# PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a, b) = (z^3 + z^2, z^4 + z^2 + 1)$

GUO-NIU HAN* AND YINING HU*

Abstract. The continued fraction conjecture claims that the continued fraction defined by the $(a, b)$-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements $(a, b)$ in $\mathbb{F}_2[x] \setminus \mathbb{F}_2$. In this paper we prove the conjecture for $(a, b) = (z^3 + z^2, z^4 + z^2 + 1)$.

## 1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \ldots)$ of elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{\ddots + \cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let $a$ and $b$ be two distinct elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$. We consider the continued fraction (1.1) associated with the $(a, b)$-Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \ldots) = (a, b, b, a, b, a, a, b, \ldots)$$

that we shall denote

$$\mathrm{CF}(a, b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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**Theorem 1.1.** *When* $(a,b)=(z^3+z^2, z^4+z^2+1)$, *the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$, *with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

*where, for* $\mathrm{CF}(a,b)$,

$$\begin{aligned}
p_0(z)&=z^{21}+z^{20}+z^{17}+z^{14}+z^{13}+z^{11}+z^9+z^8+z^6+z^5+z^3+z^2+1,\\
p_1(z)&=z^{25}+z^{23}+z^{20}+z^{19}+z^{16}+z^{13}+z^{12}+z^{10}+z^8+z^4+z^3+z^2,\\
p_2(z)&=z^{28}+z^{24}+z^{22}+z^{20}+z^{16}+z^{14}+z^6+z^4,\\
p_3(z)&=z^{25}+z^{23}+z^{20}+z^{19}+z^{16}+z^{13}+z^{12}+z^{10}+z^8+z^4+z^3+z^2,\\
p_4(z)&=z^{22}+z^{21}+z^{20}+z^{17}+z^{16}+z^{13}+z^{11}+z^9+z^5+z^4+z^3+z^2,
\end{aligned}$$

*and for* $\mathrm{CF}(b,a)$,

$$\begin{aligned}
p_0(z)&=z^{21}+z^{20}+z^{17}+z^{16}+z^{14}+z^{13}+z^{12}+z^{11}+z^9+z^8+z^6+z^5+z^3\\
&\qquad+z^2,\\
p_1(z)&=z^{25}+z^{23}+z^{20}+z^{19}+z^{16}+z^{13}+z^{12}+z^{10}+z^8+z^4+z^3+z^2,\\
p_2(z)&=z^{28}+z^{24}+z^{22}+z^{20}+z^{16}+z^{14}+z^6+z^4,\\
p_3(z)&=z^{25}+z^{23}+z^{20}+z^{19}+z^{16}+z^{13}+z^{12}+z^{10}+z^8+z^4+z^3+z^2,\\
p_4(z)&=z^{22}+z^{21}+z^{20}+z^{17}+z^{13}+z^{12}+z^{11}+z^9+z^5+z^4+z^3+z^2+1.
\end{aligned}$$

## 2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x)&=x^{\deg(t_{2^n-1})}\begin{pmatrix} t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x)&=x^{\deg(\bar t_{2^n-1})}\begin{pmatrix} \bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix} \bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(\bar t_0)}\begin{pmatrix} \bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the $(b,a)$-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\geq 0$

$$\begin{aligned}
M_{n+1}(x)&=W_n(x)\cdot M_n(x),\\
W_{n+1}(x)&=M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer $d_n$. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four $2\times 2$ matrices $M^e, M^o, W^e, W^o$ as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(175, 175, 175, 175, 175)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices $M^e, M^o, W^e, W^o$. The first one is about relations between them; the second, about the structure of the each matrix.

**Lemma 2.1.** *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

*Proof.* We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0} M^o_{0,0}$ and $W^o_{0,1} M^o_{1,0}$. We know that

$$P(x, y) = \mathrm{Res}_z \left( \phi(W^o, 0, 0)(x, z),\ z^{12} \cdot \phi(M^o, 0, 0)(x, y/z) \right)$$

is an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0} M^o_{0,0}$, that will be called $\phi_0(x, y)$. To prove that $\phi_0(x, y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x, y)$ of multiplicity $m$ and that $Q(x, y) := P(x, y)/\phi_0(x, y)^m$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0} M^o_{0,0}$ to order 630 and substitute it for $y$ in $Q(x, y)$. We get a series of valuation less than 630, which proves that $Q(x, y)$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We find the minimal polynomial $\phi_1(x, y)$ of $W^o_{0,1} M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e, 0, 0)$ is the minimal polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. We know that

$$S(x, y) = \mathrm{Res}_z \left( \phi_0(x, z),\ \phi_1(x, y + z) \right)$$

is an annihilating of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. We verify that $\phi(M^e, 0, 0)$ is an irreducible factor of $S(x, y)$ of multiplicity $\mu$, and that $Q(x, y) := S(x, y)/\phi(M^e, 0, 0)^\mu$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$ to order 770 and substitute it for $y$ in $Q(x, y)$. We get a series of valuation less than 770, and therefore $Q(x, y)$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e, 0, 0)$, we know that the two series are one and the same. □

Define

$$R^e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

**Lemma 2.2.** *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned} M^e[:14u] = {} & M^e[:7u] + x^u M^e[:6u] + x^{2u} M^e[:5u] + x^{3u} M^e[:4u] + x^{4u} M^e[:3u] \\ & + x^{5u} M^e[:2u] + x^{6u} M^e[:u] + x^u M^e[:7u] + x^{2u} M^e[:6u] \\ & + x^{3u} M^e[:5u] + x^{4u} M^e[:4u] + x^{5u} M^e[:3u] + x^{6u} M^e[:2u] \\ & + x^{7u} M^e[:u] + x^{10u} M^e[:4u] + (x^{7u} + x^{8u}) R^e, \\ W^e[:14u] = {} & W^e[:7u] + x^u W^e[:6u] + x^{2u} W^e[:5u] + x^{3u} W^e[:4u] + x^{4u} W^e[:3u] \\ & + x^{5u} W^e[:2u] + x^{6u} W^e[:u] + x^u W^e[:7u] + x^{2u} W^e[:6u] \end{aligned}$$

$$+ x^{3u}W^e[:5u] + x^{4u}W^e[:4u] + x^{5u}W^e[:3u] + x^{6u}W^e[:2u]$$
$$+ x^{7u}W^e[:u] + x^{10u}W^e[:4u] + (x^{7u} + x^{8u})R^e.$$

*For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.*

$$\begin{aligned} M^o[:14u] = {} & M^o[:7u] + x^u M^o[:6u] + x^{2u}M^o[:5u] + x^{3u}M^o[:4u] + x^{4u}M^o[:3u] \\ & + x^{5u}M^o[:2u] + x^{6u}M^o[:u] + x^u M^o[:7u] + x^{2u}M^o[:6u] \\ & + x^{3u}M^o[:5u] + x^{4u}M^o[:4u] + x^{5u}M^o[:3u] + x^{6u}M^o[:2u] \\ & + x^{7u}M^o[:u] + x^{10u}M^o[:4u] + (x^{7u} + x^{8u})R^o, \\ W^o[:14u] = {} & W^o[:7u] + x^u W^o[:6u] + x^{2u}W^o[:5u] + x^{3u}W^o[:4u] + x^{4u}W^o[:3u] \\ & + x^{5u}W^o[:2u] + x^{6u}W^o[:u] + x^u W^o[:7u] + x^{2u}W^o[:6u] \\ & + x^{3u}W^o[:5u] + x^{4u}W^o[:4u] + x^{5u}W^o[:3u] + x^{6u}W^o[:2u] \\ & + x^{7u}W^o[:u] + x^{10u}W^o[:4u] + (x^{7u} + x^{8u})R^o. \end{aligned}$$

*Proof.* To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many $k$'s into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$\begin{aligned} T[:14u] = {} & T[:7u] + x^u T[:6u] + x^{2u}T[:5u] + x^{3u}T[:4u] + x^{4u}T[:3u] + x^{5u}T[:2u] \\ & + x^{6u}T[:u] + x^u T[:7u] + x^{2u}T[:6u] + x^{3u}T[:5u] + x^{4u}T[:4u] \\ & + x^{5u}T[:3u] + x^{6u}T[:2u] + x^{7u}T[:u] + x^{10u}T[:4u]. \end{aligned}$$

The proof of the other 15 cases are similar. We break down the above identity into 7 parts:

$$\begin{aligned} 0 &= x^{7u}T[:u] + x^{6u}T[u:2u] + x^{5u}T[2u:3u] + x^{4u}T[3u:4u] \\ &\quad + x^{3u}T[4u:5u] + x^{2u}T[5u:6u] + x^u T[6u:7u] + T[7u:8u], \\ 0 &= T[8u:9u], \\ 0 &= T[9u:10u], \\ 0 &= x^{10u}T[:u] + T[10u:11u], \\ 0 &= x^{10u}T[u:2u] + T[11u:12u], \\ 0 &= x^{10u}T[2u:3u] + T[12u:13u], \\ 0 &= x^{10u}T[3u:4u] + T[13u:14u], \end{aligned}$$

which can be rewritten as

$$\begin{aligned} 0 = {} & T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] \\ & + T[[101w]_2] + T[[110w]_2] + T[[111w]_2], \end{aligned} \tag{2.7}$$

$$0 = T[[1000w]_2], \tag{2.8}$$

$$0 = T[[1001w]_2], \tag{2.9}$$

$$0 = T[[w]_2] + T[[1010w]_2], \tag{2.10}$$

$$0 = T[[1w]_2] + T[[1011w]_2], \tag{2.11}$$

$$0 = T[[10w]_2] + T[[1100w]_2], \tag{2.12}$$

$$0 = T[[11w]_2] + T[[1101w]_2], \tag{2.13}$$

for all binary word $w$ of length $2k$ and $w \neq 0^{2k}$, and

$$\begin{align} 0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] \nonumber \\ &\quad + T[[101w]_2] + T[[110w]_2] + T[[111w]_2], \tag{2.14} \\ 0 &= T[[1000w]_2], \tag{2.15} \\ 0 &= T[[1001w]_2], \tag{2.16} \\ 0 &= T[[w]_2] + T[[1010w]_2], \tag{2.17} \\ 0 &= T[[1w]_2] + T[[1011w]_2], \tag{2.18} \\ 0 &= T[[10w]_2] + T[[1100w]_2], \tag{2.19} \\ 0 &= T[[11w]_2] + T[[1101w]_2], \tag{2.20} \end{align}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates $T$ from its minimal polynomial and its first terms. This automaton has 906 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading $w$ from right to left starting from the state $s$, and $\tau$ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.20) can be written as

$$\begin{align} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \nonumber \\ &\quad + \tau(A(s,101)) + \tau(A(s,110)) + \tau(A(s,111)), \tag{2.21} \\ 0 &= \tau(A(s,1000)), \tag{2.22} \\ 0 &= \tau(A(s,1001)), \tag{2.23} \\ 0 &= \tau(A(s,\epsilon)) + \tau(A(s,1010)), \tag{2.24} \\ 0 &= \tau(A(s,1)) + \tau(A(s,1011)), \tag{2.25} \\ 0 &= \tau(A(s,10)) + \tau(A(s,1100)), \tag{2.26} \\ 0 &= \tau(A(s,11)) + \tau(A(s,1101)), \tag{2.27} \end{align}$$

for all $s \in E_{2k}$, and

$$\begin{align} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \nonumber \\ &\quad + \tau(A(s,101)) + \tau(A(s,110)) + \tau(A(s,111)), \tag{2.28} \\ 0 &= \tau(A(s,1000)), \tag{2.29} \\ 0 &= \tau(A(s,1001)), \tag{2.30} \\ 0 &= \tau(A(s,\epsilon)) + \tau(A(s,1010)), \tag{2.31} \\ 0 &= \tau(A(s,1)) + \tau(A(s,1011)), \tag{2.32} \\ 0 &= \tau(A(s,10)) + \tau(A(s,1100)), \tag{2.33} \\ 0 &= \tau(A(s,11)) + \tau(A(s,1101)), \tag{2.34} \end{align}$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{34}), E_{34}) = (A(i, 0^{22}), E_{22})$, so that we only have to verify that identities (2.21) through (2.34) hold for $2 \leq k \leq 17$, which turns out to be true. $\square$

In the following lemma, we express $M_{2k}$, $M_{2k+1}$, $W_{2k}$, and $W_{2k+1}$ in terms of $M^e$, $M^o$, $W^e$, and $W^o$.

