+ x^{28} + x^{26} + x^{25} + x^{24} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{12} + x^{11} \\
&+ x^{8} + x^{3} + 1, \\
p_9(x) &= x^{127} + x^{124} + x^{119} + x^{116} + x^{115} + x^{112} + x^{103} + x^{100} + x^{99} + x^{96} \\
&+ x^{87} + x^{84} + x^{83} + x^{80} + x^{79} + x^{76} + x^{47} + x^{44} + x^{39} + x^{36} + x^{35} \\
&+ x^{32} + x^{23} + x^{20} + x^{19} + x^{16} + x^{15} + x^{12}, \\
p_{12}(x) &= x^{127} + x^{124} + x^{115} + x^{112} + x^{111} + x^{108} + x^{107} + x^{104} + x^{95} + x^{92} \\
&+ x^{91} + x^{88} + x^{75} + x^{72} + x^{67} + x^{64} + x^{47} + x^{44} + x^{35} + x^{32} + x^{31} \\
&+ x^{28} + x^{27} + x^{24} + x^{11} + x^{8} + x^{3} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 1, 1, 1, 0, 0].

For $\phi(W^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{129} + x^{124} + x^{120} + x^{119} + x^{118} + x^{116} + x^{114} + x^{111} + x^{110} + x^{108} \\
&+ x^{107} + x^{104} + x^{102} + x^{101} + x^{96} + x^{94} + x^{93} + x^{91} + x^{88} + x^{86} \\
&+ x^{85} + x^{84} + x^{83} + x^{81} + x^{73} + x^{71} + x^{68} + x^{67} + x^{64} + x^{56} + x^{53} \\
&+ x^{50} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39} + x^{38} + x^{36}, \\
p_3(x) &= x^{127} + x^{125} + x^{124} + x^{120} + x^{115} + x^{113} + x^{111} + x^{110} + x^{109} + x^{108} \\
&+ x^{107} + x^{106} + x^{105} + x^{103} + x^{95} + x^{92} + x^{90} + x^{88} + x^{87} + x^{84} \\
&+ x^{80} + x^{75} + x^{71} + x^{70} + x^{69} + x^{65} + x^{64} + x^{63} + x^{62} + x^{60} + x^{58} \\
&+ x^{57} + x^{54} + x^{53} + x^{48} + x^{47} + x^{46} + x^{45} + x^{43} + x^{42} + x^{41} + x^{40} \\
&+ x^{39} + x^{37} + x^{35} + x^{34} + x^{31} + x^{30} + x^{29} + x^{27} + x^{26} + x^{24}, \\
p_6(x) &= x^{127} + x^{125} + x^{124} + x^{120} + x^{117} + x^{115} + x^{113} + x^{112} + x^{110} + x^{109} \\
&+ x^{107} + x^{106} + x^{104} + x^{102} + x^{100} + x^{99} + x^{98} + x^{96} + x^{93} + x^{90} \\
&+ x^{89} + x^{88} + x^{86} + x^{85} + x^{84} + x^{82} + x^{81} + x^{80} + x^{79} + x^{76} + x^{74} \\
&+ x^{73} + x^{72} + x^{71} + x^{69} + x^{67} + x^{65} + x^{62} + x^{60} + x^{58} + x^{57} + x^{49}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{43} + x^{42} + x^{41} + x^{38} + x^{36} + x^{34} + x^{32} + x^{31} + x^{30} + x^{28} + x^{24} \\
&+ x^{22} + x^{21} + x^{19} + x^{17} + x^{14} + x^{11} + x^{8} + x^{3} + 1, \\
p_9(x) &= x^{127} + x^{124} + x^{119} + x^{116} + x^{115} + x^{112} + x^{103} + x^{100} + x^{99} + x^{96} \\
&+ x^{87} + x^{84} + x^{83} + x^{80} + x^{79} + x^{76} + x^{47} + x^{44} + x^{39} + x^{36} + x^{35} \\
&+ x^{32} + x^{23} + x^{20} + x^{19} + x^{16} + x^{15} + x^{12}, \\
p_{12}(x) &= x^{127} + x^{124} + x^{115} + x^{112} + x^{111} + x^{108} + x^{107} + x^{104} + x^{95} + x^{92} \\
&+ x^{91} + x^{88} + x^{75} + x^{72} + x^{67} + x^{64} + x^{47} + x^{44} + x^{35} + x^{32} + x^{31} \\
&+ x^{28} + x^{27} + x^{24} + x^{11} + x^{8} + x^{3} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0].

For $\phi(W^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{121} + x^{120} + x^{118} + x^{115} + x^{114} + x^{113} + x^{112} + x^{111} + x^{110} \\
&+ x^{109} + x^{108} + x^{106} + x^{105} + x^{103} + x^{101} + x^{100} + x^{99} + x^{95} + x^{94} \\
&+ x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{87} + x^{85} + x^{79} + x^{76} + x^{70} + x^{69} \\
&+ x^{68} + x^{62} + x^{59} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{49} \\
&+ x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{40} + x^{39}, \\
p_3(x) &= x^{124} + x^{122} + x^{120} + x^{119} + x^{117} + x^{116} + x^{114} + x^{113} + x^{111} + x^{110} \\
&+ x^{108} + x^{107} + x^{105} + x^{104} + x^{102} + x^{101} + x^{100} + x^{99} + x^{92} + x^{90} \\
&+ x^{88} + x^{87} + x^{85} + x^{84} + x^{82} + x^{81} + x^{79} + x^{78} + x^{76} + x^{75} + x^{73} \\
&+ x^{72} + x^{70} + x^{69} + x^{67} + x^{66} + x^{64} + x^{63} + x^{61} + x^{57} + x^{56} + x^{54} \\
&+ x^{53} + x^{52} + x^{51} + x^{44} + x^{42} + x^{40} + x^{39} + x^{37} + x^{33} + x^{32} + x^{30} \\
&+ x^{29} + x^{28} + x^{27}, \\
p_6(x) &= x^{126} + x^{124} + x^{116} + x^{112} + x^{110} + x^{106} + x^{104} + x^{100} + x^{96} + x^{90} \\
&+ x^{88} + x^{84} + x^{82} + x^{72} + x^{70} + x^{68} + x^{66} + x^{58} + x^{54} + x^{52} + x^{50} \\
&+ x^{48} + x^{46} + x^{38} + x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{18}, \\
p_9(x) &= x^{128} + x^{126} + x^{124} + x^{123} + x^{121} + x^{118} + x^{117} + x^{116} + x^{114} + x^{111} \\
&+ x^{110} + x^{109} + x^{108} + x^{107} + x^{106} + x^{102} + x^{101} + x^{100} + x^{98} + x^{95} \\
&+ x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{86} + x^{85} + x^{84} + x^{82} + x^{80} + x^{79} \\
&+ x^{77} + x^{74} + x^{73} + x^{48} + x^{46} + x^{44} + x^{43} + x^{41} + x^{38} + x^{37} + x^{36} \\
&+ x^{34} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{22} + x^{21} + x^{20} + x^{18} \\
&+ x^{16} + x^{15} + x^{13} + x^{10} + x^{9}, \\
p_{12}(x) &= x^{130} + x^{128} + x^{122} + x^{120} + x^{110} + x^{108} + x^{106} + x^{104} + x^{98} + x^{96} \\
&+ x^{94} + x^{92} + x^{90} + x^{88} + x^{78} + x^{76} + x^{66} + x^{64} + x^{50} + x^{48} + x^{42} \\
&+ x^{40} + x^{30} + x^{28} + x^{26} + x^{24} + x^{14} + x^{12} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 1, 0].

