# PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a, b) = (z^3 + z^2, z^4 + 1)$

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Abstract. The continued fraction conjecture claims that the continued fraction defined by the $(a, b)$-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements $(a, b)$ in $\mathbb{F}_2[x] \setminus \mathbb{F}_2$. In this paper we prove the conjecture for $(a, b) = (z^3 + z^2, z^4 + 1)$.

## 1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \ldots)$ of elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{\ddots + \cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let $a$ and $b$ be two distinct elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$. We consider the continued fraction (1.1) associated with the $(a, b)$-Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \ldots) = (a, b, b, a, b, a, a, b, \ldots)$$

that we shall denote

$$\mathrm{CF}(a, b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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**Theorem 1.1.** *When $(a,b)=(z^3+z^2, z^4+1)$, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

*where, for* $\mathrm{CF}(a,b)$*,*

$$\begin{aligned}
p_0(z)&=z^{17}+z^{16}+z^{15}+z^{14}+z^{12}+z^{11}+z^6+z^3+z^2+1,\\
p_1(z)&=z^{21}+z^{19}+z^{18}+z^{17}+z^{14}+z^{12}+z^{11}+z^{10}+z^7+z^4+z^3+z^2,\\
p_2(z)&=z^{24}+z^{18}+z^{14}+z^{12}+z^{10}+z^8+z^6+z^4,\\
p_3(z)&=z^{21}+z^{19}+z^{18}+z^{17}+z^{14}+z^{12}+z^{11}+z^{10}+z^7+z^4+z^3+z^2,\\
p_4(z)&=z^{18}+z^{17}+z^{16}+z^{15}+z^{12}+z^{11}+z^{10}+z^3+z^2,
\end{aligned}$$

*and for* $\mathrm{CF}(b,a)$*,*

$$\begin{aligned}
p_0(z)&=z^{17}+z^{16}+z^{15}+z^{14}+z^{11}+z^6+z^4+z^3+z^2,\\
p_1(z)&=z^{21}+z^{19}+z^{18}+z^{17}+z^{14}+z^{12}+z^{11}+z^{10}+z^7+z^4+z^3+z^2,\\
p_2(z)&=z^{24}+z^{18}+z^{14}+z^{12}+z^{10}+z^8+z^6+z^4,\\
p_3(z)&=z^{21}+z^{19}+z^{18}+z^{17}+z^{14}+z^{12}+z^{11}+z^{10}+z^7+z^4+z^3+z^2,\\
p_4(z)&=z^{18}+z^{17}+z^{16}+z^{15}+z^{11}+z^{10}+z^4+z^3+z^2+1.
\end{aligned}$$

## 2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x)&=x^{\deg(t_{2^n-1})}\begin{pmatrix} t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x)&=x^{\deg(\bar t_{2^n-1})}\begin{pmatrix} \bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix} \bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(\bar t_0)}\begin{pmatrix} \bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the $(b,a)$-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\geq 0$

$$\begin{aligned}
M_{n+1}(x)&=W_n(x)\cdot M_n(x),\\
W_{n+1}(x)&=M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer $d_n$. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four $2\times 2$ matrices $M^e, M^o, W^e, W^o$ as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(175, 175, 175, 175, 175)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices $M^e, M^o, W^e, W^o$. The first one is about relations between them; the second, about the structure of the each matrix.

**Lemma 2.1.** *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

*Proof.* We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z \left(\phi(W^o,0,0)(x,z),\ z^{12}\cdot\phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity $m$ and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 630 and substitute it for $y$ in $Q(x,y)$. We get a series of valuation less than 630, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z\left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity $\mu$, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$ to order 770 and substitute it for $y$ in $Q(x,y)$. We get a series of valuation less than 770, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. □

Define

$$R^e = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

**Lemma 2.2.** *For all integers $k\geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned}
M^e[:14u] = {} & M^e[:7u] + x^u M^e[:6u] + x^{4u} M^e[:3u] + x^{5u} M^e[:2u] + x^{6u} M^e[:u]\\
& + x^u M^e[:7u] + x^{2u} M^e[:6u] + x^{5u} M^e[:3u] + x^{6u} M^e[:2u]\\
& + x^{7u} M^e[:u] + x^{2u} M^e[:7u] + x^{3u} M^e[:6u] + x^{6u} M^e[:3u]\\
& + x^{7u} M^e[:2u] + x^{8u} M^e[:u] + x^{3u} M^e[:7u] + x^{4u} M^e[:6u]\\
& + x^{7u} M^e[:3u] + x^{8u} M^e[:2u] + x^{9u} M^e[:u] + x^{6u} M^e[:7u]\\
& + x^{7u} M^e[:6u] + x^{10u} M^e[:3u] + x^{11u} M^e[:2u] + x^{12u} M^e[:u]
\end{aligned}$$

$$
\begin{aligned}
&+ x^{7u}M^e[:7u] + x^{8u}M^e[:6u] + x^{11u}M^e[:3u] + x^{12u}M^e[:2u]\\
&+ x^{13u}M^e[:u] + (x^{7u} + x^{8u} + x^{9u} + x^{10u} + x^{13u})R^e,\\
W^e[:14u] = {}& W^e[:7u] + x^{u}W^e[:6u] + x^{4u}W^e[:3u] + x^{5u}W^e[:2u] + x^{6u}W^e[:u]\\
&+ x^{u}W^e[:7u] + x^{2u}W^e[:6u] + x^{5u}W^e[:3u] + x^{6u}W^e[:2u]\\
&+ x^{7u}W^e[:u] + x^{2u}W^e[:7u] + x^{3u}W^e[:6u] + x^{6u}W^e[:3u]\\
&+ x^{7u}W^e[:2u] + x^{8u}W^e[:u] + x^{3u}W^e[:7u] + x^{4u}W^e[:6u]\\
&+ x^{7u}W^e[:3u] + x^{8u}W^e[:2u] + x^{9u}W^e[:u] + x^{6u}W^e[:7u]\\
&+ x^{7u}W^e[:6u] + x^{10u}W^e[:3u] + x^{11u}W^e[:2u] + x^{12u}W^e[:u]\\
&+ x^{7u}W^e[:7u] + x^{8u}W^e[:6u] + x^{11u}W^e[:3u] + x^{12u}W^e[:2u]\\
&+ x^{13u}W^e[:u] + (x^{7u} + x^{8u} + x^{9u} + x^{10u} + x^{13u})R^e.
\end{aligned}
$$

*For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.*

$$
\begin{aligned}
M^o[:14u] = {}& M^o[:7u] + x^{u}M^o[:6u] + x^{4u}M^o[:3u] + x^{5u}M^o[:2u] + x^{6u}M^o[:u]\\
&+ x^{u}M^o[:7u] + x^{2u}M^o[:6u] + x^{5u}M^o[:3u] + x^{6u}M^o[:2u]\\
&+ x^{7u}M^o[:u] + x^{2u}M^o[:7u] + x^{3u}M^o[:6u] + x^{6u}M^o[:3u]\\
&+ x^{7u}M^o[:2u] + x^{8u}M^o[:u] + x^{3u}M^o[:7u] + x^{4u}M^o[:6u]\\
&+ x^{7u}M^o[:3u] + x^{8u}M^o[:2u] + x^{9u}M^o[:u] + x^{6u}M^o[:7u]\\
&+ x^{7u}M^o[:6u] + x^{10u}M^o[:3u] + x^{11u}M^o[:2u] + x^{12u}M^o[:u]\\
&+ x^{7u}M^o[:7u] + x^{8u}M^o[:6u] + x^{11u}M^o[:3u] + x^{12u}M^o[:2u]\\
&+ x^{13u}M^o[:u] + (x^{7u} + x^{8u} + x^{9u} + x^{10u} + x^{13u})R^o,\\
W^o[:14u] = {}& W^o[:7u] + x^{u}W^o[:6u] + x^{4u}W^o[:3u] + x^{5u}W^o[:2u] + x^{6u}W^o[:u]\\
&+ x^{u}W^o[:7u] + x^{2u}W^o[:6u] + x^{5u}W^o[:3u] + x^{6u}W^o[:2u]\\
&+ x^{7u}W^o[:u] + x^{2u}W^o[:7u] + x^{3u}W^o[:6u] + x^{6u}W^o[:3u]\\
&+ x^{7u}W^o[:2u] + x^{8u}W^o[:u] + x^{3u}W^o[:7u] + x^{4u}W^o[:6u]\\
&+ x^{7u}W^o[:3u] + x^{8u}W^o[:2u] + x^{9u}W^o[:u] + x^{6u}W^o[:7u]\\
&+ x^{7u}W^o[:6u] + x^{10u}W^o[:3u] + x^{11u}W^o[:2u] + x^{12u}W^o[:u]\\
&+ x^{7u}W^o[:7u] + x^{8u}W^o[:6u] + x^{11u}W^o[:3u] + x^{12u}W^o[:2u]\\
&+ x^{13u}W^o[:u] + (x^{7u} + x^{8u} + x^{9u} + x^{10u} + x^{13u})R^o.
\end{aligned}
$$