**Lemma 2.3.** *For all integer $k \geq 2$, and $u = 2^{2k-1}$,*

$$\begin{aligned} M_{2k} &= M^e[:7u] + x^u M^e[:6u] + x^{2u} M^e[:5u] + x^{3u} M^e[:4u] + x^{4u} M^e[:3u] \\ &\quad + x^{5u} M^e[:2u] + x^{6u} M^e[:u] + x^{7u} R^e, \\ W_{2k} &= W^e[:7u] + x^u W^e[:6u] + x^{2u} W^e[:5u] + x^{3u} W^e[:4u] + x^{4u} W^e[:3u] \\ &\quad + x^{5u} W^e[:2u] + x^{6u} W^e[:u] + x^{7u} R^e. \end{aligned}$$

*For all integer $k \geq 2$, and $u = 2^{2k}$,*

$$\begin{aligned} M_{2k+1} &= M^o[:7u] + x^u M^o[:6u] + x^{2u} M^o[:5u] + x^{3u} M^o[:4u] + x^{4u} M^o[:3u] \\ &\quad + x^{5u} M^o[:2u] + x^{6u} M^o[:u] + x^{7u} R^o, \\ W_{2k+1} &= W^o[:7u] + x^u W^o[:6u] + x^{2u} W^o[:5u] + x^{3u} W^o[:4u] + x^{4u} W^o[:3u] \\ &\quad + x^{5u} W^o[:2u] + x^{6u} W^o[:u] + x^{7u} R^o. \end{aligned}$$

*Proof.* We call the four identities in Lemma 2.3 also by the name $M_{2k}$, $W_{2k}$, $M_{2k+1}$, and $W_{2k+1}$. For $n = 2$, the identities can be verified directly. For $n \geq 2$, we claim that

$$\begin{aligned} M_{2k} \wedge W_{2k} &\Rightarrow M_{2k+1} \wedge W_{2k+1}, \\ M_{2k+1} \wedge W_{2k+1} &\Rightarrow M_{2k+2} \wedge W_{2k+2}. \end{aligned}$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity $M_{2k+1}$ is equal to

$$\begin{aligned} W_{2k} M_{2k} = \big(&W^e[:7v] + x^v W^e[:6v] + x^{2v} W^e[:5v] + x^{3v} W^e[:4v] \\ &+ x^{4v} W^e[:3v] + x^{5v} W^e[:2v] + x^{6v} W^e[:v] + x^{7u} R^e\big) \times \big(M^e[:7v] \\ &+ x^v M^e[:6v] + x^{2v} M^e[:5v] + x^{3v} M^e[:4v] + x^{4v} M^e[:3v] \\ &+ x^{5v} M^e[:2v] + x^{6v} M^e[:v] + x^{7u} R^e\big); \end{aligned} \tag{2.35}$$

Call this expression $lhs$. Note that both sides of identity $M_{2k+1}$ have the same term of highest degree $x^{14v} R^o$. Therefore we only need to prove that their difference is $O(x^{14v})$. Using Lemma 2.1 it can be seen that the right side of identity $M_{2k+1}$ is congruent, modulo $x^{14v}$, to

$$\begin{aligned} &W^e[:14v] M^e[:14v] + x^{2v} W^e[:12v] M^e[:12v] + x^{4v} W^e[:10v] M^e[:10v] \\ &\quad + x^{6v} W^e[:8v] M^e[:8v] + x^{8v} W^e[:6v] M^e[:6v] + x^{10v} W^e[:4v] M^e[:4v] \\ &\quad + x^{12v} W^e[:2v] M^e[:2v]. \end{aligned}$$

For all $n \leq 14$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call $rhs$. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v] / X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v] / X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions $lhs$ (2.35) and $rhs$ as polynomials in $\mathbb{F}_2[a_1, ..., a_{14}, b_1, ..., b_{14}, c][X]$. Note that it is not a problem that $a_j$ commutes with $b_k$ while $W^e$ does not commute with $M^e$, because in the expressions concerned, the products of $W^e$-terms and $M^e$-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{14})$, which completes the proof. $\square$

*Proof of Theorem 1.1.* We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \le j, k \le 1$,

$$\begin{aligned} \lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k}, \\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k}, \\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k}, \\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}. \end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a, b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x, y) = \mathrm{Res}_t \left(\phi(M^e, 0, 1)(x, t), y^{12}\phi(M^e, 0, 1)(x, t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x, y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned} q_0(x) &= x^{28} + x^{26} + x^{25} + x^{23} + x^{22} + x^{20} + x^{19} + x^{17} + x^{15} + x^{14} + x^{11} + x^8 \\ &\quad + x^7, \\ q_1(x) &= x^{26} + x^{25} + x^{24} + x^{20} + x^{18} + x^{16} + x^{15} + x^{12} + x^9 + x^8 + x^5 + x^3, \\ q_2(x) &= x^{24} + x^{22} + x^{14} + x^{12} + x^8 + x^6 + x^4 + 1, \\ q_3(x) &= x^{26} + x^{25} + x^{24} + x^{20} + x^{18} + x^{16} + x^{15} + x^{12} + x^9 + x^8 + x^5 + x^3, \\ q_4(x) &= x^{26} + x^{25} + x^{24} + x^{23} + x^{19} + x^{17} + x^{15} + x^{12} + x^{11} + x^8 + x^7 + x^6. \end{aligned}$$

The polynomial $Q(x, y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x, y)$ of multiplicity $m$ and $R(x, y) := P(x, y)/Q(x, y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 216, and substitute it for $y$ in $R(z, y)$, we get a series with valuation smaller than 216, which proves that $R(z, y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z, y)$ is the minimal polynomial of $\mathrm{CF}(a, b) = f(1/z)$. $\square$

## 3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{150} + x^{146} + x^{142} + x^{138} + x^{136} + x^{134} + x^{124} + x^{122} + x^{114} + x^{110} \\
&\quad + x^{102} + x^{100} + x^{98} + x^{92} + x^{90} + x^{88} + x^{80} + x^{70} + x^{68} + x^{66} + x^{62} \\
&\quad + x^{60} + x^{58} + x^{56} + x^{54} + x^{48} + x^{42} + x^{40} + x^{36}, \\
p_3(x) &= x^{142} + x^{140} + x^{138} + x^{136} + x^{132} + x^{126} + x^{124} + x^{118} + x^{116} + x^{110} \\
&\quad + x^{92} + x^{90} + x^{88} + x^{78} + x^{76} + x^{70} + x^{68} + x^{66} + x^{64} + x^{54} + x^{50} \\
&\quad + x^{48} + x^{46} + x^{44} + x^{42} + x^{40} + x^{38} + x^{36} + x^{30} + x^{26} + x^{24} + x^{20}, \\
p_6(x) &= x^{140} + x^{138} + x^{136} + x^{132} + x^{128} + x^{124} + x^{122} + x^{120} + x^{118} + x^{116} \\
&\quad + x^{112} + x^{110} + x^{106} + x^{102} + x^{94} + x^{92} + x^{88} + x^{86} + x^{82} + x^{80} \\
&\quad + x^{76} + x^{72} + x^{70} + x^{68} + x^{58} + x^{56} + x^{54} + x^{44} + x^{42} + x^{40} + x^{36} \\
&\quad + x^{34} + x^{26} + x^{24} + x^{20} + x^{18} + x^{14} + x^{10} + x^{8} + x^{6} + x^{2} + 1, \\
p_9(x) &= x^{146} + x^{144} + x^{142} + x^{138} + x^{134} + x^{132} + x^{126} + x^{124} + x^{122} + x^{116} \\
&\quad + x^{108} + x^{106} + x^{96} + x^{92} + x^{86} + x^{84} + x^{80} + x^{78} + x^{76} + x^{74} + x^{70} \\
&\quad + x^{66} + x^{62} + x^{60} + x^{54} + x^{50} + x^{44} + x^{42} + x^{40} + x^{32} + x^{30} + x^{24} \\
&\quad + x^{22} + x^{20} + x^{18} + x^{10} + x^{8} + x^{6}, \\
p_{12}(x) &= x^{146} + x^{144} + x^{130} + x^{128} + x^{98} + x^{96} + x^{82} + x^{80} + x^{66} + x^{64} + x^{34} \\
&\quad + x^{32} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0].

For $\phi(M^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{147} + x^{145} + x^{144} + x^{142} + x^{141} + x^{140} + x^{138} + x^{137} + x^{136} + x^{135} \\
&\quad + x^{134} + x^{130} + x^{129} + x^{127} + x^{126} + x^{123} + x^{121} + x^{118} + x^{117} \\
&\quad + x^{115} + x^{113} + x^{111} + x^{107} + x^{106} + x^{103} + x^{98} + x^{96} + x^{95} + x^{94} \\
&\quad + x^{93} + x^{92} + x^{91} + x^{90} + x^{88} + x^{87} + x^{81} + x^{79} + x^{78} + x^{76} + x^{74} \\
&\quad + x^{66} + x^{64} + x^{62} + x^{61} + x^{60} + x^{58} + x^{56} + x^{54} + x^{49} + x^{45} + x^{41} \\
&\quad + x^{40} + x^{39}, \\
p_3(x) &= x^{143} + x^{141} + x^{140} + x^{139} + x^{138} + x^{137} + x^{133} + x^{131} + x^{130} + x^{129} \\
&\quad + x^{128} + x^{127} + x^{123} + x^{120} + x^{119} + x^{115} + x^{112} + x^{111} + x^{109} \\
&\quad + x^{106} + x^{105} + x^{93} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{85} + x^{82} + x^{81} \\
&\quad + x^{79} + x^{76} + x^{72} + x^{68} + x^{67} + x^{61} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} \\
&\quad + x^{53} + x^{50} + x^{49} + x^{45} + x^{43} + x^{42} + x^{41} + x^{40} + x^{37} + x^{36} + x^{34} \\
&\quad + x^{33} + x^{32} + x^{28} + x^{27},
\end{aligned}$$