For $\phi(W^o, 1, 0)$:

$$\begin{aligned}
p_0(x) = {} & x^{136} + x^{132} + x^{130} + x^{128} + x^{127} + x^{125} + x^{124} + x^{123} + x^{120} + x^{118} \\
& + x^{114} + x^{113} + x^{106} + x^{105} + x^{104} + x^{100} + x^{95} + x^{94} + x^{90} + x^{89} \\
& + x^{88} + x^{87} + x^{86} + x^{85} + x^{83} + x^{80} + x^{79} + x^{76} + x^{74} + x^{72} + x^{71} \\
& + x^{70} + x^{69} + x^{65} + x^{64} + x^{61} + x^{59} + x^{57} + x^{54} + x^{53} + x^{52} + x^{50} \\
& + x^{49} + x^{48} + x^{44} + x^{43} + x^{41} + x^{40} + x^{39}, \\
p_3(x) = {} & x^{122} + x^{117} + x^{116} + x^{113} + x^{111} + x^{109} + x^{107} + x^{106} + x^{105} + x^{103} \\
& + x^{100} + x^{99} + x^{90} + x^{85} + x^{84} + x^{81} + x^{79} + x^{77} + x^{75} + x^{74} + x^{73} \\
& + x^{71} + x^{68} + x^{67} + x^{66} + x^{61} + x^{60} + x^{58} + x^{57} + x^{55} + x^{52} + x^{51} \\
& + x^{42} + x^{37} + x^{36} + x^{34} + x^{33} + x^{31} + x^{28} + x^{27}, \\
p_6(x) = {} & x^{124} + x^{120} + x^{116} + x^{114} + x^{112} + x^{110} + x^{108} + x^{104} + x^{98} + x^{96} \\
& + x^{94} + x^{92} + x^{90} + x^{84} + x^{76} + x^{74} + x^{72} + x^{64} + x^{62} + x^{60} + x^{56} \\
& + x^{50} + x^{46} + x^{42} + x^{38} + x^{36} + x^{34} + x^{32} + x^{30} + x^{28} + x^{20} + x^{18}, \\
p_9(x) = {} & x^{126} + x^{121} + x^{120} + x^{118} + x^{117} + x^{116} + x^{115} + x^{114} + x^{112} + x^{110} \\
& + x^{109} + x^{108} + x^{107} + x^{106} + x^{104} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} \\
& + x^{96} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{88} + x^{86} + x^{85} + x^{84} + x^{83} \\
& + x^{82} + x^{80} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{46} + x^{41} + x^{40} + x^{38} \\
& + x^{37} + x^{36} + x^{35} + x^{34} + x^{32} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{24} \\
& + x^{22} + x^{21} + x^{20} + x^{19} + x^{18} + x^{16} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9}, \\
p_{12}(x) = {} & x^{128} + x^{124} + x^{116} + x^{104} + x^{96} + x^{88} + x^{76} + x^{72} + x^{68} + x^{64} + x^{48} \\
& + x^{44} + x^{36} + x^{24} + x^{12} + x^{8} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0].

For $\phi(W^o, 1, 1)$:

$$\begin{aligned}
p_0(x) = {} & x^{129} + x^{127} + x^{125} + x^{121} + x^{119} + x^{116} + x^{113} + x^{111} + x^{109} + x^{108} \\
& + x^{107} + x^{103} + x^{101} + x^{99} + x^{97} + x^{95} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} \\
& + x^{88} + x^{87} + x^{86} + x^{82} + x^{79} + x^{76} + x^{75} + x^{74} + x^{72} + x^{70} + x^{69} \\
& + x^{68} + x^{67} + x^{62} + x^{60} + x^{58} + x^{56} + x^{51} + x^{50} + x^{48} + x^{47} + x^{46} \\
& + x^{45} + x^{42}, \\
p_3(x) = {} & x^{127} + x^{125} + x^{124} + x^{123} + x^{119} + x^{116} + x^{115} + x^{109} + x^{107} + x^{106} \\
& + x^{103} + x^{102} + x^{99} + x^{96} + x^{95} + x^{92} + x^{91} + x^{90} + x^{81} + x^{80} + x^{79} \\
& + x^{78} + x^{76} + x^{75} + x^{73} + x^{71} + x^{69} + x^{65} + x^{64} + x^{62} + x^{59} + x^{58} \\
& + x^{56} + x^{53} + x^{51} + x^{47} + x^{46} + x^{45} + x^{42} + x^{41} + x^{37} + x^{36} + x^{34} \\
& + x^{33} + x^{32} + x^{31} + x^{29} + x^{26}, \\
p_6(x) = {} & x^{127} + x^{125} + x^{124} + x^{123} + x^{119} + x^{117} + x^{116} + x^{109} + x^{107} + x^{106} \\
& + x^{104} + x^{103} + x^{102} + x^{99} + x^{98} + x^{97} + x^{96} + x^{94} + x^{93} + x^{90} + x^{88}
\end{aligned}$$