*Proof.* To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many $k$'s into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$
\begin{aligned}
T[:14u] = {}& T[:7u] + x^{u}T[:6u] + x^{4u}T[:3u] + x^{5u}T[:2u] + x^{6u}T[:u] + x^{u}T[:7u]\\
&+ x^{2u}T[:6u] + x^{5u}T[:3u] + x^{6u}T[:2u] + x^{7u}T[:u] + x^{2u}T[:7u]\\
&+ x^{3u}T[:6u] + x^{6u}T[:3u] + x^{7u}T[:2u] + x^{8u}T[:u] + x^{3u}T[:7u]\\
&+ x^{4u}T[:6u] + x^{7u}T[:3u] + x^{8u}T[:2u] + x^{9u}T[:u] + x^{6u}T[:7u]\\
&+ x^{7u}T[:6u] + x^{10u}T[:3u] + x^{11u}T[:2u] + x^{12u}T[:u] + x^{7u}T[:7u]
\end{aligned}
$$

$$+ x^{8u}T[:6u] + x^{11u}T[:3u] + x^{12u}T[:2u] + x^{13u}T[:u].$$

The proof of the other 15 cases are similar. We break down the above identity into 7 parts:

$$\begin{aligned}
0 &= x^{7u}T[:u] + x^{6u}T[u:2u] + x^{5u}T[2u:3u] + x^{4u}T[3u:4u] + x^{u}T[6u:7u] \\
&\quad + T[7u:8u], \\
0 &= x^{8u}T[:u] + x^{4u}T[4u:5u] + x^{2u}T[6u:7u] + T[8u:9u], \\
0 &= x^{9u}T[:u] + x^{7u}T[2u:3u] + x^{6u}T[3u:4u] + x^{4u}T[5u:6u] \\
&\quad + x^{3u}T[6u:7u] + T[9u:10u], \\
0 &= x^{10u}T[:u] + x^{8u}T[2u:3u] + x^{6u}T[4u:5u] + T[10u:11u], \\
0 &= x^{10u}T[u:2u] + x^{8u}T[3u:4u] + x^{6u}T[5u:6u] + T[11u:12u], \\
0 &= x^{10u}T[2u:3u] + x^{8u}T[4u:5u] + x^{6u}T[6u:7u] + T[12u:13u], \\
0 &= x^{13u}T[:u] + x^{12u}T[u:2u] + x^{11u}T[2u:3u] + x^{8u}T[5u:6u] \\
&\quad + x^{7u}T[6u:7u] + T[13u:14u],
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[110w]_2] \\
&\quad + T[[111w]_2], \qquad (2.7) \\
0 &= T[[w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1000w]_2], \qquad (2.8) \\
0 &= T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2] \\
&\quad + T[[1001w]_2], \qquad (2.9) \\
0 &= T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[1010w]_2], \qquad (2.10) \\
0 &= T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1011w]_2], \qquad (2.11) \\
0 &= T[[10w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1100w]_2], \qquad (2.12) \\
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[101w]_2] + T[[110w]_2] \\
&\quad + T[[1101w]_2], \qquad (2.13)
\end{aligned}$$

for all binary word $w$ of length $2k$ and $w \neq 0^{2k}$, and

$$\begin{aligned}
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[110w]_2] \\
&\quad + T[[111w]_2], \qquad (2.14) \\
0 &= T[[w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1000w]_2], \qquad (2.15) \\
0 &= T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2] \\
&\quad + T[[1001w]_2], \qquad (2.16) \\
0 &= T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[1010w]_2], \qquad (2.17) \\
0 &= T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1011w]_2], \qquad (2.18) \\
0 &= T[[10w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1100w]_2], \qquad (2.19) \\
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[101w]_2] + T[[110w]_2] \\
&\quad + T[[1101w]_2], \qquad (2.20)
\end{aligned}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates $T$ from its minimal polynomial and its first terms. This automaton has 210 states; its transition function and output function can be found in the annex. Let $A(s,w)$ denote the state reached after reading $w$ from right to left starting from the state $s$, and $\tau$ the output function. Define

$$E_{2k} = \{A(i,w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.20) can be written as

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,110)) \\ &\quad + \tau(A(s,111)), \end{aligned} \tag{2.21}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,100)) + \tau(A(s,110)) + \tau(A(s,1000)), \tag{2.22}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101)) \\ &\quad + \tau(A(s,110)) + \tau(A(s,1001)), \end{aligned} \tag{2.23}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,1010)), \tag{2.24}$$

$$0 = \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,1011)), \tag{2.25}$$

$$0 = \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,110)) + \tau(A(s,1100)), \tag{2.26}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,101)) \\ &\quad + \tau(A(s,110)) + \tau(A(s,1101)), \end{aligned} \tag{2.27}$$

for all $s \in E_{2k}$, and

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,110)) \\ &\quad + \tau(A(s,111)), \end{aligned} \tag{2.28}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,100)) + \tau(A(s,110)) + \tau(A(s,1000)), \tag{2.29}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101)) \\ &\quad + \tau(A(s,110)) + \tau(A(s,1001)), \end{aligned} \tag{2.30}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,1010)), \tag{2.31}$$

$$0 = \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,1011)), \tag{2.32}$$

$$0 = \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,110)) + \tau(A(s,1100)), \tag{2.33}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,101)) \\ &\quad + \tau(A(s,110)) + \tau(A(s,1101)), \end{aligned} \tag{2.34}$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{22}), E_{22}) = (A(i, 0^{16}), E_{16})$, so that we only have to verify that identities (2.21) through (2.34) hold for $2 \leq k \leq 11$, which turns out to be true. $\square$

In the following lemma, we express $M_{2k}$, $M_{2k+1}$, $W_{2k}$, and $W_{2k+1}$ in terms of $M^e$, $M^o$, $W^e$, and $W^o$.

**Lemma 2.3.** *For all integer $k \geq 2$, and $u = 2^{2k-1}$,*

$$\begin{aligned} M_{2k} &= M^e[:7u] + x^u M^e[:6u] + x^{4u} M^e[:3u] + x^{5u} M^e[:2u] + x^{6u} M^e[:u] \\ &\quad + x^{7u} R^e, \\ W_{2k} &= W^e[:7u] + x^u W^e[:6u] + x^{4u} W^e[:3u] + x^{5u} W^e[:2u] + x^{6u} W^e[:u] \\ &\quad + x^{7u} R^e. \end{aligned}$$

*For all integer $k \geq 2$, and $u = 2^{2k}$,*

$$\begin{aligned}M_{2k+1} &= M^o[:7u] + x^u M^o[:6u] + x^{4u} M^o[:3u] + x^{5u} M^o[:2u] + x^{6u} M^o[:u]\\ &\quad + x^{7u} R^o,\\ W_{2k+1} &= W^o[:7u] + x^u W^o[:6u] + x^{4u} W^o[:3u] + x^{5u} W^o[:2u] + x^{6u} W^o[:u]\\ &\quad + x^{7u} R^o.\end{aligned}$$

*Proof.* We call the four identities in Lemma 2.3 also by the name $M_{2k}$, $W_{2k}$, $M_{2k+1}$, and $W_{2k+1}$. For $n = 2$, the identities can be verified directly. For $n \geq 2$, we claim that

$$\begin{aligned}M_{2k} \wedge W_{2k} &\Rightarrow M_{2k+1} \wedge W_{2k+1},\\ M_{2k+1} \wedge W_{2k+1} &\Rightarrow M_{2k+2} \wedge W_{2k+2}.\end{aligned}$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity $M_{2k+1}$ is equal to

$$\begin{aligned}W_{2k} M_{2k} &= \big(W^e[:7v] + x^v W^e[:6v] + x^{4v} W^e[:3v] + x^{5v} W^e[:2v] + x^{6v} W^e[:v]\\ &\quad + x^{7u} R^e\big) \times \big(M^e[:7v] + x^v M^e[:6v] + x^{4v} M^e[:3v] + x^{5v} M^e[:2v]\\ &\quad + x^{6v} M^e[:v] + x^{7u} R^e\big);\end{aligned}\tag{2.35}$$

Call this expression $lhs$. Note that both sides of identity $M_{2k+1}$ have the same term of highest degree $x^{14v} R^o$. Therefore we only need to prove that their difference is $O(x^{14v})$. Using Lemma 2.1 it can be seen that the right side of identity $M_{2k+1}$ is congruent, modulo $x^{14v}$, to

$$\begin{aligned}&W^e[:14v] M^e[:14v] + x^{2v} W^e[:12v] M^e[:12v] + x^{8v} W^e[:6v] M^e[:6v]\\ &\qquad + x^{10v} W^e[:4v] M^e[:4v] + x^{12v} W^e[:2v] M^e[:2v].\end{aligned}$$

For all $n \leq 14$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call $rhs$. Define

$$\begin{aligned}X &:= x^v,\\ a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n,\\ b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n,\\ c &:= R^e.\end{aligned}$$

Using the notation introduced above, we can represent the expressions $lhs$ (2.35) and $rhs$ as polynomials in $\mathbb{F}_2[a_1, ..., a_{14}, b_1, ..., b_{14}, c][X]$. Note that it is not a problem that $a_j$ commutes with $b_k$ while $W^e$ does not commute with $M^e$, because in the expressions concerned, the products of $W^e$-terms and $M^e$-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{14})$, which completes the proof. $\square$

*Proof of Theorem 1.1.* We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \leq j, k \leq 1$,

$$\lim_{n \to \infty} M_{2n,j,k} = M^e_{j,k},$$

$$\lim_{n\to\infty} M_{2n+1,j,k} = M^o_{j,k},$$
$$\lim_{n\to\infty} W_{2n,j,k} = W^e_{j,k},$$
$$\lim_{n\to\infty} W_{2n+1,j,k} = W^o_{j,k}.$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a,b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x,y) = \mathrm{Res}_t\left(\phi(M^e,0,1)(x,t), y^{12}\phi(M^e,0,1)(x,t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x,y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$q_0(x) = x^{24} + x^{22} + x^{21} + x^{18} + x^{13} + x^{12} + x^{10} + x^9 + x^8 + x^7,$$
$$q_1(x) = x^{22} + x^{21} + x^{20} + x^{17} + x^{14} + x^{13} + x^{12} + x^{10} + x^7 + x^6 + x^5 + x^3,$$
$$q_2(x) = x^{20} + x^{18} + x^{16} + x^{14} + x^{12} + x^{10} + x^6 + 1,$$
$$q_3(x) = x^{22} + x^{21} + x^{20} + x^{17} + x^{14} + x^{13} + x^{12} + x^{10} + x^7 + x^6 + x^5 + x^3,$$
$$q_4(x) = x^{22} + x^{21} + x^{14} + x^{13} + x^{12} + x^9 + x^8 + x^7 + x^6.$$