$$
\begin{aligned}
p_6(x) = {} & x^{141} + x^{140} + x^{137} + x^{136} + x^{133} + x^{132} + x^{127} + x^{126} + x^{125} + x^{124} \\
& + x^{123} + x^{122} + x^{121} + x^{120} + x^{117} + x^{116} + x^{109} + x^{108} + x^{107} \\
& + x^{106} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{95} + x^{94} + x^{93} \\
& + x^{92} + x^{87} + x^{86} + x^{83} + x^{82} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} \\
& + x^{63} + x^{62} + x^{57} + x^{56} + x^{53} + x^{52} + x^{51} + x^{50} + x^{37} + x^{36} + x^{33} \\
& + x^{32} + x^{29} + x^{28} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18}, \\
p_9(x) = {} & x^{139} + x^{136} + x^{135} + x^{133} + x^{130} + x^{127} + x^{126} + x^{124} + x^{123} + x^{122} \\
& + x^{118} + x^{117} + x^{113} + x^{111} + x^{110} + x^{109} + x^{108} + x^{104} + x^{101} \\
& + x^{100} + x^{98} + x^{97} + x^{96} + x^{95} + x^{93} + x^{90} + x^{86} + x^{82} + x^{79} + x^{78} \\
& + x^{77} + x^{76} + x^{73} + x^{72} + x^{70} + x^{68} + x^{66} + x^{64} + x^{62} + x^{61} + x^{60} \\
& + x^{57} + x^{56} + x^{54} + x^{52} + x^{50} + x^{49} + x^{48} + x^{44} + x^{40} + x^{39} + x^{27} \\
& + x^{25} + x^{24} + x^{23} + x^{22} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{13} + x^{10} \\
& + x^{9}, \\
p_{12}(x) = {} & x^{137} + x^{136} + x^{135} + x^{134} + x^{131} + x^{130} + x^{127} + x^{126} + x^{123} + x^{122} \\
& + x^{119} + x^{118} + x^{115} + x^{114} + x^{113} + x^{112} + x^{105} + x^{104} + x^{103} \\
& + x^{102} + x^{99} + x^{98} + x^{95} + x^{94} + x^{91} + x^{90} + x^{87} + x^{86} + x^{83} + x^{82} \\
& + x^{81} + x^{80} + x^{57} + x^{56} + x^{55} + x^{54} + x^{51} + x^{50} + x^{47} + x^{46} + x^{43} \\
& + x^{42} + x^{41} + x^{40} + x^{33} + x^{32} + x^{31} + x^{30} + x^{27} + x^{26} + x^{25} + x^{24} \\
& + x^{17} + x^{16} + x^{15} + x^{14} + x^{11} + x^{10} + x^{7} + x^{6} + x^{3} + x^{2} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0].

For $\phi(M^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) = {} & x^{147} + x^{145} + x^{144} + x^{142} + x^{141} + x^{140} + x^{138} + x^{137} + x^{136} + x^{135} \\
& + x^{134} + x^{130} + x^{129} + x^{127} + x^{126} + x^{123} + x^{121} + x^{118} + x^{117} \\
& + x^{115} + x^{113} + x^{111} + x^{107} + x^{106} + x^{103} + x^{98} + x^{96} + x^{95} + x^{94} \\
& + x^{93} + x^{92} + x^{91} + x^{90} + x^{88} + x^{87} + x^{81} + x^{79} + x^{78} + x^{76} + x^{74} \\
& + x^{66} + x^{64} + x^{62} + x^{61} + x^{60} + x^{58} + x^{56} + x^{54} + x^{49} + x^{45} + x^{41} \\
& + x^{40} + x^{39}, \\
p_3(x) = {} & x^{143} + x^{141} + x^{140} + x^{139} + x^{138} + x^{137} + x^{133} + x^{131} + x^{130} + x^{129} \\
& + x^{128} + x^{127} + x^{123} + x^{120} + x^{119} + x^{115} + x^{112} + x^{111} + x^{109} \\
& + x^{106} + x^{105} + x^{93} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{85} + x^{82} + x^{81} \\
& + x^{79} + x^{76} + x^{72} + x^{68} + x^{67} + x^{61} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} \\
& + x^{53} + x^{50} + x^{49} + x^{45} + x^{43} + x^{42} + x^{41} + x^{40} + x^{37} + x^{36} + x^{34} \\
& + x^{33} + x^{32} + x^{28} + x^{27}, \\
p_6(x) = {} & x^{141} + x^{140} + x^{137} + x^{136} + x^{133} + x^{132} + x^{127} + x^{126} + x^{125} + x^{124} \\
& + x^{123} + x^{122} + x^{121} + x^{120} + x^{117} + x^{116} + x^{109} + x^{108} + x^{107}
\end{aligned}
$$

$$+ x^{106} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{95} + x^{94} + x^{93} + x^{92} + x^{87} + x^{86} + x^{83} + x^{82} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{63} + x^{62} + x^{57} + x^{56} + x^{53} + x^{52} + x^{51} + x^{50} + x^{37} + x^{36} + x^{33} + x^{32} + x^{29} + x^{28} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18},$$

$$p_9(x) = x^{139} + x^{136} + x^{135} + x^{133} + x^{130} + x^{127} + x^{126} + x^{124} + x^{123} + x^{122} + x^{118} + x^{117} + x^{113} + x^{111} + x^{110} + x^{109} + x^{108} + x^{104} + x^{101} + x^{100} + x^{98} + x^{97} + x^{96} + x^{95} + x^{93} + x^{90} + x^{86} + x^{82} + x^{79} + x^{78} + x^{77} + x^{76} + x^{73} + x^{72} + x^{70} + x^{68} + x^{66} + x^{64} + x^{62} + x^{61} + x^{60} + x^{57} + x^{56} + x^{54} + x^{52} + x^{50} + x^{49} + x^{48} + x^{44} + x^{40} + x^{39} + x^{27} + x^{25} + x^{24} + x^{23} + x^{22} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{13} + x^{10} + x^{9},$$

$$p_{12}(x) = x^{137} + x^{136} + x^{135} + x^{134} + x^{131} + x^{130} + x^{127} + x^{126} + x^{123} + x^{122} + x^{119} + x^{118} + x^{115} + x^{114} + x^{113} + x^{112} + x^{105} + x^{104} + x^{103} + x^{102} + x^{99} + x^{98} + x^{95} + x^{94} + x^{91} + x^{90} + x^{87} + x^{86} + x^{83} + x^{82} + x^{81} + x^{80} + x^{57} + x^{56} + x^{55} + x^{54} + x^{51} + x^{50} + x^{47} + x^{46} + x^{43} + x^{42} + x^{41} + x^{40} + x^{33} + x^{32} + x^{31} + x^{30} + x^{27} + x^{26} + x^{25} + x^{24} + x^{17} + x^{16} + x^{15} + x^{14} + x^{11} + x^{10} + x^{7} + x^{6} + x^{3} + x^{2} + x + 1,$$

and the initial terms are [0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0].

For $\phi(M^e, 1, 1)$:

$$p_0(x) = x^{144} + x^{138} + x^{136} + x^{132} + x^{130} + x^{128} + x^{126} + x^{124} + x^{122} + x^{120} + x^{112} + x^{108} + x^{106} + x^{104} + x^{102} + x^{96} + x^{94} + x^{92} + x^{90} + x^{84} + x^{72} + x^{68} + x^{64} + x^{62} + x^{58} + x^{54} + x^{50} + x^{44} + x^{42},$$

$$p_3(x) = x^{132} + x^{130} + x^{128} + x^{122} + x^{114} + x^{110} + x^{108} + x^{104} + x^{102} + x^{100} + x^{98} + x^{94} + x^{86} + x^{82} + x^{80} + x^{78} + x^{76} + x^{74} + x^{70} + x^{68} + x^{64} + x^{62} + x^{60} + x^{58} + x^{56} + x^{52} + x^{46} + x^{44} + x^{42} + x^{38} + x^{34} + x^{32} + x^{30} + x^{26} + x^{24} + x^{22} + x^{20} + x^{18},$$

$$p_6(x) = x^{132} + x^{130} + x^{120} + x^{116} + x^{114} + x^{112} + x^{110} + x^{108} + x^{104} + x^{102} + x^{96} + x^{94} + x^{92} + x^{90} + x^{88} + x^{84} + x^{80} + x^{78} + x^{76} + x^{72} + x^{68} + x^{66} + x^{64} + x^{60} + x^{58} + x^{46} + x^{42} + x^{32} + x^{28} + x^{24} + x^{18} + x^{14} + x^{12} + x^{6} + x^{4} + 1,$$

$$p_9(x) = x^{128} + x^{122} + x^{120} + x^{118} + x^{110} + x^{100} + x^{98} + x^{96} + x^{94} + x^{90} + x^{88} + x^{86} + x^{82} + x^{80} + x^{76} + x^{74} + x^{70} + x^{66} + x^{62} + x^{54} + x^{38} + x^{34} + x^{28} + x^{26} + x^{24} + x^{22} + x^{12} + x^{6},$$

$$p_{12}(x) = x^{128} + x^{124} + x^{116} + x^{112} + x^{96} + x^{92} + x^{84} + x^{80} + x^{48} + x^{44} + x^{36} + x^{28} + x^{20} + x^{12} + x^{4} + 1,$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0].

For $\phi(W^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{144} + x^{138} + x^{132} + x^{130} + x^{126} + x^{120} + x^{114} + x^{106} + x^{104} + x^{98} \\
&\quad + x^{84} + x^{82} + x^{80} + x^{76} + x^{74} + x^{70} + x^{66} + x^{60} + x^{48} + x^{44} + x^{40} \\
&\quad + x^{38} + x^{36}, \\
p_3(x) &= x^{132} + x^{130} + x^{128} + x^{124} + x^{118} + x^{112} + x^{108} + x^{106} + x^{104} + x^{96} \\
&\quad + x^{94} + x^{92} + x^{90} + x^{88} + x^{84} + x^{80} + x^{78} + x^{74} + x^{62} + x^{60} + x^{58} \\
&\quad + x^{52} + x^{50} + x^{48} + x^{46} + x^{44} + x^{40} + x^{38} + x^{36} + x^{34} + x^{28} + x^{24} \\
&\quad + x^{22} + x^{20}, \\
p_6(x) &= x^{132} + x^{130} + x^{124} + x^{122} + x^{118} + x^{116} + x^{108} + x^{106} + x^{104} + x^{100} \\
&\quad + x^{98} + x^{94} + x^{92} + x^{88} + x^{86} + x^{82} + x^{80} + x^{78} + x^{76} + x^{72} + x^{70} \\
&\quad + x^{68} + x^{66} + x^{64} + x^{60} + x^{58} + x^{56} + x^{50} + x^{46} + x^{44} + x^{30} + x^{28} \\
&\quad + x^{26} + x^{24} + x^{20} + x^{16} + x^{14} + x^{6} + x^{4} + 1, \\
p_9(x) &= x^{128} + x^{122} + x^{120} + x^{118} + x^{110} + x^{100} + x^{98} + x^{96} + x^{94} + x^{90} \\
&\quad + x^{88} + x^{86} + x^{82} + x^{80} + x^{76} + x^{74} + x^{70} + x^{66} + x^{62} + x^{54} + x^{38} \\
&\quad + x^{34} + x^{28} + x^{26} + x^{24} + x^{22} + x^{12} + x^{6}, \\
p_{12}(x) &= x^{128} + x^{124} + x^{116} + x^{112} + x^{96} + x^{92} + x^{84} + x^{80} + x^{48} + x^{44} + x^{36} \\
&\quad + x^{28} + x^{20} + x^{12} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0].

