PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a,b)=(z^6+z^4+z^2,z+1)$

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Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a,b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a,b) in $\mathbb{F}_2[x]\setminus\mathbb{F}_2$. In this paper we prove the conjecture for $(a,b)=(z^6+z^4+z^2,z+1)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a}=(a_0(z),a_1(z),a_2(z),\ldots)$ of elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z))=\cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{a_2(z)+\cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z))=\cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{\ddots+\cfrac{1}{a_n(z)}}}}\in\mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1}+c_{-2}z^{-2}+c_{-3}z^{-3}+\cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a,b)-Thue-Morse sequence

$$\mathbf{t}=(t_0(z),t_1(z),t_2(z),\ldots)=(a,b,b,a,b,a,a,b,\ldots)$$

that we shall denote

$$\mathrm{CF}(a,b):=\mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When* $(a,b)=(z^6+z^4+z^2,z+1)$*, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

where, for $\mathrm{CF}(a,b)$*,*

$$\begin{aligned}
p_0(z)&=z^{21}+z^{20}+z^{19}+z^{18}+z^{15}+z^{12}+z^{11}+z^8+z^6+z^3+z^2+1,\\
p_1(z)&=z^{27}+z^{26}+z^{22}+z^{19}+z^{16}+z^{15}+z^{13}+z^{10}+z^9+z^8+z^7+z^6+z^5\\
&\quad+z^2,\\
p_2(z)&=z^{28}+z^{26}+z^{20}+z^{18}+z^{16}+z^{12}+z^{10}+z^4,\\
p_3(z)&=z^{27}+z^{26}+z^{22}+z^{19}+z^{16}+z^{15}+z^{13}+z^{10}+z^9+z^8+z^7+z^6+z^5\\
&\quad+z^2,\\
p_4(z)&=z^{26}+z^{21}+z^{20}+z^{19}+z^{16}+z^{15}+z^{12}+z^{11}+z^{10}+z^8+z^6+z^3+z^2,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$*,*

$$\begin{aligned}
p_0(z)&=z^{24}+z^{21}+z^{20}+z^{19}+z^{18}+z^{16}+z^{15}+z^{12}+z^{11}+z^6+z^4+z^3+z^2,\\
p_1(z)&=z^{27}+z^{26}+z^{22}+z^{19}+z^{16}+z^{15}+z^{13}+z^{10}+z^9+z^8+z^7+z^6+z^5\\
&\quad+z^2,\\
p_2(z)&=z^{28}+z^{26}+z^{20}+z^{18}+z^{16}+z^{12}+z^{10}+z^4,\\
p_3(z)&=z^{27}+z^{26}+z^{22}+z^{19}+z^{16}+z^{15}+z^{13}+z^{10}+z^9+z^8+z^7+z^6+z^5\\
&\quad+z^2,\\
p_4(z)&=z^{26}+z^{24}+z^{21}+z^{20}+z^{19}+z^{15}+z^{12}+z^{11}+z^{10}+z^6+z^4+z^3+z^2\\
&\quad+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix}P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix}:=\begin{pmatrix}t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix}t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x)&=x^{\deg(t_{2^n-1})}\begin{pmatrix}t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix}t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(t_0)}\begin{pmatrix}t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x)&=x^{\deg(\bar t_{2^n-1})}\begin{pmatrix}\bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix}\bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(\bar t_0)}\begin{pmatrix}\bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n \geq 0$

$$\begin{aligned} M_{n+1}(x) &= W_n(x) \cdot M_n(x), \\ W_{n+1}(x) &= M_n(x) \cdot W_n(x). \end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$\begin{aligned} M_n(x)_{0,1} &= x^{d_n} \tilde{P}_{2^n-1}(x), \\ M_n(x)_{0,0} &= x^{d_n} \tilde{Q}_{2^n-1}(x), \end{aligned}$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \leq i,j \leq 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \leq i,j \leq 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \leq i,j \leq 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(175, 175, 175, 175, 175)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \geq 0$. If for some $n \geq 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \geq 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z \left(\phi(W^o,0,0)(x,z),\ z^{12} \cdot \phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 630 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 630, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z \left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ to order 770 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 770, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. $\square$

Define

$$R^e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned} M^e[:14u] = {} & M^e[:7u] + x^u M^e[:6u] + x^{6u} M^e[:u] + x^u M^e[:7u] + x^{2u} M^e[:6u] \\ & + x^{7u} M^e[:u] + x^{2u} M^e[:7u] + x^{3u} M^e[:6u] + x^{8u} M^e[:u] \\ & + x^{3u} M^e[:7u] + x^{4u} M^e[:6u] + x^{9u} M^e[:u] + x^{4u} M^e[:7u] \\ & + x^{5u} M^e[:6u] + x^{10u} M^e[:u] + x^{5u} M^e[:7u] + x^{6u} M^e[:6u] \end{aligned}$$

$$
\begin{aligned}
&+ x^{11u}M^e[:u] + x^{8u}M^e[:6u]\\
&+ (x^{7u} + x^{8u} + x^{9u} + x^{10u} + x^{11u} + x^{12u})R^e,\\
W^e[:14u] = {}& W^e[:7u] + x^uW^e[:6u] + x^{6u}W^e[:u] + x^uW^e[:7u] + x^{2u}W^e[:6u]\\
&+ x^{7u}W^e[:u] + x^{2u}W^e[:7u] + x^{3u}W^e[:6u] + x^{8u}W^e[:u]\\
&+ x^{3u}W^e[:7u] + x^{4u}W^e[:6u] + x^{9u}W^e[:u] + x^{4u}W^e[:7u]\\
&+ x^{5u}W^e[:6u] + x^{10u}W^e[:u] + x^{5u}W^e[:7u] + x^{6u}W^e[:6u]\\
&+ x^{11u}W^e[:u] + x^{8u}W^e[:6u]\\
&+ (x^{7u} + x^{8u} + x^{9u} + x^{10u} + x^{11u} + x^{12u})R^e.
\end{aligned}
$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$
\begin{aligned}
M^o[:14u] = {}& M^o[:7u] + x^uM^o[:6u] + x^{6u}M^o[:u] + x^uM^o[:7u] + x^{2u}M^o[:6u]\\
&+ x^{7u}M^o[:u] + x^{2u}M^o[:7u] + x^{3u}M^o[:6u] + x^{8u}M^o[:u]\\
&+ x^{3u}M^o[:7u] + x^{4u}M^o[:6u] + x^{9u}M^o[:u] + x^{4u}M^o[:7u]\\
&+ x^{5u}M^o[:6u] + x^{10u}M^o[:u] + x^{5u}M^o[:7u] + x^{6u}M^o[:6u]\\
&+ x^{11u}M^o[:u] + x^{8u}M^o[:6u]\\
&+ (x^{7u} + x^{8u} + x^{9u} + x^{10u} + x^{11u} + x^{12u})R^o,\\
W^o[:14u] = {}& W^o[:7u] + x^uW^o[:6u] + x^{6u}W^o[:u] + x^uW^o[:7u] + x^{2u}W^o[:6u]\\
&+ x^{7u}W^o[:u] + x^{2u}W^o[:7u] + x^{3u}W^o[:6u] + x^{8u}W^o[:u]\\
&+ x^{3u}W^o[:7u] + x^{4u}W^o[:6u] + x^{9u}W^o[:u] + x^{4u}W^o[:7u]\\
&+ x^{5u}W^o[:6u] + x^{10u}W^o[:u] + x^{5u}W^o[:7u] + x^{6u}W^o[:6u]\\
&+ x^{11u}W^o[:u] + x^{8u}W^o[:6u]\\
&+ (x^{7u} + x^{8u} + x^{9u} + x^{10u} + x^{11u} + x^{12u})R^o.
\end{aligned}
$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$
\begin{aligned}
T[:14u] = {}& T[:7u] + x^uT[:6u] + x^{6u}T[:u] + x^uT[:7u] + x^{2u}T[:6u] + x^{7u}T[:u]\\
&+ x^{2u}T[:7u] + x^{3u}T[:6u] + x^{8u}T[:u] + x^{3u}T[:7u] + x^{4u}T[:6u]\\
&+ x^{9u}T[:u] + x^{4u}T[:7u] + x^{5u}T[:6u] + x^{10u}T[:u] + x^{5u}T[:7u]\\
&+ x^{6u}T[:6u] + x^{11u}T[:u] + x^{8u}T[:6u].
\end{aligned}
$$

The proof of the other 15 cases are similar. We break down the above identity into 7 parts:

$$
\begin{aligned}
0 &= x^{7u}T[:u] + x^{6u}T[u:2u] + x^uT[6u:7u] + T[7u:8u],\\
0 &= x^{6u}T[2u:3u] + x^{2u}T[6u:7u] + T[8u:9u],\\
0 &= x^{9u}T[:u] + x^{8u}T[u:2u] + x^{6u}T[3u:4u] + x^{3u}T[6u:7u] + T[9u:10u],\\
0 &= x^{10u}T[:u] + x^{8u}T[2u:3u] + x^{6u}T[4u:5u] + x^{4u}T[6u:7u]\\
&\quad + T[10u:11u],
\end{aligned}
$$

$$
\begin{aligned}
0 &= x^{11u}T[:u] + x^{8u}T[3u:4u] + x^{6u}T[5u:6u] + x^{5u}T[6u:7u] \\
&\quad + T[11u:12u], \\
0 &= x^{8u}T[4u:5u] + T[12u:13u], \\
0 &= x^{8u}T[5u:6u] + T[13u:14u],
\end{aligned}
$$

which can be rewritten as

$$0 = T[[w]_2] + T[[1w]_2] + T[[110w]_2] + T[[111w]_2], \tag{2.7}$$

$$0 = T[[10w]_2] + T[[110w]_2] + T[[1000w]_2], \tag{2.8}$$

$$0 = T[[w]_2] + T[[1w]_2] + T[[11w]_2] + T[[110w]_2] + T[[1001w]_2], \tag{2.9}$$

$$0 = T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1010w]_2], \tag{2.10}$$

$$0 = T[[w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2] + T[[1011w]_2], \tag{2.11}$$

$$0 = T[[100w]_2] + T[[1100w]_2], \tag{2.12}$$

$$0 = T[[101w]_2] + T[[1101w]_2], \tag{2.13}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$0 = T[[w]_2] + T[[1w]_2] + T[[110w]_2] + T[[111w]_2], \tag{2.14}$$

$$0 = T[[10w]_2] + T[[110w]_2] + T[[1000w]_2], \tag{2.15}$$

$$0 = T[[w]_2] + T[[1w]_2] + T[[11w]_2] + T[[110w]_2] + T[[1001w]_2], \tag{2.16}$$

$$0 = T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1010w]_2], \tag{2.17}$$

$$0 = T[[w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2] + T[[1011w]_2], \tag{2.18}$$

$$0 = T[[100w]_2] + T[[1100w]_2], \tag{2.19}$$

$$0 = T[[101w]_2] + T[[1101w]_2], \tag{2.20}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 952 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.20) can be written as

$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 1)) + \tau(A(s, 110)) + \tau(A(s, 111)), \tag{2.21}$$

$$0 = \tau(A(s, 10)) + \tau(A(s, 110)) + \tau(A(s, 1000)), \tag{2.22}$$

$$
\begin{aligned}
0 &= \tau(A(s, \epsilon)) + \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 110)) \\
&\quad + \tau(A(s, 1001)),
\end{aligned} \tag{2.23}
$$

$$
\begin{aligned}
0 &= \tau(A(s, \epsilon)) + \tau(A(s, 10)) + \tau(A(s, 100)) + \tau(A(s, 110)) \\
&\quad + \tau(A(s, 1010)),
\end{aligned} \tag{2.24}
$$

$$
\begin{aligned}
0 &= \tau(A(s, \epsilon)) + \tau(A(s, 11)) + \tau(A(s, 101)) + \tau(A(s, 110)) \\
&\quad + \tau(A(s, 1011)),
\end{aligned} \tag{2.25}
$$

$$0 = \tau(A(s, 100)) + \tau(A(s, 1100)), \tag{2.26}$$

$$0 = \tau(A(s, 101)) + \tau(A(s, 1101)), \tag{2.27}$$

for all $s \in E_{2k}$, and

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,110)) + \tau(A(s,111)), \tag{2.28}$$

$$0 = \tau(A(s,10)) + \tau(A(s,110)) + \tau(A(s,1000)), \tag{2.29}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,110)) \\ &\quad + \tau(A(s,1001)), \end{aligned} \tag{2.30}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,110)) \\ &\quad + \tau(A(s,1010)), \end{aligned} \tag{2.31}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,110)) \\ &\quad + \tau(A(s,1011)), \end{aligned} \tag{2.32}$$

$$0 = \tau(A(s,100)) + \tau(A(s,1100)), \tag{2.33}$$

$$0 = \tau(A(s,101)) + \tau(A(s,1101)), \tag{2.34}$$

for $s = A(i, 0^{2k})$. We find that $(A(i,0^{44}), E_{44}) = (A(i,0^{16}), E_{16})$, so that we only have to verify that identities (2.21) through (2.34) hold for $2 \le k \le 22$, which turns out to be true. □

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \ge 2$, and $u = 2^{2k-1}$,*

$$M_{2k} = M^e[:7u] + x^u M^e[:6u] + x^{6u} M^e[:u] + x^{7u} R^e,$$
$$W_{2k} = W^e[:7u] + x^u W^e[:6u] + x^{6u} W^e[:u] + x^{7u} R^e.$$

For all integer $k \ge 2$, and $u = 2^{2k}$,

$$M_{2k+1} = M^o[:7u] + x^u M^o[:6u] + x^{6u} M^o[:u] + x^{7u} R^o,$$
$$W_{2k+1} = W^o[:7u] + x^u W^o[:6u] + x^{6u} W^o[:u] + x^{7u} R^o.$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \ge 2$, we claim that

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1} \wedge W_{2k+1},$$
$$M_{2k+1} \wedge W_{2k+1} \Rightarrow M_{2k+2} \wedge W_{2k+2}.$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$\begin{aligned} W_{2k} M_{2k} &= \big(W^e[:7v] + x^v W^e[:6v] + x^{6v} W^e[:v] + x^{7u} R^e\big) \times \big(M^e[:7v] \\ &\quad + x^v M^e[:6v] + x^{6v} M^e[:v] + x^{7u} R^e\big); \end{aligned} \tag{2.35}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{14v} R^o$. Therefore we only need to prove that their difference is $O(x^{14v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{14v}, to

$$W^e[:14v] M^e[:14v] + x^{2v} W^e[:12v] M^e[:12v] + x^{12v} W^e[:2v] M^e[:2v].$$