$$+ x^{87} + x^{83} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{73} + x^{71} + x^{69} + x^{67}$$
$$+ x^{65} + x^{63} + x^{62} + x^{61} + x^{55} + x^{54} + x^{53} + x^{52} + x^{50} + x^{49} + x^{47}$$
$$+ x^{45} + x^{44} + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{32} + x^{31} + x^{30} + x^{29}$$
$$+ x^{28} + x^{26} + x^{25} + x^{24} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{12} + x^{11}$$
$$+ x^{8} + x^{3} + 1,$$
$$p_9(x) = x^{127} + x^{124} + x^{119} + x^{116} + x^{115} + x^{112} + x^{103} + x^{100} + x^{99} + x^{96}$$
$$+ x^{87} + x^{84} + x^{83} + x^{80} + x^{79} + x^{76} + x^{47} + x^{44} + x^{39} + x^{36} + x^{35}$$
$$+ x^{32} + x^{23} + x^{20} + x^{19} + x^{16} + x^{15} + x^{12},$$
$$p_{12}(x) = x^{127} + x^{124} + x^{115} + x^{112} + x^{111} + x^{108} + x^{107} + x^{104} + x^{95} + x^{92}$$
$$+ x^{91} + x^{88} + x^{75} + x^{72} + x^{67} + x^{64} + x^{47} + x^{44} + x^{35} + x^{32} + x^{31}$$
$$+ x^{28} + x^{27} + x^{24} + x^{11} + x^{8} + x^{3} + 1,$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 1, 1, 1, 0, 0].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	305	[463, 464]	610	[426, 114]	915	[47, 433]
1	[3, 4]	306	[465, 466]	611	[778, 165]	916	[748, 367]
2	[5, 6]	307	[422, 186]	612	[763, 737]	917	[752, 159]
3	[7, 8]	308	[410, 467]	613	[429, 167]	918	[1014, 1018]
4	[9, 10]	309	[141, 431]	614	[790, 444]	919	[1019, 1020]
5	[11, 12]	310	[47, 56]	615	[39, 118]	920	[812, 809]
6	[13, 14]	311	[57, 167]	616	[44, 791]	921	[478, 416]
7	[15, 16]	312	[468, 124]	617	[159, 158]	922	[393, 126]
8	[17, 18]	313	[441, 363]	618	[792, 740]	923	[182, 434]
9	[19, 20]	314	[469, 358]	619	[386, 45]	924	[180, 445]
10	[21, 22]	315	[470, 148]	620	[738, 150]	925	[440, 802]
11	[23, 24]	316	[447, 41]	621	[793, 474]	926	[152, 440]
12	[25, 26]	317	[471, 472]	622	[794, 124]	927	[445, 47]
13	[27, 28]	318	[158, 448]	623	[795, 749]	928	[1013, 183]
14	[26, 29]	319	[473, 368]	624	[411, 55]	929	[1021, 142]
15	[30, 31]	320	[39, 474]	625	[796, 793]	930	[772, 749]
16	[32, 33]	321	[8, 120]	626	[402, 41]	931	[826, 373]
17	[34, 35]	322	[475, 359]	627	[454, 160]	932	[458, 116]
18	[36, 37]	323	[476, 476]	628	[419, 169]	933	[468, 353]
19	[38, 39]	324	[477, 162]	629	[134, 194]	934	[773, 423]
20	[40, 41]	325	[478, 188]	630	[740, 386]	935	[1022, 767]
21	[42, 43]	326	[479, 435]	631	[794, 362]	936	[371, 113]
22	[44, 45]	327	[480, 481]	632	[797, 354]	937	[1023, 1024]
23	[46, 47]	328	[482, 400]	633	[389, 798]	938	[482, 145]
24	[48, 49]	329	[483, 484]	634	[410, 180]	939	[423, 356]

25	[50, 51]	330	[485, 486]	635	[432, 407]	940	[792, 786]
26	[52, 53]	331	[487, 177]	636	[790, 799]	941	[1021, 422]
27	[54, 55]	332	[488, 448]	637	[148, 374]	942	[1025, 354]
28	[56, 46]	333	[489, 379]	638	[382, 459]	943	[7, 388]
29	[51, 57]	334	[490, 491]	639	[793, 118]	944	[43, 158]
30	[58, 59]	335	[479, 492]	640	[456, 490]	945	[1019, 496]
31	[4, 60]	336	[166, 492]	641	[795, 51]	946	[1014, 1020]
32	[61, 62]	337	[421, 128]	642	[403, 380]	947	[1026, 831]
33	[63, 64]	338	[402, 372]	643	[389, 49]	948	[806, 829]
34	[65, 66]	339	[484, 118]	644	[378, 134]	949	[810, 833]
35	[67, 68]	340	[493, 179]	645	[800, 417]	950	[1027, 499]
36	[69, 70]	341	[494, 388]	646	[195, 450]	951	[1028, 1015]
37	[71, 72]	342	[495, 496]	647	[796, 484]	952	[835, 838]
38	[73, 74]	343	[463, 497]	648	[382, 116]	953	[1029, 1030]
39	[75, 76]	344	[498, 499]	649	[442, 133]	954	[806, 788]
40	[77, 78]	345	[489, 185]	650	[7, 426]	955	[806, 1031]
41	[79, 80]	346	[124, 500]	651	[449, 452]	956	[1027, 1012]
42	[81, 82]	347	[381, 434]	652	[453, 394]	957	[837, 1032]
43	[83, 84]	348	[392, 391]	653	[501, 454]	958	[1033, 814]
44	[85, 86]	349	[501, 452]	654	[449, 453]	959	[1034, 1035]
45	[87, 88]	350	[196, 161]	655	[394, 449]	960	[360, 737]
46	[89, 90]	351	[451, 450]	656	[394, 161]	961	[1036, 1037]
47	[91, 89]	352	[452, 196]	657	[388, 427]	962	[1038, 790]
48	[92, 93]	353	[227, 502]	658	[786, 359]	963	[1039, 1040]
49	[94, 95]	354	[503, 504]	659	[801, 378]	964	[1041, 1037]
50	[96, 97]	355	[505, 506]	660	[772, 51]	965	[1042, 766]
51	[98, 99]	356	[507, 508]	661	[413, 369]	966	[47, 180]
52	[100, 101]	357	[509, 510]	662	[430, 142]	967	[789, 367]
53	[102, 103]	358	[511, 512]	663	[400, 784]	968	[1023, 1043]
54	[104, 105]	359	[59, 513]	664	[370, 802]	969	[186, 790]
55	[106, 107]	360	[514, 515]	665	[10, 388]	970	[380, 180]
56	[108, 109]	361	[303, 516]	666	[440, 803]	971	[1044, 499]
57	[97, 110]	362	[517, 518]	667	[804, 805]	972	[1045, 1017]
58	[111, 112]	363	[519, 520]	668	[806, 807]	973	[813, 1046]
59	[40, 113]	364	[521, 522]	669	[808, 809]	974	[1047, 840]
60	[8, 114]	365	[241, 523]	670	[810, 462]	975	[1006, 1003]
61	[115, 116]	366	[524, 525]	671	[804, 811]	976	[426, 120]
62	[117, 118]	367	[526, 527]	672	[812, 466]	977	[1048, 379]
63	[119, 120]	368	[528, 529]	673	[813, 814]	978	[1049, 379]
64	[43, 32]	369	[530, 531]	674	[815, 816]	979	[1022, 173]
65	[121, 122]	370	[532, 533]	675	[38, 117]	980	[458, 747]
66	[123, 124]	371	[79, 534]	676	[157, 817]	981	[828, 1031]
67	[125, 126]	372	[77, 535]	677	[476, 450]	982	[834, 1050]
68	[127, 128]	373	[536, 537]	678	[456, 818]	983	[1028, 1051]