For $\phi(W^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{147} + x^{140} + x^{137} + x^{136} + x^{130} + x^{129} + x^{128} + x^{127} + x^{124} + x^{122} \\
&\quad + x^{121} + x^{120} + x^{118} + x^{114} + x^{112} + x^{108} + x^{106} + x^{105} + x^{104} \\
&\quad + x^{103} + x^{101} + x^{100} + x^{96} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{88} + x^{85} \\
&\quad + x^{84} + x^{83} + x^{80} + x^{78} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{71} + x^{70} \\
&\quad + x^{67} + x^{60} + x^{59} + x^{58} + x^{57} + x^{54} + x^{49} + x^{48} + x^{47} + x^{46} + x^{44} \\
&\quad + x^{41} + x^{40} + x^{39}, \\
p_3(x) &= x^{143} + x^{141} + x^{140} + x^{139} + x^{138} + x^{137} + x^{133} + x^{131} + x^{130} + x^{129} \\
&\quad + x^{128} + x^{127} + x^{123} + x^{120} + x^{119} + x^{115} + x^{112} + x^{111} + x^{109} \\
&\quad + x^{106} + x^{105} + x^{93} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{85} + x^{82} + x^{81} \\
&\quad + x^{79} + x^{76} + x^{72} + x^{68} + x^{67} + x^{61} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} \\
&\quad + x^{53} + x^{50} + x^{49} + x^{45} + x^{43} + x^{42} + x^{41} + x^{40} + x^{37} + x^{36} + x^{34} \\
&\quad + x^{33} + x^{32} + x^{28} + x^{27}, \\
p_6(x) &= x^{141} + x^{140} + x^{137} + x^{136} + x^{133} + x^{132} + x^{127} + x^{126} + x^{125} + x^{124} \\
&\quad + x^{123} + x^{122} + x^{121} + x^{120} + x^{117} + x^{116} + x^{109} + x^{108} + x^{107} \\
&\quad + x^{106} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{95} + x^{94} + x^{93}
\end{aligned}$$

$$\begin{aligned}
&+ x^{92} + x^{87} + x^{86} + x^{83} + x^{82} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68}\\
&+ x^{63} + x^{62} + x^{57} + x^{56} + x^{53} + x^{52} + x^{51} + x^{50} + x^{37} + x^{36} + x^{33}\\
&+ x^{32} + x^{29} + x^{28} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18},\\
p_9(x) &= x^{139} + x^{136} + x^{135} + x^{133} + x^{130} + x^{127} + x^{126} + x^{124} + x^{123} + x^{122}\\
&+ x^{118} + x^{117} + x^{113} + x^{111} + x^{110} + x^{109} + x^{108} + x^{104} + x^{101}\\
&+ x^{100} + x^{98} + x^{97} + x^{96} + x^{95} + x^{93} + x^{90} + x^{86} + x^{82} + x^{79} + x^{78}\\
&+ x^{77} + x^{76} + x^{73} + x^{72} + x^{70} + x^{68} + x^{66} + x^{64} + x^{62} + x^{61} + x^{60}\\
&+ x^{57} + x^{56} + x^{54} + x^{52} + x^{50} + x^{49} + x^{48} + x^{44} + x^{40} + x^{39} + x^{27}\\
&+ x^{25} + x^{24} + x^{23} + x^{22} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{13} + x^{10}\\
&+ x^{9},\\
p_{12}(x) &= x^{137} + x^{136} + x^{135} + x^{134} + x^{131} + x^{130} + x^{127} + x^{126} + x^{123} + x^{122}\\
&+ x^{119} + x^{118} + x^{115} + x^{114} + x^{113} + x^{112} + x^{105} + x^{104} + x^{103}\\
&+ x^{102} + x^{99} + x^{98} + x^{95} + x^{94} + x^{91} + x^{90} + x^{87} + x^{86} + x^{83} + x^{82}\\
&+ x^{81} + x^{80} + x^{57} + x^{56} + x^{55} + x^{54} + x^{51} + x^{50} + x^{47} + x^{46} + x^{43}\\
&+ x^{42} + x^{41} + x^{40} + x^{33} + x^{32} + x^{31} + x^{30} + x^{27} + x^{26} + x^{25} + x^{24}\\
&+ x^{17} + x^{16} + x^{15} + x^{14} + x^{11} + x^{10} + x^{7} + x^{6} + x^{3} + x^{2} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 1, 1, 0].

For $\phi(W^e, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{147} + x^{140} + x^{137} + x^{136} + x^{130} + x^{129} + x^{128} + x^{127} + x^{124} + x^{122}\\
&+ x^{121} + x^{120} + x^{118} + x^{114} + x^{112} + x^{108} + x^{106} + x^{105} + x^{104}\\
&+ x^{103} + x^{101} + x^{100} + x^{96} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{88} + x^{85}\\
&+ x^{84} + x^{83} + x^{80} + x^{78} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{71} + x^{70}\\
&+ x^{67} + x^{60} + x^{59} + x^{58} + x^{57} + x^{54} + x^{49} + x^{48} + x^{47} + x^{46} + x^{44}\\
&+ x^{41} + x^{40} + x^{39},\\
p_3(x) &= x^{143} + x^{141} + x^{140} + x^{139} + x^{138} + x^{137} + x^{133} + x^{131} + x^{130} + x^{129}\\
&+ x^{128} + x^{127} + x^{123} + x^{120} + x^{119} + x^{115} + x^{112} + x^{111} + x^{109}\\
&+ x^{106} + x^{105} + x^{93} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{85} + x^{82} + x^{81}\\
&+ x^{79} + x^{76} + x^{72} + x^{68} + x^{67} + x^{61} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55}\\
&+ x^{53} + x^{50} + x^{49} + x^{45} + x^{43} + x^{42} + x^{41} + x^{40} + x^{37} + x^{36} + x^{34}\\
&+ x^{33} + x^{32} + x^{28} + x^{27},\\
p_6(x) &= x^{141} + x^{140} + x^{137} + x^{136} + x^{133} + x^{132} + x^{127} + x^{126} + x^{125} + x^{124}\\
&+ x^{123} + x^{122} + x^{121} + x^{120} + x^{117} + x^{116} + x^{109} + x^{108} + x^{107}\\
&+ x^{106} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{95} + x^{94} + x^{93}\\
&+ x^{92} + x^{87} + x^{86} + x^{83} + x^{82} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68}\\
&+ x^{63} + x^{62} + x^{57} + x^{56} + x^{53} + x^{52} + x^{51} + x^{50} + x^{37} + x^{36} + x^{33}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{32} + x^{29} + x^{28} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18},\\
p_9(x) &= x^{139} + x^{136} + x^{135} + x^{133} + x^{130} + x^{127} + x^{126} + x^{124} + x^{123} + x^{122}\\
&+ x^{118} + x^{117} + x^{113} + x^{111} + x^{110} + x^{109} + x^{108} + x^{104} + x^{101}\\
&+ x^{100} + x^{98} + x^{97} + x^{96} + x^{95} + x^{93} + x^{90} + x^{86} + x^{82} + x^{79} + x^{78}\\
&+ x^{77} + x^{76} + x^{73} + x^{72} + x^{70} + x^{68} + x^{66} + x^{64} + x^{62} + x^{61} + x^{60}\\
&+ x^{57} + x^{56} + x^{54} + x^{52} + x^{50} + x^{49} + x^{48} + x^{44} + x^{40} + x^{39} + x^{27}\\
&+ x^{25} + x^{24} + x^{23} + x^{22} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{13} + x^{10}\\
&+ x^{9},\\
p_{12}(x) &= x^{137} + x^{136} + x^{135} + x^{134} + x^{131} + x^{130} + x^{127} + x^{126} + x^{123} + x^{122}\\
&+ x^{119} + x^{118} + x^{115} + x^{114} + x^{113} + x^{112} + x^{105} + x^{104} + x^{103}\\
&+ x^{102} + x^{99} + x^{98} + x^{95} + x^{94} + x^{91} + x^{90} + x^{87} + x^{86} + x^{83} + x^{82}\\
&+ x^{81} + x^{80} + x^{57} + x^{56} + x^{55} + x^{54} + x^{51} + x^{50} + x^{47} + x^{46} + x^{43}\\
&+ x^{42} + x^{41} + x^{40} + x^{33} + x^{32} + x^{31} + x^{30} + x^{27} + x^{26} + x^{25} + x^{24}\\
&+ x^{17} + x^{16} + x^{15} + x^{14} + x^{11} + x^{10} + x^{7} + x^{6} + x^{3} + x^{2} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 1, 1, 0].

For $\phi(W^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{150} + x^{144} + x^{128} + x^{122} + x^{116} + x^{114} + x^{110} + x^{106} + x^{104} + x^{100}\\
&+ x^{98} + x^{96} + x^{92} + x^{90} + x^{86} + x^{84} + x^{82} + x^{78} + x^{76} + x^{70} + x^{60}\\
&+ x^{58} + x^{56} + x^{50} + x^{42},\\
p_3(x) &= x^{140} + x^{138} + x^{134} + x^{132} + x^{124} + x^{120} + x^{118} + x^{116} + x^{114} + x^{112}\\
&+ x^{108} + x^{106} + x^{102} + x^{96} + x^{94} + x^{92} + x^{90} + x^{88} + x^{86} + x^{80} + x^{76}\\
&+ x^{74} + x^{70} + x^{64} + x^{54} + x^{48} + x^{36} + x^{34} + x^{28} + x^{26} + x^{22} + x^{18},\\
p_6(x) &= x^{142} + x^{140} + x^{136} + x^{130} + x^{126} + x^{124} + x^{120} + x^{114} + x^{108} + x^{102}\\
&+ x^{100} + x^{98} + x^{94} + x^{90} + x^{80} + x^{68} + x^{64} + x^{62} + x^{60} + x^{58} + x^{56}\\
&+ x^{54} + x^{50} + x^{44} + x^{38} + x^{36} + x^{34} + x^{32} + x^{26} + x^{20} + x^{12} + x^{10}\\
&+ x^{8} + x^{6} + x^{2} + 1,\\
p_9(x) &= x^{146} + x^{144} + x^{142} + x^{138} + x^{134} + x^{132} + x^{126} + x^{124} + x^{122} + x^{116}\\
&+ x^{108} + x^{106} + x^{96} + x^{92} + x^{86} + x^{84} + x^{80} + x^{78} + x^{76} + x^{74} + x^{70}\\
&+ x^{66} + x^{62} + x^{60} + x^{54} + x^{50} + x^{44} + x^{42} + x^{40} + x^{32} + x^{30} + x^{24}\\
&+ x^{22} + x^{20} + x^{18} + x^{10} + x^{8} + x^{6},\\
p_{12}(x) &= x^{146} + x^{144} + x^{130} + x^{128} + x^{98} + x^{96} + x^{82} + x^{80} + x^{66} + x^{64} + x^{34}\\
&+ x^{32} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0].

