PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a, b) = (z^6 + z^4 + z^2, z)$

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Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a, b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a, b) in $\mathbb{F}_2[x] \setminus \mathbb{F}_2$. In this paper we prove the conjecture for $(a, b) = (z^6 + z^4 + z^2, z)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \ldots)$ of elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{\ddots + \cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a, b)-Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \ldots) = (a, b, b, a, b, a, a, b, \ldots)$$

that we shall denote

$$\mathrm{CF}(a, b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When $(a,b)=(z^6+z^4+z^2,z)$, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over $\mathbb{F}_2(z)$, with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

where, for $\mathrm{CF}(a,b)$,

$$\begin{aligned}
p_0(z)&=z^{17}+z^{13}+z^{11}+z^{10}+z^9+z^7+z^6+z^5+z^2+z+1,\\
p_1(z)&=z^{23}+z^{21}+z^{18}+z^{15}+z^{14}+z^{13}+z^{12}+z^{11}+z^{10}+z^8+z^6+z^5+z^3\\
&\quad+z^2,\\
p_2(z)&=z^{24}+z^{22}+z^{16}+z^{14}+z^{12}+z^6+z^4,\\
p_3(z)&=z^{23}+z^{21}+z^{18}+z^{15}+z^{14}+z^{13}+z^{12}+z^{11}+z^{10}+z^8+z^6+z^5+z^3\\
&\quad+z^2,\\
p_4(z)&=z^{22}+z^{20}+z^{17}+z^{14}+z^{13}+z^{11}+z^{10}+z^9+z^8+z^7+z^5+z^2+z,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$,

$$\begin{aligned}
p_0(z)&=z^{20}+z^{17}+z^{13}+z^{12}+z^{11}+z^{10}+z^9+z^7+z^6+z^5+z^4+z^2+z,\\
p_1(z)&=z^{23}+z^{21}+z^{18}+z^{15}+z^{14}+z^{13}+z^{12}+z^{11}+z^{10}+z^8+z^6+z^5+z^3\\
&\quad+z^2,\\
p_2(z)&=z^{24}+z^{22}+z^{16}+z^{14}+z^{12}+z^6+z^4,\\
p_3(z)&=z^{23}+z^{21}+z^{18}+z^{15}+z^{14}+z^{13}+z^{12}+z^{11}+z^{10}+z^8+z^6+z^5+z^3\\
&\quad+z^2,\\
p_4(z)&=z^{22}+z^{17}+z^{14}+z^{13}+z^{12}+z^{11}+z^{10}+z^9+z^8+z^7+z^5+z^4+z^2+z\\
&\quad+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix}P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix}:=\begin{pmatrix}t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix}t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x)&=x^{\deg(t_{2^n-1})}\begin{pmatrix}t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix}t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(t_0)}\begin{pmatrix}t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x)&=x^{\deg(\bar t_{2^n-1})}\begin{pmatrix}\bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix}\bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(\bar t_0)}\begin{pmatrix}\bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n \geq 0$

$$M_{n+1}(x) = W_n(x) \cdot M_n(x),$$
$$W_{n+1}(x) = M_n(x) \cdot W_n(x).$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n} \tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n} \tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \leq i,j \leq 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \leq i,j \leq 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \leq i,j \leq 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(175, 175, 175, 175, 175)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \geq 0$. If for some $n \geq 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \geq 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z\left(\phi(W^o,0,0)(x,z),\ z^{12}\cdot\phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 630 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 630, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z\left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ to order 770 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 770, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. $\square$

Define

$$R^e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned} M^e[:14u] &= M^e[:7u] + x^{4u}M^e[:3u] + x^{6u}M^e[:u] + x^{4u}M^e[:7u] + x^{8u}M^e[:3u] \\ &\quad + x^{10u}M^e[:u] + x^{6u}M^e[:7u] + x^{10u}M^e[:3u] + x^{12u}M^e[:u] \\ &\quad + x^{8u}M^e[:6u] + x^{10u}M^e[:4u] + (x^{7u} + x^{11u} + x^{13u})R^e, \\ W^e[:14u] &= W^e[:7u] + x^{4u}W^e[:3u] + x^{6u}W^e[:u] + x^{4u}W^e[:7u] + x^{8u}W^e[:3u] \end{aligned}$$

$$+ x^{10u}W^e[:u] + x^{6u}W^e[:7u] + x^{10u}W^e[:3u] + x^{12u}W^e[:u] + x^{8u}W^e[:6u] + x^{10u}W^e[:4u] + (x^{7u} + x^{11u} + x^{13u})R^e.$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$\begin{aligned} M^o[:14u] = {} & M^o[:7u] + x^{4u}M^o[:3u] + x^{6u}M^o[:u] + x^{4u}M^o[:7u] + x^{8u}M^o[:3u] \\ & + x^{10u}M^o[:u] + x^{6u}M^o[:7u] + x^{10u}M^o[:3u] + x^{12u}M^o[:u] \\ & + x^{8u}M^o[:6u] + x^{10u}M^o[:4u] + (x^{7u} + x^{11u} + x^{13u})R^o, \\ W^o[:14u] = {} & W^o[:7u] + x^{4u}W^o[:3u] + x^{6u}W^o[:u] + x^{4u}W^o[:7u] + x^{8u}W^o[:3u] \\ & + x^{10u}W^o[:u] + x^{6u}W^o[:7u] + x^{10u}W^o[:3u] + x^{12u}W^o[:u] \\ & + x^{8u}W^o[:6u] + x^{10u}W^o[:4u] + (x^{7u} + x^{11u} + x^{13u})R^o. \end{aligned}$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$\begin{aligned} T[:14u] = {} & T[:7u] + x^{4u}T[:3u] + x^{6u}T[:u] + x^{4u}T[:7u] + x^{8u}T[:3u] + x^{10u}T[:u] \\ & + x^{6u}T[:7u] + x^{10u}T[:3u] + x^{12u}T[:u] + x^{8u}T[:6u] + x^{10u}T[:4u]. \end{aligned}$$

The proof of the other 15 cases are similar. We break down the above identity into 7 parts:

$$\begin{aligned} 0 &= x^{6u}T[u:2u] + x^{4u}T[3u:4u] + T[7u:8u], \\ 0 &= x^{6u}T[2u:3u] + x^{4u}T[4u:5u] + T[8u:9u], \\ 0 &= x^{6u}T[3u:4u] + x^{4u}T[5u:6u] + T[9u:10u], \\ 0 &= x^{10u}T[:u] + x^{6u}T[4u:5u] + x^{4u}T[6u:7u] + T[10u:11u], \\ 0 &= x^{8u}T[3u:4u] + x^{6u}T[5u:6u] + T[11u:12u], \\ 0 &= x^{12u}T[:u] + x^{8u}T[4u:5u] + x^{6u}T[6u:7u] + T[12u:13u], \\ 0 &= x^{10u}T[3u:4u] + x^{8u}T[5u:6u] + T[13u:14u], \end{aligned}$$

which can be rewritten as

$$0 = T[[1w]_2] + T[[11w]_2] + T[[111w]_2], \tag{2.7}$$

$$0 = T[[10w]_2] + T[[100w]_2] + T[[1000w]_2], \tag{2.8}$$

$$0 = T[[11w]_2] + T[[101w]_2] + T[[1001w]_2], \tag{2.9}$$

$$0 = T[[w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1010w]_2], \tag{2.10}$$

$$0 = T[[11w]_2] + T[[101w]_2] + T[[1011w]_2], \tag{2.11}$$

$$0 = T[[w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1100w]_2], \tag{2.12}$$

$$0 = T[[11w]_2] + T[[101w]_2] + T[[1101w]_2], \tag{2.13}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$0 = T[[1w]_2] + T[[11w]_2] + T[[111w]_2], \tag{2.14}$$

$$0 = T[[10w]_2] + T[[100w]_2] + T[[1000w]_2], \tag{2.15}$$

$$0 = T[[11w]_2] + T[[101w]_2] + T[[1001w]_2], \tag{2.16}$$

$$0 = T[[w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1010w]_2], \tag{2.17}$$

$$0 = T[[11w]_2] + T[[101w]_2] + T[[1011w]_2], \tag{2.18}$$
$$0 = T[[w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1100w]_2], \tag{2.19}$$
$$0 = T[[11w]_2] + T[[101w]_2] + T[[1101w]_2], \tag{2.20}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 309 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.20) can be written as

$$0 = \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 111)), \tag{2.21}$$
$$0 = \tau(A(s, 10)) + \tau(A(s, 100)) + \tau(A(s, 1000)), \tag{2.22}$$
$$0 = \tau(A(s, 11)) + \tau(A(s, 101)) + \tau(A(s, 1001)), \tag{2.23}$$
$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 100)) + \tau(A(s, 110)) + \tau(A(s, 1010)), \tag{2.24}$$
$$0 = \tau(A(s, 11)) + \tau(A(s, 101)) + \tau(A(s, 1011)), \tag{2.25}$$
$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 100)) + \tau(A(s, 110)) + \tau(A(s, 1100)), \tag{2.26}$$
$$0 = \tau(A(s, 11)) + \tau(A(s, 101)) + \tau(A(s, 1101)), \tag{2.27}$$

for all $s \in E_{2k}$, and

$$0 = \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 111)), \tag{2.28}$$
$$0 = \tau(A(s, 10)) + \tau(A(s, 100)) + \tau(A(s, 1000)), \tag{2.29}$$
$$0 = \tau(A(s, 11)) + \tau(A(s, 101)) + \tau(A(s, 1001)), \tag{2.30}$$
$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 100)) + \tau(A(s, 110)) + \tau(A(s, 1010)), \tag{2.31}$$
$$0 = \tau(A(s, 11)) + \tau(A(s, 101)) + \tau(A(s, 1011)), \tag{2.32}$$
$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 100)) + \tau(A(s, 110)) + \tau(A(s, 1100)), \tag{2.33}$$
$$0 = \tau(A(s, 11)) + \tau(A(s, 101)) + \tau(A(s, 1101)), \tag{2.34}$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{18}), E_{18}) = (A(i, 0^{14}), E_{14})$, so that we only have to verify that identities (2.21) through (2.34) hold for $2 \leq k \leq 9$, which turns out to be true. $\square$