For all $n \leq 14$, replace the occurrences of $W^e[: n \cdot v]$ and $M^e[: n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.35) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{14}, b_1, ..., b_{14}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{14})$, which completes the proof. □

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \leq j, k \leq 1$,

$$\begin{aligned} \lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k}, \\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k}, \\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k}, \\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}. \end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a, b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x, y) = \mathrm{Res}_t \left(\phi(M^e, 0, 1)(x, t), y^{12}\phi(M^e, 0, 1)(x, t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x, y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned} q_0(x) &= x^{28} + x^{26} + x^{25} + x^{22} + x^{20} + x^{17} + x^{16} + x^{13} + x^{10} + x^9 + x^8 + x^7, \\ q_1(x) &= x^{26} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18} + x^{15} + x^{13} + x^{12} + x^9 + x^6 \\ &\quad + x^2 + x, \\ q_2(x) &= x^{24} + x^{18} + x^{16} + x^{12} + x^{10} + x^8 + x^2 + 1, \\ q_3(x) &= x^{26} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18} + x^{15} + x^{13} + x^{12} + x^9 + x^6 \\ &\quad + x^2 + x, \\ q_4(x) &= x^{26} + x^{25} + x^{22} + x^{20} + x^{18} + x^{17} + x^{16} + x^{13} + x^{12} + x^9 + x^8 + x^7 \\ &\quad + x^2. \end{aligned}$$

The polynomial $Q(x,y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and $R(x,y) := P(x,y)/Q(x,y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 216, and substitute it for y in $R(z,y)$, we get a series with valuation smaller than 216, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b) = f(1/z)$. □

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e,0,0)$:

$$\begin{aligned}
p_0(x) &= x^{132} + x^{130} + x^{124} + x^{118} + x^{116} + x^{114} + x^{112} + x^{110} + x^{108} + x^{106} \\
&\quad + x^{104} + x^{102} + x^{100} + x^{96} + x^{90} + x^{88} + x^{86} + x^{80} + x^{78} + x^{76} + x^{70} \\
&\quad + x^{64} + x^{62} + x^{60} + x^{58} + x^{56} + x^{54} + x^{52} + x^{48} + x^{42} + x^{24} + x^{22} \\
&\quad + x^{20} + x^{16} + x^{12}, \\
p_3(x) &= x^{126} + x^{124} + x^{120} + x^{118} + x^{116} + x^{114} + x^{112} + x^{110} + x^{108} + x^{106} \\
&\quad + x^{100} + x^{98} + x^{92} + x^{78} + x^{76} + x^{70} + x^{66} + x^{60} + x^{58} + x^{52} + x^{44} \\
&\quad + x^{38} + x^{26} + x^{24} + x^{22} + x^{16} + x^{10} + x^{8}, \\
p_6(x) &= x^{126} + x^{122} + x^{112} + x^{108} + x^{106} + x^{92} + x^{90} + x^{86} + x^{82} + x^{80} + x^{76} \\
&\quad + x^{66} + x^{62} + x^{60} + x^{54} + x^{52} + x^{50} + x^{34} + x^{32} + x^{28} + x^{26} + x^{24} \\
&\quad + x^{22} + x^{16} + x^{8} + x^{4} + x^{2} + 1, \\
p_9(x) &= x^{128} + x^{118} + x^{112} + x^{110} + x^{102} + x^{96} + x^{92} + x^{88} + x^{86} + x^{82} + x^{80} \\
&\quad + x^{74} + x^{72} + x^{68} + x^{62} + x^{56} + x^{50} + x^{48} + x^{46} + x^{42} + x^{40} + x^{34} \\
&\quad + x^{26} + x^{20} + x^{18} + x^{16} + x^{14} + x^{12} + x^{6} + x^{2}, \\
p_{12}(x) &= x^{128} + x^{124} + x^{116} + x^{108} + x^{100} + x^{96} + x^{80} + x^{76} + x^{68} + x^{64} + x^{48} \\
&\quad + x^{44} + x^{36} + x^{32} + x^{16} + x^{12} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0].

For $\phi(M^e,0,1)$:

$$\begin{aligned}
p_0(x) &= x^{147} + x^{144} + x^{141} + x^{140} + x^{139} + x^{135} + x^{134} + x^{133} + x^{132} + x^{131} \\
&\quad + x^{130} + x^{129} + x^{126} + x^{123} + x^{122} + x^{119} + x^{118} + x^{117} + x^{114} \\
&\quad + x^{112} + x^{110} + x^{106} + x^{104} + x^{100} + x^{99} + x^{97} + x^{95} + x^{94} + x^{92} \\
&\quad + x^{91} + x^{88} + x^{86} + x^{84} + x^{82} + x^{81} + x^{78} + x^{76} + x^{75} + x^{74} + x^{73} \\
&\quad + x^{72} + x^{71} + x^{70} + x^{67} + x^{65} + x^{64} + x^{60} + x^{59} + x^{58} + x^{56} + x^{49} \\
&\quad + x^{48} + x^{45} + x^{44} + x^{42} + x^{41} + x^{39} + x^{38} + x^{35} + x^{33} + x^{32} + x^{31}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{27},\\
p_3(x) &= x^{143} + x^{142} + x^{141} + x^{139} + x^{136} + x^{134} + x^{131} + x^{130} + x^{129} + x^{128}\\
&+ x^{127} + x^{126} + x^{125} + x^{120} + x^{118} + x^{113} + x^{103} + x^{102} + x^{101}\\
&+ x^{99} + x^{96} + x^{94} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{80}\\
&+ x^{79} + x^{77} + x^{75} + x^{73} + x^{72} + x^{71} + x^{69} + x^{66} + x^{65} + x^{63} + x^{61}\\
&+ x^{59} + x^{58} + x^{57} + x^{55} + x^{53} + x^{49} + x^{48} + x^{46} + x^{41} + x^{39} + x^{38}\\
&+ x^{37} + x^{35} + x^{32} + x^{30} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21}\\
&+ x^{16} + x^{14} + x^{9},\\
p_6(x) &= x^{141} + x^{140} + x^{139} + x^{138} + x^{133} + x^{132} + x^{127} + x^{126} + x^{125} + x^{124}\\
&+ x^{115} + x^{114} + x^{109} + x^{108} + x^{99} + x^{98} + x^{91} + x^{90} + x^{89} + x^{88}\\
&+ x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{75}\\
&+ x^{74} + x^{73} + x^{72} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{55} + x^{54}\\
&+ x^{51} + x^{50} + x^{47} + x^{46} + x^{45} + x^{44} + x^{37} + x^{36} + x^{35} + x^{34} + x^{31}\\
&+ x^{30} + x^{27} + x^{26} + x^{23} + x^{22} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14}\\
&+ x^{13} + x^{12} + x^{7} + x^{6},\\
p_9(x) &= x^{139} + x^{138} + x^{137} + x^{135} + x^{133} + x^{131} + x^{130} + x^{128} + x^{127} + x^{126}\\
&+ x^{125} + x^{121} + x^{119} + x^{118} + x^{116} + x^{113} + x^{112} + x^{109} + x^{106}\\
&+ x^{104} + x^{103} + x^{102} + x^{100} + x^{99} + x^{95} + x^{87} + x^{86} + x^{85} + x^{82}\\
&+ x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{75} + x^{74} + x^{73} + x^{72} + x^{66} + x^{64}\\
&+ x^{59} + x^{55} + x^{54} + x^{52} + x^{51} + x^{50} + x^{48} + x^{47} + x^{45} + x^{44} + x^{43}\\
&+ x^{42} + x^{39} + x^{38} + x^{34} + x^{33} + x^{32} + x^{29} + x^{28} + x^{24} + x^{23} + x^{22}\\
&+ x^{21} + x^{20} + x^{18} + x^{16} + x^{15} + x^{13} + x^{12} + x^{8} + x^{5} + x^{3},\\
p_{12}(x) &= x^{137} + x^{136} + x^{135} + x^{134} + x^{131} + x^{130} + x^{127} + x^{126} + x^{123} + x^{122}\\
&+ x^{121} + x^{120} + x^{113} + x^{112} + x^{111} + x^{110} + x^{107} + x^{106} + x^{103}\\
&+ x^{102} + x^{99} + x^{98} + x^{97} + x^{96} + x^{89} + x^{88} + x^{87} + x^{86} + x^{83} + x^{82}\\
&+ x^{79} + x^{78} + x^{75} + x^{74} + x^{71} + x^{70} + x^{67} + x^{66} + x^{65} + x^{64} + x^{57}\\
&+ x^{56} + x^{55} + x^{54} + x^{51} + x^{50} + x^{47} + x^{46} + x^{43} + x^{42} + x^{39} + x^{38}\\
&+ x^{35} + x^{34} + x^{33} + x^{32} + x^{25} + x^{24} + x^{23} + x^{22} + x^{19} + x^{18} + x^{15}\\
&+ x^{14} + x^{11} + x^{10} + x^{7} + x^{6} + x^{3} + x^{2} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 1, 0, 1].

For $\phi(M^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{147} + x^{144} + x^{141} + x^{140} + x^{139} + x^{135} + x^{134} + x^{133} + x^{132} + x^{131}\\
&+ x^{130} + x^{129} + x^{126} + x^{123} + x^{122} + x^{119} + x^{118} + x^{117} + x^{114}\\
&+ x^{112} + x^{110} + x^{106} + x^{104} + x^{100} + x^{99} + x^{97} + x^{95} + x^{94} + x^{92}\\
&+ x^{91} + x^{88} + x^{86} + x^{84} + x^{82} + x^{81} + x^{78} + x^{76} + x^{75} + x^{74} + x^{73}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{72} + x^{71} + x^{70} + x^{67} + x^{65} + x^{64} + x^{60} + x^{59} + x^{58} + x^{56} + x^{49}\\
&+ x^{48} + x^{45} + x^{44} + x^{42} + x^{41} + x^{39} + x^{38} + x^{35} + x^{33} + x^{32} + x^{31}\\
&+ x^{27},\\
p_3(x) = {}& x^{143} + x^{142} + x^{141} + x^{139} + x^{136} + x^{134} + x^{131} + x^{130} + x^{129} + x^{128}\\
&+ x^{127} + x^{126} + x^{125} + x^{120} + x^{118} + x^{113} + x^{103} + x^{102} + x^{101}\\
&+ x^{99} + x^{96} + x^{94} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{80}\\
&+ x^{79} + x^{77} + x^{75} + x^{73} + x^{72} + x^{71} + x^{69} + x^{66} + x^{65} + x^{63} + x^{61}\\
&+ x^{59} + x^{58} + x^{57} + x^{55} + x^{53} + x^{49} + x^{48} + x^{46} + x^{41} + x^{39} + x^{38}\\
&+ x^{37} + x^{35} + x^{32} + x^{30} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21}\\
&+ x^{16} + x^{14} + x^{9},\\
p_6(x) = {}& x^{141} + x^{140} + x^{139} + x^{138} + x^{133} + x^{132} + x^{127} + x^{126} + x^{125} + x^{124}\\
&+ x^{115} + x^{114} + x^{109} + x^{108} + x^{99} + x^{98} + x^{91} + x^{90} + x^{89} + x^{88}\\
&+ x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{75}\\
&+ x^{74} + x^{73} + x^{72} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{55} + x^{54}\\
&+ x^{51} + x^{50} + x^{47} + x^{46} + x^{45} + x^{44} + x^{37} + x^{36} + x^{35} + x^{34} + x^{31}\\
&+ x^{30} + x^{27} + x^{26} + x^{23} + x^{22} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14}\\
&+ x^{13} + x^{12} + x^{7} + x^{6},\\
p_9(x) = {}& x^{139} + x^{138} + x^{137} + x^{135} + x^{133} + x^{131} + x^{130} + x^{128} + x^{127} + x^{126}\\
&+ x^{125} + x^{121} + x^{119} + x^{118} + x^{116} + x^{113} + x^{112} + x^{109} + x^{106}\\
&+ x^{104} + x^{103} + x^{102} + x^{100} + x^{99} + x^{95} + x^{87} + x^{86} + x^{85} + x^{82}\\
&+ x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{75} + x^{74} + x^{73} + x^{72} + x^{66} + x^{64}\\
&+ x^{59} + x^{55} + x^{54} + x^{52} + x^{51} + x^{50} + x^{48} + x^{47} + x^{45} + x^{44} + x^{43}\\
&+ x^{42} + x^{39} + x^{38} + x^{34} + x^{33} + x^{32} + x^{29} + x^{28} + x^{24} + x^{23} + x^{22}\\
&+ x^{21} + x^{20} + x^{18} + x^{16} + x^{15} + x^{13} + x^{12} + x^{8} + x^{5} + x^{3},\\
p_{12}(x) = {}& x^{137} + x^{136} + x^{135} + x^{134} + x^{131} + x^{130} + x^{127} + x^{126} + x^{123} + x^{122}\\
&+ x^{121} + x^{120} + x^{113} + x^{112} + x^{111} + x^{110} + x^{107} + x^{106} + x^{103}\\
&+ x^{102} + x^{99} + x^{98} + x^{97} + x^{96} + x^{89} + x^{88} + x^{87} + x^{86} + x^{83} + x^{82}\\
&+ x^{79} + x^{78} + x^{75} + x^{74} + x^{71} + x^{70} + x^{67} + x^{66} + x^{65} + x^{64} + x^{57}\\
&+ x^{56} + x^{55} + x^{54} + x^{51} + x^{50} + x^{47} + x^{46} + x^{43} + x^{42} + x^{39} + x^{38}\\
&+ x^{35} + x^{34} + x^{33} + x^{32} + x^{25} + x^{24} + x^{23} + x^{22} + x^{19} + x^{18} + x^{15}\\
&+ x^{14} + x^{11} + x^{10} + x^{7} + x^{6} + x^{3} + x^{2} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 1, 0, 1].