For $\phi(M^o, 0, 0)$:

$$
p_0(x) = x^{147} + x^{145} + x^{144} + x^{143} + x^{142} + x^{140} + x^{138} + x^{136} + x^{132} + x^{131}
$$

$$\begin{aligned}
&+ x^{130} + x^{127} + x^{125} + x^{124} + x^{123} + x^{122} + x^{121} + x^{119} + x^{118} \\
&+ x^{117} + x^{116} + x^{115} + x^{114} + x^{111} + x^{110} + x^{109} + x^{106} + x^{102} \\
&+ x^{101} + x^{99} + x^{98} + x^{91} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} \\
&+ x^{80} + x^{78} + x^{75} + x^{70} + x^{67} + x^{66} + x^{65} + x^{64} + x^{62} + x^{61} + x^{60} \\
&+ x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{48} + x^{47} + x^{46} + x^{42} + x^{41} + x^{40} \\
&+ x^{37} + x^{36}, \\
p_3(x) = {}& x^{145} + x^{144} + x^{142} + x^{141} + x^{135} + x^{134} + x^{133} + x^{131} + x^{129} + x^{128} \\
&+ x^{127} + x^{126} + x^{123} + x^{119} + x^{118} + x^{114} + x^{108} + x^{103} + x^{102} \\
&+ x^{97} + x^{96} + x^{95} + x^{94} + x^{92} + x^{91} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} \\
&+ x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{74} + x^{73} + x^{72} + x^{70} + x^{69} + x^{68} \\
&+ x^{67} + x^{66} + x^{60} + x^{59} + x^{57} + x^{56} + x^{55} + x^{54} + x^{52} + x^{51} + x^{50} \\
&+ x^{45} + x^{42} + x^{39} + x^{37} + x^{35} + x^{30} + x^{25} + x^{24}, \\
p_6(x) = {}& x^{145} + x^{144} + x^{142} + x^{141} + x^{134} + x^{133} + x^{132} + x^{129} + x^{127} + x^{121} \\
&+ x^{119} + x^{118} + x^{112} + x^{109} + x^{99} + x^{97} + x^{96} + x^{95} + x^{94} + x^{89} \\
&+ x^{84} + x^{80} + x^{78} + x^{77} + x^{76} + x^{75} + x^{72} + x^{71} + x^{68} + x^{67} + x^{65} \\
&+ x^{61} + x^{59} + x^{58} + x^{56} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{45} + x^{44} \\
&+ x^{43} + x^{42} + x^{38} + x^{37} + x^{30} + x^{25} + x^{23} + x^{21} + x^{20} + x^{18} + x^{17} \\
&+ x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^{7} + x^{6} + x^{3} + x^{2} + x + 1, \\
p_9(x) = {}& x^{145} + x^{144} + x^{143} + x^{142} + x^{141} + x^{140} + x^{133} + x^{132} + x^{131} + x^{130} \\
&+ x^{127} + x^{126} + x^{125} + x^{124} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} \\
&+ x^{108} + x^{101} + x^{100} + x^{99} + x^{98} + x^{95} + x^{94} + x^{93} + x^{92} + x^{65} + x^{64} \\
&+ x^{63} + x^{62} + x^{61} + x^{60} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{37} \\
&+ x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{21} + x^{20} + x^{19} + x^{18} + x^{15} + x^{14} \\
&+ x^{13} + x^{12}, \\
p_{12}(x) = {}& x^{145} + x^{144} + x^{143} + x^{142} + x^{141} + x^{140} + x^{137} + x^{136} + x^{135} + x^{134} \\
&+ x^{133} + x^{132} + x^{125} + x^{124} + x^{123} + x^{122} + x^{119} + x^{118} + x^{115} \\
&+ x^{114} + x^{111} + x^{110} + x^{109} + x^{108} + x^{105} + x^{104} + x^{103} + x^{102} \\
&+ x^{101} + x^{100} + x^{93} + x^{92} + x^{91} + x^{90} + x^{87} + x^{86} + x^{83} + x^{82} + x^{81} \\
&+ x^{80} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{57} + x^{56} + x^{55} + x^{54} \\
&+ x^{53} + x^{52} + x^{49} + x^{48} + x^{47} + x^{46} + x^{43} + x^{42} + x^{41} + x^{40} + x^{37} \\
&+ x^{36} + x^{35} + x^{34} + x^{31} + x^{30} + x^{27} + x^{26} + x^{25} + x^{24} + x^{21} + x^{20} \\
&+ x^{19} + x^{18} + x^{17} + x^{16} + x^{13} + x^{12} + x^{11} + x^{10} + x^{7} + x^{6} + x^{3} + x^{2} \\
&+ x + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 1, 1, 0, 1, 0, 0, 0, 0, 0].

For $\phi(M^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{152} + x^{146} + x^{145} + x^{144} + x^{142} + x^{140} + x^{136} + x^{135} + x^{134} + x^{128}\\
&\quad + x^{125} + x^{122} + x^{119} + x^{118} + x^{114} + x^{113} + x^{112} + x^{110} + x^{107}\\
&\quad + x^{105} + x^{103} + x^{102} + x^{101} + x^{97} + x^{94} + x^{93} + x^{86} + x^{82} + x^{81}\\
&\quad + x^{78} + x^{77} + x^{76} + x^{71} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62}\\
&\quad + x^{57} + x^{50} + x^{49} + x^{48} + x^{46} + x^{45} + x^{43} + x^{42} + x^{39},\\
p_3(x) &= x^{138} + x^{137} + x^{135} + x^{133} + x^{132} + x^{131} + x^{129} + x^{127} + x^{126} + x^{125}\\
&\quad + x^{123} + x^{122} + x^{120} + x^{116} + x^{114} + x^{110} + x^{108} + x^{106} + x^{105}\\
&\quad + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{96} + x^{94} + x^{89} + x^{88}\\
&\quad + x^{87} + x^{85} + x^{83} + x^{82} + x^{81} + x^{79} + x^{77} + x^{76} + x^{75} + x^{74} + x^{70}\\
&\quad + x^{68} + x^{64} + x^{62} + x^{57} + x^{56} + x^{55} + x^{53} + x^{51} + x^{50} + x^{49} + x^{47}\\
&\quad + x^{45} + x^{44} + x^{43} + x^{41} + x^{39} + x^{38} + x^{37} + x^{35} + x^{33} + x^{32} + x^{31}\\
&\quad + x^{29} + x^{27},\\
p_6(x) &= x^{140} + x^{138} + x^{134} + x^{132} + x^{126} + x^{122} + x^{120} + x^{118} + x^{114} + x^{110}\\
&\quad + x^{104} + x^{102} + x^{100} + x^{92} + x^{86} + x^{84} + x^{80} + x^{76} + x^{74} + x^{72} + x^{70}\\
&\quad + x^{64} + x^{60} + x^{50} + x^{36} + x^{34} + x^{30} + x^{28} + x^{22} + x^{18},\\
p_9(x) &= x^{142} + x^{141} + x^{140} + x^{135} + x^{134} + x^{133} + x^{132} + x^{130} + x^{129} + x^{126}\\
&\quad + x^{123} + x^{121} + x^{110} + x^{109} + x^{108} + x^{103} + x^{102} + x^{101} + x^{100}\\
&\quad + x^{98} + x^{97} + x^{94} + x^{91} + x^{89} + x^{62} + x^{61} + x^{60} + x^{55} + x^{54} + x^{53}\\
&\quad + x^{52} + x^{50} + x^{49} + x^{45} + x^{44} + x^{43} + x^{41} + x^{39} + x^{38} + x^{37} + x^{36}\\
&\quad + x^{34} + x^{33} + x^{29} + x^{28} + x^{27} + x^{25} + x^{23} + x^{22} + x^{21} + x^{20} + x^{18}\\
&\quad + x^{17} + x^{14} + x^{11} + x^{9},\\
p_{12}(x) &= x^{144} + x^{140} + x^{136} + x^{132} + x^{124} + x^{108} + x^{104} + x^{100} + x^{92} + x^{80}\\
&\quad + x^{64} + x^{60} + x^{56} + x^{52} + x^{48} + x^{40} + x^{36} + x^{24} + x^{20} + x^{16} + x^{12} + 1\\
&\quad ,
\end{aligned}$$

and the initial terms are [0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 1, 1, 1].

For $\phi(M^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{152} + x^{149} + x^{145} + x^{143} + x^{141} + x^{140} + x^{137} + x^{133} + x^{131} + x^{127}\\
&\quad + x^{126} + x^{122} + x^{120} + x^{119} + x^{118} + x^{116} + x^{113} + x^{112} + x^{110}\\
&\quad + x^{108} + x^{107} + x^{103} + x^{100} + x^{99} + x^{98} + x^{97} + x^{95} + x^{94} + x^{93}\\
&\quad + x^{90} + x^{88} + x^{87} + x^{86} + x^{83} + x^{80} + x^{79} + x^{75} + x^{74} + x^{70} + x^{68}\\
&\quad + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{58} + x^{56} + x^{55} + x^{53}\\
&\quad + x^{49} + x^{48} + x^{47} + x^{45} + x^{44} + x^{42} + x^{39},\\
p_3(x) &= x^{150} + x^{149} + x^{147} + x^{146} + x^{144} + x^{140} + x^{138} + x^{134} + x^{132} + x^{128}\\
&\quad + x^{125} + x^{123} + x^{118} + x^{117} + x^{115} + x^{114} + x^{112} + x^{110} + x^{109}
\end{aligned}$$

$$+ x^{108} + x^{107} + x^{104} + x^{102} + x^{98} + x^{96} + x^{92} + x^{90} + x^{86} + x^{84} + x^{80} + x^{77} + x^{75} + x^{70} + x^{69} + x^{67} + x^{66} + x^{64} + x^{60} + x^{58} + x^{53} + x^{52} + x^{51} + x^{50} + x^{45} + x^{44} + x^{43} + x^{42} + x^{38} + x^{36} + x^{32} + x^{29} + x^{27},$$

$$p_6(x) = x^{152} + x^{150} + x^{148} + x^{144} + x^{142} + x^{140} + x^{138} + x^{136} + x^{134} + x^{128} + x^{122} + x^{118} + x^{112} + x^{110} + x^{106} + x^{102} + x^{100} + x^{98} + x^{94} + x^{92} + x^{86} + x^{82} + x^{76} + x^{74} + x^{72} + x^{68} + x^{66} + x^{60} + x^{54} + x^{50} + x^{48} + x^{46} + x^{44} + x^{40} + x^{38} + x^{36} + x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{18},$$

$$p_9(x) = x^{154} + x^{153} + x^{152} + x^{150} + x^{149} + x^{148} + x^{147} + x^{146} + x^{145} + x^{144} + x^{143} + x^{140} + x^{138} + x^{136} + x^{135} + x^{132} + x^{129} + x^{127} + x^{126} + x^{125} + x^{123} + x^{122} + x^{120} + x^{118} + x^{117} + x^{116} + x^{115} + x^{114} + x^{113} + x^{112} + x^{111} + x^{108} + x^{106} + x^{104} + x^{103} + x^{100} + x^{97} + x^{95} + x^{94} + x^{93} + x^{91} + x^{89} + x^{74} + x^{73} + x^{72} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{60} + x^{57} + x^{55} + x^{54} + x^{53} + x^{51} + x^{50} + x^{48} + x^{46} + x^{45} + x^{44} + x^{43} + x^{39} + x^{38} + x^{37} + x^{35} + x^{34} + x^{32} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{20} + x^{17} + x^{15} + x^{14} + x^{13} + x^{11} + x^{9},$$

$$p_{12}(x) = x^{156} + x^{136} + x^{116} + x^{112} + x^{104} + x^{92} + x^{84} + x^{80} + x^{76} + x^{60} + x^{56} + x^{40} + x^{36} + x^{32} + x^{28} + x^{24} + x^{20} + x^{16} + x^{12} + x^{4} + 1,$$

and the initial terms are [0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0].