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \geq 2$, and $u = 2^{2k-1}$,*

$$M_{2k} = M^e[:7u] + x^{4u} M^e[:3u] + x^{6u} M^e[:u] + x^{7u} R^e,$$
$$W_{2k} = W^e[:7u] + x^{4u} W^e[:3u] + x^{6u} W^e[:u] + x^{7u} R^e.$$

For all integer $k \geq 2$, and $u = 2^{2k}$,

$$M_{2k+1} = M^o[:7u] + x^{4u} M^o[:3u] + x^{6u} M^o[:u] + x^{7u} R^o,$$
$$W_{2k+1} = W^o[:7u] + x^{4u} W^o[:3u] + x^{6u} W^o[:u] + x^{7u} R^o.$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \geq 2$, we claim that

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1} \wedge W_{2k+1},$$
$$M_{2k+1} \wedge W_{2k+1} \Rightarrow M_{2k+2} \wedge W_{2k+2}.$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$\begin{aligned} W_{2k}M_{2k} = &\left(W^e[:7v] + x^{4v}W^e[:3v] + x^{6v}W^e[:v] + x^{7u}R^e\right) \times \left(M^e[:7v]\right. \\ &\left. + x^{4v}M^e[:3v] + x^{6v}M^e[:v] + x^{7u}R^e\right); \end{aligned} \tag{2.35}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{14v}R^o$. Therefore we only need to prove that their difference is $O(x^{14v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{14v}, to

$$W^e[:14v]M^e[:14v] + x^{8v}W^e[:6v]M^e[:6v] + x^{12v}W^e[:2v]M^e[:2v].$$

For all $n \leq 14$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.35) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{14}, b_1, ..., b_{14}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{14})$, which completes the proof. □

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \leq j, k \leq 1$,

$$\begin{aligned} \lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k}, \\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k}, \\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k}, \\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}. \end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a, b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e,0,1)$ and $\phi(M^e,0,0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x,y) = \mathrm{Res}_t\left(\phi(M^e,0,1)(x,t), y^{12}\phi(M^e,0,1)(x,t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x,y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned}
q_0(x) &= x^{24} + x^{23} + x^{22} + x^{19} + x^{18} + x^{17} + x^{15} + x^{14} + x^{13} + x^{11} + x^7,\\
q_1(x) &= x^{22} + x^{21} + x^{19} + x^{18} + x^{16} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^9 + x^6\\
&\quad + x^3 + x,\\
q_2(x) &= x^{20} + x^{18} + x^{12} + x^{10} + x^8 + x^2 + 1,\\
q_3(x) &= x^{22} + x^{21} + x^{19} + x^{18} + x^{16} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^9 + x^6\\
&\quad + x^3 + x,\\
q_4(x) &= x^{23} + x^{22} + x^{19} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{11} + x^{10} + x^7 + x^4\\
&\quad + x^2.
\end{aligned}$$

The polynomial $Q(x,y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and $R(x,y) := P(x,y)/Q(x,y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 216, and substitute it for y in $R(z,y)$, we get a series with valuation smaller than 216, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b) = f(1/z)$. $\square$

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e,0,0)$:

$$\begin{aligned}
p_0(x) &= x^{114} + x^{112} + x^{106} + x^{98} + x^{96} + x^{94} + x^{92} + x^{86} + x^{82} + x^{80} + x^{78}\\
&\quad + x^{76} + x^{74} + x^{72} + x^{70} + x^{64} + x^{56} + x^{54} + x^{52} + x^{44} + x^{42} + x^{34}\\
&\quad + x^{32} + x^{30} + x^{26} + x^{24} + x^{22} + x^{20} + x^{12},\\
p_3(x) &= x^{106} + x^{104} + x^{102} + x^{100} + x^{94} + x^{90} + x^{88} + x^{80} + x^{76} + x^{68} + x^{66}\\
&\quad + x^{62} + x^{56} + x^{54} + x^{50} + x^{44} + x^{40} + x^{34} + x^{30} + x^{22} + x^{20} + x^{16}\\
&\quad + x^{14} + x^{12} + x^{10} + x^8,\\
p_6(x) &= x^{106} + x^{104} + x^{98} + x^{96} + x^{92} + x^{90} + x^{86} + x^{80} + x^{74} + x^{70} + x^{62}\\
&\quad + x^{36} + x^{32} + x^{30} + x^{26} + x^{16} + x^{14} + x^{10} + x^8 + x^6 + x^2 + 1,\\
p_9(x) &= x^{104} + x^{98} + x^{96} + x^{92} + x^{90} + x^{88} + x^{86} + x^{80} + x^{78} + x^{74} + x^{70}
\end{aligned}$$

$$+ x^{68} + x^{52} + x^{50} + x^{48} + x^{44} + x^{40} + x^{32} + x^{30} + x^{28} + x^{26} + x^{24} + x^{22} + x^{20} + x^{14} + x^{10} + x^{4} + x^{2},$$

$$p_{12}(x) = x^{104} + x^{100} + x^{96} + x^{88} + x^{84} + x^{80} + x^{72} + x^{68} + x^{64} + x^{56} + x^{52} + x^{48} + x^{40} + x^{36} + x^{32} + x^{24} + x^{20} + x^{16} + x^{8} + x^{4} + 1,$$

and the initial terms are [1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0].

For $\phi(M^e, 0, 1)$:

$$p_0(x) = x^{129} + x^{127} + x^{126} + x^{124} + x^{122} + x^{116} + x^{113} + x^{112} + x^{110} + x^{109} + x^{104} + x^{102} + x^{101} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{90} + x^{85} + x^{82} + x^{79} + x^{78} + x^{76} + x^{74} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{62} + x^{61} + x^{60} + x^{59} + x^{57} + x^{52} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{35} + x^{32} + x^{27},$$

$$p_3(x) = x^{122} + x^{121} + x^{120} + x^{115} + x^{114} + x^{113} + x^{112} + x^{110} + x^{109} + x^{105} + x^{104} + x^{103} + x^{102} + x^{98} + x^{96} + x^{91} + x^{86} + x^{85} + x^{83} + x^{82} + x^{79} + x^{77} + x^{72} + x^{71} + x^{70} + x^{66} + x^{64} + x^{59} + x^{58} + x^{57} + x^{56} + x^{54} + x^{53} + x^{49} + x^{48} + x^{47} + x^{46} + x^{42} + x^{40} + x^{35} + x^{30} + x^{29} + x^{27} + x^{26} + x^{23} + x^{21} + x^{16} + x^{15} + x^{14} + x^{9},$$

$$p_6(x) = x^{120} + x^{116} + x^{112} + x^{106} + x^{104} + x^{102} + x^{96} + x^{92} + x^{88} + x^{82} + x^{78} + x^{66} + x^{64} + x^{58} + x^{54} + x^{52} + x^{44} + x^{42} + x^{38} + x^{34} + x^{30} + x^{28} + x^{24} + x^{20} + x^{18} + x^{16} + x^{14} + x^{12} + x^{8} + x^{6},$$

$$p_9(x) = x^{118} + x^{117} + x^{115} + x^{114} + x^{112} + x^{108} + x^{105} + x^{103} + x^{102} + x^{101} + x^{99} + x^{97} + x^{90} + x^{89} + x^{87} + x^{86} + x^{84} + x^{82} + x^{81} + x^{78} + x^{76} + x^{74} + x^{73} + x^{69} + x^{66} + x^{65} + x^{64} + x^{60} + x^{58} + x^{56} + x^{55} + x^{54} + x^{53} + x^{49} + x^{44} + x^{43} + x^{41} + x^{39} + x^{38} + x^{37} + x^{36} + x^{34} + x^{33} + x^{29} + x^{26} + x^{25} + x^{23} + x^{21} + x^{18} + x^{17} + x^{15} + x^{14} + x^{12} + x^{10} + x^{9} + x^{8} + x^{7} + x^{3},$$

$$p_{12}(x) = x^{116} + x^{114} + x^{112} + x^{108} + x^{106} + x^{104} + x^{92} + x^{90} + x^{88} + x^{76} + x^{74} + x^{72} + x^{60} + x^{58} + x^{56} + x^{44} + x^{42} + x^{40} + x^{28} + x^{26} + x^{24} + x^{12} + x^{10} + x^{8} + x^{4} + x^{2} + 1,$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 1, 1, 0].