The polynomial $Q(x,y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x,y)$ of multiplicity $m$ and $R(x,y) := P(x,y)/Q(x,y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 216, and substitute it for $y$ in $R(z,y)$, we get a series with valuation smaller than 216, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b) = f(1/z)$. □

## 3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e, 0, 0)$:

$$\begin{aligned} p_0(x) = {} & x^{132} + x^{130} + x^{128} + x^{126} + x^{116} + x^{114} + x^{110} + x^{108} + x^{106} + x^{104} \\ & + x^{100} + x^{94} + x^{92} + x^{90} + x^{84} + x^{82} + x^{80} + x^{76} + x^{74} + x^{72} + x^{70} \\ & + x^{68} + x^{66} + x^{64} + x^{60} + x^{58} + x^{52} + x^{50} + x^{48} + x^{38} + x^{36}, \\ p_3(x) = {} & x^{114} + x^{112} + x^{110} + x^{108} + x^{104} + x^{88} + x^{72} + x^{66} + x^{64} + x^{62} + x^{60} \\ & + x^{58} + x^{56} + x^{54} + x^{52} + x^{50} + x^{46} + x^{44} + x^{40} + x^{34} + x^{30} + x^{28} \\ & + x^{26} + x^{24} + x^{22} + x^{20}, \end{aligned}$$

$$\begin{aligned}
p_6(x) &= x^{124} + x^{122} + x^{114} + x^{108} + x^{104} + x^{100} + x^{96} + x^{92} + x^{90} + x^{86} \\
&\quad + x^{84} + x^{82} + x^{78} + x^{72} + x^{70} + x^{66} + x^{62} + x^{60} + x^{58} + x^{54} + x^{52} \\
&\quad + x^{46} + x^{44} + x^{42} + x^{40} + x^{36} + x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{24} \\
&\quad + x^{16} + x^{10} + x^{6} + 1, \\
p_9(x) &= x^{128} + x^{124} + x^{122} + x^{106} + x^{104} + x^{98} + x^{96} + x^{94} + x^{88} + x^{76} + x^{74} \\
&\quad + x^{68} + x^{66} + x^{60} + x^{46} + x^{44} + x^{42} + x^{40} + x^{38} + x^{36} + x^{30} + x^{26} \\
&\quad + x^{24} + x^{14} + x^{10} + x^{6}, \\
p_{12}(x) &= x^{128} + x^{112} + x^{64} + x^{48} + x^{32} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0].

For $\phi(M^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{129} + x^{126} + x^{124} + x^{122} + x^{119} + x^{118} + x^{117} + x^{115} + x^{114} + x^{113} \\
&\quad + x^{112} + x^{110} + x^{109} + x^{108} + x^{106} + x^{103} + x^{102} + x^{101} + x^{100} \\
&\quad + x^{96} + x^{94} + x^{93} + x^{92} + x^{91} + x^{89} + x^{86} + x^{83} + x^{82} + x^{80} + x^{78} \\
&\quad + x^{75} + x^{73} + x^{71} + x^{67} + x^{66} + x^{65} + x^{62} + x^{61} + x^{60} + x^{59} + x^{56} \\
&\quad + x^{55} + x^{53} + x^{51} + x^{50} + x^{48} + x^{44} + x^{40} + x^{39}, \\
p_3(x) &= x^{125} + x^{123} + x^{122} + x^{121} + x^{113} + x^{110} + x^{109} + x^{108} + x^{106} + x^{102} \\
&\quad + x^{101} + x^{100} + x^{98} + x^{97} + x^{89} + x^{86} + x^{84} + x^{83} + x^{81} + x^{78} + x^{76} \\
&\quad + x^{75} + x^{73} + x^{70} + x^{68} + x^{67} + x^{65} + x^{62} + x^{60} + x^{59} + x^{57} + x^{54} \\
&\quad + x^{52} + x^{51} + x^{49} + x^{46} + x^{44} + x^{43} + x^{41} + x^{38} + x^{37} + x^{36} + x^{34} \\
&\quad + x^{30} + x^{28} + x^{27}, \\
p_6(x) &= x^{123} + x^{122} + x^{117} + x^{116} + x^{113} + x^{112} + x^{105} + x^{104} + x^{103} + x^{102} \\
&\quad + x^{101} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{89} \\
&\quad + x^{88} + x^{87} + x^{86} + x^{83} + x^{82} + x^{77} + x^{76} + x^{73} + x^{72} + x^{71} + x^{70} \\
&\quad + x^{63} + x^{62} + x^{59} + x^{58} + x^{57} + x^{56} + x^{49} + x^{48} + x^{45} + x^{44} + x^{43} \\
&\quad + x^{42} + x^{31} + x^{30} + x^{25} + x^{24} + x^{21} + x^{20} + x^{19} + x^{18}, \\
p_9(x) &= x^{121} + x^{118} + x^{117} + x^{116} + x^{114} + x^{111} + x^{110} + x^{109} + x^{105} + x^{102} \\
&\quad + x^{101} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{92} + x^{88} + x^{87} \\
&\quad + x^{86} + x^{84} + x^{81} + x^{80} + x^{79} + x^{75} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} \\
&\quad + x^{67} + x^{66} + x^{65} + x^{64} + x^{62} + x^{58} + x^{56} + x^{53} + x^{52} + x^{51} + x^{49} \\
&\quad + x^{47} + x^{46} + x^{42} + x^{40} + x^{39} + x^{33} + x^{31} + x^{30} + x^{27} + x^{26} + x^{25} \\
&\quad + x^{23} + x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{10} \\
&\quad + x^{9}, \\
p_{12}(x) &= x^{119} + x^{118} + x^{117} + x^{116} + x^{115} + x^{114} + x^{113} + x^{112} + x^{55} + x^{54} \\
&\quad + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} \\
&\quad + x^{18} + x^{17} + x^{16} + x^{7} + x^{6} + x^{5} + x^{4} + x^{3} + x^{2} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 0, 0, 1, 0, 0, 1].

For $\phi(M^e, 1, 0)$:

$$\begin{aligned}
p_0(x) = {} & x^{129} + x^{126} + x^{124} + x^{122} + x^{119} + x^{118} + x^{117} + x^{115} + x^{114} + x^{113} \\
& + x^{112} + x^{110} + x^{109} + x^{108} + x^{106} + x^{103} + x^{102} + x^{101} + x^{100} \\
& + x^{96} + x^{94} + x^{93} + x^{92} + x^{91} + x^{89} + x^{86} + x^{83} + x^{82} + x^{80} + x^{78} \\
& + x^{75} + x^{73} + x^{71} + x^{67} + x^{66} + x^{65} + x^{62} + x^{61} + x^{60} + x^{59} + x^{56} \\
& + x^{55} + x^{53} + x^{51} + x^{50} + x^{48} + x^{44} + x^{40} + x^{39}, \\
p_3(x) = {} & x^{125} + x^{123} + x^{122} + x^{121} + x^{113} + x^{110} + x^{109} + x^{108} + x^{106} + x^{102} \\
& + x^{101} + x^{100} + x^{98} + x^{97} + x^{89} + x^{86} + x^{84} + x^{83} + x^{81} + x^{78} + x^{76} \\
& + x^{75} + x^{73} + x^{70} + x^{68} + x^{67} + x^{65} + x^{62} + x^{60} + x^{59} + x^{57} + x^{54} \\
& + x^{52} + x^{51} + x^{49} + x^{46} + x^{44} + x^{43} + x^{41} + x^{38} + x^{37} + x^{36} + x^{34} \\
& + x^{30} + x^{28} + x^{27}, \\
p_6(x) = {} & x^{123} + x^{122} + x^{117} + x^{116} + x^{113} + x^{112} + x^{105} + x^{104} + x^{103} + x^{102} \\
& + x^{101} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{89} \\
& + x^{88} + x^{87} + x^{86} + x^{83} + x^{82} + x^{77} + x^{76} + x^{73} + x^{72} + x^{71} + x^{70} \\
& + x^{63} + x^{62} + x^{59} + x^{58} + x^{57} + x^{56} + x^{49} + x^{48} + x^{45} + x^{44} + x^{43} \\
& + x^{42} + x^{31} + x^{30} + x^{25} + x^{24} + x^{21} + x^{20} + x^{19} + x^{18}, \\
p_9(x) = {} & x^{121} + x^{118} + x^{117} + x^{116} + x^{114} + x^{111} + x^{110} + x^{109} + x^{105} + x^{102} \\
& + x^{101} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{92} + x^{88} + x^{87} \\
& + x^{86} + x^{84} + x^{81} + x^{80} + x^{79} + x^{75} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} \\
& + x^{67} + x^{66} + x^{65} + x^{64} + x^{62} + x^{58} + x^{56} + x^{53} + x^{52} + x^{51} + x^{49} \\
& + x^{47} + x^{46} + x^{42} + x^{40} + x^{39} + x^{33} + x^{31} + x^{30} + x^{27} + x^{26} + x^{25} \\
& + x^{23} + x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{10} \\
& + x^{9}, \\
p_{12}(x) = {} & x^{119} + x^{118} + x^{117} + x^{116} + x^{115} + x^{114} + x^{113} + x^{112} + x^{55} + x^{54} \\
& + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} \\
& + x^{18} + x^{17} + x^{16} + x^{7} + x^{6} + x^{5} + x^{4} + x^{3} + x^{2} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 0, 0, 1, 0, 0, 1].