For $\phi(M^o, 1, 1)$:

$$p_0(x) = x^{147} + x^{143} + x^{139} + x^{136} + x^{134} + x^{133} + x^{131} + x^{130} + x^{128} + x^{127} + x^{125} + x^{117} + x^{115} + x^{114} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} + x^{106} + x^{100} + x^{98} + x^{97} + x^{96} + x^{95} + x^{93} + x^{92} + x^{90} + x^{86} + x^{81} + x^{79} + x^{78} + x^{77} + x^{75} + x^{73} + x^{70} + x^{69} + x^{68} + x^{67} + x^{65} + x^{63} + x^{60} + x^{58} + x^{57} + x^{56} + x^{55} + x^{52} + x^{49} + x^{48} + x^{47} + x^{43} + x^{42},$$

$$p_3(x) = x^{145} + x^{144} + x^{142} + x^{140} + x^{133} + x^{130} + x^{129} + x^{128} + x^{127} + x^{126} + x^{125} + x^{124} + x^{123} + x^{121} + x^{120} + x^{117} + x^{116} + x^{114} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{107} + x^{106} + x^{105} + x^{104} + x^{97} + x^{96} + x^{95} + x^{94} + x^{92} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{84} + x^{78} + x^{77} + x^{76} + x^{75} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{65} + x^{64} + x^{60} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{43} + x^{39} + x^{37} + x^{35} + x^{33} + x^{32} + x^{30} + x^{27} + x^{26},$$

$$p_6(x) = x^{145} + x^{144} + x^{142} + x^{140} + x^{135} + x^{131} + x^{130} + x^{128} + x^{126} + x^{120}$$

$$
\begin{aligned}
&+ x^{117} + x^{116} + x^{115} + x^{114} + x^{112} + x^{111} + x^{110} + x^{108} + x^{105} \\
&+ x^{104} + x^{103} + x^{102} + x^{99} + x^{93} + x^{92} + x^{89} + x^{87} + x^{86} + x^{85} + x^{81} \\
&+ x^{78} + x^{77} + x^{76} + x^{75} + x^{72} + x^{71} + x^{69} + x^{66} + x^{64} + x^{63} + x^{62} \\
&+ x^{61} + x^{59} + x^{58} + x^{57} + x^{50} + x^{47} + x^{46} + x^{43} + x^{42} + x^{38} + x^{36} \\
&+ x^{31} + x^{29} + x^{28} + x^{27} + x^{26} + x^{24} + x^{22} + x^{21} + x^{20} + x^{18} + x^{17} \\
&+ x^{16} + x^{11} + x^{10} + x^{7} + x^{6} + x^{3} + x^{2} + x + 1, \\
p_9(x) = &\, x^{145} + x^{144} + x^{143} + x^{142} + x^{141} + x^{140} + x^{133} + x^{132} + x^{131} + x^{130} \\
&+ x^{127} + x^{126} + x^{125} + x^{124} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} \\
&+ x^{108} + x^{101} + x^{100} + x^{99} + x^{98} + x^{95} + x^{94} + x^{93} + x^{92} + x^{65} + x^{64} \\
&+ x^{63} + x^{62} + x^{61} + x^{60} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{37} \\
&+ x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{21} + x^{20} + x^{19} + x^{18} + x^{15} + x^{14} \\
&+ x^{13} + x^{12}, \\
p_{12}(x) = &\, x^{145} + x^{144} + x^{143} + x^{142} + x^{141} + x^{140} + x^{137} + x^{136} + x^{135} + x^{134} \\
&+ x^{133} + x^{132} + x^{125} + x^{124} + x^{123} + x^{122} + x^{119} + x^{118} + x^{115} \\
&+ x^{114} + x^{111} + x^{110} + x^{109} + x^{108} + x^{105} + x^{104} + x^{103} + x^{102} \\
&+ x^{101} + x^{100} + x^{93} + x^{92} + x^{91} + x^{90} + x^{87} + x^{86} + x^{83} + x^{82} + x^{81} \\
&+ x^{80} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{57} + x^{56} + x^{55} + x^{54} \\
&+ x^{53} + x^{52} + x^{49} + x^{48} + x^{47} + x^{46} + x^{43} + x^{42} + x^{41} + x^{40} + x^{37} \\
&+ x^{36} + x^{35} + x^{34} + x^{31} + x^{30} + x^{27} + x^{26} + x^{25} + x^{24} + x^{21} + x^{20} \\
&+ x^{19} + x^{18} + x^{17} + x^{16} + x^{13} + x^{12} + x^{11} + x^{10} + x^{7} + x^{6} + x^{3} + x^{2} \\
&+ x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 1, 0].

For $\phi(W^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) = &\, x^{147} + x^{145} + x^{144} + x^{143} + x^{142} + x^{140} + x^{138} + x^{136} + x^{132} + x^{131} \\
&+ x^{130} + x^{127} + x^{125} + x^{124} + x^{123} + x^{122} + x^{121} + x^{119} + x^{118} \\
&+ x^{117} + x^{116} + x^{115} + x^{114} + x^{111} + x^{110} + x^{109} + x^{106} + x^{102} \\
&+ x^{101} + x^{99} + x^{98} + x^{91} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} \\
&+ x^{80} + x^{78} + x^{75} + x^{70} + x^{67} + x^{66} + x^{65} + x^{64} + x^{62} + x^{61} + x^{60} \\
&+ x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{48} + x^{47} + x^{46} + x^{42} + x^{41} + x^{40} \\
&+ x^{37} + x^{36}, \\
p_3(x) = &\, x^{145} + x^{144} + x^{142} + x^{141} + x^{135} + x^{134} + x^{133} + x^{131} + x^{129} + x^{128} \\
&+ x^{127} + x^{126} + x^{123} + x^{119} + x^{118} + x^{114} + x^{108} + x^{103} + x^{102} \\
&+ x^{97} + x^{96} + x^{95} + x^{94} + x^{92} + x^{91} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} \\
&+ x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{74} + x^{73} + x^{72} + x^{70} + x^{69} + x^{68} \\
&+ x^{67} + x^{66} + x^{60} + x^{59} + x^{57} + x^{56} + x^{55} + x^{54} + x^{52} + x^{51} + x^{50}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{45} + x^{42} + x^{39} + x^{37} + x^{35} + x^{30} + x^{25} + x^{24},\\
p_6(x) &= x^{145} + x^{144} + x^{142} + x^{141} + x^{134} + x^{133} + x^{132} + x^{129} + x^{127} + x^{121}\\
&+ x^{119} + x^{118} + x^{112} + x^{109} + x^{99} + x^{97} + x^{96} + x^{95} + x^{94} + x^{89}\\
&+ x^{84} + x^{80} + x^{78} + x^{77} + x^{76} + x^{75} + x^{72} + x^{71} + x^{68} + x^{67} + x^{65}\\
&+ x^{61} + x^{59} + x^{58} + x^{56} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{45} + x^{44}\\
&+ x^{43} + x^{42} + x^{38} + x^{37} + x^{30} + x^{25} + x^{23} + x^{21} + x^{20} + x^{18} + x^{17}\\
&+ x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^{7} + x^{6} + x^{3} + x^{2} + x + 1,\\
p_9(x) &= x^{145} + x^{144} + x^{143} + x^{142} + x^{141} + x^{140} + x^{133} + x^{132} + x^{131} + x^{130}\\
&+ x^{127} + x^{126} + x^{125} + x^{124} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109}\\
&+ x^{108} + x^{101} + x^{100} + x^{99} + x^{98} + x^{95} + x^{94} + x^{93} + x^{92} + x^{65} + x^{64}\\
&+ x^{63} + x^{62} + x^{61} + x^{60} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{37}\\
&+ x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{21} + x^{20} + x^{19} + x^{18} + x^{15} + x^{14}\\
&+ x^{13} + x^{12},\\
p_{12}(x) &= x^{145} + x^{144} + x^{143} + x^{142} + x^{141} + x^{140} + x^{137} + x^{136} + x^{135} + x^{134}\\
&+ x^{133} + x^{132} + x^{125} + x^{124} + x^{123} + x^{122} + x^{119} + x^{118} + x^{115}\\
&+ x^{114} + x^{111} + x^{110} + x^{109} + x^{108} + x^{105} + x^{104} + x^{103} + x^{102}\\
&+ x^{101} + x^{100} + x^{93} + x^{92} + x^{91} + x^{90} + x^{87} + x^{86} + x^{83} + x^{82} + x^{81}\\
&+ x^{80} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{57} + x^{56} + x^{55} + x^{54}\\
&+ x^{53} + x^{52} + x^{49} + x^{48} + x^{47} + x^{46} + x^{43} + x^{42} + x^{41} + x^{40} + x^{37}\\
&+ x^{36} + x^{35} + x^{34} + x^{31} + x^{30} + x^{27} + x^{26} + x^{25} + x^{24} + x^{21} + x^{20}\\
&+ x^{19} + x^{18} + x^{17} + x^{16} + x^{13} + x^{12} + x^{11} + x^{10} + x^{7} + x^{6} + x^{3} + x^{2}\\
&+ x + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 1, 1, 0, 1, 0, 0, 0, 0, 0].