For $\phi(M^e, 1, 0)$:

$$p_0(x) = x^{129} + x^{127} + x^{126} + x^{124} + x^{122} + x^{116} + x^{113} + x^{112} + x^{110} + x^{109} + x^{104} + x^{102} + x^{101} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{90} + x^{85} + x^{82} + x^{79} + x^{78} + x^{76} + x^{74} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{62} + x^{61} + x^{60} + x^{59} + x^{57} + x^{52} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{35} + x^{32} + x^{27},$$

$$
\begin{aligned}
p_3(x) &= x^{122} + x^{121} + x^{120} + x^{115} + x^{114} + x^{113} + x^{112} + x^{110} + x^{109} + x^{105} \\
&\quad + x^{104} + x^{103} + x^{102} + x^{98} + x^{96} + x^{91} + x^{86} + x^{85} + x^{83} + x^{82} + x^{79} \\
&\quad + x^{77} + x^{72} + x^{71} + x^{70} + x^{66} + x^{64} + x^{59} + x^{58} + x^{57} + x^{56} + x^{54} \\
&\quad + x^{53} + x^{49} + x^{48} + x^{47} + x^{46} + x^{42} + x^{40} + x^{35} + x^{30} + x^{29} + x^{27} \\
&\quad + x^{26} + x^{23} + x^{21} + x^{16} + x^{15} + x^{14} + x^{9}, \\
p_6(x) &= x^{120} + x^{116} + x^{112} + x^{106} + x^{104} + x^{102} + x^{96} + x^{92} + x^{88} + x^{82} \\
&\quad + x^{78} + x^{66} + x^{64} + x^{58} + x^{54} + x^{52} + x^{44} + x^{42} + x^{38} + x^{34} + x^{30} \\
&\quad + x^{28} + x^{24} + x^{20} + x^{18} + x^{16} + x^{14} + x^{12} + x^{8} + x^{6}, \\
p_9(x) &= x^{118} + x^{117} + x^{115} + x^{114} + x^{112} + x^{108} + x^{105} + x^{103} + x^{102} + x^{101} \\
&\quad + x^{99} + x^{97} + x^{90} + x^{89} + x^{87} + x^{86} + x^{84} + x^{82} + x^{81} + x^{78} + x^{76} \\
&\quad + x^{74} + x^{73} + x^{69} + x^{66} + x^{65} + x^{64} + x^{60} + x^{58} + x^{56} + x^{55} + x^{54} \\
&\quad + x^{53} + x^{49} + x^{44} + x^{43} + x^{41} + x^{39} + x^{38} + x^{37} + x^{36} + x^{34} + x^{33} \\
&\quad + x^{29} + x^{26} + x^{25} + x^{23} + x^{21} + x^{18} + x^{17} + x^{15} + x^{14} + x^{12} + x^{10} \\
&\quad + x^{9} + x^{8} + x^{7} + x^{3}, \\
p_{12}(x) &= x^{116} + x^{114} + x^{112} + x^{108} + x^{106} + x^{104} + x^{92} + x^{90} + x^{88} + x^{76} \\
&\quad + x^{74} + x^{72} + x^{60} + x^{58} + x^{56} + x^{44} + x^{42} + x^{40} + x^{28} + x^{26} + x^{24} \\
&\quad + x^{12} + x^{10} + x^{8} + x^{4} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 1, 1, 0].

For $\phi(M^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{144} + x^{142} + x^{138} + x^{128} + x^{126} + x^{124} + x^{120} + x^{116} + x^{106} + x^{102} \\
&\quad + x^{100} + x^{98} + x^{96} + x^{88} + x^{84} + x^{82} + x^{78} + x^{76} + x^{66} + x^{64} + x^{62} \\
&\quad + x^{56} + x^{54} + x^{50} + x^{42}, \\
p_3(x) &= x^{132} + x^{124} + x^{118} + x^{112} + x^{110} + x^{108} + x^{102} + x^{100} + x^{94} + x^{92} \\
&\quad + x^{84} + x^{80} + x^{68} + x^{62} + x^{52} + x^{48} + x^{44} + x^{40} + x^{36} + x^{32} + x^{30} \\
&\quad + x^{28} + x^{22} + x^{20} + x^{16} + x^{14} + x^{8} + x^{6}, \\
p_6(x) &= x^{132} + x^{128} + x^{126} + x^{120} + x^{118} + x^{116} + x^{112} + x^{110} + x^{106} + x^{104} \\
&\quad + x^{102} + x^{92} + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{80} + x^{78} + x^{70} + x^{64} \\
&\quad + x^{54} + x^{52} + x^{48} + x^{46} + x^{44} + x^{42} + x^{32} + x^{30} + x^{24} + x^{22} + x^{18} \\
&\quad + x^{16} + x^{12} + x^{10} + x^{6} + x^{2} + 1, \\
p_9(x) &= x^{128} + x^{124} + x^{122} + x^{120} + x^{118} + x^{116} + x^{114} + x^{110} + x^{108} + x^{106} \\
&\quad + x^{104} + x^{102} + x^{100} + x^{94} + x^{84} + x^{76} + x^{74} + x^{72} + x^{68} + x^{64} + x^{62} \\
&\quad + x^{56} + x^{52} + x^{50} + x^{44} + x^{40} + x^{32} + x^{30} + x^{28} + x^{24} + x^{22} + x^{20} \\
&\quad + x^{18} + x^{16} + x^{14} + x^{10} + x^{8} + x^{6} + x^{4} + x^{2}, \\
p_{12}(x) &= x^{128} + x^{96} + x^{80} + x^{64} + x^{48} + x^{32} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0].

For $\phi(W^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{144} + x^{142} + x^{138} + x^{136} + x^{128} + x^{120} + x^{116} + x^{114} + x^{112} + x^{110} \\
&\quad + x^{106} + x^{104} + x^{100} + x^{98} + x^{96} + x^{92} + x^{90} + x^{88} + x^{84} + x^{76} + x^{68} \\
&\quad + x^{64} + x^{62} + x^{60} + x^{56} + x^{54} + x^{52} + x^{50} + x^{48} + x^{46} + x^{44} + x^{40} \\
&\quad + x^{38} + x^{36} + x^{34} + x^{30} + x^{28} + x^{22} + x^{12}, \\
p_3(x) &= x^{132} + x^{122} + x^{116} + x^{112} + x^{100} + x^{98} + x^{96} + x^{92} + x^{90} + x^{88} + x^{80} \\
&\quad + x^{74} + x^{68} + x^{64} + x^{58} + x^{56} + x^{52} + x^{50} + x^{40} + x^{32} + x^{28} + x^{26} \\
&\quad + x^{24} + x^{8}, \\
p_6(x) &= x^{132} + x^{128} + x^{126} + x^{124} + x^{122} + x^{116} + x^{112} + x^{106} + x^{104} + x^{100} \\
&\quad + x^{98} + x^{94} + x^{88} + x^{86} + x^{82} + x^{80} + x^{78} + x^{74} + x^{72} + x^{70} + x^{62} \\
&\quad + x^{60} + x^{58} + x^{54} + x^{52} + x^{50} + x^{48} + x^{46} + x^{42} + x^{36} + x^{32} + x^{28} \\
&\quad + x^{26} + x^{24} + x^{18} + x^{14} + x^{10} + x^{4} + x^{2} + 1, \\
p_9(x) &= x^{128} + x^{124} + x^{122} + x^{120} + x^{118} + x^{116} + x^{114} + x^{110} + x^{108} + x^{106} \\
&\quad + x^{104} + x^{102} + x^{100} + x^{94} + x^{84} + x^{76} + x^{74} + x^{72} + x^{68} + x^{64} + x^{62} \\
&\quad + x^{56} + x^{52} + x^{50} + x^{44} + x^{40} + x^{32} + x^{30} + x^{28} + x^{24} + x^{22} + x^{20} \\
&\quad + x^{18} + x^{16} + x^{14} + x^{10} + x^{8} + x^{6} + x^{4} + x^{2}, \\
p_{12}(x) &= x^{128} + x^{96} + x^{80} + x^{64} + x^{48} + x^{32} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0].