For $\phi(M^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) = {}& x^{162} + x^{160} + x^{158} + x^{154} + x^{144} + x^{138} + x^{136} + x^{132} + x^{126} + x^{122}\\
&+ x^{116} + x^{114} + x^{112} + x^{110} + x^{108} + x^{106} + x^{104} + x^{102} + x^{100}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{96} + x^{92} + x^{88} + x^{86} + x^{82} + x^{80} + x^{76} + x^{70} + x^{66} + x^{58} + x^{56}\\
&+ x^{50} + x^{46} + x^{42},\\
p_3(x) &= x^{150} + x^{148} + x^{142} + x^{140} + x^{134} + x^{132} + x^{130} + x^{124} + x^{122} + x^{116}\\
&+ x^{110} + x^{108} + x^{104} + x^{100} + x^{96} + x^{92} + x^{90} + x^{88} + x^{82} + x^{80}\\
&+ x^{78} + x^{76} + x^{74} + x^{72} + x^{70} + x^{68} + x^{64} + x^{60} + x^{46} + x^{44} + x^{38}\\
&+ x^{36} + x^{34} + x^{32} + x^{26} + x^{20} + x^{8} + x^{6},\\
p_6(x) &= x^{150} + x^{148} + x^{146} + x^{144} + x^{140} + x^{136} + x^{132} + x^{130} + x^{124} + x^{120}\\
&+ x^{118} + x^{112} + x^{110} + x^{108} + x^{104} + x^{102} + x^{98} + x^{94} + x^{92} + x^{90}\\
&+ x^{88} + x^{82} + x^{78} + x^{76} + x^{74} + x^{68} + x^{66} + x^{60} + x^{54} + x^{52} + x^{48}\\
&+ x^{46} + x^{36} + x^{34} + x^{28} + x^{24} + x^{18} + x^{12} + x^{6} + 1,\\
p_9(x) &= x^{146} + x^{144} + x^{142} + x^{140} + x^{138} + x^{134} + x^{132} + x^{130} + x^{124} + x^{116}\\
&+ x^{106} + x^{104} + x^{100} + x^{96} + x^{94} + x^{90} + x^{74} + x^{70} + x^{68} + x^{66} + x^{64}\\
&+ x^{60} + x^{58} + x^{56} + x^{48} + x^{46} + x^{44} + x^{38} + x^{36} + x^{34} + x^{32} + x^{30}\\
&+ x^{28} + x^{26} + x^{22} + x^{18} + x^{14} + x^{12} + x^{4} + x^{2},\\
p_{12}(x) &= x^{146} + x^{144} + x^{82} + x^{80} + x^{50} + x^{48} + x^{18} + x^{16} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0].

For $\phi(W^e, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{162} + x^{160} + x^{158} + x^{152} + x^{144} + x^{142} + x^{136} + x^{128} + x^{118} + x^{116}\\
&+ x^{112} + x^{110} + x^{102} + x^{100} + x^{96} + x^{92} + x^{90} + x^{88} + x^{86} + x^{84}\\
&+ x^{82} + x^{80} + x^{76} + x^{74} + x^{72} + x^{70} + x^{68} + x^{60} + x^{58} + x^{52} + x^{50}\\
&+ x^{46} + x^{44} + x^{36} + x^{32} + x^{28} + x^{22} + x^{18} + x^{16} + x^{14} + x^{12},\\
p_3(x) &= x^{150} + x^{148} + x^{136} + x^{134} + x^{132} + x^{130} + x^{126} + x^{124} + x^{120} + x^{118}\\
&+ x^{116} + x^{112} + x^{110} + x^{108} + x^{106} + x^{104} + x^{98} + x^{96} + x^{78} + x^{76}\\
&+ x^{74} + x^{70} + x^{68} + x^{66} + x^{64} + x^{56} + x^{54} + x^{52} + x^{50} + x^{42} + x^{40}\\
&+ x^{38} + x^{36} + x^{30} + x^{28} + x^{24} + x^{22} + x^{20} + x^{18} + x^{16} + x^{10} + x^{8},\\
p_6(x) &= x^{150} + x^{148} + x^{146} + x^{144} + x^{142} + x^{138} + x^{136} + x^{132} + x^{130} + x^{122}\\
&+ x^{118} + x^{116} + x^{114} + x^{112} + x^{110} + x^{108} + x^{98} + x^{96} + x^{92} + x^{80}\\
&+ x^{78} + x^{76} + x^{68} + x^{66} + x^{64} + x^{62} + x^{60} + x^{58} + x^{44} + x^{42} + x^{40}\\
&+ x^{36} + x^{32} + x^{30} + x^{26} + x^{20} + x^{16} + x^{12} + x^{8} + x^{6} + x^{4} + 1,\\
p_9(x) &= x^{146} + x^{144} + x^{142} + x^{140} + x^{138} + x^{134} + x^{132} + x^{130} + x^{124} + x^{116}\\
&+ x^{106} + x^{104} + x^{100} + x^{96} + x^{94} + x^{90} + x^{74} + x^{70} + x^{68} + x^{66} + x^{64}\\
&+ x^{60} + x^{58} + x^{56} + x^{48} + x^{46} + x^{44} + x^{38} + x^{36} + x^{34} + x^{32} + x^{30}\\
&+ x^{28} + x^{26} + x^{22} + x^{18} + x^{14} + x^{12} + x^{4} + x^{2},\\
p_{12}(x) &= x^{146} + x^{144} + x^{82} + x^{80} + x^{50} + x^{48} + x^{18} + x^{16} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) = {} & x^{147} + x^{145} + x^{143} + x^{141} + x^{140} + x^{139} + x^{138} + x^{136} + x^{135} + x^{133} \\
& + x^{132} + x^{130} + x^{129} + x^{125} + x^{124} + x^{123} + x^{120} + x^{118} + x^{115} \\
& + x^{113} + x^{111} + x^{110} + x^{107} + x^{106} + x^{105} + x^{104} + x^{103} + x^{102} \\
& + x^{101} + x^{99} + x^{95} + x^{94} + x^{92} + x^{84} + x^{82} + x^{79} + x^{78} + x^{75} + x^{74} \\
& + x^{73} + x^{71} + x^{70} + x^{69} + x^{66} + x^{64} + x^{63} + x^{62} + x^{61} + x^{56} + x^{54} \\
& + x^{52} + x^{51} + x^{50} + x^{48} + x^{42} + x^{41} + x^{40} + x^{38} + x^{36} + x^{35} + x^{33} \\
& + x^{32} + x^{30} + x^{29} + x^{26} + x^{23} + x^{21} + x^{19} + x^{18}, \\
p_3(x) = {} & x^{143} + x^{142} + x^{141} + x^{139} + x^{136} + x^{134} + x^{131} + x^{130} + x^{129} + x^{128} \\
& + x^{127} + x^{126} + x^{125} + x^{120} + x^{118} + x^{113} + x^{103} + x^{102} + x^{101} \\
& + x^{99} + x^{96} + x^{94} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{80} \\
& + x^{79} + x^{77} + x^{75} + x^{73} + x^{72} + x^{71} + x^{69} + x^{66} + x^{65} + x^{63} + x^{61} \\
& + x^{59} + x^{58} + x^{57} + x^{55} + x^{53} + x^{49} + x^{48} + x^{46} + x^{41} + x^{39} + x^{38} \\
& + x^{37} + x^{35} + x^{32} + x^{30} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} \\
& + x^{16} + x^{14} + x^{9}, \\
p_6(x) = {} & x^{141} + x^{140} + x^{139} + x^{138} + x^{133} + x^{132} + x^{127} + x^{126} + x^{125} + x^{124} \\
& + x^{115} + x^{114} + x^{109} + x^{108} + x^{99} + x^{98} + x^{91} + x^{90} + x^{89} + x^{88} \\
& + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{75} \\
& + x^{74} + x^{73} + x^{72} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{55} + x^{54} \\
& + x^{51} + x^{50} + x^{47} + x^{46} + x^{45} + x^{44} + x^{37} + x^{36} + x^{35} + x^{34} + x^{31} \\
& + x^{30} + x^{27} + x^{26} + x^{23} + x^{22} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} \\
& + x^{13} + x^{12} + x^{7} + x^{6}, \\
p_9(x) = {} & x^{139} + x^{138} + x^{137} + x^{135} + x^{133} + x^{131} + x^{130} + x^{128} + x^{127} + x^{126} \\
& + x^{125} + x^{121} + x^{119} + x^{118} + x^{116} + x^{113} + x^{112} + x^{109} + x^{106} \\
& + x^{104} + x^{103} + x^{102} + x^{100} + x^{99} + x^{95} + x^{87} + x^{86} + x^{85} + x^{82} \\
& + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{75} + x^{74} + x^{73} + x^{72} + x^{66} + x^{64} \\
& + x^{59} + x^{55} + x^{54} + x^{52} + x^{51} + x^{50} + x^{48} + x^{47} + x^{45} + x^{44} + x^{43} \\
& + x^{42} + x^{39} + x^{38} + x^{34} + x^{33} + x^{32} + x^{29} + x^{28} + x^{24} + x^{23} + x^{22} \\
& + x^{21} + x^{20} + x^{18} + x^{16} + x^{15} + x^{13} + x^{12} + x^{8} + x^{5} + x^{3}, \\
p_{12}(x) = {} & x^{137} + x^{136} + x^{135} + x^{134} + x^{131} + x^{130} + x^{127} + x^{126} + x^{123} + x^{122} \\
& + x^{121} + x^{120} + x^{113} + x^{112} + x^{111} + x^{110} + x^{107} + x^{106} + x^{103} \\
& + x^{102} + x^{99} + x^{98} + x^{97} + x^{96} + x^{89} + x^{88} + x^{87} + x^{86} + x^{83} + x^{82} \\
& + x^{79} + x^{78} + x^{75} + x^{74} + x^{71} + x^{70} + x^{67} + x^{66} + x^{65} + x^{64} + x^{57} \\
& + x^{56} + x^{55} + x^{54} + x^{51} + x^{50} + x^{47} + x^{46} + x^{43} + x^{42} + x^{39} + x^{38} \\
& + x^{35} + x^{34} + x^{33} + x^{32} + x^{25} + x^{24} + x^{23} + x^{22} + x^{19} + x^{18} + x^{15}
\end{aligned}
$$

$$+ x^{14} + x^{11} + x^{10} + x^{7} + x^{6} + x^{3} + x^{2} + x + 1,$$

and the initial terms are [0, 1, 1, 1, 1, 0, 0, 0, 1, 1, 1, 1, 0, 1, 0, 1].

For $\phi(W^e, 1, 0)$:

$$\begin{aligned}
p_0(x) = {} & x^{147} + x^{145} + x^{143} + x^{141} + x^{140} + x^{139} + x^{138} + x^{136} + x^{135} + x^{133} \\
& + x^{132} + x^{130} + x^{129} + x^{125} + x^{124} + x^{123} + x^{120} + x^{118} + x^{115} \\
& + x^{113} + x^{111} + x^{110} + x^{107} + x^{106} + x^{105} + x^{104} + x^{103} + x^{102} \\
& + x^{101} + x^{99} + x^{95} + x^{94} + x^{92} + x^{84} + x^{82} + x^{79} + x^{78} + x^{75} + x^{74} \\
& + x^{73} + x^{71} + x^{70} + x^{69} + x^{66} + x^{64} + x^{63} + x^{62} + x^{61} + x^{56} + x^{54} \\
& + x^{52} + x^{51} + x^{50} + x^{48} + x^{42} + x^{41} + x^{40} + x^{38} + x^{36} + x^{35} + x^{33} \\
& + x^{32} + x^{30} + x^{29} + x^{26} + x^{23} + x^{21} + x^{19} + x^{18}, \\
p_3(x) = {} & x^{143} + x^{142} + x^{141} + x^{139} + x^{136} + x^{134} + x^{131} + x^{130} + x^{129} + x^{128} \\
& + x^{127} + x^{126} + x^{125} + x^{120} + x^{118} + x^{113} + x^{103} + x^{102} + x^{101} \\
& + x^{99} + x^{96} + x^{94} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{80} \\
& + x^{79} + x^{77} + x^{75} + x^{73} + x^{72} + x^{71} + x^{69} + x^{66} + x^{65} + x^{63} + x^{61} \\
& + x^{59} + x^{58} + x^{57} + x^{55} + x^{53} + x^{49} + x^{48} + x^{46} + x^{41} + x^{39} + x^{38} \\
& + x^{37} + x^{35} + x^{32} + x^{30} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} \\
& + x^{16} + x^{14} + x^{9}, \\
p_6(x) = {} & x^{141} + x^{140} + x^{139} + x^{138} + x^{133} + x^{132} + x^{127} + x^{126} + x^{125} + x^{124} \\
& + x^{115} + x^{114} + x^{109} + x^{108} + x^{99} + x^{98} + x^{91} + x^{90} + x^{89} + x^{88} \\
& + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{75} \\
& + x^{74} + x^{73} + x^{72} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{55} + x^{54} \\
& + x^{51} + x^{50} + x^{47} + x^{46} + x^{45} + x^{44} + x^{37} + x^{36} + x^{35} + x^{34} + x^{31} \\
& + x^{30} + x^{27} + x^{26} + x^{23} + x^{22} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} \\
& + x^{13} + x^{12} + x^{7} + x^{6}, \\
p_9(x) = {} & x^{139} + x^{138} + x^{137} + x^{135} + x^{133} + x^{131} + x^{130} + x^{128} + x^{127} + x^{126} \\
& + x^{125} + x^{121} + x^{119} + x^{118} + x^{116} + x^{113} + x^{112} + x^{109} + x^{106} \\
& + x^{104} + x^{103} + x^{102} + x^{100} + x^{99} + x^{95} + x^{87} + x^{86} + x^{85} + x^{82} \\
& + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{75} + x^{74} + x^{73} + x^{72} + x^{66} + x^{64} \\
& + x^{59} + x^{55} + x^{54} + x^{52} + x^{51} + x^{50} + x^{48} + x^{47} + x^{45} + x^{44} + x^{43} \\
& + x^{42} + x^{39} + x^{38} + x^{34} + x^{33} + x^{32} + x^{29} + x^{28} + x^{24} + x^{23} + x^{22} \\
& + x^{21} + x^{20} + x^{18} + x^{16} + x^{15} + x^{13} + x^{12} + x^{8} + x^{5} + x^{3}, \\
p_{12}(x) = {} & x^{137} + x^{136} + x^{135} + x^{134} + x^{131} + x^{130} + x^{127} + x^{126} + x^{123} + x^{122} \\
& + x^{121} + x^{120} + x^{113} + x^{112} + x^{111} + x^{110} + x^{107} + x^{106} + x^{103} \\
& + x^{102} + x^{99} + x^{98} + x^{97} + x^{96} + x^{89} + x^{88} + x^{87} + x^{86} + x^{83} + x^{82} \\
& + x^{79} + x^{78} + x^{75} + x^{74} + x^{71} + x^{70} + x^{67} + x^{66} + x^{65} + x^{64} + x^{57}
\end{aligned}$$