69	[46, 129]	374	[538, 539]	679	[797, 138]	984	[1028, 1052]
70	[130, 131]	375	[318, 540]	680	[742, 819]	985	[369, 1053]
71	[132, 51]	376	[541, 63]	681	[181, 47]	986	[1014, 1054]
72	[133, 134]	377	[542, 345]	682	[123, 750]	987	[1055, 1050]
73	[135, 136]	378	[543, 347]	683	[111, 36]	988	[1016, 1056]
74	[137, 138]	379	[544, 224]	684	[30, 475]	989	[835, 1057]
75	[129, 56]	380	[545, 546]	685	[141, 422]	990	[1058, 1059]
76	[139, 140]	381	[547, 298]	686	[132, 749]	991	[360, 190]
77	[121, 126]	382	[279, 548]	687	[820, 170]	992	[1060, 1061]
78	[141, 142]	383	[268, 549]	688	[151, 472]	993	[1062, 194]
79	[143, 144]	384	[550, 551]	689	[440, 179]	994	[1063, 1064]
80	[145, 146]	385	[198, 552]	690	[196, 501]	995	[1065, 1066]
81	[147, 148]	386	[553, 554]	691	[501, 453]	996	[1067, 790]
82	[149, 150]	387	[555, 556]	692	[449, 163]	997	[1048, 1068]
83	[151, 152]	388	[557, 5]	693	[397, 782]	998	[443, 193]
84	[44, 153]	389	[558, 559]	694	[801, 133]	999	[366, 392]
85	[154, 155]	390	[560, 561]	695	[402, 425]	1000	[1069, 1070]
86	[156, 157]	391	[562, 563]	696	[785, 385]	1001	[311, 1071]
87	[158, 33]	392	[525, 562]	697	[463, 805]	1002	[1072, 543]
88	[159, 32]	393	[330, 564]	698	[768, 821]	1003	[1073, 1074]
89	[160, 161]	394	[294, 565]	699	[822, 823]	1004	[720, 1075]
90	[162, 163]	395	[559, 566]	700	[797, 824]	1005	[1076, 1077]
91	[160, 162]	396	[567, 568]	701	[124, 751]	1006	[1078, 1079]
92	[164, 165]	397	[569, 570]	702	[141, 820]	1007	[604, 1080]
93	[166, 167]	398	[571, 572]	703	[433, 445]	1008	[870, 1081]
94	[168, 169]	399	[573, 262]	704	[54, 131]	1009	[705, 25]
95	[170, 171]	400	[574, 575]	705	[135, 52]	1010	[1082, 1083]
96	[172, 173]	401	[576, 577]	706	[400, 146]	1011	[925, 949]
97	[174, 175]	402	[200, 73]	707	[825, 117]	1012	[880, 1084]
98	[176, 177]	403	[578, 579]	708	[473, 486]	1013	[1085, 1086]
99	[178, 179]	404	[580, 581]	709	[473, 155]	1014	[87, 1087]
100	[180, 181]	405	[582, 583]	710	[112, 37]	1015	[1088, 1089]
101	[182, 183]	406	[584, 507]	711	[826, 413]	1016	[1090, 1091]
102	[184, 185]	407	[585, 340]	712	[371, 41]	1017	[1092, 1093]
103	[142, 186]	408	[308, 586]	713	[827, 407]	1018	[1094, 1095]
104	[187, 188]	409	[587, 333]	714	[818, 491]	1019	[60, 1096]
105	[189, 190]	410	[588, 589]	715	[388, 460]	1020	[670, 1097]
106	[191, 192]	411	[590, 591]	716	[494, 426]	1021	[1098, 1099]
107	[193, 194]	412	[592, 530]	717	[149, 35]	1022	[1100, 1101]
108	[195, 195]	413	[593, 594]	718	[11, 157]	1023	[259, 1102]
109	[163, 196]	414	[205, 595]	719	[471, 152]	1024	[886, 1103]
110	[173, 157]	415	[596, 597]	720	[781, 791]	1025	[1104, 600]
111	[197, 198]	416	[598, 599]	721	[828, 829]	1026	[1105, 972]
112	[28, 199]	417	[600, 601]	722	[828, 807]	1027	[897, 1106]

113	[200, 201]	418	[602, 603]	723	[830, 831]	1028	[1107, 1108]
114	[16, 202]	419	[509, 604]	724	[832, 833]	1029	[1109, 1110]
115	[203, 204]	420	[605, 511]	725	[834, 771]	1030	[70, 1111]
116	[205, 206]	421	[221, 606]	726	[835, 836]	1031	[1087, 1112]
117	[207, 208]	422	[607, 608]	727	[837, 838]	1032	[1113, 1114]
118	[74, 209]	423	[609, 610]	728	[839, 840]	1033	[1115, 279]
119	[210, 211]	424	[611, 612]	729	[827, 423]	1034	[336, 1116]
120	[212, 213]	425	[613, 614]	730	[362, 500]	1035	[296, 1117]
121	[214, 215]	426	[20, 615]	731	[394, 501]	1036	[715, 1118]
122	[216, 217]	427	[616, 617]	732	[501, 163]	1037	[1119, 1120]
123	[218, 219]	428	[618, 619]	733	[841, 842]	1038	[1121, 1122]
124	[220, 221]	429	[620, 621]	734	[843, 844]	1039	[105, 941]
125	[222, 223]	430	[622, 623]	735	[250, 253]	1040	[693, 1123]
126	[224, 225]	431	[575, 624]	736	[591, 845]	1041	[1082, 1124]
127	[226, 227]	432	[625, 626]	737	[846, 847]	1042	[1125, 1126]
128	[228, 229]	433	[627, 91]	738	[848, 592]	1043	[1118, 1127]
129	[230, 231]	434	[628, 629]	739	[849, 850]	1044	[1128, 1129]
130	[232, 233]	435	[619, 630]	740	[851, 852]	1045	[1130, 268]
131	[234, 235]	436	[631, 632]	741	[853, 854]	1046	[1131, 1132]
132	[236, 237]	437	[633, 634]	742	[855, 856]	1047	[1133, 1134]
133	[238, 239]	438	[278, 635]	743	[857, 858]	1048	[1135, 1136]
134	[240, 241]	439	[277, 636]	744	[859, 860]	1049	[1137, 569]
135	[242, 102]	440	[316, 637]	745	[861, 862]	1050	[1138, 1139]
136	[243, 244]	441	[638, 639]	746	[863, 864]	1051	[1140, 1141]
137	[245, 246]	442	[640, 641]	747	[848, 865]	1052	[1142, 1143]
138	[247, 248]	443	[642, 643]	748	[866, 867]	1053	[879, 1144]
139	[249, 250]	444	[644, 645]	749	[868, 869]	1054	[1145, 1146]
140	[251, 252]	445	[546, 646]	750	[606, 870]	1055	[958, 1128]
141	[253, 254]	446	[647, 648]	751	[219, 228]	1056	[1147, 1148]
142	[255, 256]	447	[613, 649]	752	[675, 61]	1057	[1149, 1150]
143	[257, 258]	448	[288, 650]	753	[871, 729]	1058	[298, 1151]
144	[259, 260]	449	[651, 351]	754	[872, 557]	1059	[612, 1152]
145	[101, 261]	450	[652, 653]	755	[867, 871]	1060	[910, 1153]
146	[262, 263]	451	[451, 451]	756	[873, 240]	1061	[1154, 1155]
147	[264, 265]	452	[654, 293]	757	[874, 875]	1062	[1156, 863]
148	[266, 267]	453	[655, 656]	758	[876, 877]	1063	[567, 929]
149	[268, 269]	454	[352, 291]	759	[313, 576]	1064	[867, 1157]
150	[270, 271]	455	[657, 658]	760	[878, 879]	1065	[602, 1158]
151	[272, 273]	456	[659, 660]	761	[880, 881]	1066	[1159, 1160]
152	[274, 275]	457	[246, 661]	762	[882, 883]	1067	[1161, 644]
153	[60, 276]	458	[528, 662]	763	[884, 885]	1068	[1162, 216]
154	[277, 278]	459	[65, 663]	764	[619, 886]	1069	[430, 820]
155	[279, 280]	460	[664, 665]	765	[887, 888]	1070	[1048, 185]
156	[281, 282]	461	[666, 667]	766	[239, 889]	1071	[145, 784]