For $\phi(W^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{129} + x^{127} + x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{114} + x^{113} \\
&\quad + x^{112} + x^{110} + x^{108} + x^{107} + x^{105} + x^{104} + x^{103} + x^{101} + x^{99} + x^{98} \\
&\quad + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{87} + x^{85} \\
&\quad + x^{83} + x^{81} + x^{80} + x^{79} + x^{78} + x^{75} + x^{73} + x^{72} + x^{69} + x^{68} + x^{64} \\
&\quad + x^{62} + x^{61} + x^{59} + x^{58} + x^{55} + x^{53} + x^{51} + x^{49} + x^{48} + x^{47} + x^{44} \\
&\quad + x^{43} + x^{41} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} \\
&\quad + x^{29} + x^{27} + x^{26} + x^{25} + x^{24} + x^{22} + x^{21} + x^{20} + x^{18}, \\
p_3(x) &= x^{122} + x^{121} + x^{120} + x^{115} + x^{114} + x^{113} + x^{112} + x^{110} + x^{109} + x^{105} \\
&\quad + x^{104} + x^{103} + x^{102} + x^{98} + x^{96} + x^{91} + x^{86} + x^{85} + x^{83} + x^{82} + x^{79} \\
&\quad + x^{77} + x^{72} + x^{71} + x^{70} + x^{66} + x^{64} + x^{59} + x^{58} + x^{57} + x^{56} + x^{54} \\
&\quad + x^{53} + x^{49} + x^{48} + x^{47} + x^{46} + x^{42} + x^{40} + x^{35} + x^{30} + x^{29} + x^{27} \\
&\quad + x^{26} + x^{23} + x^{21} + x^{16} + x^{15} + x^{14} + x^{9}, \\
p_6(x) &= x^{120} + x^{116} + x^{112} + x^{106} + x^{104} + x^{102} + x^{96} + x^{92} + x^{88} + x^{82} \\
&\quad + x^{78} + x^{66} + x^{64} + x^{58} + x^{54} + x^{52} + x^{44} + x^{42} + x^{38} + x^{34} + x^{30} \\
&\quad + x^{28} + x^{24} + x^{20} + x^{18} + x^{16} + x^{14} + x^{12} + x^{8} + x^{6}, \\
p_9(x) &= x^{118} + x^{117} + x^{115} + x^{114} + x^{112} + x^{108} + x^{105} + x^{103} + x^{102} + x^{101}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{99} + x^{97} + x^{90} + x^{89} + x^{87} + x^{86} + x^{84} + x^{82} + x^{81} + x^{78} + x^{76}\\
&+ x^{74} + x^{73} + x^{69} + x^{66} + x^{65} + x^{64} + x^{60} + x^{58} + x^{56} + x^{55} + x^{54}\\
&+ x^{53} + x^{49} + x^{44} + x^{43} + x^{41} + x^{39} + x^{38} + x^{37} + x^{36} + x^{34} + x^{33}\\
&+ x^{29} + x^{26} + x^{25} + x^{23} + x^{21} + x^{18} + x^{17} + x^{15} + x^{14} + x^{12} + x^{10}\\
&+ x^{9} + x^{8} + x^{7} + x^{3},\\
p_{12}(x) &= x^{116} + x^{114} + x^{112} + x^{108} + x^{106} + x^{104} + x^{92} + x^{90} + x^{88} + x^{76}\\
&+ x^{74} + x^{72} + x^{60} + x^{58} + x^{56} + x^{44} + x^{42} + x^{40} + x^{28} + x^{26} + x^{24}\\
&+ x^{12} + x^{10} + x^{8} + x^{4} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 1, 0, 1, 1, 0, 0].

For $\phi(W^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{129} + x^{127} + x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{114} + x^{113}\\
&+ x^{112} + x^{110} + x^{108} + x^{107} + x^{105} + x^{104} + x^{103} + x^{101} + x^{99} + x^{98}\\
&+ x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{87} + x^{85}\\
&+ x^{83} + x^{81} + x^{80} + x^{79} + x^{78} + x^{75} + x^{73} + x^{72} + x^{69} + x^{68} + x^{64}\\
&+ x^{62} + x^{61} + x^{59} + x^{58} + x^{55} + x^{53} + x^{51} + x^{49} + x^{48} + x^{47} + x^{44}\\
&+ x^{43} + x^{41} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31}\\
&+ x^{29} + x^{27} + x^{26} + x^{25} + x^{24} + x^{22} + x^{21} + x^{20} + x^{18},\\
p_3(x) &= x^{122} + x^{121} + x^{120} + x^{115} + x^{114} + x^{113} + x^{112} + x^{110} + x^{109} + x^{105}\\
&+ x^{104} + x^{103} + x^{102} + x^{98} + x^{96} + x^{91} + x^{86} + x^{85} + x^{83} + x^{82} + x^{79}\\
&+ x^{77} + x^{72} + x^{71} + x^{70} + x^{66} + x^{64} + x^{59} + x^{58} + x^{57} + x^{56} + x^{54}\\
&+ x^{53} + x^{49} + x^{48} + x^{47} + x^{46} + x^{42} + x^{40} + x^{35} + x^{30} + x^{29} + x^{27}\\
&+ x^{26} + x^{23} + x^{21} + x^{16} + x^{15} + x^{14} + x^{9},\\
p_6(x) &= x^{120} + x^{116} + x^{112} + x^{106} + x^{104} + x^{102} + x^{96} + x^{92} + x^{88} + x^{82}\\
&+ x^{78} + x^{66} + x^{64} + x^{58} + x^{54} + x^{52} + x^{44} + x^{42} + x^{38} + x^{34} + x^{30}\\
&+ x^{28} + x^{24} + x^{20} + x^{18} + x^{16} + x^{14} + x^{12} + x^{8} + x^{6},\\
p_9(x) &= x^{118} + x^{117} + x^{115} + x^{114} + x^{112} + x^{108} + x^{105} + x^{103} + x^{102} + x^{101}\\
&+ x^{99} + x^{97} + x^{90} + x^{89} + x^{87} + x^{86} + x^{84} + x^{82} + x^{81} + x^{78} + x^{76}\\
&+ x^{74} + x^{73} + x^{69} + x^{66} + x^{65} + x^{64} + x^{60} + x^{58} + x^{56} + x^{55} + x^{54}\\
&+ x^{53} + x^{49} + x^{44} + x^{43} + x^{41} + x^{39} + x^{38} + x^{37} + x^{36} + x^{34} + x^{33}\\
&+ x^{29} + x^{26} + x^{25} + x^{23} + x^{21} + x^{18} + x^{17} + x^{15} + x^{14} + x^{12} + x^{10}\\
&+ x^{9} + x^{8} + x^{7} + x^{3},\\
p_{12}(x) &= x^{116} + x^{114} + x^{112} + x^{108} + x^{106} + x^{104} + x^{92} + x^{90} + x^{88} + x^{76}\\
&+ x^{74} + x^{72} + x^{60} + x^{58} + x^{56} + x^{44} + x^{42} + x^{40} + x^{28} + x^{26} + x^{24}\\
&+ x^{12} + x^{10} + x^{8} + x^{4} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 1, 0, 1, 1, 0, 0].

For $\phi(W^e,1,1)$:

$$\begin{aligned}
p_0(x) &= x^{114}+x^{112}+x^{108}+x^{106}+x^{104}+x^{102}+x^{92}+x^{90}+x^{88}+x^{74}\\
&\quad +x^{72}+x^{68}+x^{66}+x^{64}+x^{62}+x^{60}+x^{58}+x^{54}+x^{52}+x^{48}+x^{46}\\
&\quad +x^{42}+x^{40}+x^{36}+x^{34}+x^{32}+x^{30}+x^{28}+x^{24},\\
p_3(x) &= x^{106}+x^{104}+x^{102}+x^{98}+x^{96}+x^{94}+x^{90}+x^{84}+x^{80}+x^{70}+x^{66}\\
&\quad +x^{64}+x^{62}+x^{60}+x^{58}+x^{52}+x^{48}+x^{46}+x^{44}+x^{40}+x^{38}+x^{36}\\
&\quad +x^{26}+x^{24}+x^{20}+x^{18}+x^{14}+x^{12}+x^{8}+x^{6},\\
p_6(x) &= x^{106}+x^{104}+x^{100}+x^{96}+x^{92}+x^{90}+x^{88}+x^{86}+x^{84}+x^{80}+x^{76}\\
&\quad +x^{74}+x^{64}+x^{62}+x^{60}+x^{58}+x^{54}+x^{50}+x^{46}+x^{44}+x^{38}+x^{36}\\
&\quad +x^{34}+x^{28}+x^{22}+x^{20}+x^{18}+x^{16}+x^{14}+x^{4}+x^{2}+1,\\
p_9(x) &= x^{104}+x^{98}+x^{96}+x^{92}+x^{90}+x^{88}+x^{86}+x^{80}+x^{78}+x^{74}+x^{70}\\
&\quad +x^{68}+x^{52}+x^{50}+x^{48}+x^{44}+x^{40}+x^{32}+x^{30}+x^{28}+x^{26}+x^{24}\\
&\quad +x^{22}+x^{20}+x^{14}+x^{10}+x^{4}+x^{2},\\
p_{12}(x) &= x^{104}+x^{100}+x^{96}+x^{88}+x^{84}+x^{80}+x^{72}+x^{68}+x^{64}+x^{56}+x^{52}\\
&\quad +x^{48}+x^{40}+x^{36}+x^{32}+x^{24}+x^{20}+x^{16}+x^{8}+x^{4}+1,
\end{aligned}$$

and the initial terms are [0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0].