$$+ x^{56} + x^{55} + x^{54} + x^{51} + x^{50} + x^{47} + x^{46} + x^{43} + x^{42} + x^{39} + x^{38}$$
$$+ x^{35} + x^{34} + x^{33} + x^{32} + x^{25} + x^{24} + x^{23} + x^{22} + x^{19} + x^{18} + x^{15}$$
$$+ x^{14} + x^{11} + x^{10} + x^{7} + x^{6} + x^{3} + x^{2} + x + 1,$$

and the initial terms are [0, 1, 1, 1, 1, 0, 0, 0, 1, 1, 1, 1, 0, 1, 0, 1].
For $\phi(W^e, 1, 1)$:

$$\begin{aligned}
p_0(x) = {} & x^{132} + x^{130} + x^{128} + x^{122} + x^{120} + x^{112} + x^{110} + x^{104} + x^{100} + x^{98} \\
& + x^{96} + x^{92} + x^{90} + x^{84} + x^{82} + x^{80} + x^{74} + x^{72} + x^{66} + x^{64} + x^{58} \\
& + x^{50} + x^{46} + x^{44} + x^{42} + x^{36} + x^{30} + x^{28} + x^{24}, \\
p_3(x) = {} & x^{126} + x^{114} + x^{110} + x^{106} + x^{104} + x^{102} + x^{92} + x^{88} + x^{82} + x^{76} \\
& + x^{70} + x^{64} + x^{58} + x^{54} + x^{52} + x^{48} + x^{38} + x^{36} + x^{24} + x^{18} + x^{12} \\
& + x^{6}, \\
p_6(x) = {} & x^{126} + x^{124} + x^{122} + x^{118} + x^{106} + x^{102} + x^{100} + x^{98} + x^{96} + x^{92} \\
& + x^{90} + x^{88} + x^{86} + x^{84} + x^{80} + x^{78} + x^{68} + x^{62} + x^{56} + x^{50} + x^{48} \\
& + x^{44} + x^{36} + x^{34} + x^{32} + x^{24} + x^{18} + x^{16} + x^{10} + x^{6} + x^{2} + 1, \\
p_9(x) = {} & x^{128} + x^{118} + x^{112} + x^{110} + x^{102} + x^{96} + x^{92} + x^{88} + x^{86} + x^{82} + x^{80} \\
& + x^{74} + x^{72} + x^{68} + x^{62} + x^{56} + x^{50} + x^{48} + x^{46} + x^{42} + x^{40} + x^{34} \\
& + x^{26} + x^{20} + x^{18} + x^{16} + x^{14} + x^{12} + x^{6} + x^{2}, \\
p_{12}(x) = {} & x^{128} + x^{124} + x^{116} + x^{108} + x^{100} + x^{96} + x^{80} + x^{76} + x^{68} + x^{64} + x^{48} \\
& + x^{44} + x^{36} + x^{32} + x^{16} + x^{12} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0].
For $\phi(M^o, 0, 0)$:

$$\begin{aligned}
p_0(x) = {} & x^{147} + x^{144} + x^{143} + x^{140} + x^{138} + x^{137} + x^{131} + x^{130} + x^{127} + x^{124} \\
& + x^{123} + x^{119} + x^{117} + x^{115} + x^{114} + x^{112} + x^{111} + x^{110} + x^{109} \\
& + x^{104} + x^{102} + x^{101} + x^{93} + x^{92} + x^{87} + x^{85} + x^{79} + x^{78} + x^{76} + x^{74} \\
& + x^{72} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} + x^{59} + x^{56} \\
& + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{47} + x^{44} + x^{43} + x^{41} + x^{37} + x^{36} \\
& + x^{35} + x^{32} + x^{31} + x^{29} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{20} + x^{18} \\
& + x^{13} + x^{12}, \\
p_3(x) = {} & x^{145} + x^{144} + x^{142} + x^{141} + x^{140} + x^{136} + x^{135} + x^{131} + x^{129} + x^{127} \\
& + x^{125} + x^{124} + x^{123} + x^{122} + x^{120} + x^{118} + x^{117} + x^{116} + x^{115} \\
& + x^{114} + x^{109} + x^{108} + x^{107} + x^{106} + x^{103} + x^{99} + x^{98} + x^{96} + x^{94} \\
& + x^{91} + x^{89} + x^{87} + x^{85} + x^{84} + x^{83} + x^{82} + x^{80} + x^{72} + x^{71} + x^{70} \\
& + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{58} + x^{51} + x^{50} + x^{49} + x^{46} + x^{43} \\
& + x^{42} + x^{39} + x^{32} + x^{30} + x^{29} + x^{28} + x^{27} + x^{25} + x^{22} + x^{19} + x^{18} \\
& + x^{17} + x^{15} + x^{13} + x^{12} + x^{9} + x^{8},
\end{aligned}$$

$$\begin{aligned}
p_6(x) &= x^{145} + x^{144} + x^{142} + x^{141} + x^{140} + x^{136} + x^{133} + x^{132} + x^{130} + x^{129} \\
&+ x^{128} + x^{127} + x^{126} + x^{123} + x^{122} + x^{119} + x^{116} + x^{115} + x^{114} \\
&+ x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} + x^{102} + x^{100} + x^{99} + x^{98} \\
&+ x^{97} + x^{96} + x^{94} + x^{93} + x^{92} + x^{90} + x^{89} + x^{87} + x^{83} + x^{78} + x^{73} \\
&+ x^{72} + x^{71} + x^{70} + x^{67} + x^{65} + x^{62} + x^{59} + x^{56} + x^{55} + x^{52} + x^{51} \\
&+ x^{47} + x^{46} + x^{44} + x^{40} + x^{38} + x^{35} + x^{32} + x^{31} + x^{29} + x^{26} + x^{25} \\
&+ x^{24} + x^{22} + x^{21} + x^{19} + x^{13} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6} + x^{5} + x^{4} \\
&+ x^{3} + x^{2} + x + 1, \\
p_9(x) &= x^{145} + x^{144} + x^{143} + x^{142} + x^{139} + x^{138} + x^{135} + x^{134} + x^{133} + x^{132} \\
&+ x^{125} + x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{113} + x^{112} + x^{111} \\
&+ x^{110} + x^{109} + x^{108} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} \\
&+ x^{97} + x^{96} + x^{95} + x^{94} + x^{91} + x^{90} + x^{87} + x^{86} + x^{85} + x^{84} + x^{81} \\
&+ x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} \\
&+ x^{65} + x^{64} + x^{63} + x^{62} + x^{59} + x^{58} + x^{55} + x^{54} + x^{53} + x^{52} + x^{49} \\
&+ x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} \\
&+ x^{33} + x^{32} + x^{31} + x^{30} + x^{27} + x^{26} + x^{23} + x^{22} + x^{21} + x^{20} + x^{17} \\
&+ x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{9} + x^{8} + x^{7} + x^{6} + x^{5} + x^{4}, \\
p_{12}(x) &= x^{145} + x^{144} + x^{143} + x^{142} + x^{139} + x^{138} + x^{137} + x^{136} + x^{133} + x^{132} \\
&+ x^{131} + x^{130} + x^{127} + x^{126} + x^{125} + x^{124} + x^{109} + x^{108} + x^{107} \\
&+ x^{106} + x^{105} + x^{104} + x^{101} + x^{100} + x^{99} + x^{98} + x^{95} + x^{94} + x^{91} \\
&+ x^{90} + x^{89} + x^{88} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{77} + x^{76} \\
&+ x^{75} + x^{74} + x^{73} + x^{72} + x^{69} + x^{68} + x^{67} + x^{66} + x^{63} + x^{62} + x^{59} \\
&+ x^{58} + x^{57} + x^{56} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{45} + x^{44} \\
&+ x^{43} + x^{42} + x^{41} + x^{40} + x^{37} + x^{36} + x^{35} + x^{34} + x^{31} + x^{30} + x^{27} \\
&+ x^{26} + x^{25} + x^{24} + x^{21} + x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{13} + x^{12} \\
&+ x^{11} + x^{10} + x^{9} + x^{8} + x^{5} + x^{4} + x^{3} + x^{2} + x + 1,
\end{aligned}$$

and the initial terms are [1, 0, 1, 0, 1, 0, 1, 1, 1, 1, 1, 1, 0, 0, 1, 1].

For $\phi(M^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{164} + x^{160} + x^{157} + x^{154} + x^{152} + x^{151} + x^{150} + x^{149} + x^{147} + x^{146} \\
&+ x^{145} + x^{144} + x^{139} + x^{138} + x^{137} + x^{136} + x^{134} + x^{133} + x^{125} \\
&+ x^{117} + x^{116} + x^{110} + x^{109} + x^{104} + x^{103} + x^{98} + x^{96} + x^{95} + x^{92} \\
&+ x^{91} + x^{90} + x^{83} + x^{82} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{71} + x^{68} \\
&+ x^{65} + x^{63} + x^{62} + x^{61} + x^{60} + x^{58} + x^{55} + x^{54} + x^{52} + x^{46} + x^{45} \\
&+ x^{42} + x^{41} + x^{40} + x^{39} + x^{37} + x^{33} + x^{32} + x^{30} + x^{29} + x^{28} + x^{27}, \\
p_3(x) &= x^{150} + x^{147} + x^{145} + x^{142} + x^{140} + x^{136} + x^{131} + x^{130} + x^{129} + x^{128}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{126} + x^{122} + x^{120} + x^{118} + x^{116} + x^{115} + x^{114} + x^{113} + x^{110}\\
&+ x^{107} + x^{105} + x^{102} + x^{100} + x^{96} + x^{91} + x^{90} + x^{89} + x^{88} + x^{83}\\
&+ x^{82} + x^{81} + x^{80} + x^{78} + x^{74} + x^{72} + x^{70} + x^{68} + x^{67} + x^{66} + x^{65}\\
&+ x^{62} + x^{59} + x^{57} + x^{52} + x^{51} + x^{49} + x^{48} + x^{44} + x^{42} + x^{38} + x^{36}\\
&+ x^{32} + x^{27} + x^{26} + x^{25} + x^{24} + x^{22} + x^{18} + x^{16} + x^{14} + x^{12} + x^{11}\\
&+ x^{10} + x^{9},\\
p_6(x) &= x^{152} + x^{148} + x^{146} + x^{138} + x^{130} + x^{128} + x^{126} + x^{118} + x^{116} + x^{104}\\
&+ x^{102} + x^{100} + x^{98} + x^{96} + x^{94} + x^{90} + x^{84} + x^{82} + x^{80} + x^{76} + x^{74}\\
&+ x^{72} + x^{64} + x^{62} + x^{32} + x^{26} + x^{22} + x^{16} + x^{8} + x^{6},\\
p_9(x) &= x^{154} + x^{151} + x^{149} + x^{148} + x^{145} + x^{140} + x^{139} + x^{138} + x^{137} + x^{136}\\
&+ x^{135} + x^{133} + x^{132} + x^{130} + x^{128} + x^{127} + x^{125} + x^{123} + x^{121}\\
&+ x^{116} + x^{115} + x^{114} + x^{113} + x^{112} + x^{111} + x^{109} + x^{108} + x^{106}\\
&+ x^{104} + x^{103} + x^{101} + x^{99} + x^{97} + x^{92} + x^{91} + x^{89} + x^{88} + x^{82} + x^{81}\\
&+ x^{80} + x^{79} + x^{77} + x^{76} + x^{74} + x^{72} + x^{71} + x^{69} + x^{67} + x^{65} + x^{60}\\
&+ x^{59} + x^{57} + x^{56} + x^{50} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44} + x^{42} + x^{40}\\
&+ x^{39} + x^{37} + x^{35} + x^{33} + x^{28} + x^{27} + x^{25} + x^{24} + x^{18} + x^{17} + x^{16}\\
&+ x^{15} + x^{13} + x^{12} + x^{8} + x^{4} + x^{3},\\
p_{12}(x) &= x^{156} + x^{152} + x^{148} + x^{132} + x^{128} + x^{124} + x^{120} + x^{108} + x^{104} + x^{100}\\
&+ x^{96} + x^{92} + x^{88} + x^{80} + x^{76} + x^{72} + x^{68} + x^{64} + x^{60} + x^{56} + x^{48}\\
&+ x^{44} + x^{40} + x^{36} + x^{32} + x^{28} + x^{24} + x^{16} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 1].