For $\phi(W^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{152} + x^{149} + x^{145} + x^{143} + x^{141} + x^{140} + x^{137} + x^{133} + x^{131} + x^{127}\\
&+ x^{126} + x^{122} + x^{120} + x^{119} + x^{118} + x^{116} + x^{113} + x^{112} + x^{110}\\
&+ x^{108} + x^{107} + x^{103} + x^{100} + x^{99} + x^{98} + x^{97} + x^{95} + x^{94} + x^{93}\\
&+ x^{90} + x^{88} + x^{87} + x^{86} + x^{83} + x^{80} + x^{79} + x^{75} + x^{74} + x^{70} + x^{68}\\
&+ x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{58} + x^{56} + x^{55} + x^{53}\\
&+ x^{49} + x^{48} + x^{47} + x^{45} + x^{44} + x^{42} + x^{39},\\
p_3(x) &= x^{150} + x^{149} + x^{147} + x^{146} + x^{144} + x^{140} + x^{138} + x^{134} + x^{132} + x^{128}\\
&+ x^{125} + x^{123} + x^{118} + x^{117} + x^{115} + x^{114} + x^{112} + x^{110} + x^{109}\\
&+ x^{108} + x^{107} + x^{104} + x^{102} + x^{98} + x^{96} + x^{92} + x^{90} + x^{86} + x^{84}\\
&+ x^{80} + x^{77} + x^{75} + x^{70} + x^{69} + x^{67} + x^{66} + x^{64} + x^{60} + x^{58} + x^{53}\\
&+ x^{52} + x^{51} + x^{50} + x^{45} + x^{44} + x^{43} + x^{42} + x^{38} + x^{36} + x^{32} + x^{29}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{27},\\
p_6(x) &= x^{152}+x^{150}+x^{148}+x^{144}+x^{142}+x^{140}+x^{138}+x^{136}+x^{134}+x^{128}\\
&+x^{122}+x^{118}+x^{112}+x^{110}+x^{106}+x^{102}+x^{100}+x^{98}+x^{94}+x^{92}\\
&+x^{86}+x^{82}+x^{76}+x^{74}+x^{72}+x^{68}+x^{66}+x^{60}+x^{54}+x^{50}+x^{48}\\
&+x^{46}+x^{44}+x^{40}+x^{38}+x^{36}+x^{34}+x^{32}+x^{30}+x^{28}+x^{26}+x^{18},\\
p_9(x) &= x^{154}+x^{153}+x^{152}+x^{150}+x^{149}+x^{148}+x^{147}+x^{146}+x^{145}+x^{144}\\
&+x^{143}+x^{140}+x^{138}+x^{136}+x^{135}+x^{132}+x^{129}+x^{127}+x^{126}\\
&+x^{125}+x^{123}+x^{122}+x^{120}+x^{118}+x^{117}+x^{116}+x^{115}+x^{114}\\
&+x^{113}+x^{112}+x^{111}+x^{108}+x^{106}+x^{104}+x^{103}+x^{100}+x^{97}+x^{95}\\
&+x^{94}+x^{93}+x^{91}+x^{89}+x^{74}+x^{73}+x^{72}+x^{70}+x^{69}+x^{68}+x^{67}\\
&+x^{66}+x^{65}+x^{64}+x^{63}+x^{60}+x^{57}+x^{55}+x^{54}+x^{53}+x^{51}+x^{50}\\
&+x^{48}+x^{46}+x^{45}+x^{44}+x^{43}+x^{39}+x^{38}+x^{37}+x^{35}+x^{34}+x^{32}\\
&+x^{30}+x^{29}+x^{28}+x^{27}+x^{26}+x^{25}+x^{24}+x^{23}+x^{20}+x^{17}+x^{15}\\
&+x^{14}+x^{13}+x^{11}+x^{9},\\
p_{12}(x) &= x^{156}+x^{136}+x^{116}+x^{112}+x^{104}+x^{92}+x^{84}+x^{80}+x^{76}+x^{60}+x^{56}\\
&+x^{40}+x^{36}+x^{32}+x^{28}+x^{24}+x^{20}+x^{16}+x^{12}+x^{4}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0].

For $\phi(W^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{152}+x^{146}+x^{145}+x^{144}+x^{142}+x^{140}+x^{136}+x^{135}+x^{134}+x^{128}\\
&+x^{125}+x^{122}+x^{119}+x^{118}+x^{114}+x^{113}+x^{112}+x^{110}+x^{107}\\
&+x^{105}+x^{103}+x^{102}+x^{101}+x^{97}+x^{94}+x^{93}+x^{86}+x^{82}+x^{81}\\
&+x^{78}+x^{77}+x^{76}+x^{71}+x^{68}+x^{67}+x^{66}+x^{65}+x^{64}+x^{63}+x^{62}\\
&+x^{57}+x^{50}+x^{49}+x^{48}+x^{46}+x^{45}+x^{43}+x^{42}+x^{39},\\
p_3(x) &= x^{138}+x^{137}+x^{135}+x^{133}+x^{132}+x^{131}+x^{129}+x^{127}+x^{126}+x^{125}\\
&+x^{123}+x^{122}+x^{120}+x^{116}+x^{114}+x^{110}+x^{108}+x^{106}+x^{105}\\
&+x^{104}+x^{103}+x^{102}+x^{101}+x^{100}+x^{99}+x^{96}+x^{94}+x^{89}+x^{88}\\
&+x^{87}+x^{85}+x^{83}+x^{82}+x^{81}+x^{79}+x^{77}+x^{76}+x^{75}+x^{74}+x^{70}\\
&+x^{68}+x^{64}+x^{62}+x^{57}+x^{56}+x^{55}+x^{53}+x^{51}+x^{50}+x^{49}+x^{47}\\
&+x^{45}+x^{44}+x^{43}+x^{41}+x^{39}+x^{38}+x^{37}+x^{35}+x^{33}+x^{32}+x^{31}\\
&+x^{29}+x^{27},\\
p_6(x) &= x^{140}+x^{138}+x^{134}+x^{132}+x^{126}+x^{122}+x^{120}+x^{118}+x^{114}+x^{110}\\
&+x^{104}+x^{102}+x^{100}+x^{92}+x^{86}+x^{84}+x^{80}+x^{76}+x^{74}+x^{72}+x^{70}\\
&+x^{64}+x^{60}+x^{50}+x^{36}+x^{34}+x^{30}+x^{28}+x^{22}+x^{18},\\
p_9(x) &= x^{142}+x^{141}+x^{140}+x^{135}+x^{134}+x^{133}+x^{132}+x^{130}+x^{129}+x^{126}\\
&+x^{123}+x^{121}+x^{110}+x^{109}+x^{108}+x^{103}+x^{102}+x^{101}+x^{100}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{98}+x^{97}+x^{94}+x^{91}+x^{89}+x^{62}+x^{61}+x^{60}+x^{55}+x^{54}+x^{53}\\
&+x^{52}+x^{50}+x^{49}+x^{45}+x^{44}+x^{43}+x^{41}+x^{39}+x^{38}+x^{37}+x^{36}\\
&+x^{34}+x^{33}+x^{29}+x^{28}+x^{27}+x^{25}+x^{23}+x^{22}+x^{21}+x^{20}+x^{18}\\
&+x^{17}+x^{14}+x^{11}+x^{9},\\
p_{12}(x)=&\,x^{144}+x^{140}+x^{136}+x^{132}+x^{124}+x^{108}+x^{104}+x^{100}+x^{92}+x^{80}\\
&+x^{64}+x^{60}+x^{56}+x^{52}+x^{48}+x^{40}+x^{36}+x^{24}+x^{20}+x^{16}+x^{12}+1\\
&,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 1, 1, 1].

For $\phi(W^o, 1, 1)$:

$$
\begin{aligned}
p_0(x)=&\,x^{147}+x^{143}+x^{139}+x^{136}+x^{134}+x^{133}+x^{131}+x^{130}+x^{128}+x^{127}\\
&+x^{125}+x^{117}+x^{115}+x^{114}+x^{113}+x^{112}+x^{111}+x^{110}+x^{109}\\
&+x^{108}+x^{106}+x^{100}+x^{98}+x^{97}+x^{96}+x^{95}+x^{93}+x^{92}+x^{90}+x^{86}\\
&+x^{81}+x^{79}+x^{78}+x^{77}+x^{75}+x^{73}+x^{70}+x^{69}+x^{68}+x^{67}+x^{65}\\
&+x^{63}+x^{60}+x^{58}+x^{57}+x^{56}+x^{55}+x^{52}+x^{49}+x^{48}+x^{47}+x^{43}\\
&+x^{42},\\
p_3(x)=&\,x^{145}+x^{144}+x^{142}+x^{140}+x^{133}+x^{130}+x^{129}+x^{128}+x^{127}+x^{126}\\
&+x^{125}+x^{124}+x^{123}+x^{121}+x^{120}+x^{117}+x^{116}+x^{114}+x^{113}\\
&+x^{112}+x^{111}+x^{110}+x^{109}+x^{107}+x^{106}+x^{105}+x^{104}+x^{97}+x^{96}\\
&+x^{95}+x^{94}+x^{92}+x^{90}+x^{89}+x^{88}+x^{87}+x^{86}+x^{84}+x^{78}+x^{77}\\
&+x^{76}+x^{75}+x^{73}+x^{72}+x^{71}+x^{69}+x^{68}+x^{65}+x^{64}+x^{60}+x^{58}\\
&+x^{57}+x^{56}+x^{55}+x^{54}+x^{53}+x^{49}+x^{48}+x^{47}+x^{46}+x^{45}+x^{43}\\
&+x^{39}+x^{37}+x^{35}+x^{33}+x^{32}+x^{30}+x^{27}+x^{26},\\
p_6(x)=&\,x^{145}+x^{144}+x^{142}+x^{140}+x^{135}+x^{131}+x^{130}+x^{128}+x^{126}+x^{120}\\
&+x^{117}+x^{116}+x^{115}+x^{114}+x^{112}+x^{111}+x^{110}+x^{108}+x^{105}\\
&+x^{104}+x^{103}+x^{102}+x^{99}+x^{93}+x^{92}+x^{89}+x^{87}+x^{86}+x^{85}+x^{81}\\
&+x^{78}+x^{77}+x^{76}+x^{75}+x^{72}+x^{71}+x^{69}+x^{66}+x^{64}+x^{63}+x^{62}\\
&+x^{61}+x^{59}+x^{58}+x^{57}+x^{50}+x^{47}+x^{46}+x^{43}+x^{42}+x^{38}+x^{36}\\
&+x^{31}+x^{29}+x^{28}+x^{27}+x^{26}+x^{24}+x^{22}+x^{21}+x^{20}+x^{18}+x^{17}\\
&+x^{16}+x^{11}+x^{10}+x^{7}+x^{6}+x^{3}+x^{2}+x+1,\\
p_9(x)=&\,x^{145}+x^{144}+x^{143}+x^{142}+x^{141}+x^{140}+x^{133}+x^{132}+x^{131}+x^{130}\\
&+x^{127}+x^{126}+x^{125}+x^{124}+x^{113}+x^{112}+x^{111}+x^{110}+x^{109}\\
&+x^{108}+x^{101}+x^{100}+x^{99}+x^{98}+x^{95}+x^{94}+x^{93}+x^{92}+x^{65}+x^{64}\\
&+x^{63}+x^{62}+x^{61}+x^{60}+x^{53}+x^{52}+x^{51}+x^{50}+x^{49}+x^{48}+x^{37}\\
&+x^{36}+x^{35}+x^{34}+x^{33}+x^{32}+x^{21}+x^{20}+x^{19}+x^{18}+x^{15}+x^{14}\\
&+x^{13}+x^{12},
\end{aligned}
$$

$$\begin{aligned}p_{12}(x) = {} & x^{145}+x^{144}+x^{143}+x^{142}+x^{141}+x^{140}+x^{137}+x^{136}+x^{135}+x^{134}\\ & +x^{133}+x^{132}+x^{125}+x^{124}+x^{123}+x^{122}+x^{119}+x^{118}+x^{115}\\ & +x^{114}+x^{111}+x^{110}+x^{109}+x^{108}+x^{105}+x^{104}+x^{103}+x^{102}\\ & +x^{101}+x^{100}+x^{93}+x^{92}+x^{91}+x^{90}+x^{87}+x^{86}+x^{83}+x^{82}+x^{81}\\ & +x^{80}+x^{65}+x^{64}+x^{63}+x^{62}+x^{61}+x^{60}+x^{57}+x^{56}+x^{55}+x^{54}\\ & +x^{53}+x^{52}+x^{49}+x^{48}+x^{47}+x^{46}+x^{43}+x^{42}+x^{41}+x^{40}+x^{37}\\ & +x^{36}+x^{35}+x^{34}+x^{31}+x^{30}+x^{27}+x^{26}+x^{25}+x^{24}+x^{21}+x^{20}\\ & +x^{19}+x^{18}+x^{17}+x^{16}+x^{13}+x^{12}+x^{11}+x^{10}+x^{7}+x^{6}+x^{3}+x^{2}\\ & +x+1,\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 1, 0].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