For $\phi(M^o,0,0)$:

$$\begin{aligned}
p_0(x) &= x^{129}+x^{127}+x^{126}+x^{125}+x^{122}+x^{121}+x^{120}+x^{118}+x^{117}+x^{116}\\
&\quad +x^{113}+x^{108}+x^{106}+x^{105}+x^{104}+x^{103}+x^{102}+x^{99}+x^{98}+x^{95}\\
&\quad +x^{94}+x^{91}+x^{89}+x^{88}+x^{85}+x^{84}+x^{83}+x^{73}+x^{72}+x^{70}+x^{68}\\
&\quad +x^{67}+x^{66}+x^{61}+x^{58}+x^{56}+x^{52}+x^{51}+x^{50}+x^{49}+x^{48}+x^{47}\\
&\quad +x^{46}+x^{44}+x^{43}+x^{41}+x^{39}+x^{37}+x^{36}+x^{30}+x^{29}+x^{27}+x^{25}\\
&\quad +x^{24}+x^{23}+x^{21}+x^{20}+x^{19}+x^{18}+x^{16}+x^{12},\\
p_3(x) &= x^{124}+x^{123}+x^{122}+x^{121}+x^{120}+x^{118}+x^{116}+x^{115}+x^{113}+x^{112}\\
&\quad +x^{111}+x^{105}+x^{103}+x^{99}+x^{98}+x^{97}+x^{94}+x^{88}+x^{87}+x^{86}+x^{83}\\
&\quad +x^{78}+x^{76}+x^{74}+x^{73}+x^{71}+x^{68}+x^{67}+x^{66}+x^{65}+x^{64}+x^{62}\\
&\quad +x^{59}+x^{57}+x^{55}+x^{52}+x^{49}+x^{48}+x^{47}+x^{44}+x^{43}+x^{42}+x^{41}\\
&\quad +x^{40}+x^{38}+x^{36}+x^{31}+x^{30}+x^{28}+x^{27}+x^{24}+x^{22}+x^{18}+x^{17}\\
&\quad +x^{16}+x^{15}+x^{12}+x^{8},\\
p_6(x) &= x^{124}+x^{123}+x^{122}+x^{121}+x^{120}+x^{118}+x^{116}+x^{109}+x^{108}+x^{107}\\
&\quad +x^{102}+x^{100}+x^{99}+x^{98}+x^{97}+x^{92}+x^{85}+x^{84}+x^{81}+x^{80}+x^{79}\\
&\quad +x^{77}+x^{76}+x^{71}+x^{64}+x^{61}+x^{60}+x^{58}+x^{57}+x^{54}+x^{52}+x^{51}\\
&\quad +x^{49}+x^{48}+x^{45}+x^{44}+x^{43}+x^{42}+x^{40}+x^{38}+x^{37}+x^{36}+x^{33}\\
&\quad +x^{32}+x^{30}+x^{28}+x^{23}+x^{22}+x^{20}+x^{17}+x^{16}+x^{15}+x^{14}+x^{13}\\
&\quad +x^{12}+x^{11}+x^{10}+x^{8}+x^{2}+1,
\end{aligned}$$

$$\begin{aligned}
p_9(x) &= x^{124}+x^{122}+x^{118}+x^{116}+x^{112}+x^{110}+x^{106}+x^{104}+x^{96}+x^{94}\\
&\quad +x^{90}+x^{88}+x^{80}+x^{78}+x^{74}+x^{72}+x^{64}+x^{62}+x^{58}+x^{56}+x^{48}\\
&\quad +x^{46}+x^{42}+x^{40}+x^{32}+x^{30}+x^{26}+x^{24}+x^{16}+x^{14}+x^{12}+x^{8}\\
&\quad +x^{6}+x^{4},\\
p_{12}(x) &= x^{124}+x^{122}+x^{118}+x^{114}+x^{110}+x^{108}+x^{96}+x^{94}+x^{92}+x^{80}\\
&\quad +x^{78}+x^{76}+x^{64}+x^{62}+x^{60}+x^{48}+x^{46}+x^{44}+x^{32}+x^{30}+x^{28}\\
&\quad +x^{16}+x^{14}+x^{10}+x^{6}+x^{2}+1,
\end{aligned}$$

and the initial terms are [1, 0, 1, 0, 0, 0, 0, 1, 1, 0, 1, 1, 1, 1, 1, 1].
For $\phi(M^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{144}+x^{142}+x^{140}+x^{139}+x^{138}+x^{137}+x^{135}+x^{133}+x^{130}+x^{129}\\
&\quad +x^{127}+x^{125}+x^{123}+x^{120}+x^{117}+x^{114}+x^{113}+x^{112}+x^{110}\\
&\quad +x^{109}+x^{108}+x^{107}+x^{105}+x^{103}+x^{102}+x^{101}+x^{97}+x^{96}+x^{93}\\
&\quad +x^{91}+x^{90}+x^{87}+x^{85}+x^{83}+x^{82}+x^{81}+x^{78}+x^{77}+x^{76}+x^{75}\\
&\quad +x^{73}+x^{72}+x^{70}+x^{67}+x^{65}+x^{64}+x^{63}+x^{62}+x^{61}+x^{60}+x^{59}\\
&\quad +x^{58}+x^{57}+x^{55}+x^{53}+x^{51}+x^{47}+x^{46}+x^{44}+x^{38}+x^{37}+x^{35}\\
&\quad +x^{34}+x^{33}+x^{32}+x^{31}+x^{27},\\
p_3(x) &= x^{130}+x^{129}+x^{128}+x^{124}+x^{122}+x^{118}+x^{116}+x^{114}+x^{113}+x^{110}\\
&\quad +x^{108}+x^{106}+x^{105}+x^{102}+x^{100}+x^{96}+x^{94}+x^{89}+x^{82}+x^{81}\\
&\quad +x^{80}+x^{76}+x^{74}+x^{70}+x^{68}+x^{66}+x^{65}+x^{62}+x^{60}+x^{58}+x^{57}\\
&\quad +x^{54}+x^{52}+x^{50}+x^{49}+x^{46}+x^{44}+x^{40}+x^{38}+x^{33}+x^{26}+x^{25}\\
&\quad +x^{24}+x^{20}+x^{17}+x^{16}+x^{14}+x^{9},\\
p_6(x) &= x^{132}+x^{130}+x^{128}+x^{124}+x^{122}+x^{120}+x^{118}+x^{116}+x^{114}+x^{112}\\
&\quad +x^{110}+x^{108}+x^{106}+x^{104}+x^{102}+x^{100}+x^{98}+x^{96}+x^{94}+x^{88}\\
&\quad +x^{82}+x^{80}+x^{74}+x^{72}+x^{66}+x^{62}+x^{36}+x^{34}+x^{30}+x^{28}+x^{22}\\
&\quad +x^{18}+x^{16}+x^{14}+x^{10}+x^{6},\\
p_9(x) &= x^{134}+x^{133}+x^{131}+x^{129}+x^{122}+x^{121}+x^{120}+x^{118}+x^{117}+x^{114}\\
&\quad +x^{112}+x^{110}+x^{109}+x^{105}+x^{102}+x^{101}+x^{99}+x^{98}+x^{96}+x^{94}\\
&\quad +x^{93}+x^{89}+x^{86}+x^{85}+x^{83}+x^{82}+x^{80}+x^{78}+x^{77}+x^{73}+x^{70}\\
&\quad +x^{69}+x^{67}+x^{66}+x^{64}+x^{62}+x^{61}+x^{57}+x^{54}+x^{53}+x^{51}+x^{50}\\
&\quad +x^{48}+x^{46}+x^{45}+x^{41}+x^{38}+x^{37}+x^{35}+x^{34}+x^{32}+x^{30}+x^{29}\\
&\quad +x^{25}+x^{18}+x^{17}+x^{16}+x^{14}+x^{13}+x^{10}+x^{8}+x^{3},\\
p_{12}(x) &= x^{136}+x^{132}+x^{112}+x^{100}+x^{84}+x^{68}+x^{52}+x^{36}+x^{24}+x^{4}+1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 1].
For $\phi(M^o, 1, 0)$:

$$p_0(x) = x^{122}+x^{120}+x^{119}+x^{117}+x^{116}+x^{115}+x^{111}+x^{109}+x^{108}+x^{107}$$

$$
\begin{aligned}
&+ x^{106} + x^{105} + x^{102} + x^{100} + x^{92} + x^{91} + x^{89} + x^{87} + x^{86} + x^{84}\\
&+ x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{76} + x^{75} + x^{73} + x^{69} + x^{68} + x^{65}\\
&+ x^{64} + x^{62} + x^{61} + x^{60} + x^{56} + x^{53} + x^{52} + x^{51} + x^{50} + x^{48} + x^{47}\\
&+ x^{45} + x^{40} + x^{39} + x^{36} + x^{33} + x^{28} + x^{26} + x^{22} + x^{21} + x^{20} + x^{18},\\
p_3(x) &= x^{114} + x^{113} + x^{112} + x^{108} + x^{105} + x^{104} + x^{102} + x^{97} + x^{90} + x^{89}\\
&+ x^{88} + x^{84} + x^{82} + x^{78} + x^{76} + x^{72} + x^{70} + x^{65} + x^{58} + x^{57} + x^{56}\\
&+ x^{52} + x^{49} + x^{48} + x^{46} + x^{41} + x^{34} + x^{33} + x^{32} + x^{28} + x^{26} + x^{22}\\
&+ x^{20} + x^{16} + x^{14} + x^{9},\\
p_6(x) &= x^{116} + x^{114} + x^{112} + x^{102} + x^{92} + x^{90} + x^{88} + x^{78} + x^{76} + x^{74} + x^{70}\\
&+ x^{68} + x^{64} + x^{54} + x^{52} + x^{50} + x^{48} + x^{44} + x^{42} + x^{40} + x^{38} + x^{30}\\
&+ x^{28} + x^{26} + x^{24} + x^{16} + x^{10} + x^{6},\\
p_9(x) &= x^{118} + x^{117} + x^{115} + x^{113} + x^{110} + x^{109} + x^{107} + x^{106} + x^{104} + x^{102}\\
&+ x^{101} + x^{97} + x^{94} + x^{93} + x^{91} + x^{90} + x^{88} + x^{86} + x^{85} + x^{81} + x^{78}\\
&+ x^{77} + x^{75} + x^{74} + x^{72} + x^{70} + x^{69} + x^{65} + x^{62} + x^{61} + x^{59} + x^{58}\\
&+ x^{56} + x^{54} + x^{53} + x^{49} + x^{46} + x^{45} + x^{43} + x^{42} + x^{40} + x^{38} + x^{37}\\
&+ x^{33} + x^{30} + x^{29} + x^{27} + x^{26} + x^{24} + x^{22} + x^{21} + x^{17} + x^{14} + x^{13}\\
&+ x^{11} + x^{10} + x^{8} + x^{3},\\
p_{12}(x) &= x^{120} + x^{116} + x^{112} + x^{108} + x^{92} + x^{76} + x^{60} + x^{44} + x^{28} + x^{12} + x^{8}\\
&+ x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1].