For $\phi(M^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{140} + x^{137} + x^{134} + x^{133} + x^{127} + x^{126} + x^{124} + x^{122} + x^{119} + x^{117}\\
&+ x^{116} + x^{114} + x^{113} + x^{111} + x^{109} + x^{106} + x^{105} + x^{104} + x^{97} + x^{96}\\
&+ x^{95} + x^{91} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{77} + x^{76}\\
&+ x^{75} + x^{70} + x^{69} + x^{67} + x^{64} + x^{63} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57}\\
&+ x^{56} + x^{55} + x^{54} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{46} + x^{45} + x^{44}\\
&+ x^{40} + x^{38} + x^{36} + x^{35} + x^{34} + x^{32} + x^{31} + x^{30} + x^{28} + x^{25} + x^{24}\\
&+ x^{22} + x^{21} + x^{18},\\
p_3(x) &= x^{138} + x^{135} + x^{134} + x^{133} + x^{131} + x^{129} + x^{128} + x^{127} + x^{125} + x^{123}\\
&+ x^{122} + x^{121} + x^{119} + x^{117} + x^{116} + x^{115} + x^{114} + x^{113} + x^{98} + x^{95}\\
&+ x^{94} + x^{93} + x^{91} + x^{89} + x^{88} + x^{87} + x^{85} + x^{83} + x^{82} + x^{81} + x^{79}\\
&+ x^{77} + x^{76} + x^{75} + x^{73} + x^{71} + x^{70} + x^{69} + x^{67} + x^{66} + x^{65} + x^{64}\\
&+ x^{62} + x^{58} + x^{56} + x^{52} + x^{47} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{34}\\
&+ x^{31} + x^{30} + x^{29} + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} + x^{19} + x^{18} + x^{17}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{15} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9},\\
p_6(x) &= x^{140} + x^{134} + x^{130} + x^{124} + x^{114} + x^{112} + x^{110} + x^{108} + x^{98} + x^{96}\\
&+ x^{94} + x^{88} + x^{86} + x^{82} + x^{80} + x^{72} + x^{62} + x^{58} + x^{54} + x^{52} + x^{48}\\
&+ x^{46} + x^{44} + x^{42} + x^{40} + x^{34} + x^{30} + x^{28} + x^{24} + x^{22} + x^{18} + x^{14}\\
&+ x^{12} + x^{10} + x^{8} + x^{6},\\
p_9(x) &= x^{142} + x^{139} + x^{138} + x^{137} + x^{136} + x^{135} + x^{134} + x^{132} + x^{131} + x^{130}\\
&+ x^{126} + x^{125} + x^{124} + x^{122} + x^{121} + x^{120} + x^{118} + x^{114} + x^{109}\\
&+ x^{108} + x^{107} + x^{103} + x^{100} + x^{99} + x^{94} + x^{91} + x^{90} + x^{89} + x^{88}\\
&+ x^{87} + x^{86} + x^{84} + x^{83} + x^{82} + x^{77} + x^{76} + x^{75} + x^{71} + x^{68} + x^{67}\\
&+ x^{62} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{52} + x^{51} + x^{50} + x^{45}\\
&+ x^{44} + x^{43} + x^{39} + x^{36} + x^{35} + x^{30} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23}\\
&+ x^{22} + x^{20} + x^{19} + x^{18} + x^{13} + x^{12} + x^{11} + x^{7} + x^{4} + x^{3},\\
p_{12}(x) &= x^{144} + x^{136} + x^{132} + x^{124} + x^{108} + x^{104} + x^{100} + x^{88} + x^{84} + x^{80}\\
&+ x^{76} + x^{72} + x^{68} + x^{56} + x^{52} + x^{48} + x^{44} + x^{40} + x^{36} + x^{24} + x^{20}\\
&+ x^{16} + x^{12} + x^{8} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 1, 1, 0].

For $\phi(M^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{147} + x^{145} + x^{141} + x^{139} + x^{137} + x^{136} + x^{135} + x^{134} + x^{133} + x^{132}\\
&+ x^{131} + x^{129} + x^{128} + x^{127} + x^{126} + x^{125} + x^{122} + x^{120} + x^{114}\\
&+ x^{112} + x^{110} + x^{109} + x^{105} + x^{104} + x^{101} + x^{99} + x^{96} + x^{94} + x^{91}\\
&+ x^{90} + x^{89} + x^{86} + x^{85} + x^{83} + x^{81} + x^{78} + x^{75} + x^{73} + x^{72} + x^{69}\\
&+ x^{68} + x^{67} + x^{66} + x^{65} + x^{63} + x^{62} + x^{58} + x^{57} + x^{55} + x^{54} + x^{53}\\
&+ x^{51} + x^{48} + x^{44} + x^{38} + x^{36} + x^{33},\\
p_3(x) &= x^{145} + x^{144} + x^{142} + x^{139} + x^{138} + x^{137} + x^{134} + x^{131} + x^{129} + x^{127}\\
&+ x^{123} + x^{122} + x^{121} + x^{119} + x^{117} + x^{116} + x^{115} + x^{114} + x^{113}\\
&+ x^{112} + x^{109} + x^{108} + x^{107} + x^{106} + x^{103} + x^{101} + x^{100} + x^{97} + x^{95}\\
&+ x^{91} + x^{89} + x^{87} + x^{83} + x^{82} + x^{81} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75}\\
&+ x^{74} + x^{72} + x^{66} + x^{63} + x^{62} + x^{61} + x^{60} + x^{58} + x^{57} + x^{56} + x^{55}\\
&+ x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{47} + x^{43} + x^{42} + x^{41} + x^{40}\\
&+ x^{39} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{31} + x^{29} + x^{28} + x^{27} + x^{25}\\
&+ x^{22} + x^{21} + x^{20} + x^{19} + x^{18} + x^{16} + x^{14} + x^{13} + x^{12},\\
p_6(x) &= x^{145} + x^{144} + x^{142} + x^{139} + x^{138} + x^{137} + x^{135} + x^{134} + x^{131} + x^{125}\\
&+ x^{124} + x^{123} + x^{122} + x^{119} + x^{117} + x^{115} + x^{114} + x^{113} + x^{112}\\
&+ x^{109} + x^{108} + x^{102} + x^{101} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{91}\\
&+ x^{89} + x^{87} + x^{85} + x^{84} + x^{83} + x^{79} + x^{75} + x^{74} + x^{71} + x^{70} + x^{67}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{64} + x^{62} + x^{61} + x^{60} + x^{59} + x^{56} + x^{55} + x^{53} + x^{50} + x^{49} + x^{48} \\
&+ x^{47} + x^{46} + x^{45} + x^{43} + x^{42} + x^{41} + x^{38} + x^{35} + x^{32} + x^{31} + x^{28} \\
&+ x^{27} + x^{22} + x^{21} + x^{19} + x^{17} + x^{16} + x^{12} + x^{11} + x^{9} + x^{8} + x^{7} + x^{6} \\
&+ x^{3} + x^{2} + x + 1, \\
p_9(x) &= x^{145} + x^{144} + x^{143} + x^{142} + x^{139} + x^{138} + x^{135} + x^{134} + x^{133} + x^{132} \\
&+ x^{125} + x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{113} + x^{112} + x^{111} \\
&+ x^{110} + x^{109} + x^{108} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} \\
&+ x^{97} + x^{96} + x^{95} + x^{94} + x^{91} + x^{90} + x^{87} + x^{86} + x^{85} + x^{84} + x^{81} \\
&+ x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} \\
&+ x^{65} + x^{64} + x^{63} + x^{62} + x^{59} + x^{58} + x^{55} + x^{54} + x^{53} + x^{52} + x^{49} \\
&+ x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} \\
&+ x^{33} + x^{32} + x^{31} + x^{30} + x^{27} + x^{26} + x^{23} + x^{22} + x^{21} + x^{20} + x^{17} \\
&+ x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{9} + x^{8} + x^{7} + x^{6} + x^{5} + x^{4}, \\
p_{12}(x) &= x^{145} + x^{144} + x^{143} + x^{142} + x^{139} + x^{138} + x^{137} + x^{136} + x^{133} + x^{132} \\
&+ x^{131} + x^{130} + x^{127} + x^{126} + x^{125} + x^{124} + x^{109} + x^{108} + x^{107} \\
&+ x^{106} + x^{105} + x^{104} + x^{101} + x^{100} + x^{99} + x^{98} + x^{95} + x^{94} + x^{91} \\
&+ x^{90} + x^{89} + x^{88} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{77} + x^{76} \\
&+ x^{75} + x^{74} + x^{73} + x^{72} + x^{69} + x^{68} + x^{67} + x^{66} + x^{63} + x^{62} + x^{59} \\
&+ x^{58} + x^{57} + x^{56} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{45} + x^{44} \\
&+ x^{43} + x^{42} + x^{41} + x^{40} + x^{37} + x^{36} + x^{35} + x^{34} + x^{31} + x^{30} + x^{27} \\
&+ x^{26} + x^{25} + x^{24} + x^{21} + x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{13} + x^{12} \\
&+ x^{11} + x^{10} + x^{9} + x^{8} + x^{5} + x^{4} + x^{3} + x^{2} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 1, 1].

For $\phi(W^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{147} + x^{144} + x^{143} + x^{140} + x^{138} + x^{137} + x^{131} + x^{130} + x^{127} + x^{124} \\
&+ x^{123} + x^{119} + x^{117} + x^{115} + x^{114} + x^{112} + x^{111} + x^{110} + x^{109} \\
&+ x^{104} + x^{102} + x^{101} + x^{93} + x^{92} + x^{87} + x^{85} + x^{79} + x^{78} + x^{76} + x^{74} \\
&+ x^{72} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} + x^{59} + x^{56} \\
&+ x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{47} + x^{44} + x^{43} + x^{41} + x^{37} + x^{36} \\
&+ x^{35} + x^{32} + x^{31} + x^{29} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{20} + x^{18} \\
&+ x^{13} + x^{12}, \\
p_3(x) &= x^{145} + x^{144} + x^{142} + x^{141} + x^{140} + x^{136} + x^{135} + x^{131} + x^{129} + x^{127} \\
&+ x^{125} + x^{124} + x^{123} + x^{122} + x^{120} + x^{118} + x^{117} + x^{116} + x^{115} \\
&+ x^{114} + x^{109} + x^{108} + x^{107} + x^{106} + x^{103} + x^{99} + x^{98} + x^{96} + x^{94} \\
&+ x^{91} + x^{89} + x^{87} + x^{85} + x^{84} + x^{83} + x^{82} + x^{80} + x^{72} + x^{71} + x^{70}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{69}+x^{68}+x^{67}+x^{65}+x^{64}+x^{58}+x^{51}+x^{50}+x^{49}+x^{46}+x^{43}\\
&+x^{42}+x^{39}+x^{32}+x^{30}+x^{29}+x^{28}+x^{27}+x^{25}+x^{22}+x^{19}+x^{18}\\
&+x^{17}+x^{15}+x^{13}+x^{12}+x^{9}+x^{8},\\
p_6(x)={}&x^{145}+x^{144}+x^{142}+x^{141}+x^{140}+x^{136}+x^{133}+x^{132}+x^{130}+x^{129}\\
&+x^{128}+x^{127}+x^{126}+x^{123}+x^{122}+x^{119}+x^{116}+x^{115}+x^{114}\\
&+x^{113}+x^{112}+x^{111}+x^{110}+x^{109}+x^{108}+x^{102}+x^{100}+x^{99}+x^{98}\\
&+x^{97}+x^{96}+x^{94}+x^{93}+x^{92}+x^{90}+x^{89}+x^{87}+x^{83}+x^{78}+x^{73}\\
&+x^{72}+x^{71}+x^{70}+x^{67}+x^{65}+x^{62}+x^{59}+x^{56}+x^{55}+x^{52}+x^{51}\\
&+x^{47}+x^{46}+x^{44}+x^{40}+x^{38}+x^{35}+x^{32}+x^{31}+x^{29}+x^{26}+x^{25}\\
&+x^{24}+x^{22}+x^{21}+x^{19}+x^{13}+x^{10}+x^{9}+x^{8}+x^{7}+x^{6}+x^{5}+x^{4}\\
&+x^{3}+x^{2}+x+1,\\
p_9(x)={}&x^{145}+x^{144}+x^{143}+x^{142}+x^{139}+x^{138}+x^{135}+x^{134}+x^{133}+x^{132}\\
&+x^{125}+x^{124}+x^{123}+x^{122}+x^{121}+x^{120}+x^{113}+x^{112}+x^{111}\\
&+x^{110}+x^{109}+x^{108}+x^{105}+x^{104}+x^{103}+x^{102}+x^{101}+x^{100}\\
&+x^{97}+x^{96}+x^{95}+x^{94}+x^{91}+x^{90}+x^{87}+x^{86}+x^{85}+x^{84}+x^{81}\\
&+x^{80}+x^{79}+x^{78}+x^{77}+x^{76}+x^{73}+x^{72}+x^{71}+x^{70}+x^{69}+x^{68}\\
&+x^{65}+x^{64}+x^{63}+x^{62}+x^{59}+x^{58}+x^{55}+x^{54}+x^{53}+x^{52}+x^{49}\\
&+x^{48}+x^{47}+x^{46}+x^{45}+x^{44}+x^{41}+x^{40}+x^{39}+x^{38}+x^{37}+x^{36}\\
&+x^{33}+x^{32}+x^{31}+x^{30}+x^{27}+x^{26}+x^{23}+x^{22}+x^{21}+x^{20}+x^{17}\\
&+x^{16}+x^{15}+x^{14}+x^{13}+x^{12}+x^{9}+x^{8}+x^{7}+x^{6}+x^{5}+x^{4},\\
p_{12}(x)={}&x^{145}+x^{144}+x^{143}+x^{142}+x^{139}+x^{138}+x^{137}+x^{136}+x^{133}+x^{132}\\
&+x^{131}+x^{130}+x^{127}+x^{126}+x^{125}+x^{124}+x^{109}+x^{108}+x^{107}\\
&+x^{106}+x^{105}+x^{104}+x^{101}+x^{100}+x^{99}+x^{98}+x^{95}+x^{94}+x^{91}\\
&+x^{90}+x^{89}+x^{88}+x^{85}+x^{84}+x^{83}+x^{82}+x^{81}+x^{80}+x^{77}+x^{76}\\
&+x^{75}+x^{74}+x^{73}+x^{72}+x^{69}+x^{68}+x^{67}+x^{66}+x^{63}+x^{62}+x^{59}\\
&+x^{58}+x^{57}+x^{56}+x^{53}+x^{52}+x^{51}+x^{50}+x^{49}+x^{48}+x^{45}+x^{44}\\
&+x^{43}+x^{42}+x^{41}+x^{40}+x^{37}+x^{36}+x^{35}+x^{34}+x^{31}+x^{30}+x^{27}\\
&+x^{26}+x^{25}+x^{24}+x^{21}+x^{20}+x^{19}+x^{18}+x^{17}+x^{16}+x^{13}+x^{12}\\
&+x^{11}+x^{10}+x^{9}+x^{8}+x^{5}+x^{4}+x^{3}+x^{2}+x+1,
\end{aligned}
$$

and the initial terms are [1, 0, 1, 0, 1, 0, 1, 1, 1, 1, 1, 1, 0, 0, 1, 1].