For $\phi(M^e, 1, 1)$:

$$\begin{aligned}
p_0(x) = {} & x^{126} + x^{124} + x^{120} + x^{114} + x^{108} + x^{106} + x^{104} + x^{102} + x^{100} + x^{92} \\
& + x^{88} + x^{86} + x^{84} + x^{78} + x^{76} + x^{72} + x^{68} + x^{66} + x^{62} + x^{60} + x^{56} \\
& + x^{52} + x^{42}, \\
p_3(x) = {} & x^{114} + x^{108} + x^{106} + x^{104} + x^{100} + x^{98} + x^{96} + x^{94} + x^{92} + x^{88} + x^{86} \\
& + x^{82} + x^{76} + x^{72} + x^{68} + x^{64} + x^{62} + x^{54} + x^{46} + x^{36} + x^{32} + x^{30} \\
& + x^{26} + x^{18},
\end{aligned}$$

$$\begin{aligned}
p_6(x) &= x^{114} + x^{110} + x^{106} + x^{100} + x^{94} + x^{84} + x^{82} + x^{78} + x^{76} + x^{72} + x^{66} \\
&\quad + x^{64} + x^{60} + x^{58} + x^{56} + x^{54} + x^{50} + x^{46} + x^{42} + x^{40} + x^{38} + x^{32} \\
&\quad + x^{28} + x^{18} + x^{10} + x^{4} + x^{2} + 1, \\
p_9(x) &= x^{110} + x^{108} + x^{104} + x^{102} + x^{100} + x^{98} + x^{96} + x^{90} + x^{88} + x^{80} + x^{78} \\
&\quad + x^{74} + x^{72} + x^{70} + x^{68} + x^{66} + x^{60} + x^{58} + x^{50} + x^{48} + x^{46} + x^{42} \\
&\quad + x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{24} + x^{20} + x^{18} + x^{16} + x^{14} + x^{8} \\
&\quad + x^{6}, \\
p_{12}(x) &= x^{110} + x^{108} + x^{106} + x^{104} + x^{102} + x^{100} + x^{98} + x^{96} + x^{94} + x^{92} \\
&\quad + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{80} + x^{78} + x^{76} + x^{74} + x^{72} + x^{70} \\
&\quad + x^{68} + x^{66} + x^{64} + x^{62} + x^{60} + x^{58} + x^{56} + x^{54} + x^{52} + x^{50} + x^{48} \\
&\quad + x^{14} + x^{12} + x^{10} + x^{8} + x^{6} + x^{4} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0].

For $\phi(W^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{124} + x^{120} + x^{118} + x^{116} + x^{114} + x^{108} + x^{106} + x^{102} + x^{100} \\
&\quad + x^{96} + x^{94} + x^{88} + x^{82} + x^{80} + x^{78} + x^{72} + x^{66} + x^{62} + x^{58} + x^{54} \\
&\quad + x^{50} + x^{36}, \\
p_3(x) &= x^{114} + x^{108} + x^{104} + x^{102} + x^{96} + x^{92} + x^{90} + x^{86} + x^{82} + x^{80} + x^{72} \\
&\quad + x^{70} + x^{66} + x^{64} + x^{56} + x^{52} + x^{48} + x^{46} + x^{42} + x^{40} + x^{38} + x^{36} \\
&\quad + x^{34} + x^{32} + x^{30} + x^{28} + x^{24} + x^{20}, \\
p_6(x) &= x^{114} + x^{110} + x^{100} + x^{98} + x^{94} + x^{92} + x^{84} + x^{82} + x^{80} + x^{76} + x^{74} \\
&\quad + x^{70} + x^{68} + x^{64} + x^{56} + x^{54} + x^{50} + x^{48} + x^{44} + x^{30} + x^{28} + x^{20} \\
&\quad + x^{18} + x^{16} + x^{14} + x^{12} + x^{10} + x^{4} + x^{2} + 1, \\
p_9(x) &= x^{110} + x^{108} + x^{104} + x^{102} + x^{100} + x^{98} + x^{96} + x^{90} + x^{88} + x^{80} + x^{78} \\
&\quad + x^{74} + x^{72} + x^{70} + x^{68} + x^{66} + x^{60} + x^{58} + x^{50} + x^{48} + x^{46} + x^{42} \\
&\quad + x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{24} + x^{20} + x^{18} + x^{16} + x^{14} + x^{8} \\
&\quad + x^{6}, \\
p_{12}(x) &= x^{110} + x^{108} + x^{106} + x^{104} + x^{102} + x^{100} + x^{98} + x^{96} + x^{94} + x^{92} \\
&\quad + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{80} + x^{78} + x^{76} + x^{74} + x^{72} + x^{70} \\
&\quad + x^{68} + x^{66} + x^{64} + x^{62} + x^{60} + x^{58} + x^{56} + x^{54} + x^{52} + x^{50} + x^{48} \\
&\quad + x^{14} + x^{12} + x^{10} + x^{8} + x^{6} + x^{4} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0].

For $\phi(W^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{129} + x^{127} + x^{122} + x^{121} + x^{119} + x^{117} + x^{116} + x^{115} + x^{114} + x^{112} \\
&\quad + x^{111} + x^{107} + x^{106} + x^{105} + x^{102} + x^{101} + x^{98} + x^{96} + x^{93} + x^{92} \\
&\quad + x^{90} + x^{89} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{80} + x^{79} + x^{77} + x^{72}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{71} + x^{69} + x^{66} + x^{63} + x^{58} + x^{57} + x^{56} + x^{55} + x^{53} + x^{52} + x^{51} \\
&+ x^{47} + x^{46} + x^{45} + x^{40} + x^{39}, \\
p_3(x) &= x^{125} + x^{123} + x^{122} + x^{121} + x^{113} + x^{110} + x^{109} + x^{108} + x^{106} + x^{102} \\
&+ x^{101} + x^{100} + x^{98} + x^{97} + x^{89} + x^{86} + x^{84} + x^{83} + x^{81} + x^{78} + x^{76} \\
&+ x^{75} + x^{73} + x^{70} + x^{68} + x^{67} + x^{65} + x^{62} + x^{60} + x^{59} + x^{57} + x^{54} \\
&+ x^{52} + x^{51} + x^{49} + x^{46} + x^{44} + x^{43} + x^{41} + x^{38} + x^{37} + x^{36} + x^{34} \\
&+ x^{30} + x^{28} + x^{27}, \\
p_6(x) &= x^{123} + x^{122} + x^{117} + x^{116} + x^{113} + x^{112} + x^{105} + x^{104} + x^{103} + x^{102} \\
&+ x^{101} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{89} \\
&+ x^{88} + x^{87} + x^{86} + x^{83} + x^{82} + x^{77} + x^{76} + x^{73} + x^{72} + x^{71} + x^{70} \\
&+ x^{63} + x^{62} + x^{59} + x^{58} + x^{57} + x^{56} + x^{49} + x^{48} + x^{45} + x^{44} + x^{43} \\
&+ x^{42} + x^{31} + x^{30} + x^{25} + x^{24} + x^{21} + x^{20} + x^{19} + x^{18}, \\
p_9(x) &= x^{121} + x^{118} + x^{117} + x^{116} + x^{114} + x^{111} + x^{110} + x^{109} + x^{105} + x^{102} \\
&+ x^{101} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{92} + x^{88} + x^{87} \\
&+ x^{86} + x^{84} + x^{81} + x^{80} + x^{79} + x^{75} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} \\
&+ x^{67} + x^{66} + x^{65} + x^{64} + x^{62} + x^{58} + x^{56} + x^{53} + x^{52} + x^{51} + x^{49} \\
&+ x^{47} + x^{46} + x^{42} + x^{40} + x^{39} + x^{33} + x^{31} + x^{30} + x^{27} + x^{26} + x^{25} \\
&+ x^{23} + x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{10} \\
&+ x^{9}, \\
p_{12}(x) &= x^{119} + x^{118} + x^{117} + x^{116} + x^{115} + x^{114} + x^{113} + x^{112} + x^{55} + x^{54} \\
&+ x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} \\
&+ x^{18} + x^{17} + x^{16} + x^{7} + x^{6} + x^{5} + x^{4} + x^{3} + x^{2} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 1, 1, 1, 1, 1].

For $\phi(W^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{129} + x^{127} + x^{122} + x^{121} + x^{119} + x^{117} + x^{116} + x^{115} + x^{114} + x^{112} \\
&+ x^{111} + x^{107} + x^{106} + x^{105} + x^{102} + x^{101} + x^{98} + x^{96} + x^{93} + x^{92} \\
&+ x^{90} + x^{89} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{80} + x^{79} + x^{77} + x^{72} \\
&+ x^{71} + x^{69} + x^{66} + x^{63} + x^{58} + x^{57} + x^{56} + x^{55} + x^{53} + x^{52} + x^{51} \\
&+ x^{47} + x^{46} + x^{45} + x^{40} + x^{39}, \\
p_3(x) &= x^{125} + x^{123} + x^{122} + x^{121} + x^{113} + x^{110} + x^{109} + x^{108} + x^{106} + x^{102} \\
&+ x^{101} + x^{100} + x^{98} + x^{97} + x^{89} + x^{86} + x^{84} + x^{83} + x^{81} + x^{78} + x^{76} \\
&+ x^{75} + x^{73} + x^{70} + x^{68} + x^{67} + x^{65} + x^{62} + x^{60} + x^{59} + x^{57} + x^{54} \\
&+ x^{52} + x^{51} + x^{49} + x^{46} + x^{44} + x^{43} + x^{41} + x^{38} + x^{37} + x^{36} + x^{34} \\
&+ x^{30} + x^{28} + x^{27}, \\
p_6(x) &= x^{123} + x^{122} + x^{117} + x^{116} + x^{113} + x^{112} + x^{105} + x^{104} + x^{103} + x^{102}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{101}+x^{100}+x^{99}+x^{98}+x^{97}+x^{96}+x^{95}+x^{94}+x^{93}+x^{92}+x^{89}\\
&+x^{88}+x^{87}+x^{86}+x^{83}+x^{82}+x^{77}+x^{76}+x^{73}+x^{72}+x^{71}+x^{70}\\
&+x^{63}+x^{62}+x^{59}+x^{58}+x^{57}+x^{56}+x^{49}+x^{48}+x^{45}+x^{44}+x^{43}\\
&+x^{42}+x^{31}+x^{30}+x^{25}+x^{24}+x^{21}+x^{20}+x^{19}+x^{18},\\
p_9(x)=&x^{121}+x^{118}+x^{117}+x^{116}+x^{114}+x^{111}+x^{110}+x^{109}+x^{105}+x^{102}\\
&+x^{101}+x^{100}+x^{99}+x^{98}+x^{97}+x^{96}+x^{95}+x^{94}+x^{92}+x^{88}+x^{87}\\
&+x^{86}+x^{84}+x^{81}+x^{80}+x^{79}+x^{75}+x^{72}+x^{71}+x^{70}+x^{69}+x^{68}\\
&+x^{67}+x^{66}+x^{65}+x^{64}+x^{62}+x^{58}+x^{56}+x^{53}+x^{52}+x^{51}+x^{49}\\
&+x^{47}+x^{46}+x^{42}+x^{40}+x^{39}+x^{33}+x^{31}+x^{30}+x^{27}+x^{26}+x^{25}\\
&+x^{23}+x^{20}+x^{19}+x^{18}+x^{17}+x^{16}+x^{15}+x^{14}+x^{13}+x^{12}+x^{10}\\
&+x^{9},\\
p_{12}(x)=&x^{119}+x^{118}+x^{117}+x^{116}+x^{115}+x^{114}+x^{113}+x^{112}+x^{55}+x^{54}\\
&+x^{53}+x^{52}+x^{51}+x^{50}+x^{49}+x^{48}+x^{23}+x^{22}+x^{21}+x^{20}+x^{19}\\
&+x^{18}+x^{17}+x^{16}+x^{7}+x^{6}+x^{5}+x^{4}+x^{3}+x^{2}+x+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 1, 1, 1, 1, 1].