For $\phi(M^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{129} + x^{127} + x^{125} + x^{123} + x^{120} + x^{118} + x^{117} + x^{114} + x^{113} + x^{110}\\
&+ x^{109} + x^{106} + x^{103} + x^{100} + x^{99} + x^{98} + x^{96} + x^{95} + x^{93} + x^{92}\\
&+ x^{91} + x^{90} + x^{86} + x^{85} + x^{84} + x^{81} + x^{79} + x^{76} + x^{75} + x^{74} + x^{73}\\
&+ x^{71} + x^{68} + x^{66} + x^{65} + x^{64} + x^{63} + x^{61} + x^{57} + x^{56} + x^{54} + x^{53}\\
&+ x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{41} + x^{39} + x^{38} + x^{36} + x^{35} + x^{33},\\
p_3(x) &= x^{124} + x^{123} + x^{122} + x^{121} + x^{118} + x^{116} + x^{115} + x^{114} + x^{113} + x^{111}\\
&+ x^{108} + x^{105} + x^{104} + x^{103} + x^{102} + x^{99} + x^{98} + x^{97} + x^{94} + x^{90}\\
&+ x^{88} + x^{87} + x^{86} + x^{84} + x^{83} + x^{82} + x^{74} + x^{73} + x^{71} + x^{70} + x^{68}\\
&+ x^{67} + x^{66} + x^{65} + x^{64} + x^{62} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{49}\\
&+ x^{47} + x^{46} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{38} + x^{36} + x^{34} + x^{31}\\
&+ x^{30} + x^{27} + x^{26} + x^{24} + x^{20} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{12},\\
p_6(x) &= x^{124} + x^{123} + x^{122} + x^{121} + x^{118} + x^{116} + x^{114} + x^{109} + x^{108} + x^{107}\\
&+ x^{106} + x^{102} + x^{99} + x^{98} + x^{97} + x^{96} + x^{92} + x^{90} + x^{85} + x^{84} + x^{82}\\
&+ x^{81} + x^{79} + x^{77} + x^{76} + x^{74} + x^{71} + x^{70} + x^{68} + x^{66} + x^{61} + x^{60}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{57} + x^{56} + x^{54} + x^{51} + x^{50} + x^{49} + x^{45} + x^{44} + x^{43} + x^{42} + x^{40}\\
&+ x^{38} + x^{37} + x^{34} + x^{33} + x^{30} + x^{26} + x^{23} + x^{22} + x^{18} + x^{17} + x^{15}\\
&+ x^{13} + x^{12} + x^{11} + x^{4} + x^{2} + 1,\\
p_9(x) = {}& x^{124} + x^{122} + x^{118} + x^{116} + x^{112} + x^{110} + x^{106} + x^{104} + x^{96} + x^{94}\\
&+ x^{90} + x^{88} + x^{80} + x^{78} + x^{74} + x^{72} + x^{64} + x^{62} + x^{58} + x^{56} + x^{48}\\
&+ x^{46} + x^{42} + x^{40} + x^{32} + x^{30} + x^{26} + x^{24} + x^{16} + x^{14} + x^{12} + x^{8}\\
&+ x^{6} + x^{4},\\
p_{12}(x) = {}& x^{124} + x^{122} + x^{118} + x^{114} + x^{110} + x^{108} + x^{96} + x^{94} + x^{92} + x^{80}\\
&+ x^{78} + x^{76} + x^{64} + x^{62} + x^{60} + x^{48} + x^{46} + x^{44} + x^{32} + x^{30} + x^{28}\\
&+ x^{16} + x^{14} + x^{10} + x^{6} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 1, 1].

For $\phi(W^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) = {}& x^{129} + x^{127} + x^{126} + x^{125} + x^{122} + x^{121} + x^{120} + x^{118} + x^{117} + x^{116}\\
&+ x^{113} + x^{108} + x^{106} + x^{105} + x^{104} + x^{103} + x^{102} + x^{99} + x^{98} + x^{95}\\
&+ x^{94} + x^{91} + x^{89} + x^{88} + x^{85} + x^{84} + x^{83} + x^{73} + x^{72} + x^{70} + x^{68}\\
&+ x^{67} + x^{66} + x^{61} + x^{58} + x^{56} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47}\\
&+ x^{46} + x^{44} + x^{43} + x^{41} + x^{39} + x^{37} + x^{36} + x^{30} + x^{29} + x^{27} + x^{25}\\
&+ x^{24} + x^{23} + x^{21} + x^{20} + x^{19} + x^{18} + x^{16} + x^{12},\\
p_3(x) = {}& x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{118} + x^{116} + x^{115} + x^{113} + x^{112}\\
&+ x^{111} + x^{105} + x^{103} + x^{99} + x^{98} + x^{97} + x^{94} + x^{88} + x^{87} + x^{86} + x^{83}\\
&+ x^{78} + x^{76} + x^{74} + x^{73} + x^{71} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{62}\\
&+ x^{59} + x^{57} + x^{55} + x^{52} + x^{49} + x^{48} + x^{47} + x^{44} + x^{43} + x^{42} + x^{41}\\
&+ x^{40} + x^{38} + x^{36} + x^{31} + x^{30} + x^{28} + x^{27} + x^{24} + x^{22} + x^{18} + x^{17}\\
&+ x^{16} + x^{15} + x^{12} + x^{8},\\
p_6(x) = {}& x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{118} + x^{116} + x^{109} + x^{108} + x^{107}\\
&+ x^{102} + x^{100} + x^{99} + x^{98} + x^{97} + x^{92} + x^{85} + x^{84} + x^{81} + x^{80} + x^{79}\\
&+ x^{77} + x^{76} + x^{71} + x^{64} + x^{61} + x^{60} + x^{58} + x^{57} + x^{54} + x^{52} + x^{51}\\
&+ x^{49} + x^{48} + x^{45} + x^{44} + x^{43} + x^{42} + x^{40} + x^{38} + x^{37} + x^{36} + x^{33}\\
&+ x^{32} + x^{30} + x^{28} + x^{23} + x^{22} + x^{20} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13}\\
&+ x^{12} + x^{11} + x^{10} + x^{8} + x^{2} + 1,\\
p_9(x) = {}& x^{124} + x^{122} + x^{118} + x^{116} + x^{112} + x^{110} + x^{106} + x^{104} + x^{96} + x^{94}\\
&+ x^{90} + x^{88} + x^{80} + x^{78} + x^{74} + x^{72} + x^{64} + x^{62} + x^{58} + x^{56} + x^{48}\\
&+ x^{46} + x^{42} + x^{40} + x^{32} + x^{30} + x^{26} + x^{24} + x^{16} + x^{14} + x^{12} + x^{8}\\
&+ x^{6} + x^{4},\\
p_{12}(x) = {}& x^{124} + x^{122} + x^{118} + x^{114} + x^{110} + x^{108} + x^{96} + x^{94} + x^{92} + x^{80}
\end{aligned}
$$

$$+ x^{78} + x^{76} + x^{64} + x^{62} + x^{60} + x^{48} + x^{46} + x^{44} + x^{32} + x^{30} + x^{28} + x^{16} + x^{14} + x^{10} + x^{6} + x^{2} + 1,$$

and the initial terms are [1, 0, 1, 0, 0, 0, 0, 1, 1, 0, 1, 1, 1, 1, 1, 1].

For $\phi(W^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{122} + x^{120} + x^{119} + x^{117} + x^{116} + x^{115} + x^{111} + x^{109} + x^{108} + x^{107} \\
&\quad + x^{106} + x^{105} + x^{102} + x^{100} + x^{92} + x^{91} + x^{89} + x^{87} + x^{86} + x^{84} \\
&\quad + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{76} + x^{75} + x^{73} + x^{69} + x^{68} + x^{65} \\
&\quad + x^{64} + x^{62} + x^{61} + x^{60} + x^{56} + x^{53} + x^{52} + x^{51} + x^{50} + x^{48} + x^{47} \\
&\quad + x^{45} + x^{40} + x^{39} + x^{36} + x^{33} + x^{28} + x^{26} + x^{22} + x^{21} + x^{20} + x^{18}, \\
p_3(x) &= x^{114} + x^{113} + x^{112} + x^{108} + x^{105} + x^{104} + x^{102} + x^{97} + x^{90} + x^{89} \\
&\quad + x^{88} + x^{84} + x^{82} + x^{78} + x^{76} + x^{72} + x^{70} + x^{65} + x^{58} + x^{57} + x^{56} \\
&\quad + x^{52} + x^{49} + x^{48} + x^{46} + x^{41} + x^{34} + x^{33} + x^{32} + x^{28} + x^{26} + x^{22} \\
&\quad + x^{20} + x^{16} + x^{14} + x^{9}, \\
p_6(x) &= x^{116} + x^{114} + x^{112} + x^{102} + x^{92} + x^{90} + x^{88} + x^{78} + x^{76} + x^{74} + x^{70} \\
&\quad + x^{68} + x^{64} + x^{54} + x^{52} + x^{50} + x^{48} + x^{44} + x^{42} + x^{40} + x^{38} + x^{30} \\
&\quad + x^{28} + x^{26} + x^{24} + x^{16} + x^{10} + x^{6}, \\
p_9(x) &= x^{118} + x^{117} + x^{115} + x^{113} + x^{110} + x^{109} + x^{107} + x^{106} + x^{104} + x^{102} \\
&\quad + x^{101} + x^{97} + x^{94} + x^{93} + x^{91} + x^{90} + x^{88} + x^{86} + x^{85} + x^{81} + x^{78} \\
&\quad + x^{77} + x^{75} + x^{74} + x^{72} + x^{70} + x^{69} + x^{65} + x^{62} + x^{61} + x^{59} + x^{58} \\
&\quad + x^{56} + x^{54} + x^{53} + x^{49} + x^{46} + x^{45} + x^{43} + x^{42} + x^{40} + x^{38} + x^{37} \\
&\quad + x^{33} + x^{30} + x^{29} + x^{27} + x^{26} + x^{24} + x^{22} + x^{21} + x^{17} + x^{14} + x^{13} \\
&\quad + x^{11} + x^{10} + x^{8} + x^{3}, \\
p_{12}(x) &= x^{120} + x^{116} + x^{112} + x^{108} + x^{92} + x^{76} + x^{60} + x^{44} + x^{28} + x^{12} + x^{8} \\
&\quad + x^{4} + 1,
\end{aligned}$$