For $\phi(W^o, 0, 1)$:

$$
\begin{aligned}
p_0(x)={}&x^{140}+x^{137}+x^{134}+x^{133}+x^{127}+x^{126}+x^{124}+x^{122}+x^{119}+x^{117}\\
&+x^{116}+x^{114}+x^{113}+x^{111}+x^{109}+x^{106}+x^{105}+x^{104}+x^{97}+x^{96}\\
&+x^{95}+x^{91}+x^{87}+x^{86}+x^{85}+x^{84}+x^{83}+x^{82}+x^{81}+x^{77}+x^{76}\\
&+x^{75}+x^{70}+x^{69}+x^{67}+x^{64}+x^{63}+x^{61}+x^{60}+x^{59}+x^{58}+x^{57}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{56} + x^{55} + x^{54} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{46} + x^{45} + x^{44}\\
&+ x^{40} + x^{38} + x^{36} + x^{35} + x^{34} + x^{32} + x^{31} + x^{30} + x^{28} + x^{25} + x^{24}\\
&+ x^{22} + x^{21} + x^{18},\\
p_3(x) &= x^{138} + x^{135} + x^{134} + x^{133} + x^{131} + x^{129} + x^{128} + x^{127} + x^{125} + x^{123}\\
&+ x^{122} + x^{121} + x^{119} + x^{117} + x^{116} + x^{115} + x^{114} + x^{113} + x^{98} + x^{95}\\
&+ x^{94} + x^{93} + x^{91} + x^{89} + x^{88} + x^{87} + x^{85} + x^{83} + x^{82} + x^{81} + x^{79}\\
&+ x^{77} + x^{76} + x^{75} + x^{73} + x^{71} + x^{70} + x^{69} + x^{67} + x^{66} + x^{65} + x^{64}\\
&+ x^{62} + x^{58} + x^{56} + x^{52} + x^{47} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{34}\\
&+ x^{31} + x^{30} + x^{29} + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} + x^{19} + x^{18} + x^{17}\\
&+ x^{15} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9},\\
p_6(x) &= x^{140} + x^{134} + x^{130} + x^{124} + x^{114} + x^{112} + x^{110} + x^{108} + x^{98} + x^{96}\\
&+ x^{94} + x^{88} + x^{86} + x^{82} + x^{80} + x^{72} + x^{62} + x^{58} + x^{54} + x^{52} + x^{48}\\
&+ x^{46} + x^{44} + x^{42} + x^{40} + x^{34} + x^{30} + x^{28} + x^{24} + x^{22} + x^{18} + x^{14}\\
&+ x^{12} + x^{10} + x^{8} + x^{6},\\
p_9(x) &= x^{142} + x^{139} + x^{138} + x^{137} + x^{136} + x^{135} + x^{134} + x^{132} + x^{131} + x^{130}\\
&+ x^{126} + x^{125} + x^{124} + x^{122} + x^{121} + x^{120} + x^{118} + x^{114} + x^{109}\\
&+ x^{108} + x^{107} + x^{103} + x^{100} + x^{99} + x^{94} + x^{91} + x^{90} + x^{89} + x^{88}\\
&+ x^{87} + x^{86} + x^{84} + x^{83} + x^{82} + x^{77} + x^{76} + x^{75} + x^{71} + x^{68} + x^{67}\\
&+ x^{62} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{52} + x^{51} + x^{50} + x^{45}\\
&+ x^{44} + x^{43} + x^{39} + x^{36} + x^{35} + x^{30} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23}\\
&+ x^{22} + x^{20} + x^{19} + x^{18} + x^{13} + x^{12} + x^{11} + x^{7} + x^{4} + x^{3},\\
p_{12}(x) &= x^{144} + x^{136} + x^{132} + x^{124} + x^{108} + x^{104} + x^{100} + x^{88} + x^{84} + x^{80}\\
&+ x^{76} + x^{72} + x^{68} + x^{56} + x^{52} + x^{48} + x^{44} + x^{40} + x^{36} + x^{24} + x^{20}\\
&+ x^{16} + x^{12} + x^{8} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 1, 1, 0].

For $\phi(W^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{164} + x^{160} + x^{157} + x^{154} + x^{152} + x^{151} + x^{150} + x^{149} + x^{147} + x^{146}\\
&+ x^{145} + x^{144} + x^{139} + x^{138} + x^{137} + x^{136} + x^{134} + x^{133} + x^{125}\\
&+ x^{117} + x^{116} + x^{110} + x^{109} + x^{104} + x^{103} + x^{98} + x^{96} + x^{95} + x^{92}\\
&+ x^{91} + x^{90} + x^{83} + x^{82} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{71} + x^{68}\\
&+ x^{65} + x^{63} + x^{62} + x^{61} + x^{60} + x^{58} + x^{55} + x^{54} + x^{52} + x^{46} + x^{45}\\
&+ x^{42} + x^{41} + x^{40} + x^{39} + x^{37} + x^{33} + x^{32} + x^{30} + x^{29} + x^{28} + x^{27},\\
p_3(x) &= x^{150} + x^{147} + x^{145} + x^{142} + x^{140} + x^{136} + x^{131} + x^{130} + x^{129} + x^{128}\\
&+ x^{126} + x^{122} + x^{120} + x^{118} + x^{116} + x^{115} + x^{114} + x^{113} + x^{110}\\
&+ x^{107} + x^{105} + x^{102} + x^{100} + x^{96} + x^{91} + x^{90} + x^{89} + x^{88} + x^{83}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{82} + x^{81} + x^{80} + x^{78} + x^{74} + x^{72} + x^{70} + x^{68} + x^{67} + x^{66} + x^{65}\\
&+ x^{62} + x^{59} + x^{57} + x^{52} + x^{51} + x^{49} + x^{48} + x^{44} + x^{42} + x^{38} + x^{36}\\
&+ x^{32} + x^{27} + x^{26} + x^{25} + x^{24} + x^{22} + x^{18} + x^{16} + x^{14} + x^{12} + x^{11}\\
&+ x^{10} + x^{9},\\
p_6(x) &= x^{152} + x^{148} + x^{146} + x^{138} + x^{130} + x^{128} + x^{126} + x^{118} + x^{116} + x^{104}\\
&+ x^{102} + x^{100} + x^{98} + x^{96} + x^{94} + x^{90} + x^{84} + x^{82} + x^{80} + x^{76} + x^{74}\\
&+ x^{72} + x^{64} + x^{62} + x^{32} + x^{26} + x^{22} + x^{16} + x^{8} + x^{6},\\
p_9(x) &= x^{154} + x^{151} + x^{149} + x^{148} + x^{145} + x^{140} + x^{139} + x^{138} + x^{137} + x^{136}\\
&+ x^{135} + x^{133} + x^{132} + x^{130} + x^{128} + x^{127} + x^{125} + x^{123} + x^{121}\\
&+ x^{116} + x^{115} + x^{114} + x^{113} + x^{112} + x^{111} + x^{109} + x^{108} + x^{106}\\
&+ x^{104} + x^{103} + x^{101} + x^{99} + x^{97} + x^{92} + x^{91} + x^{89} + x^{88} + x^{82} + x^{81}\\
&+ x^{80} + x^{79} + x^{77} + x^{76} + x^{74} + x^{72} + x^{71} + x^{69} + x^{67} + x^{65} + x^{60}\\
&+ x^{59} + x^{57} + x^{56} + x^{50} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44} + x^{42} + x^{40}\\
&+ x^{39} + x^{37} + x^{35} + x^{33} + x^{28} + x^{27} + x^{25} + x^{24} + x^{18} + x^{17} + x^{16}\\
&+ x^{15} + x^{13} + x^{12} + x^{8} + x^{4} + x^{3},\\
p_{12}(x) &= x^{156} + x^{152} + x^{148} + x^{132} + x^{128} + x^{124} + x^{120} + x^{108} + x^{104} + x^{100}\\
&+ x^{96} + x^{92} + x^{88} + x^{80} + x^{76} + x^{72} + x^{68} + x^{64} + x^{60} + x^{56} + x^{48}\\
&+ x^{44} + x^{40} + x^{36} + x^{32} + x^{28} + x^{24} + x^{16} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 1].

For $\phi(W^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{147} + x^{145} + x^{141} + x^{139} + x^{137} + x^{136} + x^{135} + x^{134} + x^{133} + x^{132}\\
&+ x^{131} + x^{129} + x^{128} + x^{127} + x^{126} + x^{125} + x^{122} + x^{120} + x^{114}\\
&+ x^{112} + x^{110} + x^{109} + x^{105} + x^{104} + x^{101} + x^{99} + x^{96} + x^{94} + x^{91}\\
&+ x^{90} + x^{89} + x^{86} + x^{85} + x^{83} + x^{81} + x^{78} + x^{75} + x^{73} + x^{72} + x^{69}\\
&+ x^{68} + x^{67} + x^{66} + x^{65} + x^{63} + x^{62} + x^{58} + x^{57} + x^{55} + x^{54} + x^{53}\\
&+ x^{51} + x^{48} + x^{44} + x^{38} + x^{36} + x^{33},\\
p_3(x) &= x^{145} + x^{144} + x^{142} + x^{139} + x^{138} + x^{137} + x^{134} + x^{131} + x^{129} + x^{127}\\
&+ x^{123} + x^{122} + x^{121} + x^{119} + x^{117} + x^{116} + x^{115} + x^{114} + x^{113}\\
&+ x^{112} + x^{109} + x^{108} + x^{107} + x^{106} + x^{103} + x^{101} + x^{100} + x^{97} + x^{95}\\
&+ x^{91} + x^{89} + x^{87} + x^{83} + x^{82} + x^{81} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75}\\
&+ x^{74} + x^{72} + x^{66} + x^{63} + x^{62} + x^{61} + x^{60} + x^{58} + x^{57} + x^{56} + x^{55}\\
&+ x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{47} + x^{43} + x^{42} + x^{41} + x^{40}\\
&+ x^{39} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{31} + x^{29} + x^{28} + x^{27} + x^{25}\\
&+ x^{22} + x^{21} + x^{20} + x^{19} + x^{18} + x^{16} + x^{14} + x^{13} + x^{12},\\
p_6(x) &= x^{145} + x^{144} + x^{142} + x^{139} + x^{138} + x^{137} + x^{135} + x^{134} + x^{131} + x^{125}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{124} + x^{123} + x^{122} + x^{119} + x^{117} + x^{115} + x^{114} + x^{113} + x^{112}\\
&+ x^{109} + x^{108} + x^{102} + x^{101} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{91}\\
&+ x^{89} + x^{87} + x^{85} + x^{84} + x^{83} + x^{79} + x^{75} + x^{74} + x^{71} + x^{70} + x^{67}\\
&+ x^{64} + x^{62} + x^{61} + x^{60} + x^{59} + x^{56} + x^{55} + x^{53} + x^{50} + x^{49} + x^{48}\\
&+ x^{47} + x^{46} + x^{45} + x^{43} + x^{42} + x^{41} + x^{38} + x^{35} + x^{32} + x^{31} + x^{28}\\
&+ x^{27} + x^{22} + x^{21} + x^{19} + x^{17} + x^{16} + x^{12} + x^{11} + x^{9} + x^{8} + x^{7} + x^{6}\\
&+ x^{3} + x^{2} + x + 1,\\
p_9(x) = {}& x^{145} + x^{144} + x^{143} + x^{142} + x^{139} + x^{138} + x^{135} + x^{134} + x^{133} + x^{132}\\
&+ x^{125} + x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{113} + x^{112} + x^{111}\\
&+ x^{110} + x^{109} + x^{108} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100}\\
&+ x^{97} + x^{96} + x^{95} + x^{94} + x^{91} + x^{90} + x^{87} + x^{86} + x^{85} + x^{84} + x^{81}\\
&+ x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68}\\
&+ x^{65} + x^{64} + x^{63} + x^{62} + x^{59} + x^{58} + x^{55} + x^{54} + x^{53} + x^{52} + x^{49}\\
&+ x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36}\\
&+ x^{33} + x^{32} + x^{31} + x^{30} + x^{27} + x^{26} + x^{23} + x^{22} + x^{21} + x^{20} + x^{17}\\
&+ x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{9} + x^{8} + x^{7} + x^{6} + x^{5} + x^{4},\\
p_{12}(x) = {}& x^{145} + x^{144} + x^{143} + x^{142} + x^{139} + x^{138} + x^{137} + x^{136} + x^{133} + x^{132}\\
&+ x^{131} + x^{130} + x^{127} + x^{126} + x^{125} + x^{124} + x^{109} + x^{108} + x^{107}\\
&+ x^{106} + x^{105} + x^{104} + x^{101} + x^{100} + x^{99} + x^{98} + x^{95} + x^{94} + x^{91}\\
&+ x^{90} + x^{89} + x^{88} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{77} + x^{76}\\
&+ x^{75} + x^{74} + x^{73} + x^{72} + x^{69} + x^{68} + x^{67} + x^{66} + x^{63} + x^{62} + x^{59}\\
&+ x^{58} + x^{57} + x^{56} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{45} + x^{44}\\
&+ x^{43} + x^{42} + x^{41} + x^{40} + x^{37} + x^{36} + x^{35} + x^{34} + x^{31} + x^{30} + x^{27}\\
&+ x^{26} + x^{25} + x^{24} + x^{21} + x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{13} + x^{12}\\
&+ x^{11} + x^{10} + x^{9} + x^{8} + x^{5} + x^{4} + x^{3} + x^{2} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 1, 1].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	191	[333, 173]	382	[549, 29]	573	[431, 457]	764	[892, 893]
1	[3, 4]	192	[334, 36]	383	[550, 551]	574	[738, 49]	765	[894, 895]
2	[5, 6]	193	[335, 185]	384	[552, 553]	575	[463, 739]	766	[896, 897]
3	[7, 8]	194	[333, 336]	385	[554, 555]	576	[740, 431]	767	[898, 257]
4	[9, 10]	195	[337, 338]	386	[76, 556]	577	[412, 12]	768	[899, 900]
5	[11, 12]	196	[339, 340]	387	[557, 318]	578	[465, 340]	769	[699, 174]
6	[13, 14]	197	[341, 342]	388	[558, 559]	579	[444, 378]	770	[901, 157]