For $\phi(W^e, 1, 1)$:

$$
\begin{aligned}
p_0(x)=&x^{132}+x^{130}+x^{126}+x^{124}+x^{122}+x^{112}+x^{110}+x^{98}+x^{90}+x^{86}\\
&+x^{84}+x^{82}+x^{76}+x^{74}+x^{68}+x^{64}+x^{62}+x^{60}+x^{54}+x^{44}+x^{42},\\
p_3(x)=&x^{124}+x^{122}+x^{120}+x^{108}+x^{106}+x^{104}+x^{88}+x^{76}+x^{74}+x^{68}\\
&+x^{66}+x^{64}+x^{56}+x^{52}+x^{50}+x^{48}+x^{36}+x^{34}+x^{28}+x^{26}+x^{20}\\
&+x^{18},\\
p_6(x)=&x^{116}+x^{110}+x^{108}+x^{106}+x^{104}+x^{96}+x^{92}+x^{90}+x^{86}+x^{84}+x^{82}\\
&+x^{78}+x^{76}+x^{74}+x^{72}+x^{70}+x^{66}+x^{60}+x^{52}+x^{44}+x^{42}+x^{40}\\
&+x^{34}+x^{32}+x^{26}+x^{24}+x^{22}+x^{20}+x^{18}+x^{16}+x^{12}+x^{10}+x^{6}+1,\\
p_9(x)=&x^{128}+x^{124}+x^{122}+x^{106}+x^{104}+x^{98}+x^{96}+x^{94}+x^{88}+x^{76}+x^{74}\\
&+x^{68}+x^{66}+x^{60}+x^{46}+x^{44}+x^{42}+x^{40}+x^{38}+x^{36}+x^{30}+x^{26}\\
&+x^{24}+x^{14}+x^{10}+x^{6},\\
p_{12}(x)=&x^{128}+x^{112}+x^{64}+x^{48}+x^{32}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0].

For $\phi(M^o, 0, 0)$:

$$
\begin{aligned}
p_0(x)=&x^{129}+x^{126}+x^{125}+x^{124}+x^{122}+x^{121}+x^{119}+x^{117}+x^{116}+x^{115}\\
&+x^{114}+x^{113}+x^{111}+x^{110}+x^{109}+x^{102}+x^{101}+x^{100}+x^{97}+x^{96}\\
&+x^{93}+x^{91}+x^{90}+x^{88}+x^{87}+x^{85}+x^{82}+x^{81}+x^{76}+x^{72}+x^{70}\\
&+x^{67}+x^{66}+x^{65}+x^{61}+x^{59}+x^{57}+x^{56}+x^{55}+x^{53}+x^{52}+x^{50}\\
&+x^{46}+x^{45}+x^{42}+x^{37}+x^{36},
\end{aligned}
$$

$$
\begin{aligned}
p_3(x) &= x^{127} + x^{126} + x^{124} + x^{123} + x^{122} + x^{117} + x^{116} + x^{112} + x^{111} + x^{109} \\
&\quad + x^{107} + x^{106} + x^{104} + x^{103} + x^{101} + x^{97} + x^{96} + x^{93} + x^{92} + x^{88} \\
&\quad + x^{87} + x^{85} + x^{84} + x^{83} + x^{82} + x^{80} + x^{79} + x^{75} + x^{74} + x^{73} + x^{71} \\
&\quad + x^{69} + x^{68} + x^{64} + x^{63} + x^{59} + x^{58} + x^{57} + x^{54} + x^{53} + x^{52} + x^{51} \\
&\quad + x^{50} + x^{48} + x^{46} + x^{43} + x^{42} + x^{41} + x^{38} + x^{37} + x^{32} + x^{30} + x^{29} \\
&\quad + x^{28} + x^{25} + x^{24}, \\
p_6(x) &= x^{127} + x^{126} + x^{124} + x^{123} + x^{122} + x^{116} + x^{115} + x^{114} + x^{113} + x^{112} \\
&\quad + x^{111} + x^{105} + x^{104} + x^{101} + x^{99} + x^{95} + x^{94} + x^{93} + x^{92} + x^{90} \\
&\quad + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{80} + x^{79} + x^{77} + x^{76} \\
&\quad + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{61} \\
&\quad + x^{60} + x^{58} + x^{57} + x^{56} + x^{55} + x^{53} + x^{52} + x^{49} + x^{44} + x^{42} + x^{33} \\
&\quad + x^{31} + x^{30} + x^{29} + x^{28} + x^{24} + x^{23} + x^{22} + x^{20} + x^{19} + x^{13} + x^{12} \\
&\quad + x^{7} + x^{6} + x^{5} + x^{4} + x^{3} + x^{2} + x + 1, \\
p_9(x) &= x^{127} + x^{126} + x^{125} + x^{124} + x^{119} + x^{118} + x^{117} + x^{116} + x^{115} + x^{114} \\
&\quad + x^{113} + x^{112} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} \\
&\quad + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{71} + x^{70} + x^{69} \\
&\quad + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{31} + x^{30} \\
&\quad + x^{29} + x^{28} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} \\
&\quad + x^{14} + x^{13} + x^{12}, \\
p_{12}(x) &= x^{127} + x^{126} + x^{125} + x^{124} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} \\
&\quad + x^{97} + x^{96} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{71} \\
&\quad + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} \\
&\quad + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{31} + x^{30} + x^{29} \\
&\quad + x^{28} + x^{15} + x^{14} + x^{13} + x^{12} + x^{7} + x^{6} + x^{5} + x^{4} + x^{3} + x^{2} + x + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 1, 1, 1, 0].

For $\phi(M^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{128} + x^{122} + x^{121} + x^{120} + x^{119} + x^{116} + x^{115} + x^{114} + x^{112} + x^{105} \\
&\quad + x^{104} + x^{102} + x^{100} + x^{99} + x^{98} + x^{97} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} \\
&\quad + x^{89} + x^{88} + x^{86} + x^{83} + x^{80} + x^{75} + x^{74} + x^{72} + x^{71} + x^{68} + x^{66} \\
&\quad + x^{65} + x^{62} + x^{59} + x^{55} + x^{52} + x^{51} + x^{50} + x^{49} + x^{46} + x^{44} + x^{41} \\
&\quad + x^{39}, \\
p_3(x) &= x^{114} + x^{113} + x^{111} + x^{110} + x^{109} + x^{107} + x^{106} + x^{105} + x^{103} + x^{102} \\
&\quad + x^{101} + x^{99} + x^{90} + x^{89} + x^{87} + x^{86} + x^{85} + x^{83} + x^{82} + x^{81} + x^{79} \\
&\quad + x^{78} + x^{77} + x^{75} + x^{74} + x^{73} + x^{71} + x^{70} + x^{69} + x^{67} + x^{66} + x^{65} \\
&\quad + x^{63} + x^{62} + x^{61} + x^{59} + x^{58} + x^{57} + x^{55} + x^{54} + x^{53} + x^{51} + x^{50}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{49} + x^{47} + x^{46} + x^{45} + x^{43} + x^{42} + x^{41} + x^{39} + x^{38} + x^{37} + x^{35}\\
&+ x^{34} + x^{33} + x^{31} + x^{30} + x^{29} + x^{27},\\
p_6(x) &= x^{116} + x^{114} + x^{110} + x^{104} + x^{102} + x^{98} + x^{96} + x^{94} + x^{90} + x^{88} + x^{86}\\
&+ x^{84} + x^{80} + x^{74} + x^{72} + x^{70} + x^{68} + x^{64} + x^{60} + x^{54} + x^{48} + x^{46}\\
&+ x^{42} + x^{24} + x^{22} + x^{18},\\
p_9(x) &= x^{118} + x^{117} + x^{116} + x^{114} + x^{111} + x^{108} + x^{107} + x^{105} + x^{102} + x^{101}\\
&+ x^{100} + x^{98} + x^{95} + x^{92} + x^{91} + x^{89} + x^{86} + x^{85} + x^{84} + x^{82} + x^{79}\\
&+ x^{76} + x^{75} + x^{73} + x^{70} + x^{69} + x^{68} + x^{66} + x^{63} + x^{60} + x^{59} + x^{57}\\
&+ x^{22} + x^{21} + x^{20} + x^{18} + x^{15} + x^{12} + x^{11} + x^{9},\\
p_{12}(x) &= x^{120} + x^{116} + x^{112} + x^{108} + x^{104} + x^{100} + x^{92} + x^{88} + x^{84} + x^{76}\\
&+ x^{72} + x^{68} + x^{60} + x^{48} + x^{24} + x^{20} + x^{16} + x^{12} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 1, 1, 0].