and the initial terms are [0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1].

For $\phi(W^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{144} + x^{142} + x^{140} + x^{139} + x^{138} + x^{137} + x^{135} + x^{133} + x^{130} + x^{129} \\
&\quad + x^{127} + x^{125} + x^{123} + x^{120} + x^{117} + x^{114} + x^{113} + x^{112} + x^{110} \\
&\quad + x^{109} + x^{108} + x^{107} + x^{105} + x^{103} + x^{102} + x^{101} + x^{97} + x^{96} + x^{93} \\
&\quad + x^{91} + x^{90} + x^{87} + x^{85} + x^{83} + x^{82} + x^{81} + x^{78} + x^{77} + x^{76} + x^{75} \\
&\quad + x^{73} + x^{72} + x^{70} + x^{67} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} \\
&\quad + x^{58} + x^{57} + x^{55} + x^{53} + x^{51} + x^{47} + x^{46} + x^{44} + x^{38} + x^{37} + x^{35} \\
&\quad + x^{34} + x^{33} + x^{32} + x^{31} + x^{27}, \\
p_3(x) &= x^{130} + x^{129} + x^{128} + x^{124} + x^{122} + x^{118} + x^{116} + x^{114} + x^{113} + x^{110} \\
&\quad + x^{108} + x^{106} + x^{105} + x^{102} + x^{100} + x^{96} + x^{94} + x^{89} + x^{82} + x^{81}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{80} + x^{76} + x^{74} + x^{70} + x^{68} + x^{66} + x^{65} + x^{62} + x^{60} + x^{58} + x^{57}\\
&+ x^{54} + x^{52} + x^{50} + x^{49} + x^{46} + x^{44} + x^{40} + x^{38} + x^{33} + x^{26} + x^{25}\\
&+ x^{24} + x^{20} + x^{17} + x^{16} + x^{14} + x^{9},\\
p_6(x) = {}& x^{132} + x^{130} + x^{128} + x^{124} + x^{122} + x^{120} + x^{118} + x^{116} + x^{114} + x^{112}\\
&+ x^{110} + x^{108} + x^{106} + x^{104} + x^{102} + x^{100} + x^{98} + x^{96} + x^{94} + x^{88}\\
&+ x^{82} + x^{80} + x^{74} + x^{72} + x^{66} + x^{62} + x^{36} + x^{34} + x^{30} + x^{28} + x^{22}\\
&+ x^{18} + x^{16} + x^{14} + x^{10} + x^{6},\\
p_9(x) = {}& x^{134} + x^{133} + x^{131} + x^{129} + x^{122} + x^{121} + x^{120} + x^{118} + x^{117} + x^{114}\\
&+ x^{112} + x^{110} + x^{109} + x^{105} + x^{102} + x^{101} + x^{99} + x^{98} + x^{96} + x^{94}\\
&+ x^{93} + x^{89} + x^{86} + x^{85} + x^{83} + x^{82} + x^{80} + x^{78} + x^{77} + x^{73} + x^{70}\\
&+ x^{69} + x^{67} + x^{66} + x^{64} + x^{62} + x^{61} + x^{57} + x^{54} + x^{53} + x^{51} + x^{50}\\
&+ x^{48} + x^{46} + x^{45} + x^{41} + x^{38} + x^{37} + x^{35} + x^{34} + x^{32} + x^{30} + x^{29}\\
&+ x^{25} + x^{18} + x^{17} + x^{16} + x^{14} + x^{13} + x^{10} + x^{8} + x^{3},\\
p_{12}(x) = {}& x^{136} + x^{132} + x^{112} + x^{100} + x^{84} + x^{68} + x^{52} + x^{36} + x^{24} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 1].

For $\phi(W^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) = {}& x^{129} + x^{127} + x^{125} + x^{123} + x^{120} + x^{118} + x^{117} + x^{114} + x^{113} + x^{110}\\
&+ x^{109} + x^{106} + x^{103} + x^{100} + x^{99} + x^{98} + x^{96} + x^{95} + x^{93} + x^{92}\\
&+ x^{91} + x^{90} + x^{86} + x^{85} + x^{84} + x^{81} + x^{79} + x^{76} + x^{75} + x^{74} + x^{73}\\
&+ x^{71} + x^{68} + x^{66} + x^{65} + x^{64} + x^{63} + x^{61} + x^{57} + x^{56} + x^{54} + x^{53}\\
&+ x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{41} + x^{39} + x^{38} + x^{36} + x^{35} + x^{33},\\
p_3(x) = {}& x^{124} + x^{123} + x^{122} + x^{121} + x^{118} + x^{116} + x^{115} + x^{114} + x^{113} + x^{111}\\
&+ x^{108} + x^{105} + x^{104} + x^{103} + x^{102} + x^{99} + x^{98} + x^{97} + x^{94} + x^{90}\\
&+ x^{88} + x^{87} + x^{86} + x^{84} + x^{83} + x^{82} + x^{74} + x^{73} + x^{71} + x^{70} + x^{68}\\
&+ x^{67} + x^{66} + x^{65} + x^{64} + x^{62} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{49}\\
&+ x^{47} + x^{46} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{38} + x^{36} + x^{34} + x^{31}\\
&+ x^{30} + x^{27} + x^{26} + x^{24} + x^{20} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{12},\\
p_6(x) = {}& x^{124} + x^{123} + x^{122} + x^{121} + x^{118} + x^{116} + x^{114} + x^{109} + x^{108} + x^{107}\\
&+ x^{106} + x^{102} + x^{99} + x^{98} + x^{97} + x^{96} + x^{92} + x^{90} + x^{85} + x^{84} + x^{82}\\
&+ x^{81} + x^{79} + x^{77} + x^{76} + x^{74} + x^{71} + x^{70} + x^{68} + x^{66} + x^{61} + x^{60}\\
&+ x^{57} + x^{56} + x^{54} + x^{51} + x^{50} + x^{49} + x^{45} + x^{44} + x^{43} + x^{42} + x^{40}\\
&+ x^{38} + x^{37} + x^{34} + x^{33} + x^{30} + x^{26} + x^{23} + x^{22} + x^{18} + x^{17} + x^{15}\\
&+ x^{13} + x^{12} + x^{11} + x^{4} + x^{2} + 1,\\
p_9(x) = {}& x^{124} + x^{122} + x^{118} + x^{116} + x^{112} + x^{110} + x^{106} + x^{104} + x^{96} + x^{94}\\
&+ x^{90} + x^{88} + x^{80} + x^{78} + x^{74} + x^{72} + x^{64} + x^{62} + x^{58} + x^{56} + x^{48}
\end{aligned}
$$

$$+ x^{46} + x^{42} + x^{40} + x^{32} + x^{30} + x^{26} + x^{24} + x^{16} + x^{14} + x^{12} + x^{8} + x^{6} + x^{4},$$

$$p_{12}(x) = x^{124} + x^{122} + x^{118} + x^{114} + x^{110} + x^{108} + x^{96} + x^{94} + x^{92} + x^{80} + x^{78} + x^{76} + x^{64} + x^{62} + x^{60} + x^{48} + x^{46} + x^{44} + x^{32} + x^{30} + x^{28} + x^{16} + x^{14} + x^{10} + x^{6} + x^{2} + 1,$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 1, 1].