7	[15, 16]	198	[133, 343]	389	[560, 561]	580	[741, 135]	771	[405, 128]
8	[17, 18]	199	[344, 345]	390	[562, 563]	581	[710, 175]	772	[745, 348]
9	[19, 20]	200	[346, 347]	391	[564, 565]	582	[401, 35]	773	[691, 340]
10	[21, 22]	201	[337, 348]	392	[566, 567]	583	[401, 171]	774	[460, 366]
11	[23, 24]	202	[349, 14]	393	[568, 569]	584	[415, 123]	775	[152, 162]
12	[25, 26]	203	[350, 351]	394	[570, 571]	585	[742, 364]	776	[352, 370]
13	[27, 28]	204	[163, 163]	395	[572, 573]	586	[8, 375]	777	[902, 152]
14	[29, 3]	205	[352, 183]	396	[574, 575]	587	[48, 413]	778	[901, 903]
15	[30, 31]	206	[351, 183]	397	[576, 280]	588	[454, 739]	779	[385, 434]
16	[32, 33]	207	[353, 354]	398	[577, 578]	589	[452, 428]	780	[737, 417]
17	[34, 35]	208	[355, 356]	399	[516, 579]	590	[436, 717]	781	[717, 437]
18	[36, 37]	209	[9, 357]	400	[580, 581]	591	[732, 451]	782	[397, 452]
19	[38, 39]	210	[358, 148]	401	[582, 583]	592	[127, 743]	783	[765, 766]
20	[40, 41]	211	[359, 348]	402	[584, 99]	593	[702, 451]	784	[745, 44]
21	[42, 43]	212	[149, 360]	403	[585, 215]	594	[444, 136]	785	[353, 457]
22	[44, 8]	213	[36, 361]	404	[551, 586]	595	[335, 179]	786	[904, 43]
23	[45, 46]	214	[130, 362]	405	[587, 588]	596	[744, 431]	787	[751, 739]
24	[47, 33]	215	[363, 343]	406	[589, 590]	597	[745, 460]	788	[410, 452]
25	[48, 49]	216	[364, 152]	407	[591, 105]	598	[383, 148]	789	[410, 369]
26	[50, 51]	217	[365, 177]	408	[592, 593]	599	[746, 707]	790	[729, 444]
27	[52, 53]	218	[366, 35]	409	[594, 595]	600	[363, 134]	791	[698, 722]
28	[54, 55]	219	[367, 368]	410	[596, 597]	601	[169, 126]	792	[767, 358]
29	[56, 49]	220	[336, 121]	411	[598, 506]	602	[732, 422]	793	[710, 343]
30	[57, 58]	221	[369, 182]	412	[599, 600]	603	[393, 173]	794	[757, 441]
31	[59, 60]	222	[351, 370]	413	[22, 601]	604	[747, 154]	795	[165, 367]
32	[61, 62]	223	[371, 13]	414	[602, 603]	605	[742, 397]	796	[452, 186]
33	[63, 64]	224	[372, 347]	415	[110, 311]	606	[748, 138]	797	[136, 379]
34	[65, 66]	225	[49, 373]	416	[604, 605]	607	[373, 365]	798	[350, 163]
35	[67, 68]	226	[132, 374]	417	[606, 607]	608	[454, 727]	799	[708, 134]
36	[69, 70]	227	[51, 375]	418	[608, 609]	609	[402, 460]	800	[901, 179]
37	[71, 72]	228	[154, 376]	419	[610, 611]	610	[372, 704]	801	[700, 716]
38	[73, 74]	229	[377, 181]	420	[612, 613]	611	[749, 457]	802	[437, 429]
39	[75, 76]	230	[169, 362]	421	[614, 615]	612	[119, 118]	803	[905, 367]
40	[77, 78]	231	[378, 379]	422	[6, 616]	613	[429, 717]	804	[741, 167]
41	[79, 80]	232	[164, 183]	423	[617, 618]	614	[713, 750]	805	[154, 368]
42	[81, 82]	233	[380, 157]	424	[619, 620]	615	[734, 739]	806	[335, 903]
43	[83, 84]	234	[381, 117]	425	[621, 622]	616	[353, 432]	807	[353, 725]
44	[85, 86]	235	[382, 349]	426	[623, 624]	617	[751, 752]	808	[416, 430]
45	[87, 88]	236	[139, 357]	427	[625, 626]	618	[372, 373]	809	[697, 364]
46	[89, 90]	237	[383, 347]	428	[627, 628]	619	[115, 354]	810	[708, 407]
47	[91, 92]	238	[11, 384]	429	[206, 629]	620	[339, 112]	811	[734, 752]
48	[93, 94]	239	[385, 340]	430	[630, 631]	621	[730, 418]	812	[147, 415]
49	[95, 96]	240	[336, 136]	431	[632, 633]	622	[397, 152]	813	[412, 356]
50	[97, 98]	241	[184, 386]	432	[634, 635]	623	[753, 353]	814	[358, 704]

51	[99, 100]	242	[387, 167]	433	[636, 637]	624	[383, 373]	815	[738, 694]
52	[101, 102]	243	[388, 389]	434	[638, 639]	625	[754, 425]	816	[737, 345]
53	[103, 104]	244	[188, 134]	435	[640, 641]	626	[344, 448]	817	[694, 148]
54	[105, 106]	245	[390, 391]	436	[642, 643]	627	[402, 7]	818	[141, 725]
55	[107, 108]	246	[127, 368]	437	[613, 644]	628	[755, 142]	819	[158, 174]
56	[109, 110]	247	[338, 34]	438	[645, 321]	629	[183, 163]	820	[387, 730]
57	[111, 112]	248	[392, 54]	439	[646, 205]	630	[373, 123]	821	[49, 415]
58	[113, 10]	249	[393, 336]	440	[647, 4]	631	[132, 362]	822	[906, 759]
59	[114, 115]	250	[394, 395]	441	[648, 572]	632	[756, 391]	823	[174, 407]
60	[112, 50]	251	[396, 397]	442	[649, 491]	633	[154, 722]	824	[348, 401]
61	[116, 117]	252	[398, 399]	443	[650, 651]	634	[442, 358]	825	[435, 142]
62	[118, 119]	253	[400, 401]	444	[652, 653]	635	[704, 125]	826	[48, 358]
63	[38, 120]	254	[402, 50]	445	[654, 655]	636	[757, 120]	827	[907, 425]
64	[121, 122]	255	[403, 404]	446	[656, 657]	637	[742, 410]	828	[128, 452]
65	[123, 124]	256	[366, 171]	447	[658, 659]	638	[148, 125]	829	[764, 908]
66	[125, 126]	257	[405, 41]	448	[660, 255]	639	[125, 362]	830	[711, 358]
67	[127, 33]	258	[406, 407]	449	[661, 662]	640	[758, 759]	831	[765, 448]
68	[128, 129]	259	[408, 53]	450	[663, 664]	641	[159, 343]	832	[909, 149]
69	[130, 131]	260	[367, 33]	451	[665, 5]	642	[437, 118]	833	[910, 127]
70	[132, 126]	261	[409, 406]	452	[666, 667]	643	[420, 429]	834	[42, 750]
71	[133, 134]	262	[30, 410]	453	[668, 669]	644	[118, 429]	835	[458, 145]
72	[135, 136]	263	[411, 412]	454	[670, 671]	645	[420, 439]	836	[41, 129]
73	[137, 138]	264	[413, 148]	455	[672, 474]	646	[438, 351]	837	[380, 395]
74	[139, 10]	265	[414, 145]	456	[673, 674]	647	[760, 703]	838	[189, 128]
75	[140, 14]	266	[413, 415]	457	[675, 676]	648	[149, 761]	839	[755, 354]
76	[141, 142]	267	[416, 112]	458	[677, 678]	649	[762, 367]	840	[111, 430]
77	[143, 49]	268	[344, 417]	459	[679, 680]	650	[430, 7]	841	[173, 181]
78	[144, 145]	269	[418, 419]	460	[681, 67]	651	[711, 372]	842	[156, 386]
79	[146, 13]	270	[420, 119]	461	[682, 683]	652	[763, 349]	843	[172, 336]
80	[147, 148]	271	[421, 422]	462	[684, 685]	653	[340, 44]	844	[911, 188]
81	[149, 14]	272	[423, 160]	463	[686, 687]	654	[51, 446]	845	[396, 410]
82	[150, 49]	273	[424, 425]	464	[688, 689]	655	[34, 171]	846	[754, 736]
83	[151, 152]	274	[113, 357]	465	[690, 85]	656	[708, 155]	847	[714, 132]
84	[135, 121]	275	[412, 384]	466	[135, 381]	657	[730, 378]	848	[694, 704]
85	[153, 128]	276	[426, 381]	467	[691, 337]	658	[764, 120]	849	[339, 337]
86	[154, 55]	277	[427, 179]	468	[692, 459]	659	[153, 41]	850	[751, 455]
87	[133, 155]	278	[423, 428]	469	[166, 135]	660	[760, 138]	851	[56, 413]
88	[156, 157]	279	[119, 429]	470	[398, 462]	661	[372, 415]	852	[427, 386]
89	[158, 159]	280	[430, 50]	471	[144, 459]	662	[143, 694]	853	[161, 186]
90	[152, 160]	281	[431, 432]	472	[693, 141]	663	[748, 703]	854	[359, 338]
91	[161, 162]	282	[433, 10]	473	[694, 415]	664	[765, 417]	855	[111, 402]
92	[163, 164]	283	[434, 44]	474	[347, 125]	665	[749, 354]	856	[7, 401]
93	[165, 54]	284	[340, 7]	475	[396, 31]	666	[447, 766]	857	[695, 368]
94	[166, 167]	285	[435, 432]	476	[695, 55]	667	[430, 348]	858	[394, 185]

95	[168, 169]	286	[436, 437]	477	[696, 127]	668	[464, 12]	859	[173, 121]
96	[170, 171]	287	[438, 164]	478	[697, 397]	669	[721, 402]	860	[421, 441]
97	[172, 173]	288	[439, 437]	479	[338, 36]	670	[178, 157]	861	[349, 384]
98	[174, 175]	289	[370, 370]	480	[445, 375]	671	[30, 397]	862	[731, 402]
99	[176, 124]	290	[167, 418]	481	[698, 33]	672	[424, 736]	863	[912, 913]
100	[123, 177]	291	[440, 441]	482	[38, 451]	673	[767, 383]	864	[32, 368]
101	[159, 155]	292	[442, 383]	483	[699, 159]	674	[403, 125]	865	[173, 418]
102	[178, 179]	293	[443, 149]	484	[700, 182]	675	[150, 358]	866	[902, 161]
103	[180, 127]	294	[402, 44]	485	[136, 701]	676	[415, 132]	867	[450, 422]
104	[181, 117]	295	[387, 444]	486	[702, 39]	677	[421, 451]	868	[166, 730]
105	[129, 182]	296	[47, 368]	487	[697, 410]	678	[741, 730]	869	[735, 425]
106	[183, 164]	297	[445, 361]	488	[137, 703]	679	[461, 399]	870	[168, 132]
107	[184, 185]	298	[365, 124]	489	[704, 404]	680	[460, 34]	871	[434, 348]
108	[152, 186]	299	[8, 446]	490	[8, 37]	681	[40, 128]	872	[416, 402]
109	[187, 188]	300	[445, 37]	491	[705, 336]	682	[431, 142]	873	[13, 761]
110	[189, 41]	301	[31, 369]	492	[159, 134]	683	[767, 694]	874	[738, 372]
111	[190, 191]	302	[150, 383]	493	[706, 707]	684	[41, 452]	875	[914, 913]
112	[192, 17]	303	[447, 448]	494	[708, 175]	685	[151, 129]	876	[695, 722]
113	[193, 194]	304	[449, 141]	495	[163, 438]	686	[440, 451]	877	[188, 915]
114	[195, 196]	305	[346, 373]	496	[119, 119]	687	[705, 173]	878	[764, 441]
115	[197, 198]	306	[370, 183]	497	[118, 439]	688	[768, 46]	879	[916, 138]
116	[199, 200]	307	[450, 451]	498	[350, 164]	689	[698, 55]	880	[447, 417]
117	[201, 202]	308	[452, 160]	499	[183, 438]	690	[180, 154]	881	[406, 134]
118	[203, 204]	309	[125, 374]	500	[709, 342]	691	[769, 97]	882	[184, 157]
119	[205, 206]	310	[170, 35]	501	[710, 407]	692	[770, 771]	883	[747, 127]
120	[207, 208]	311	[188, 343]	502	[711, 383]	693	[772, 773]	884	[719, 701]
121	[209, 210]	312	[41, 161]	503	[373, 132]	694	[774, 217]	885	[141, 456]
122	[211, 212]	313	[453, 399]	504	[712, 389]	695	[775, 776]	886	[56, 358]
123	[213, 214]	314	[454, 455]	505	[406, 175]	696	[777, 778]	887	[416, 434]
124	[215, 216]	315	[431, 456]	506	[465, 112]	697	[779, 780]	888	[714, 365]
125	[217, 218]	316	[435, 457]	507	[148, 404]	698	[234, 781]	889	[336, 418]
126	[219, 220]	317	[31, 152]	508	[713, 43]	699	[782, 307]	890	[413, 168]
127	[221, 222]	318	[458, 459]	509	[334, 51]	700	[783, 784]	891	[731, 430]
128	[223, 224]	319	[340, 460]	510	[714, 169]	701	[541, 785]	892	[917, 43]
129	[78, 225]	320	[13, 12]	511	[51, 37]	702	[786, 787]	893	[692, 145]
130	[226, 227]	321	[164, 370]	512	[36, 375]	703	[788, 789]	894	[427, 185]
131	[228, 229]	322	[461, 462]	513	[385, 112]	704	[790, 791]	895	[729, 167]
132	[96, 230]	323	[463, 455]	514	[168, 365]	705	[792, 209]	896	[916, 703]
133	[231, 232]	324	[464, 384]	515	[406, 155]	706	[793, 794]	897	[415, 365]
134	[233, 234]	325	[464, 360]	516	[426, 136]	707	[795, 796]	898	[733, 406]
135	[235, 201]	326	[115, 142]	517	[444, 181]	708	[797, 798]	899	[32, 743]
136	[236, 237]	327	[377, 136]	518	[715, 716]	709	[799, 800]	900	[440, 908]
137	[238, 239]	328	[394, 157]	519	[439, 717]	710	[801, 802]	901	[918, 919]
138	[72, 240]	329	[181, 122]	520	[718, 115]	711	[803, 804]	902	[920, 921]