For $\phi(M^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{132} + x^{129} + x^{128} + x^{127} + x^{126} + x^{118} + x^{117} + x^{115} + x^{112} + x^{111}\\
&+ x^{110} + x^{108} + x^{107} + x^{105} + x^{98} + x^{93} + x^{92} + x^{91} + x^{89} + x^{86}\\
&+ x^{81} + x^{76} + x^{74} + x^{71} + x^{70} + x^{68} + x^{67} + x^{65} + x^{63} + x^{61} + x^{60}\\
&+ x^{55} + x^{54} + x^{53} + x^{52} + x^{50} + x^{46} + x^{41} + x^{39},\\
p_3(x) &= x^{130} + x^{129} + x^{127} + x^{126} + x^{125} + x^{123} + x^{122} + x^{121} + x^{119} + x^{118}\\
&+ x^{117} + x^{115} + x^{114} + x^{113} + x^{111} + x^{110} + x^{109} + x^{107} + x^{98} + x^{97}\\
&+ x^{95} + x^{94} + x^{93} + x^{91} + x^{42} + x^{41} + x^{39} + x^{38} + x^{37} + x^{35} + x^{34}\\
&+ x^{33} + x^{31} + x^{30} + x^{29} + x^{27},\\
p_6(x) &= x^{132} + x^{130} + x^{126} + x^{120} + x^{118} + x^{116} + x^{112} + x^{106} + x^{100} + x^{98}\\
&+ x^{94} + x^{76} + x^{74} + x^{72} + x^{68} + x^{62} + x^{60} + x^{58} + x^{54} + x^{48} + x^{46}\\
&+ x^{42} + x^{40} + x^{38} + x^{34} + x^{24} + x^{22} + x^{18},\\
p_9(x) &= x^{134} + x^{133} + x^{132} + x^{130} + x^{127} + x^{124} + x^{123} + x^{121} + x^{70} + x^{69}\\
&+ x^{68} + x^{66} + x^{63} + x^{60} + x^{59} + x^{57} + x^{38} + x^{37} + x^{36} + x^{34} + x^{31}\\
&+ x^{28} + x^{27} + x^{25} + x^{22} + x^{21} + x^{20} + x^{18} + x^{15} + x^{12} + x^{11} + x^{9},\\
p_{12}(x) &= x^{136} + x^{132} + x^{128} + x^{124} + x^{112} + x^{72} + x^{68} + x^{64} + x^{60} + x^{48} + x^{40}\\
&+ x^{36} + x^{32} + x^{28} + x^{24} + x^{20} + x^{12} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0].

For $\phi(M^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{129} + x^{127} + x^{125} + x^{123} + x^{121} + x^{118} + x^{116} + x^{115} + x^{114} + x^{112}\\
&+ x^{111} + x^{103} + x^{101} + x^{98} + x^{95} + x^{91} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84}\\
&+ x^{82} + x^{79} + x^{75} + x^{72} + x^{70} + x^{68} + x^{67} + x^{65} + x^{64} + x^{59} + x^{58}\\
&+ x^{57} + x^{54} + x^{53} + x^{52} + x^{50} + x^{49} + x^{47} + x^{43} + x^{42},
\end{aligned}
$$

$$
\begin{aligned}
p_3(x) &= x^{127} + x^{126} + x^{124} + x^{119} + x^{118} + x^{117} + x^{116} + x^{112} + x^{110} + x^{108} \\
&\quad + x^{107} + x^{106} + x^{105} + x^{103} + x^{100} + x^{95} + x^{94} + x^{93} + x^{92} + x^{88} \\
&\quad + x^{86} + x^{81} + x^{78} + x^{77} + x^{76} + x^{72} + x^{70} + x^{67} + x^{66} + x^{65} + x^{62} \\
&\quad + x^{61} + x^{60} + x^{56} + x^{55} + x^{49} + x^{47} + x^{45} + x^{44} + x^{40} + x^{39} + x^{36} \\
&\quad + x^{33} + x^{30} + x^{27} + x^{26}, \\
p_6(x) &= x^{127} + x^{126} + x^{124} + x^{119} + x^{118} + x^{116} + x^{113} + x^{112} + x^{110} + x^{109} \\
&\quad + x^{108} + x^{101} + x^{98} + x^{97} + x^{96} + x^{90} + x^{87} + x^{86} + x^{83} + x^{81} + x^{78} \\
&\quad + x^{77} + x^{76} + x^{75} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{66} + x^{62} + x^{61} \\
&\quad + x^{60} + x^{58} + x^{57} + x^{56} + x^{55} + x^{53} + x^{52} + x^{48} + x^{47} + x^{46} + x^{44} \\
&\quad + x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{33} + x^{29} + x^{28} + x^{27} \\
&\quad + x^{26} + x^{24} + x^{20} + x^{19} + x^{17} + x^{16} + x^{15} + x^{14} + x^{7} + x^{6} + x^{5} + x^{4} \\
&\quad + x^{3} + x^{2} + x + 1, \\
p_9(x) &= x^{127} + x^{126} + x^{125} + x^{124} + x^{119} + x^{118} + x^{117} + x^{116} + x^{115} + x^{114} \\
&\quad + x^{113} + x^{112} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} \\
&\quad + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{71} + x^{70} + x^{69} \\
&\quad + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{31} + x^{30} \\
&\quad + x^{29} + x^{28} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} \\
&\quad + x^{14} + x^{13} + x^{12}, \\
p_{12}(x) &= x^{127} + x^{126} + x^{125} + x^{124} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} \\
&\quad + x^{97} + x^{96} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{71} \\
&\quad + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} \\
&\quad + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{31} + x^{30} + x^{29} \\
&\quad + x^{28} + x^{15} + x^{14} + x^{13} + x^{12} + x^{7} + x^{6} + x^{5} + x^{4} + x^{3} + x^{2} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 1, 0, 0].

For $\phi(W^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{129} + x^{126} + x^{125} + x^{124} + x^{122} + x^{121} + x^{119} + x^{117} + x^{116} + x^{115} \\
&\quad + x^{114} + x^{113} + x^{111} + x^{110} + x^{109} + x^{102} + x^{101} + x^{100} + x^{97} + x^{96} \\
&\quad + x^{93} + x^{91} + x^{90} + x^{88} + x^{87} + x^{85} + x^{82} + x^{81} + x^{76} + x^{72} + x^{70} \\
&\quad + x^{67} + x^{66} + x^{65} + x^{61} + x^{59} + x^{57} + x^{56} + x^{55} + x^{53} + x^{52} + x^{50} \\
&\quad + x^{46} + x^{45} + x^{42} + x^{37} + x^{36}, \\
p_3(x) &= x^{127} + x^{126} + x^{124} + x^{123} + x^{122} + x^{117} + x^{116} + x^{112} + x^{111} + x^{109} \\
&\quad + x^{107} + x^{106} + x^{104} + x^{103} + x^{101} + x^{97} + x^{96} + x^{93} + x^{92} + x^{88} \\
&\quad + x^{87} + x^{85} + x^{84} + x^{83} + x^{82} + x^{80} + x^{79} + x^{75} + x^{74} + x^{73} + x^{71} \\
&\quad + x^{69} + x^{68} + x^{64} + x^{63} + x^{59} + x^{58} + x^{57} + x^{54} + x^{53} + x^{52} + x^{51} \\
&\quad + x^{50} + x^{48} + x^{46} + x^{43} + x^{42} + x^{41} + x^{38} + x^{37} + x^{32} + x^{30} + x^{29}
\end{aligned}
$$

$$\begin{aligned}
&+x^{28}+x^{25}+x^{24},\\
p_6(x) &= x^{127}+x^{126}+x^{124}+x^{123}+x^{122}+x^{116}+x^{115}+x^{114}+x^{113}+x^{112}\\
&+x^{111}+x^{105}+x^{104}+x^{101}+x^{99}+x^{95}+x^{94}+x^{93}+x^{92}+x^{90}\\
&+x^{89}+x^{88}+x^{87}+x^{86}+x^{85}+x^{84}+x^{83}+x^{80}+x^{79}+x^{77}+x^{76}\\
&+x^{74}+x^{73}+x^{72}+x^{71}+x^{70}+x^{68}+x^{67}+x^{65}+x^{64}+x^{63}+x^{61}\\
&+x^{60}+x^{58}+x^{57}+x^{56}+x^{55}+x^{53}+x^{52}+x^{49}+x^{44}+x^{42}+x^{33}\\
&+x^{31}+x^{30}+x^{29}+x^{28}+x^{24}+x^{23}+x^{22}+x^{20}+x^{19}+x^{13}+x^{12}\\
&+x^{7}+x^{6}+x^{5}+x^{4}+x^{3}+x^{2}+x+1,\\
p_9(x) &= x^{127}+x^{126}+x^{125}+x^{124}+x^{119}+x^{118}+x^{117}+x^{116}+x^{115}+x^{114}\\
&+x^{113}+x^{112}+x^{103}+x^{102}+x^{101}+x^{100}+x^{99}+x^{98}+x^{97}+x^{96}\\
&+x^{87}+x^{86}+x^{85}+x^{84}+x^{83}+x^{82}+x^{81}+x^{80}+x^{71}+x^{70}+x^{69}\\
&+x^{68}+x^{67}+x^{66}+x^{65}+x^{64}+x^{63}+x^{62}+x^{61}+x^{60}+x^{31}+x^{30}\\
&+x^{29}+x^{28}+x^{23}+x^{22}+x^{21}+x^{20}+x^{19}+x^{18}+x^{17}+x^{16}+x^{15}\\
&+x^{14}+x^{13}+x^{12},\\
p_{12}(x) &= x^{127}+x^{126}+x^{125}+x^{124}+x^{103}+x^{102}+x^{101}+x^{100}+x^{99}+x^{98}\\
&+x^{97}+x^{96}+x^{87}+x^{86}+x^{85}+x^{84}+x^{83}+x^{82}+x^{81}+x^{80}+x^{71}\\
&+x^{70}+x^{69}+x^{68}+x^{67}+x^{66}+x^{65}+x^{64}+x^{63}+x^{62}+x^{61}+x^{60}\\
&+x^{55}+x^{54}+x^{53}+x^{52}+x^{51}+x^{50}+x^{49}+x^{48}+x^{31}+x^{30}+x^{29}\\
&+x^{28}+x^{15}+x^{14}+x^{13}+x^{12}+x^{7}+x^{6}+x^{5}+x^{4}+x^{3}+x^{2}+x+1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 1, 1, 1, 0].