157	[283, 284]	462	[668, 669]	767	[890, 891]	1072	[411, 744]
158	[285, 286]	463	[212, 670]	768	[892, 893]	1073	[1005, 31]
159	[287, 288]	464	[671, 672]	769	[894, 895]	1074	[370, 375]
160	[289, 290]	465	[673, 674]	770	[896, 897]	1075	[31, 386]
161	[291, 292]	466	[675, 676]	771	[898, 899]	1076	[438, 11]
162	[293, 294]	467	[324, 677]	772	[900, 620]	1077	[154, 175]
163	[290, 295]	468	[678, 679]	773	[901, 514]	1078	[1000, 36]
164	[296, 297]	469	[680, 681]	774	[612, 902]	1079	[473, 175]
165	[298, 299]	470	[682, 538]	775	[105, 903]	1080	[822, 831]
166	[300, 301]	471	[683, 532]	776	[548, 71]	1081	[467, 445]
167	[99, 302]	472	[684, 321]	777	[904, 713]	1082	[488, 739]
168	[303, 304]	473	[270, 685]	778	[905, 542]	1083	[1003, 32]
169	[305, 306]	474	[686, 687]	779	[506, 906]	1084	[374, 1053]
170	[254, 307]	475	[688, 689]	780	[248, 907]	1085	[164, 779]
171	[256, 308]	476	[690, 652]	781	[908, 909]	1086	[397, 122]
172	[309, 283]	477	[691, 692]	782	[910, 911]	1087	[1163, 811]
173	[310, 27]	478	[693, 694]	783	[912, 913]	1088	[1060, 1164]
174	[311, 312]	479	[695, 696]	784	[103, 914]	1089	[751, 421]
175	[313, 314]	480	[216, 697]	785	[915, 916]	1090	[1058, 1165]
176	[315, 316]	481	[698, 699]	786	[917, 87]	1091	[143, 418]
177	[317, 318]	482	[700, 701]	787	[288, 918]	1092	[1036, 1166]
178	[319, 320]	483	[702, 686]	788	[919, 920]	1093	[491, 145]
179	[321, 322]	484	[703, 704]	789	[921, 922]	1094	[1019, 1054]
180	[90, 230]	485	[67, 705]	790	[923, 924]	1095	[1044, 1012]
181	[323, 324]	486	[203, 706]	791	[925, 926]	1096	[1019, 1018]
182	[325, 326]	487	[707, 708]	792	[898, 553]	1097	[808, 466]
183	[327, 328]	488	[709, 710]	793	[927, 928]	1098	[801, 408]
184	[329, 330]	489	[711, 712]	794	[929, 930]	1099	[1025, 138]
185	[331, 332]	490	[348, 574]	795	[931, 932]	1100	[468, 750]
186	[333, 334]	491	[713, 714]	796	[933, 934]	1101	[1049, 185]
187	[335, 242]	492	[715, 716]	797	[935, 936]	1102	[463, 811]
188	[336, 337]	493	[717, 718]	798	[937, 938]	1103	[770, 1050]
189	[338, 339]	494	[719, 212]	799	[307, 939]	1104	[776, 785]
190	[340, 341]	495	[720, 721]	800	[940, 602]	1105	[837, 836]
191	[224, 342]	496	[722, 723]	801	[941, 942]	1106	[1028, 816]
192	[343, 344]	497	[724, 725]	802	[892, 943]	1107	[1034, 1167]
193	[345, 346]	498	[726, 727]	803	[666, 944]	1108	[143, 780]
194	[347, 348]	499	[19, 728]	804	[664, 945]	1109	[773, 1168]
195	[349, 350]	500	[729, 730]	805	[946, 947]	1110	[52, 491]
196	[351, 352]	501	[731, 732]	806	[857, 948]	1111	[129, 180]
197	[123, 353]	502	[389, 367]	807	[949, 950]	1112	[1026, 823]
198	[137, 354]	503	[733, 494]	808	[951, 952]	1113	[1169, 1170]
199	[355, 183]	504	[734, 448]	809	[81, 953]	1114	[491, 400]
200	[356, 357]	505	[735, 736]	810	[321, 954]	1115	[1039, 1171]