Below is the transition function and output function of an 2-automaton that generates $T = M^{o}_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	62	[50, 54]	124	[190, 191]	186	[118, 12]	248	[290, 207]
1	[3, 4]	63	[8, 111]	125	[192, 193]	187	[124, 240]	249	[291, 168]
2	[5, 6]	64	[112, 111]	126	[194, 150]	188	[119, 122]	250	[292, 293]
3	[7, 8]	65	[113, 49]	127	[18, 195]	189	[243, 10]	251	[294, 80]
4	[9, 10]	66	[108, 114]	128	[196, 197]	190	[100, 241]	252	[146, 282]
5	[11, 12]	67	[115, 12]	129	[198, 199]	191	[131, 224]	253	[115, 110]
6	[13, 14]	68	[116, 45]	130	[20, 200]	192	[41, 134]	254	[119, 244]
7	[15, 16]	69	[117, 108]	131	[201, 202]	193	[228, 103]	255	[250, 47]
8	[17, 18]	70	[118, 110]	132	[70, 203]	194	[32, 143]	256	[48, 2]
9	[19, 20]	71	[38, 119]	133	[204, 205]	195	[35, 244]	257	[232, 40]
10	[21, 22]	72	[8, 120]	134	[206, 207]	196	[245, 124]	258	[120, 224]
11	[23, 24]	73	[121, 122]	135	[208, 209]	197	[11, 143]	259	[118, 227]
12	[25, 26]	74	[123, 99]	136	[210, 211]	198	[246, 134]	260	[54, 14]
13	[27, 28]	75	[53, 13]	137	[212, 213]	199	[131, 139]	261	[137, 36]
14	[29, 27]	76	[124, 125]	138	[214, 215]	200	[38, 132]	262	[137, 120]
15	[30, 31]	77	[126, 127]	139	[85, 216]	201	[236, 247]	263	[13, 51]
16	[32, 33]	78	[128, 112]	140	[217, 156]	202	[116, 106]	264	[112, 120]
17	[34, 35]	79	[51, 53]	141	[218, 219]	203	[108, 237]	265	[8, 35]
18	[34, 36]	80	[129, 130]	142	[220, 218]	204	[248, 244]	266	[245, 226]
19	[37, 38]	81	[99, 131]	143	[221, 222]	205	[249, 137]	267	[118, 247]
20	[39, 40]	82	[132, 127]	144	[144, 144]	206	[144, 50]	268	[295, 43]
21	[41, 42]	83	[133, 134]	145	[99, 132]	207	[226, 240]	269	[123, 112]
22	[36, 43]	84	[133, 135]	146	[137, 35]	208	[54, 54]	270	[296, 252]
23	[44, 45]	85	[38, 103]	147	[223, 224]	209	[47, 130]	271	[225, 38]
24	[46, 47]	86	[136, 137]	148	[225, 137]	210	[250, 129]	272	[246, 229]
25	[48, 49]	87	[41, 102]	149	[50, 52]	211	[236, 110]	273	[297, 228]
26	[50, 51]	88	[138, 135]	150	[226, 125]	212	[230, 103]	274	[298, 125]
27	[52, 50]	89	[120, 139]	151	[52, 144]	213	[112, 36]	275	[107, 124]
28	[52, 53]	90	[140, 141]	152	[47, 141]	214	[251, 104]	276	[233, 40]
29	[54, 52]	91	[142, 47]	153	[109, 227]	215	[249, 38]	277	[100, 102]
30	[55, 56]	92	[113, 143]	154	[10, 45]	216	[111, 252]	278	[246, 42]
31	[57, 58]	93	[51, 13]	155	[228, 132]	217	[138, 42]	279	[14, 14]

32	[59, 60]	94	[53, 50]	156	[35, 122]	218	[232, 102]	280	[129, 141]
33	[61, 62]	95	[51, 144]	157	[41, 135]	219	[230, 119]	281	[39, 241]
34	[63, 64]	96	[144, 13]	158	[111, 127]	220	[7, 230]	282	[132, 252]
35	[65, 66]	97	[53, 53]	159	[138, 229]	221	[115, 2]	283	[299, 230]
36	[67, 68]	98	[14, 51]	160	[230, 131]	222	[14, 54]	284	[136, 8]
37	[69, 70]	99	[145, 146]	161	[231, 112]	223	[253, 152]	285	[133, 101]
38	[71, 72]	100	[147, 148]	162	[232, 134]	224	[219, 254]	286	[300, 139]
39	[73, 74]	101	[149, 150]	163	[233, 101]	225	[255, 256]	287	[128, 99]
40	[75, 76]	102	[151, 152]	164	[36, 104]	226	[257, 258]	288	[32, 247]
41	[77, 78]	103	[153, 154]	165	[234, 130]	227	[259, 260]	289	[13, 52]
42	[79, 80]	104	[155, 156]	166	[235, 108]	228	[261, 262]	290	[11, 227]
43	[81, 82]	105	[157, 158]	167	[236, 33]	229	[263, 187]	291	[117, 31]
44	[83, 82]	106	[159, 160]	168	[11, 2]	230	[264, 265]	292	[231, 34]
45	[84, 85]	107	[161, 162]	169	[31, 237]	231	[266, 267]	293	[100, 135]
46	[86, 87]	108	[163, 164]	170	[228, 131]	232	[268, 269]	294	[48, 33]
47	[88, 89]	109	[165, 166]	171	[34, 111]	233	[270, 271]	295	[301, 209]
48	[90, 91]	110	[167, 98]	172	[238, 237]	234	[272, 188]	296	[302, 76]
49	[92, 29]	111	[168, 169]	173	[142, 116]	235	[273, 192]	297	[303, 153]
50	[93, 94]	112	[170, 171]	174	[99, 119]	236	[274, 275]	298	[304, 195]
51	[6, 95]	113	[172, 173]	175	[239, 240]	237	[180, 155]	299	[305, 201]
52	[62, 96]	114	[56, 174]	176	[46, 116]	238	[276, 216]	300	[306, 280]
53	[97, 93]	115	[175, 176]	177	[133, 229]	239	[277, 254]	301	[236, 49]
54	[98, 97]	116	[177, 178]	178	[103, 43]	240	[278, 81]	302	[113, 247]
55	[7, 99]	117	[179, 180]	179	[37, 228]	241	[279, 280]	303	[307, 10]
56	[100, 101]	118	[181, 182]	180	[233, 241]	242	[281, 282]	304	[232, 101]
57	[39, 102]	119	[16, 183]	181	[242, 114]	243	[283, 159]	305	[243, 116]
58	[103, 104]	120	[184, 185]	182	[235, 124]	244	[64, 158]	306	[109, 2]
59	[105, 106]	121	[186, 187]	183	[31, 114]	245	[284, 285]	307	[308, 84]
60	[107, 108]	122	[72, 188]	184	[48, 143]	246	[286, 287]	308	[297, 38]
61	[109, 110]	123	[189, 184]	185	[10, 106]	247	[288, 289]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	45	1	90	0	135	0	180	1	225	1	270	1
1	0	46	1	91	0	136	1	181	0	226	1	271	1
2	1	47	0	92	1	137	1	182	0	227	0	272	1
3	0	48	0	93	1	138	0	183	0	228	1	273	0
4	0	49	1	94	0	139	0	184	0	229	1	274	1
5	1	50	1	95	1	140	0	185	0	230	1	275	1
6	1	51	1	96	0	141	1	186	0	231	1	276	1
7	0	52	1	97	0	142	0	187	0	232	1	277	0
8	0	53	0	98	0	143	0	188	0	233	1	278	1
9	0	54	0	99	0	144	0	189	0	234	1	279	0
10	0	55	0	100	0	145	0	190	0	235	0	280	1

11	1	56	0	101	1	146	1	191	1	236	1	281	0
12	0	57	0	102	1	147	0	192	0	237	1	282	0
13	1	58	1	103	1	148	1	193	1	238	1	283	0
14	0	59	0	104	1	149	1	194	0	239	0	284	1
15	0	60	1	105	0	150	1	195	1	240	1	285	1
16	0	61	1	106	0	151	1	196	1	241	0	286	1
17	0	62	1	107	1	152	0	197	1	242	0	287	1
18	0	63	0	108	1	153	1	198	1	243	0	288	0
19	0	64	1	109	1	154	0	199	1	244	1	289	1
20	0	65	1	110	1	155	1	200	0	245	1	290	1
21	0	66	1	111	1	156	1	201	1	246	1	291	0
22	0	67	0	112	1	157	0	202	1	247	0	292	1
23	1	68	1	113	1	158	1	203	1	248	1	293	0
24	1	69	0	114	0	159	0	204	1	249	0	294	0
25	0	70	0	115	0	160	1	205	0	250	1	295	1
26	1	71	0	116	1	161	1	206	0	251	0	296	1
27	1	72	0	117	0	162	1	207	1	252	1	297	0
28	1	73	0	118	0	163	1	208	0	253	0	298	1
29	0	74	0	119	0	164	0	209	0	254	0	299	0
30	0	75	0	120	0	165	1	210	1	255	1	300	1
31	0	76	0	121	0	166	0	211	1	256	0	301	1
32	0	77	0	122	0	167	1	212	1	257	1	302	1
33	1	78	1	123	0	168	1	213	1	258	0	303	0
34	0	79	1	124	0	169	0	214	0	259	0	304	1
35	1	80	1	125	0	170	1	215	0	260	0	305	0
36	0	81	0	126	0	171	0	216	1	261	1	306	1
37	0	82	0	127	0	172	1	217	0	262	1	307	0
38	0	83	1	128	1	173	0	218	1	263	1	308	0
39	0	84	1	129	1	174	0	219	1	264	1		
40	0	85	0	130	0	175	0	220	0	265	0		
41	0	86	1	131	1	176	1	221	0	266	1		
42	1	87	0	132	0	177	1	222	0	267	0		
43	0	88	0	133	1	178	1	223	0	268	1		
44	1	89	0	134	0	179	0	224	1	269	0		

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