For $\phi(W^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{132}+x^{129}+x^{128}+x^{127}+x^{126}+x^{118}+x^{117}+x^{115}+x^{112}+x^{111}\\
&+x^{110}+x^{108}+x^{107}+x^{105}+x^{98}+x^{93}+x^{92}+x^{91}+x^{89}+x^{86}\\
&+x^{81}+x^{76}+x^{74}+x^{71}+x^{70}+x^{68}+x^{67}+x^{65}+x^{63}+x^{61}+x^{60}\\
&+x^{55}+x^{54}+x^{53}+x^{52}+x^{50}+x^{46}+x^{41}+x^{39},\\
p_3(x) &= x^{130}+x^{129}+x^{127}+x^{126}+x^{125}+x^{123}+x^{122}+x^{121}+x^{119}+x^{118}\\
&+x^{117}+x^{115}+x^{114}+x^{113}+x^{111}+x^{110}+x^{109}+x^{107}+x^{98}+x^{97}\\
&+x^{95}+x^{94}+x^{93}+x^{91}+x^{42}+x^{41}+x^{39}+x^{38}+x^{37}+x^{35}+x^{34}\\
&+x^{33}+x^{31}+x^{30}+x^{29}+x^{27},\\
p_6(x) &= x^{132}+x^{130}+x^{126}+x^{120}+x^{118}+x^{116}+x^{112}+x^{106}+x^{100}+x^{98}\\
&+x^{94}+x^{76}+x^{74}+x^{72}+x^{68}+x^{62}+x^{60}+x^{58}+x^{54}+x^{48}+x^{46}\\
&+x^{42}+x^{40}+x^{38}+x^{34}+x^{24}+x^{22}+x^{18},\\
p_9(x) &= x^{134}+x^{133}+x^{132}+x^{130}+x^{127}+x^{124}+x^{123}+x^{121}+x^{70}+x^{69}\\
&+x^{68}+x^{66}+x^{63}+x^{60}+x^{59}+x^{57}+x^{38}+x^{37}+x^{36}+x^{34}+x^{31}\\
&+x^{28}+x^{27}+x^{25}+x^{22}+x^{21}+x^{20}+x^{18}+x^{15}+x^{12}+x^{11}+x^{9},
\end{aligned}$$

$$\begin{aligned} p_{12}(x) = {} & x^{136} + x^{132} + x^{128} + x^{124} + x^{112} + x^{72} + x^{68} + x^{64} + x^{60} + x^{48} + x^{40} \\ & + x^{36} + x^{32} + x^{28} + x^{24} + x^{20} + x^{12} + 1, \end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0].

For $\phi(W^o, 1, 0)$:

$$\begin{aligned} p_0(x) = {} & x^{128} + x^{122} + x^{121} + x^{120} + x^{119} + x^{116} + x^{115} + x^{114} + x^{112} + x^{105} \\ & + x^{104} + x^{102} + x^{100} + x^{99} + x^{98} + x^{97} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} \\ & + x^{89} + x^{88} + x^{86} + x^{83} + x^{80} + x^{75} + x^{74} + x^{72} + x^{71} + x^{68} + x^{66} \\ & + x^{65} + x^{62} + x^{59} + x^{55} + x^{52} + x^{51} + x^{50} + x^{49} + x^{46} + x^{44} + x^{41} \\ & + x^{39}, \\ p_3(x) = {} & x^{114} + x^{113} + x^{111} + x^{110} + x^{109} + x^{107} + x^{106} + x^{105} + x^{103} + x^{102} \\ & + x^{101} + x^{99} + x^{90} + x^{89} + x^{87} + x^{86} + x^{85} + x^{83} + x^{82} + x^{81} + x^{79} \\ & + x^{78} + x^{77} + x^{75} + x^{74} + x^{73} + x^{71} + x^{70} + x^{69} + x^{67} + x^{66} + x^{65} \\ & + x^{63} + x^{62} + x^{61} + x^{59} + x^{58} + x^{57} + x^{55} + x^{54} + x^{53} + x^{51} + x^{50} \\ & + x^{49} + x^{47} + x^{46} + x^{45} + x^{43} + x^{42} + x^{41} + x^{39} + x^{38} + x^{37} + x^{35} \\ & + x^{34} + x^{33} + x^{31} + x^{30} + x^{29} + x^{27}, \\ p_6(x) = {} & x^{116} + x^{114} + x^{110} + x^{104} + x^{102} + x^{98} + x^{96} + x^{94} + x^{90} + x^{88} + x^{86} \\ & + x^{84} + x^{80} + x^{74} + x^{72} + x^{70} + x^{68} + x^{64} + x^{60} + x^{54} + x^{48} + x^{46} \\ & + x^{42} + x^{24} + x^{22} + x^{18}, \\ p_9(x) = {} & x^{118} + x^{117} + x^{116} + x^{114} + x^{111} + x^{108} + x^{107} + x^{105} + x^{102} + x^{101} \\ & + x^{100} + x^{98} + x^{95} + x^{92} + x^{91} + x^{89} + x^{86} + x^{85} + x^{84} + x^{82} + x^{79} \\ & + x^{76} + x^{75} + x^{73} + x^{70} + x^{69} + x^{68} + x^{66} + x^{63} + x^{60} + x^{59} + x^{57} \\ & + x^{22} + x^{21} + x^{20} + x^{18} + x^{15} + x^{12} + x^{11} + x^{9}, \\ p_{12}(x) = {} & x^{120} + x^{116} + x^{112} + x^{108} + x^{104} + x^{100} + x^{92} + x^{88} + x^{84} + x^{76} \\ & + x^{72} + x^{68} + x^{60} + x^{48} + x^{24} + x^{20} + x^{16} + x^{12} + 1, \end{aligned}$$

and the initial terms are [0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 1, 1, 0].

For $\phi(W^o, 1, 1)$:

$$\begin{aligned} p_0(x) = {} & x^{129} + x^{127} + x^{125} + x^{123} + x^{121} + x^{118} + x^{116} + x^{115} + x^{114} + x^{112} \\ & + x^{111} + x^{103} + x^{101} + x^{98} + x^{95} + x^{91} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} \\ & + x^{82} + x^{79} + x^{75} + x^{72} + x^{70} + x^{68} + x^{67} + x^{65} + x^{64} + x^{59} + x^{58} \\ & + x^{57} + x^{54} + x^{53} + x^{52} + x^{50} + x^{49} + x^{47} + x^{43} + x^{42}, \\ p_3(x) = {} & x^{127} + x^{126} + x^{124} + x^{119} + x^{118} + x^{117} + x^{116} + x^{112} + x^{110} + x^{108} \\ & + x^{107} + x^{106} + x^{105} + x^{103} + x^{100} + x^{95} + x^{94} + x^{93} + x^{92} + x^{88} \\ & + x^{86} + x^{81} + x^{78} + x^{77} + x^{76} + x^{72} + x^{70} + x^{67} + x^{66} + x^{65} + x^{62} \\ & + x^{61} + x^{60} + x^{56} + x^{55} + x^{49} + x^{47} + x^{45} + x^{44} + x^{40} + x^{39} + x^{36} \\ & + x^{33} + x^{30} + x^{27} + x^{26}, \end{aligned}$$

$$\begin{aligned}
p_6(x) = {} & x^{127} + x^{126} + x^{124} + x^{119} + x^{118} + x^{116} + x^{113} + x^{112} + x^{110} + x^{109} \\
& + x^{108} + x^{101} + x^{98} + x^{97} + x^{96} + x^{90} + x^{87} + x^{86} + x^{83} + x^{81} + x^{78} \\
& + x^{77} + x^{76} + x^{75} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{66} + x^{62} + x^{61} \\
& + x^{60} + x^{58} + x^{57} + x^{56} + x^{55} + x^{53} + x^{52} + x^{48} + x^{47} + x^{46} + x^{44} \\
& + x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{33} + x^{29} + x^{28} + x^{27} \\
& + x^{26} + x^{24} + x^{20} + x^{19} + x^{17} + x^{16} + x^{15} + x^{14} + x^{7} + x^{6} + x^{5} + x^{4} \\
& + x^{3} + x^{2} + x + 1, \\
p_9(x) = {} & x^{127} + x^{126} + x^{125} + x^{124} + x^{119} + x^{118} + x^{117} + x^{116} + x^{115} + x^{114} \\
& + x^{113} + x^{112} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} \\
& + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{71} + x^{70} + x^{69} \\
& + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{31} + x^{30} \\
& + x^{29} + x^{28} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} \\
& + x^{14} + x^{13} + x^{12}, \\
p_{12}(x) = {} & x^{127} + x^{126} + x^{125} + x^{124} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} \\
& + x^{97} + x^{96} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{71} \\
& + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} \\
& + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{31} + x^{30} + x^{29} \\
& + x^{28} + x^{15} + x^{14} + x^{13} + x^{12} + x^{7} + x^{6} + x^{5} + x^{4} + x^{3} + x^{2} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 1, 0, 0].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