201	[127, 358]	506	[189, 737]	811	[955, 956]	1116	[777, 52]
202	[31, 359]	507	[738, 439]	812	[957, 958]	1117	[1008, 128]
203	[360, 361]	508	[384, 37]	813	[959, 960]	1118	[1011, 1018]
204	[123, 362]	509	[734, 739]	814	[961, 962]	1119	[461, 821]
205	[143, 363]	510	[740, 359]	815	[963, 613]	1120	[1172, 466]
206	[364, 365]	511	[422, 170]	816	[964, 965]	1121	[795, 741]
207	[366, 366]	512	[741, 429]	817	[24, 966]	1122	[362, 751]
208	[48, 367]	513	[112, 385]	818	[634, 967]	1123	[1173, 444]
209	[136, 53]	514	[174, 486]	819	[968, 969]	1124	[1174, 811]
210	[154, 368]	515	[373, 369]	820	[970, 923]	1125	[795, 428]
211	[148, 369]	516	[177, 370]	821	[667, 971]	1126	[818, 53]
212	[119, 114]	517	[467, 403]	822	[972, 973]	1127	[465, 809]
213	[152, 370]	518	[742, 131]	823	[315, 974]	1128	[1033, 1046]
214	[371, 372]	519	[734, 743]	824	[975, 976]	1129	[1045, 1056]
215	[373, 374]	520	[387, 426]	825	[649, 977]	1130	[1063, 1175]
216	[178, 375]	521	[742, 744]	826	[269, 978]	1131	[1176, 1177]
217	[376, 158]	522	[180, 46]	827	[979, 980]	1132	[751, 1008]
218	[377, 378]	523	[407, 356]	828	[849, 724]	1133	[773, 1178]
219	[184, 379]	524	[454, 477]	829	[981, 982]	1134	[750, 500]
220	[181, 380]	525	[162, 454]	830	[983, 984]	1135	[401, 785]
221	[381, 183]	526	[191, 481]	831	[707, 985]	1136	[382, 747]
222	[382, 383]	527	[745, 421]	832	[318, 894]	1137	[172, 767]
223	[384, 385]	528	[57, 435]	833	[986, 987]	1138	[825, 39]
224	[386, 153]	529	[364, 746]	834	[988, 989]	1139	[1179, 1030]
225	[387, 388]	530	[446, 138]	835	[990, 991]	1140	[1169, 1180]
226	[389, 390]	531	[443, 134]	836	[992, 993]	1141	[1038, 171]
227	[391, 392]	532	[115, 747]	837	[994, 67]	1142	[1065, 1177]
228	[378, 193]	533	[36, 385]	838	[995, 996]	1143	[53, 145]
229	[379, 393]	534	[468, 362]	839	[997, 998]	1144	[757, 392]
230	[161, 163]	535	[437, 746]	840	[208, 999]	1145	[1163, 805]
231	[163, 394]	536	[433, 46]	841	[1000, 112]	1146	[1055, 771]
232	[395, 396]	537	[748, 49]	842	[34, 439]	1147	[1176, 1066]
233	[397, 126]	538	[50, 749]	843	[149, 1001]	1148	[1062, 766]
234	[168, 398]	539	[750, 751]	844	[760, 369]	1149	[1041, 1166]
235	[399, 400]	540	[472, 370]	845	[1002, 799]	1150	[500, 1008]
236	[401, 384]	541	[752, 43]	846	[359, 45]	1151	[777, 136]
237	[402, 113]	542	[753, 124]	847	[1003, 158]	1152	[1173, 799]
238	[180, 403]	543	[392, 392]	848	[356, 361]	1153	[768, 462]
239	[404, 405]	544	[754, 387]	849	[44, 1004]	1154	[1181, 788]
240	[406, 407]	545	[476, 195]	850	[472, 440]	1155	[1172, 809]
241	[408, 193]	546	[161, 454]	851	[1005, 475]	1156	[1025, 824]
242	[409, 142]	547	[755, 396]	852	[781, 45]	1157	[1182, 146]
243	[403, 410]	548	[756, 133]	853	[1006, 376]	1158	[1181, 829]
244	[411, 131]	549	[469, 128]	854	[488, 33]	1159	[461, 769]

245	[412, 413]	550	[757, 757]	855	[774, 736]	1160	[1183, 499]
246	[414, 150]	551	[758, 140]	856	[479, 167]	1161	[1025, 800]
247	[176, 415]	552	[353, 751]	857	[370, 803]	1162	[1184, 376]
248	[359, 153]	553	[759, 116]	858	[475, 386]	1163	[616, 1145]
249	[187, 416]	554	[760, 374]	859	[764, 1007]	1164	[342, 1185]
250	[417, 418]	555	[761, 7]	860	[420, 1008]	1165	[250, 1186]
251	[419, 398]	556	[370, 179]	861	[797, 800]	1166	[1187, 1188]
252	[420, 421]	557	[759, 747]	862	[490, 53]	1167	[905, 1189]
253	[409, 422]	558	[755, 762]	863	[142, 170]	1168	[1190, 846]
254	[406, 423]	559	[763, 190]	864	[824, 417]	1169	[340, 887]
255	[181, 410]	560	[764, 765]	865	[171, 799]	1170	[1191, 1192]
256	[424, 405]	561	[134, 766]	866	[735, 775]	1171	[335, 1193]
257	[371, 425]	562	[477, 161]	867	[417, 780]	1172	[1129, 988]
258	[173, 12]	563	[454, 394]	868	[733, 387]	1173	[1194, 1195]
259	[426, 427]	564	[484, 474]	869	[388, 114]	1174	[16, 1196]
260	[376, 32]	565	[394, 162]	870	[404, 140]	1175	[559, 1197]
261	[181, 129]	566	[456, 136]	871	[753, 362]	1176	[248, 1198]
262	[408, 134]	567	[393, 455]	872	[1009, 11]	1177	[1199, 1200]
263	[428, 429]	568	[766, 746]	873	[377, 408]	1178	[1201, 1202]
264	[430, 431]	569	[414, 439]	874	[195, 476]	1179	[1203, 1204]
265	[432, 423]	570	[767, 157]	875	[195, 451]	1180	[1158, 1205]
266	[129, 433]	571	[768, 769]	876	[395, 762]	1181	[63, 671]
267	[355, 434]	572	[770, 771]	877	[166, 435]	1182	[1206, 970]
268	[429, 435]	573	[772, 428]	878	[366, 757]	1183	[1207, 673]
269	[436, 52]	574	[424, 140]	879	[130, 55]	1184	[1208, 285]
270	[360, 357]	575	[391, 391]	880	[147, 760]	1185	[1209, 1050]
271	[437, 365]	576	[436, 136]	881	[154, 486]	1186	[194, 365]
272	[438, 156]	577	[773, 407]	882	[166, 1010]	1187	[1174, 497]
273	[149, 439]	578	[163, 477]	883	[1002, 444]	1188	[1209, 771]
274	[42, 159]	579	[454, 196]	884	[458, 383]	1189	[1182, 784]
275	[440, 375]	580	[774, 775]	885	[767, 12]	1190	[1184, 1003]
276	[10, 426]	581	[393, 122]	886	[787, 829]	1191	[495, 1018]
277	[441, 144]	582	[480, 192]	887	[1011, 496]	1192	[1210, 831]
278	[442, 443]	583	[186, 171]	888	[498, 1012]	1193	[194, 746]
279	[125, 122]	584	[776, 384]	889	[403, 129]	1194	[789, 798]
280	[171, 444]	585	[754, 494]	890	[757, 366]	1195	[403, 47]
281	[56, 445]	586	[742, 55]	891	[1013, 434]	1196	[828, 788]
282	[139, 405]	587	[777, 490]	892	[781, 1004]	1197	[1008, 358]
283	[446, 354]	588	[163, 160]	893	[1014, 496]	1198	[768, 833]
284	[431, 170]	589	[476, 451]	894	[832, 462]	1199	[1174, 464]
285	[447, 113]	590	[778, 779]	895	[830, 823]	1200	[1210, 823]
286	[156, 12]	591	[457, 780]	896	[815, 1015]	1201	[1211, 786]
287	[9, 7]	592	[756, 443]	897	[1016, 1017]	1202	[734, 33]
288	[32, 448]	593	[445, 129]	898	[470, 760]	1203	[1048, 1212]