139	[241, 242]	330	[438, 163]	521	[434, 460]	712	[805, 806]	903	[922, 923]
140	[243, 244]	331	[465, 434]	522	[404, 124]	713	[807, 808]	904	[924, 925]
141	[245, 246]	332	[466, 233]	523	[169, 131]	714	[809, 810]	905	[926, 927]
142	[196, 247]	333	[467, 468]	524	[176, 177]	715	[811, 812]	906	[928, 929]
143	[248, 249]	334	[469, 71]	525	[719, 419]	716	[264, 813]	907	[930, 931]
144	[250, 251]	335	[470, 471]	526	[429, 437]	717	[717, 717]	908	[932, 933]
145	[252, 253]	336	[472, 473]	527	[702, 120]	718	[814, 815]	909	[934, 513]
146	[254, 25]	337	[474, 65]	528	[418, 720]	719	[816, 817]	910	[935, 63]
147	[255, 256]	338	[475, 476]	529	[112, 348]	720	[473, 818]	911	[936, 937]
148	[257, 258]	339	[477, 478]	530	[437, 119]	721	[819, 820]	912	[938, 939]
149	[259, 260]	340	[479, 480]	531	[346, 148]	722	[567, 801]	913	[940, 941]
150	[261, 262]	341	[481, 482]	532	[721, 430]	723	[821, 675]	914	[942, 943]
151	[263, 264]	342	[483, 484]	533	[50, 366]	724	[822, 823]	915	[944, 221]
152	[265, 266]	343	[328, 485]	534	[130, 126]	725	[773, 824]	916	[945, 946]
153	[267, 268]	344	[486, 487]	535	[54, 722]	726	[825, 826]	917	[947, 948]
154	[269, 270]	345	[488, 489]	536	[695, 376]	727	[827, 638]	918	[907, 736]
155	[271, 272]	346	[489, 490]	537	[167, 121]	728	[828, 829]	919	[433, 357]
156	[273, 274]	347	[491, 219]	538	[173, 378]	729	[830, 831]	920	[949, 412]
157	[275, 75]	348	[191, 492]	539	[335, 386]	730	[832, 211]	921	[49, 347]
158	[276, 277]	349	[493, 494]	540	[723, 353]	731	[833, 681]	922	[435, 456]
159	[278, 279]	350	[495, 496]	541	[358, 415]	732	[834, 835]	923	[755, 725]
160	[280, 281]	351	[497, 498]	542	[337, 50]	733	[836, 591]	924	[355, 14]
161	[282, 283]	352	[499, 286]	543	[49, 704]	734	[837, 838]	925	[143, 372]
162	[284, 285]	353	[500, 501]	544	[724, 725]	735	[839, 840]	926	[364, 700]
163	[286, 287]	354	[502, 503]	545	[726, 399]	736	[216, 841]	927	[178, 903]
164	[288, 289]	355	[504, 505]	546	[414, 459]	737	[842, 843]	928	[54, 33]
165	[290, 291]	356	[506, 507]	547	[463, 727]	738	[844, 845]	929	[421, 908]
166	[292, 236]	357	[508, 509]	548	[359, 50]	739	[846, 847]	930	[140, 761]
167	[293, 294]	358	[510, 511]	549	[112, 7]	740	[848, 634]	931	[442, 49]
168	[295, 296]	359	[253, 512]	550	[348, 8]	741	[849, 850]	932	[355, 761]
169	[297, 298]	360	[513, 514]	551	[404, 177]	742	[851, 265]	933	[749, 432]
170	[299, 300]	361	[515, 516]	552	[111, 434]	743	[852, 853]	934	[147, 168]
171	[16, 301]	362	[296, 517]	553	[347, 404]	744	[854, 855]	935	[730, 121]
172	[302, 303]	363	[518, 519]	554	[728, 159]	745	[856, 582]	936	[410, 129]
173	[304, 305]	364	[520, 521]	555	[729, 730]	746	[857, 858]	937	[757, 908]
174	[64, 306]	365	[256, 522]	556	[115, 725]	747	[859, 860]	938	[363, 915]
175	[307, 308]	366	[523, 524]	557	[731, 340]	748	[861, 862]	939	[450, 120]
176	[309, 310]	367	[525, 526]	558	[174, 343]	749	[863, 864]	940	[392, 367]
177	[311, 312]	368	[527, 528]	559	[732, 120]	750	[865, 866]	941	[129, 716]
178	[313, 314]	369	[58, 529]	560	[733, 188]	751	[867, 868]	942	[47, 376]
179	[315, 316]	370	[289, 530]	561	[116, 122]	752	[869, 870]	943	[427, 903]
180	[317, 107]	371	[531, 532]	562	[367, 376]	753	[871, 872]	944	[440, 422]
181	[318, 319]	372	[533, 534]	563	[380, 185]	754	[873, 874]	945	[724, 456]
182	[80, 320]	373	[249, 535]	564	[409, 188]	755	[875, 876]	946	[711, 694]

183	[321, 288]	374	[536, 537]	565	[378, 720]	756	[877, 878]	947	[749, 725]
184	[322, 323]	375	[492, 538]	566	[151, 700]	757	[879, 880]	948	[691, 112]
185	[324, 325]	376	[539, 518]	567	[156, 185]	758	[881, 882]	949	[950, 951]
186	[60, 326]	377	[540, 541]	568	[721, 340]	759	[883, 884]	950	[340, 50]
187	[327, 328]	378	[468, 542]	569	[734, 455]	760	[885, 886]	951	[767, 372]
188	[329, 330]	379	[543, 544]	570	[735, 736]	761	[887, 888]		
189	[331, 282]	380	[545, 546]	571	[737, 448]	762	[889, 527]		
190	[332, 159]	381	[547, 548]	572	[724, 142]	763	[890, 891]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	137	1	274	1	411	0	548	1	685	0	822	0
1	0	138	0	275	1	412	1	549	0	686	1	823	1
2	1	139	0	276	1	413	1	550	0	687	0	824	0
3	0	140	1	277	0	414	1	551	0	688	0	825	1
4	1	141	1	278	0	415	0	552	0	689	1	826	1
5	1	142	1	279	0	416	1	553	0	690	0	827	1
6	0	143	0	280	1	417	0	554	1	691	0	828	1
7	0	144	0	281	0	418	0	555	1	692	1	829	1
8	0	145	1	282	0	419	1	556	1	693	0	830	1
9	1	146	0	283	1	420	0	557	0	694	0	831	0
10	1	147	0	284	1	421	0	558	1	695	1	832	0
11	1	148	1	285	1	422	0	559	1	696	1	833	0
12	1	149	1	286	1	423	0	560	0	697	1	834	1
13	0	150	1	287	0	424	1	561	0	698	1	835	0
14	0	151	0	288	0	425	0	562	1	699	0	836	0
15	0	152	1	289	1	426	1	563	1	700	0	837	1
16	0	153	1	290	1	427	0	564	1	701	1	838	0
17	0	154	0	291	1	428	0	565	1	702	0	839	0
18	0	155	0	292	0	429	1	566	0	703	1	840	0
19	1	156	1	293	1	430	1	567	1	704	1	841	1
20	0	157	1	294	0	431	0	568	1	705	0	842	1
21	1	158	1	295	0	432	0	569	1	706	0	843	1
22	1	159	0	296	0	433	0	570	0	707	1	844	1
23	1	160	1	297	1	434	1	571	1	708	0	845	1
24	0	161	0	298	1	435	1	572	0	709	0	846	0
25	1	162	1	299	0	436	1	573	0	710	0	847	1
26	1	163	1	300	1	437	1	574	1	711	1	848	0
27	0	164	0	301	0	438	0	575	1	712	0	849	1
28	0	165	1	302	1	439	0	576	0	713	0	850	0
29	0	166	0	303	1	440	1	577	1	714	1	851	0
30	0	167	1	304	1	441	1	578	0	715	1	852	0
31	0	168	0	305	1	442	0	579	1	716	1	853	0
32	0	169	1	306	1	443	1	580	1	717	0	854	1

33	1	170	0	307	0	444	1	581	0	718	1	855	0
34	0	171	0	308	1	445	1	582	1	719	1	856	0
35	1	172	1	309	1	446	0	583	1	720	0	857	1
36	0	173	1	310	0	447	1	584	0	721	1	858	0
37	1	174	1	311	0	448	1	585	0	722	1	859	1
38	1	175	0	312	0	449	1	586	0	723	0	860	0
39	1	176	1	313	0	450	0	587	1	724	0	861	0
40	0	177	0	314	0	451	1	588	0	725	0	862	0
41	0	178	0	315	0	452	1	589	1	726	1	863	1
42	1	179	0	316	1	453	0	590	1	727	1	864	0
43	0	180	0	317	0	454	0	591	1	728	1	865	1
44	1	181	0	318	0	455	1	592	1	729	1	866	1
45	1	182	0	319	1	456	0	593	0	730	0	867	0
46	1	183	0	320	0	457	1	594	1	731	0	868	0
47	0	184	0	321	0	458	0	595	1	732	1	869	0
48	1	185	0	322	0	459	0	596	1	733	0	870	0
49	0	186	0	323	1	460	0	597	0	734	1	871	1
50	1	187	0	324	0	461	0	598	0	735	0	872	1
51	1	188	0	325	0	462	0	599	1	736	1	873	0
52	0	189	0	326	1	463	1	600	1	737	1	874	1
53	0	190	0	327	0	464	0	601	1	738	1	875	0
54	0	191	0	328	0	465	0	602	1	739	0	876	1
55	0	192	0	329	0	466	0	603	1	740	0	877	0
56	0	193	1	330	0	467	0	604	1	741	1	878	1
57	0	194	0	331	0	468	1	605	0	742	0	879	0
58	1	195	0	332	0	469	0	606	0	743	0	880	1
59	0	196	1	333	0	470	1	607	1	744	1	881	1
60	0	197	1	334	0	471	0	608	0	745	0	882	0
61	0	198	1	335	1	472	0	609	0	746	1	883	1
62	1	199	0	336	0	473	0	610	1	747	1	884	1
63	1	200	1	337	0	474	0	611	1	748	0	885	1
64	1	201	0	338	1	475	1	612	0	749	1	886	0
65	0	202	0	339	1	476	1	613	1	750	1	887	1
66	1	203	1	340	1	477	1	614	0	751	0	888	1
67	1	204	1	341	1	478	1	615	1	752	0	889	0
68	1	205	0	342	0	479	1	616	0	753	1	890	1
69	0	206	1	343	0	480	1	617	0	754	0	891	0
70	0	207	0	344	0	481	1	618	1	755	0	892	1
71	1	208	0	345	1	482	1	619	1	756	0	893	1
72	0	209	1	346	1	483	0	620	1	757	0	894	0
73	1	210	1	347	0	484	0	621	0	758	1	895	1
74	0	211	1	348	0	485	0	622	0	759	1	896	0
75	1	212	1	349	0	486	0	623	1	760	1	897	0
76	1	213	0	350	1	487	1	624	0	761	1	898	0

77	0	214	0	351	1	488	1	625	0	762	0	899	0
78	0	215	1	352	0	489	1	626	0	763	1	900	1
79	0	216	1	353	0	490	0	627	0	764	1	901	1
80	0	217	1	354	1	491	0	628	0	765	0	902	1
81	1	218	1	355	0	492	0	629	0	766	0	903	1
82	1	219	1	356	0	493	0	630	1	767	0	904	0
83	0	220	0	357	0	494	0	631	0	768	0	905	1
84	0	221	1	358	1	495	1	632	0	769	0	906	0
85	1	222	1	359	1	496	0	633	0	770	1	907	1
86	0	223	1	360	1	497	1	634	0	771	1	908	0
87	1	224	1	361	1	498	1	635	1	772	0	909	0
88	1	225	0	362	0	499	0	636	0	773	0	910	0
89	1	226	0	363	1	500	0	637	0	774	0	911	1
90	1	227	1	364	1	501	0	638	1	775	1	912	1
91	0	228	0	365	1	502	1	639	1	776	0	913	0
92	1	229	0	366	1	503	1	640	1	777	1	914	0
93	1	230	1	367	1	504	0	641	0	778	1	915	1
94	0	231	1	368	0	505	1	642	1	779	1	916	0
95	0	232	0	369	1	506	0	643	0	780	1	917	1
96	0	233	1	370	1	507	1	644	1	781	0	918	1
97	1	234	1	371	1	508	0	645	0	782	0	919	0
98	1	235	0	372	1	509	0	646	0	783	0	920	1
99	1	236	0	373	1	510	1	647	1	784	0	921	0
100	0	237	0	374	1	511	1	648	1	785	0	922	1
101	0	238	1	375	0	512	0	649	0	786	0	923	0
102	0	239	1	376	1	513	1	650	1	787	0	924	0
103	0	240	0	377	0	514	0	651	1	788	1	925	0
104	0	241	0	378	1	515	1	652	1	789	1	926	1
105	0	242	0	379	0	516	1	653	1	790	1	927	0
106	0	243	1	380	1	517	1	654	1	791	1	928	0
107	0	244	0	381	1	518	1	655	0	792	0	929	0
108	1	245	1	382	0	519	0	656	0	793	0	930	1
109	0	246	1	383	0	520	1	657	0	794	0	931	0
110	0	247	1	384	0	521	1	658	1	795	1	932	0
111	0	248	0	385	1	522	0	659	1	796	1	933	1
112	0	249	1	386	1	523	1	660	1	797	0	934	0
113	1	250	0	387	0	524	1	661	1	798	1	935	0
114	0	251	1	388	1	525	1	662	0	799	0	936	1
115	1	252	1	389	0	526	1	663	0	800	1	937	0
116	0	253	1	390	1	527	0	664	0	801	0	938	1
117	0	254	0	391	1	528	0	665	1	802	1	939	0
118	1	255	0	392	0	529	0	666	1	803	1	940	0
119	0	256	1	393	1	530	1	667	1	804	1	941	0
120	0	257	1	394	0	531	1	668	0	805	0	942	0

121	1	258	1	395	0	532	1	669	1	806	1	943	0
122	1	259	1	396	1	533	1	670	0	807	0	944	1
123	0	260	1	397	0	534	0	671	0	808	1	945	0
124	1	261	1	398	1	535	0	672	1	809	1	946	1
125	1	262	0	399	1	536	1	673	0	810	0	947	1
126	1	263	0	400	1	537	1	674	0	811	1	948	0
127	1	264	1	401	1	538	1	675	1	812	0	949	1
128	1	265	1	402	0	539	1	676	0	813	1	950	1
129	0	266	1	403	0	540	0	677	0	814	1	951	0
130	0	267	1	404	0	541	1	678	1	815	1		
131	0	268	0	405	1	542	0	679	0	816	1		
132	0	269	0	406	1	543	0	680	0	817	0		
133	1	270	0	407	1	544	0	681	0	818	1		
134	1	271	0	408	1	545	1	682	0	819	1		
135	0	272	0	409	1	546	1	683	0	820	0		
136	0	273	1	410	1	547	1	684	0	821	0		

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