| $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | [1, 2] | 43 | [52, 73] | 86 | [118, 119] | 129 | [161, 102] | 172 | [79, 8] |
| 1 | [3, 4] | 44 | [74, 75] | 87 | [23, 120] | 130 | [93, 44] | 173 | [88, 164] |
| 2 | [5, 6] | 45 | [76, 77] | 88 | [121, 122] | 131 | [85, 31] | 174 | [101, 10] |
| 3 | [7, 8] | 46 | [78, 74] | 89 | [123, 124] | 132 | [97, 87] | 175 | [43, 37] |
| 4 | [9, 10] | 47 | [79, 30] | 90 | [125, 126] | 133 | [96, 159] | 176 | [196, 93] |
| 5 | [11, 10] | 48 | [8, 28] | 91 | [4, 127] | 134 | [162, 43] | 177 | [156, 163] |
| 6 | [12, 13] | 49 | [31, 7] | 92 | [128, 129] | 135 | [34, 163] | 178 | [96, 44] |
| 7 | [14, 15] | 50 | [29, 8] | 93 | [68, 130] | 136 | [36, 159] | 179 | [10, 104] |
| 8 | [16, 17] | 51 | [80, 81] | 94 | [131, 132] | 137 | [89, 159] | 180 | [102, 87] |
| 9 | [18, 19] | 52 | [82, 42] | 95 | [133, 134] | 138 | [158, 44] | 181 | [161, 10] |
| 10 | [20, 21] | 53 | [83, 84] | 96 | [135, 136] | 139 | [156, 35] | 182 | [197, 155] |
| 11 | [22, 23] | 54 | [85, 7] | 97 | [137, 138] | 140 | [46, 89] | 183 | [45, 195] |
| 12 | [24, 25] | 55 | [86, 87] | 98 | [139, 140] | 141 | [97, 164] | 184 | [161, 88] |
| 13 | [26, 27] | 56 | [9, 88] | 99 | [141, 142] | 142 | [105, 88] | 185 | [8, 7] |
| 14 | [28, 29] | 57 | [89, 90] | 100 | [143, 144] | 143 | [92, 95] | 186 | [36, 44] |
| 15 | [30, 31] | 58 | [91, 89] | 101 | [145, 146] | 144 | [165, 36] | 187 | [162, 93] |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 16 | [31, 31] | 59 | [92, 42] | 102 | [147, 148] | 145 | [166, 165] | 188 | [99, 163] |
| 17 | [30, 7] | 60 | [93, 94] | 103 | [149, 150] | 146 | [34, 157] | 189 | [165, 89] |
| 18 | [32, 33] | 61 | [30, 28] | 104 | [19, 151] | 147 | [93, 90] | 190 | [11, 97] |
| 19 | [34, 35] | 62 | [88, 87] | 105 | [152, 26] | 148 | [89, 44] | 191 | [33, 36] |
| 20 | [36, 37] | 63 | [45, 95] | 106 | [7, 30] | 149 | [81, 37] | 192 | [99, 157] |
| 21 | [38, 39] | 64 | [96, 94] | 107 | [153, 97] | 150 | [89, 94] | 193 | [165, 93] |
| 22 | [40, 33] | 65 | [7, 29] | 108 | [41, 95] | 151 | [33, 43] | 194 | [198, 123] |
| 23 | [41, 42] | 66 | [97, 98] | 109 | [36, 39] | 152 | [167, 46] | 195 | [199, 200] |
| 24 | [43, 44] | 67 | [46, 93] | 110 | [10, 100] | 153 | [168, 139] | 196 | [201, 3] |
| 25 | [43, 39] | 68 | [99, 35] | 111 | [105, 103] | 154 | [169, 170] | 197 | [202, 203] |
| 26 | [45, 42] | 69 | [84, 100] | 112 | [154, 155] | 155 | [171, 172] | 198 | [91, 81] |
| 27 | [46, 36] | 70 | [101, 102] | 113 | [156, 157] | 156 | [173, 174] | 199 | [158, 90] |
| 28 | [47, 16] | 71 | [38, 37] | 114 | [38, 44] | 157 | [175, 176] | 200 | [196, 43] |
| 29 | [29, 29] | 72 | [91, 36] | 115 | [158, 94] | 158 | [177, 178] | 201 | [204, 84] |
| 30 | [48, 49] | 73 | [81, 39] | 116 | [85, 28] | 159 | [127, 179] | 202 | [196, 96] |
| 31 | [15, 50] | 74 | [79, 29] | 117 | [28, 8] | 160 | [180, 181] | 203 | [160, 157] |
| 32 | [51, 52] | 75 | [10, 13] | 118 | [158, 159] | 161 | [182, 183] | 204 | [205, 206] |
| 33 | [53, 54] | 76 | [103, 104] | 119 | [43, 159] | 162 | [184, 185] | 205 | [207, 165] |
| 34 | [55, 56] | 77 | [9, 103] | 120 | [33, 93] | 163 | [186, 187] | 206 | [41, 195] |
| 35 | [57, 58] | 78 | [105, 10] | 121 | [96, 90] | 164 | [188, 189] | 207 | [208, 209] |
| 36 | [59, 60] | 79 | [106, 14] | 122 | [38, 159] | 165 | [190, 131] | 208 | [162, 96] |
| 37 | [61, 62] | 80 | [107, 65] | 123 | [160, 35] | 166 | [191, 192] | 209 | [82, 195] |
| 38 | [63, 64] | 81 | [108, 109] | 124 | [81, 159] | 167 | [193, 59] | | |
| 39 | [65, 66] | 82 | [110, 111] | 125 | [29, 30] | 168 | [194, 46] | | |
| 40 | [67, 68] | 83 | [112, 113] | 126 | [88, 98] | 169 | [155, 36] | | |
| 41 | [69, 70] | 84 | [114, 115] | 127 | [8, 31] | 170 | [92, 195] | | |
| 42 | [71, 72] | 85 | [116, 117] | 128 | [12, 104] | 171 | [101, 88] | | |

Output function $n \mapsto \tau(n)$:

| $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 31 | 1 | 62 | 1 | 93 | 1 | 124 | 0 | 155 | 1 | 186 | 0 |
| 1 | 0 | 32 | 0 | 63 | 0 | 94 | 1 | 125 | 0 | 156 | 1 | 187 | 1 |
| 2 | 0 | 33 | 1 | 64 | 1 | 95 | 1 | 126 | 1 | 157 | 0 | 188 | 1 |
| 3 | 0 | 34 | 1 | 65 | 0 | 96 | 1 | 127 | 1 | 158 | 1 | 189 | 0 |
| 4 | 0 | 35 | 1 | 66 | 1 | 97 | 1 | 128 | 0 | 159 | 1 | 190 | 0 |
| 5 | 0 | 36 | 0 | 67 | 0 | 98 | 1 | 129 | 1 | 160 | 1 | 191 | 1 |
| 6 | 0 | 37 | 1 | 68 | 1 | 99 | 1 | 130 | 1 | 161 | 1 | 192 | 1 |
| 7 | 0 | 38 | 0 | 69 | 0 | 100 | 0 | 131 | 1 | 162 | 1 | 193 | 0 |
| 8 | 1 | 39 | 0 | 70 | 1 | 101 | 1 | 132 | 1 | 163 | 0 | 194 | 0 |
| 9 | 0 | 40 | 0 | 71 | 0 | 102 | 1 | 133 | 1 | 164 | 1 | 195 | 1 |
| 10 | 0 | 41 | 0 | 72 | 0 | 103 | 0 | 134 | 1 | 165 | 0 | 196 | 1 |
| 11 | 0 | 42 | 0 | 73 | 0 | 104 | 1 | 135 | 1 | 166 | 1 | 197 | 1 |
| 12 | 0 | 43 | 0 | 74 | 0 | 105 | 0 | 136 | 0 | 167 | 0 | 198 | 0 |
| 13 | 0 | 44 | 0 | 75 | 0 | 106 | 0 | 137 | 1 | 168 | 0 | 199 | 1 |

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 14 | 0 | 45 | 0 | 76 | 0 | 107 | 0 | 138 | 1 | 169 | 1 | 200 | 1 |
| 15 | 1 | 46 | 0 | 77 | 0 | 108 | 0 | 139 | 1 | 170 | 0 | 201 | 1 |
| 16 | 1 | 47 | 0 | 78 | 0 | 109 | 0 | 140 | 0 | 171 | 1 | 202 | 1 |
| 17 | 1 | 48 | 1 | 79 | 0 | 110 | 0 | 141 | 1 | 172 | 0 | 203 | 1 |
| 18 | 0 | 49 | 1 | 80 | 0 | 111 | 0 | 142 | 0 | 173 | 1 | 204 | 1 |
| 19 | 1 | 50 | 0 | 81 | 0 | 112 | 1 | 143 | 0 | 174 | 1 | 205 | 1 |
| 20 | 0 | 51 | 0 | 82 | 0 | 113 | 1 | 144 | 0 | 175 | 0 | 206 | 0 |
| 21 | 0 | 52 | 0 | 83 | 1 | 114 | 0 | 145 | 1 | 176 | 1 | 207 | 1 |
| 22 | 0 | 53 | 1 | 84 | 0 | 115 | 1 | 146 | 1 | 177 | 1 | 208 | 1 |
| 23 | 0 | 54 | 1 | 85 | 1 | 116 | 1 | 147 | 1 | 178 | 1 | 209 | 0 |
| 24 | 0 | 55 | 1 | 86 | 1 | 117 | 0 | 148 | 1 | 179 | 0 | | |
| 25 | 0 | 56 | 0 | 87 | 0 | 118 | 1 | 149 | 0 | 180 | 1 | | |
| 26 | 0 | 57 | 1 | 88 | 1 | 119 | 0 | 150 | 1 | 181 | 1 | | |
| 27 | 0 | 58 | 0 | 89 | 1 | 120 | 1 | 151 | 1 | 182 | 1 | | |
| 28 | 0 | 59 | 0 | 90 | 0 | 121 | 1 | 152 | 0 | 183 | 0 | | |
| 29 | 0 | 60 | 1 | 91 | 0 | 122 | 0 | 153 | 0 | 184 | 1 | | |
| 30 | 1 | 61 | 1 | 92 | 0 | 123 | 1 | 154 | 1 | 185 | 1 | | |

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