289	[196, 449]	594	[758, 405]	899	[12, 817]	1204	[431, 186]
290	[450, 450]	595	[430, 422]	900	[483, 793]	1205	[1183, 1012]
291	[450, 451]	596	[761, 10]	901	[412, 373]	1206	[411, 819]
292	[196, 162]	597	[781, 153]	902	[801, 443]	1207	[815, 1051]
293	[452, 394]	598	[393, 782]	903	[766, 365]	1208	[1009, 156]
294	[453, 196]	599	[783, 784]	904	[777, 818]	1209	[1213, 1214]
295	[449, 454]	600	[485, 175]	905	[479, 1010]	1210	[973, 1215]
296	[397, 455]	601	[785, 37]	906	[783, 146]	1211	[1138, 1216]
297	[456, 52]	602	[493, 375]	907	[415, 440]	1212	[1217, 1218]
298	[457, 418]	603	[786, 386]	908	[34, 1001]	1213	[835, 1032]
299	[421, 358]	604	[787, 788]	909	[117, 474]	1214	[815, 1052]
300	[458, 459]	605	[772, 741]	910	[488, 743]	1215	[837, 1057]
301	[413, 374]	606	[380, 467]	911	[415, 370]	1216	[34, 150]
302	[177, 440]	607	[467, 181]	912	[789, 390]	1217	[1211, 740]
303	[426, 460]	608	[789, 49]	913	[467, 46]	1218	[388, 120]
304	[461, 462]	609	[487, 415]	914	[185, 393]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	204	0	408	1	612	0	816	0	1020	0
1	0	205	0	409	0	613	1	817	0	1021	1
2	0	206	0	410	1	614	0	818	1	1022	1
3	0	207	0	411	1	615	1	819	0	1023	0
4	0	208	0	412	0	616	1	820	0	1024	0
5	0	209	1	413	1	617	0	821	0	1025	0
6	1	210	1	414	0	618	0	822	1	1026	1
7	0	211	1	415	1	619	1	823	0	1027	0
8	0	212	1	416	1	620	1	824	1	1028	0
9	0	213	1	417	1	621	1	825	1	1029	0
10	1	214	0	418	1	622	1	826	1	1030	0
11	0	215	0	419	1	623	1	827	1	1031	1
12	0	216	0	420	0	624	1	828	1	1032	1
13	1	217	0	421	1	625	1	829	1	1033	1
14	0	218	0	422	1	626	1	830	0	1034	0
15	0	219	0	423	1	627	0	831	1	1035	0
16	0	220	0	424	1	628	1	832	1	1036	1
17	0	221	1	425	1	629	0	833	1	1037	1
18	0	222	1	426	0	630	1	834	0	1038	1
19	0	223	1	427	1	631	1	835	0	1039	1
20	0	224	1	428	0	632	1	836	0	1040	0
21	1	225	1	429	1	633	1	837	1	1041	0
22	1	226	1	430	1	634	1	838	1	1042	1
23	0	227	1	431	1	635	1	839	1	1043	1
24	0	228	0	432	1	636	0	840	0	1044	1

25	0	229	0	433	0	637	1	841	1	1045	1
26	0	230	1	434	1	638	1	842	0	1046	0
27	1	231	1	435	1	639	1	843	1	1047	0
28	1	232	0	436	1	640	1	844	0	1048	1
29	0	233	0	437	1	641	1	845	1	1049	0
30	0	234	0	438	1	642	1	846	0	1050	1
31	0	235	0	439	1	643	1	847	1	1051	1
32	0	236	1	440	1	644	0	848	1	1052	1
33	1	237	1	441	1	645	0	849	1	1053	0
34	0	238	0	442	1	646	1	850	0	1054	1
35	1	239	0	443	1	647	1	851	1	1055	1
36	0	240	0	444	0	648	1	852	0	1056	0
37	1	241	1	445	1	649	1	853	1	1057	0
38	0	242	0	446	1	650	0	854	0	1058	0
39	1	243	1	447	1	651	0	855	0	1059	0
40	0	244	1	448	0	652	1	856	1	1060	0
41	0	245	0	449	0	653	1	857	0	1061	1
42	1	246	0	450	1	654	0	858	1	1062	0
43	1	247	0	451	0	655	1	859	1	1063	1
44	1	248	0	452	0	656	1	860	0	1064	1
45	1	249	1	453	1	657	1	861	1	1065	1
46	0	250	1	454	0	658	0	862	0	1066	1
47	0	251	1	455	1	659	1	863	0	1067	0
48	0	252	0	456	1	660	0	864	1	1068	0
49	0	253	0	457	0	661	1	865	1	1069	1
50	0	254	0	458	0	662	1	866	1	1070	1
51	0	255	0	459	0	663	1	867	1	1071	0
52	0	256	1	460	0	664	0	868	1	1072	1
53	0	257	0	461	1	665	1	869	1	1073	1
54	1	258	0	462	0	666	1	870	0	1074	0
55	1	259	0	463	1	667	0	871	0	1075	0
56	1	260	0	464	0	668	0	872	0	1076	1
57	0	261	0	465	0	669	0	873	0	1077	1
58	0	262	1	466	0	670	0	874	1	1078	1
59	0	263	0	467	1	671	0	875	1	1079	0
60	0	264	1	468	1	672	1	876	0	1080	1
61	0	265	1	469	0	673	0	877	0	1081	1
62	0	266	1	470	0	674	1	878	0	1082	0
63	1	267	1	471	0	675	0	879	0	1083	1
64	1	268	1	472	0	676	1	880	1	1084	0
65	0	269	1	473	0	677	0	881	1	1085	0
66	0	270	0	474	1	678	1	882	0	1086	0
67	1	271	1	475	1	679	1	883	1	1087	1
68	1	272	1	476	0	680	0	884	0	1088	0

69	0	273	1	477	1	681	0	885	1	1089	0
70	0	274	1	478	0	682	0	886	0	1090	0
71	1	275	1	479	1	683	0	887	1	1091	0
72	0	276	1	480	0	684	0	888	0	1092	1
73	0	277	1	481	0	685	0	889	1	1093	1
74	0	278	1	482	1	686	1	890	1	1094	0
75	1	279	1	483	0	687	0	891	0	1095	1
76	1	280	1	484	0	688	1	892	0	1096	0
77	0	281	1	485	1	689	1	893	1	1097	0
78	0	282	1	486	0	690	0	894	1	1098	1
79	0	283	1	487	1	691	1	895	0	1099	0
80	0	284	1	488	0	692	0	896	1	1100	1
81	1	285	1	489	1	693	0	897	0	1101	0
82	1	286	1	490	0	694	1	898	0	1102	1
83	1	287	0	491	1	695	1	899	0	1103	1
84	1	288	0	492	1	696	0	900	0	1104	0
85	1	289	0	493	1	697	1	901	0	1105	1
86	1	290	1	494	0	698	0	902	1	1106	0
87	1	291	1	495	0	699	1	903	0	1107	0
88	0	292	0	496	1	700	1	904	0	1108	0
89	0	293	0	497	1	701	0	905	1	1109	0
90	0	294	1	498	0	702	0	906	0	1110	0
91	0	295	0	499	0	703	0	907	1	1111	1
92	0	296	0	500	1	704	1	908	0	1112	1
93	0	297	1	501	1	705	0	909	0	1113	1
94	0	298	0	502	1	706	1	910	0	1114	1
95	0	299	1	503	1	707	1	911	1	1115	1
96	0	300	0	504	1	708	0	912	0	1116	0
97	0	301	1	505	1	709	0	913	1	1117	0
98	0	302	0	506	1	710	1	914	1	1118	1
99	0	303	0	507	1	711	1	915	0	1119	1
100	0	304	1	508	1	712	0	916	1	1120	1
101	0	305	1	509	1	713	1	917	0	1121	1
102	0	306	0	510	1	714	1	918	1	1122	1
103	0	307	1	511	1	715	1	919	0	1123	0
104	1	308	1	512	1	716	0	920	1	1124	0
105	1	309	0	513	1	717	1	921	0	1125	1
106	1	310	0	514	0	718	0	922	1	1126	1
107	1	311	0	515	0	719	0	923	0	1127	0
108	1	312	1	516	0	720	0	924	0	1128	1
109	1	313	1	517	1	721	1	925	1	1129	1
110	0	314	0	518	0	722	1	926	1	1130	1
111	0	315	0	519	1	723	0	927	1	1131	0
112	1	316	1	520	1	724	1	928	0	1132	0

113	1	317	0	521	0	725	0	929	1	1133	0
114	0	318	1	522	0	726	0	930	0	1134	0
115	0	319	0	523	0	727	1	931	1	1135	1
116	0	320	1	524	0	728	1	932	0	1136	1
117	0	321	0	525	0	729	1	933	1	1137	0
118	0	322	1	526	1	730	1	934	0	1138	1
119	1	323	0	527	1	731	1	935	1	1139	1
120	1	324	1	528	0	732	1	936	0	1140	1
121	0	325	0	529	0	733	1	937	0	1141	1
122	0	326	1	530	1	734	1	938	1	1142	1
123	0	327	0	531	1	735	1	939	1	1143	0
124	0	328	1	532	0	736	0	940	0	1144	1
125	1	329	0	533	0	737	0	941	1	1145	1
126	1	330	1	534	1	738	1	942	0	1146	1
127	1	331	1	535	1	739	1	943	0	1147	0
128	0	332	0	536	0	740	1	944	1	1148	0
129	1	333	1	537	1	741	1	945	0	1149	0
130	0	334	0	538	0	742	0	946	1	1150	1
131	0	335	1	539	0	743	0	947	1	1151	0
132	1	336	0	540	0	744	1	948	0	1152	0
133	0	337	1	541	0	745	1	949	0	1153	0
134	0	338	1	542	0	746	0	950	0	1154	1
135	0	339	0	543	0	747	1	951	0	1155	1
136	1	340	1	544	0	748	1	952	0	1156	0
137	0	341	0	545	0	749	1	953	0	1157	1
138	0	342	0	546	1	750	0	954	0	1158	1
139	1	343	1	547	1	751	0	955	0	1159	1
140	1	344	0	548	0	752	0	956	0	1160	1
141	0	345	1	549	0	753	0	957	1	1161	0
142	0	346	0	550	1	754	0	958	1	1162	0
143	0	347	1	551	0	755	1	959	0	1163	1
144	0	348	0	552	1	756	0	960	0	1164	0
145	0	349	1	553	1	757	1	961	1	1165	1
146	1	350	0	554	0	758	0	962	1	1166	0
147	1	351	0	555	1	759	1	963	1	1167	1
148	1	352	0	556	0	760	0	964	0	1168	0
149	1	353	1	557	1	761	1	965	1	1169	1
150	0	354	1	558	1	762	0	966	0	1170	0
151	1	355	1	559	0	763	0	967	0	1171	1
152	1	356	1	560	1	764	1	968	0	1172	1
153	0	357	1	561	0	765	1	969	1	1173	0
154	1	358	1	562	1	766	0	970	0	1174	0
155	1	359	0	563	0	767	1	971	1	1175	0
156	1	360	0	564	0	768	0	972	1	1176	0

157	1	361	0	565	1	769	1	973	0	1177	0
158	1	362	1	566	1	770	1	974	0	1178	1
159	0	363	1	567	1	771	0	975	1	1179	1
160	0	364	0	568	0	772	0	976	0	1180	1
161	1	365	1	569	0	773	0	977	1	1181	1
162	0	366	0	570	1	774	0	978	0	1182	1
163	1	367	1	571	0	775	1	979	1	1183	1
164	0	368	0	572	1	776	0	980	0	1184	0
165	0	369	1	573	0	777	0	981	1	1185	0
166	0	370	0	574	1	778	1	982	0	1186	1
167	0	371	0	575	1	779	1	983	0	1187	0
168	0	372	0	576	1	780	0	984	0	1188	0
169	1	373	0	577	0	781	0	985	1	1189	1
170	0	374	0	578	1	782	0	986	1	1190	0
171	1	375	1	579	0	783	0	987	1	1191	0
172	0	376	0	580	0	784	0	988	0	1192	0
173	0	377	0	581	1	785	0	989	0	1193	1
174	0	378	0	582	0	786	0	990	0	1194	0
175	1	379	0	583	1	787	0	991	0	1195	1
176	0	380	0	584	0	788	0	992	0	1196	1
177	0	381	1	585	0	789	0	993	0	1197	0
178	0	382	1	586	0	790	0	994	1	1198	0
179	0	383	1	587	0	791	1	995	1	1199	0
180	0	384	1	588	1	792	0	996	0	1200	0
181	0	385	0	589	0	793	1	997	1	1201	1
182	0	386	1	590	1	794	1	998	1	1202	1
183	0	387	1	591	0	795	1	999	0	1203	1
184	0	388	1	592	0	796	1	1000	1	1204	1
185	1	389	1	593	1	797	1	1001	0	1205	1
186	1	390	1	594	0	798	0	1002	1	1206	1
187	1	391	1	595	1	799	1	1003	1	1207	1
188	0	392	0	596	1	800	0	1004	0	1208	0
189	1	393	1	597	0	801	1	1005	1	1209	0
190	1	394	1	598	1	802	0	1006	1	1210	0
191	1	395	0	599	0	803	1	1007	0	1211	1
192	1	396	1	600	1	804	0	1008	0	1212	1
193	1	397	0	601	0	805	1	1009	0	1213	0
194	1	398	0	602	1	806	0	1010	0	1214	1
195	1	399	0	603	0	807	0	1011	1	1215	1
196	0	400	1	604	0	808	0	1012	1	1216	0
197	0	401	1	605	0	809	1	1013	0	1217	1
198	0	402	1	606	0	810	0	1014	1	1218	1
199	1	403	1	607	1	811	0	1015	0		
200	1	404	0	608	0	812	1	1016	0		

201	1	405	0	609	1	813	0	1017	1		
202	0	406	0	610	0	814	1	1018	0		
203	0	407	0	611	1	815	1	1019	0		

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