| $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | [1, 2] | 182 | [301, 302] | 364 | [494, 495] | 546 | [629, 629] | 728 | [409, 788] |
| 1 | [3, 4] | 183 | [183, 183] | 365 | [496, 497] | 547 | [636, 182] | 729 | [322, 381] |
| 2 | [5, 6] | 184 | [303, 304] | 366 | [498, 499] | 548 | [167, 637] | 730 | [401, 642] |
| 3 | [7, 8] | 185 | [241, 194] | 367 | [500, 501] | 549 | [172, 638] | 731 | [789, 382] |
| 4 | [9, 10] | 186 | [116, 305] | 368 | [502, 503] | 550 | [628, 589] | 732 | [398, 54] |
| 5 | [11, 12] | 187 | [306, 250] | 369 | [238, 504] | 551 | [573, 366] | 733 | [627, 394] |
| 6 | [13, 14] | 188 | [307, 308] | 370 | [505, 506] | 552 | [639, 153] | 734 | [588, 628] |
| 7 | [15, 16] | 189 | [309, 310] | 371 | [507, 508] | 553 | [640, 314] | 735 | [589, 629] |
| 8 | [17, 18] | 190 | [270, 311] | 372 | [509, 510] | 554 | [328, 150] | 736 | [601, 183] |
| 9 | [19, 20] | 191 | [74, 312] | 373 | [511, 512] | 555 | [629, 589] | 737 | [182, 616] |
| 10 | [21, 22] | 192 | [224, 313] | 374 | [513, 502] | 556 | [399, 171] | 738 | [356, 52] |
| 11 | [23, 24] | 193 | [59, 143] | 375 | [514, 515] | 557 | [629, 588] | 739 | [630, 790] |
| 12 | [25, 26] | 194 | [59, 146] | 376 | [516, 517] | 558 | [400, 172] | 740 | [769, 154] |
| 13 | [27, 28] | 195 | [314, 14] | 377 | [518, 519] | 559 | [641, 642] | 741 | [151, 602] |
| 14 | [29, 30] | 196 | [126, 315] | 378 | [520, 5] | 560 | [398, 341] | 742 | [124, 591] |
| 15 | [31, 32] | 197 | [12, 137] | 379 | [521, 522] | 561 | [389, 347] | 743 | [388, 149] |
| 16 | [33, 34] | 198 | [316, 36] | 380 | [523, 524] | 562 | [391, 404] | 744 | [630, 176] |
| 17 | [35, 36] | 199 | [317, 156] | 381 | [525, 526] | 563 | [141, 643] | 745 | [183, 392] |
| 18 | [37, 38] | 200 | [318, 319] | 382 | [527, 528] | 564 | [327, 619] | 746 | [392, 601] |
| 19 | [39, 40] | 201 | [320, 186] | 383 | [529, 530] | 565 | [358, 178] | 747 | [333, 176] |
| 20 | [41, 37] | 202 | [118, 38] | 384 | [282, 531] | 566 | [644, 645] | 748 | [391, 399] |
| 21 | [42, 43] | 203 | [119, 38] | 385 | [532, 533] | 567 | [342, 646] | 749 | [117, 37] |
| 22 | [44, 45] | 204 | [160, 123] | 386 | [534, 110] | 568 | [403, 353] | 750 | [120, 319] |
| 23 | [46, 32] | 205 | [145, 60] | 387 | [107, 489] | 569 | [647, 465] | 751 | [791, 58] |
| 24 | [47, 48] | 206 | [321, 322] | 388 | [535, 536] | 570 | [648, 649] | 752 | [385, 767] |
| 25 | [49, 50] | 207 | [323, 324] | 389 | [537, 457] | 571 | [650, 213] | 753 | [792, 145] |
| 26 | [51, 52] | 208 | [9, 145] | 390 | [538, 539] | 572 | [651, 652] | 754 | [793, 175] |

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|---|---|---|---|---|---|---|---|---|---|
| 27 | [53, 54] | 209 | [325, 326] | 391 | [540, 283] | 573 | [653, 654] | 755 | [136, 371] |
| 28 | [55, 56] | 210 | [130, 157] | 392 | [541, 303] | 574 | [470, 655] | 756 | [132, 44] |
| 29 | [57, 58] | 211 | [121, 327] | 393 | [279, 256] | 575 | [297, 656] | 757 | [360, 155] |
| 30 | [59, 60] | 212 | [328, 14] | 394 | [542, 300] | 576 | [657, 658] | 758 | [167, 396] |
| 31 | [61, 62] | 213 | [329, 330] | 395 | [427, 543] | 577 | [659, 559] | 759 | [163, 776] |
| 32 | [63, 64] | 214 | [331, 124] | 396 | [544, 545] | 578 | [66, 101] | 760 | [38, 794] |
| 33 | [65, 66] | 215 | [319, 332] | 397 | [546, 547] | 579 | [660, 661] | 761 | [12, 795] |
| 34 | [67, 68] | 216 | [333, 334] | 398 | [548, 230] | 580 | [195, 662] | 762 | [132, 174] |
| 35 | [69, 70] | 217 | [335, 126] | 399 | [549, 550] | 581 | [663, 664] | 763 | [704, 673] |
| 36 | [71, 72] | 218 | [144, 34] | 400 | [284, 285] | 582 | [239, 665] | 764 | [796, 500] |
| 37 | [73, 74] | 219 | [49, 336] | 401 | [551, 552] | 583 | [666, 667] | 765 | [797, 756] |
| 38 | [75, 76] | 220 | [337, 338] | 402 | [553, 554] | 584 | [668, 669] | 766 | [798, 799] |
| 39 | [77, 78] | 221 | [8, 339] | 403 | [252, 291] | 585 | [670, 671] | 767 | [100, 800] |
| 40 | [79, 80] | 222 | [340, 45] | 404 | [555, 556] | 586 | [672, 428] | 768 | [801, 802] |
| 41 | [81, 82] | 223 | [159, 40] | 405 | [557, 558] | 587 | [673, 674] | 769 | [803, 804] |
| 42 | [83, 84] | 224 | [321, 324] | 406 | [559, 560] | 588 | [675, 676] | 770 | [805, 806] |
| 43 | [85, 86] | 225 | [49, 341] | 407 | [561, 562] | 589 | [304, 677] | 771 | [807, 808] |
| 44 | [86, 87] | 226 | [342, 343] | 408 | [563, 564] | 590 | [678, 433] | 772 | [262, 809] |
| 45 | [88, 89] | 227 | [344, 345] | 409 | [565, 566] | 591 | [679, 680] | 773 | [810, 811] |
| 46 | [90, 91] | 228 | [153, 346] | 410 | [567, 568] | 592 | [681, 682] | 774 | [93, 812] |
| 47 | [92, 93] | 229 | [39, 130] | 411 | [510, 514] | 593 | [683, 444] | 775 | [813, 814] |
| 48 | [94, 95] | 230 | [347, 348] | 412 | [179, 357] | 594 | [684, 516] | 776 | [815, 659] |
| 49 | [96, 27] | 231 | [349, 56] | 413 | [38, 191] | 595 | [685, 686] | 777 | [76, 461] |
| 50 | [97, 98] | 232 | [350, 351] | 414 | [325, 321] | 596 | [249, 687] | 778 | [816, 209] |
| 51 | [99, 100] | 233 | [352, 60] | 415 | [326, 324] | 597 | [688, 689] | 779 | [817, 298] |
| 52 | [101, 102] | 234 | [179, 353] | 416 | [121, 370] | 598 | [690, 691] | 780 | [818, 819] |
| 53 | [103, 104] | 235 | [132, 354] | 417 | [569, 142] | 599 | [692, 693] | 781 | [562, 668] |
| 54 | [105, 106] | 236 | [185, 34] | 418 | [145, 138] | 600 | [694, 695] | 782 | [820, 821] |
| 55 | [78, 107] | 237 | [41, 118] | 419 | [570, 321] | 601 | [696, 697] | 783 | [822, 823] |
| 56 | [108, 109] | 238 | [323, 165] | 420 | [571, 37] | 602 | [564, 698] | 784 | [236, 544] |
| 57 | [110, 111] | 239 | [131, 346] | 421 | [42, 7] | 603 | [699, 700] | 785 | [824, 232] |
| 58 | [112, 113] | 240 | [355, 59] | 422 | [339, 192] | 604 | [701, 507] | 786 | [413, 251] |
| 59 | [114, 115] | 241 | [185, 186] | 423 | [185, 572] | 605 | [702, 703] | 787 | [825, 227] |
| 60 | [20, 116] | 242 | [39, 156] | 424 | [573, 574] | 606 | [656, 704] | 788 | [226, 826] |
| 61 | [117, 118] | 243 | [119, 356] | 425 | [374, 44] | 607 | [664, 705] | 789 | [827, 828] |
| 62 | [117, 119] | 244 | [170, 357] | 426 | [145, 143] | 608 | [458, 706] | 790 | [829, 430] |
| 63 | [120, 121] | 245 | [358, 359] | 427 | [575, 371] | 609 | [707, 708] | 791 | [830, 553] |
| 64 | [122, 123] | 246 | [327, 360] | 428 | [573, 576] | 610 | [709, 289] | 792 | [831, 442] |
| 65 | [124, 125] | 247 | [144, 186] | 429 | [127, 122] | 611 | [710, 29] | 793 | [832, 833] |
| 66 | [126, 127] | 248 | [53, 336] | 430 | [319, 327] | 612 | [711, 712] | 794 | [834, 835] |
| 67 | [128, 129] | 249 | [322, 166] | 431 | [577, 148] | 613 | [713, 714] | 795 | [836, 837] |
| 68 | [130, 55] | 250 | [361, 362] | 432 | [7, 331] | 614 | [26, 653] | 796 | [141, 373] |
| 69 | [131, 132] | 251 | [49, 54] | 433 | [570, 323] | 615 | [715, 716] | 797 | [372, 387] |
| 70 | [133, 134] | 252 | [363, 364] | 434 | [578, 356] | 616 | [717, 718] | 798 | [360, 838] |

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|---|---|---|---|---|---|---|---|---|---|
| 71 | [131, 135] | 253 | [365, 366] | 435 | [579, 580] | 617 | [719, 63] | 799 | [367, 327] |
| 72 | [136, 137] | 254 | [120, 367] | 436 | [383, 407] | 618 | [720, 721] | 800 | [160, 192] |
| 73 | [59, 138] | 255 | [340, 368] | 437 | [581, 359] | 619 | [476, 722] | 801 | [383, 839] |
| 74 | [139, 140] | 256 | [360, 369] | 438 | [135, 340] | 620 | [205, 525] | 802 | [597, 381] |
| 75 | [141, 142] | 257 | [352, 146] | 439 | [174, 45] | 621 | [512, 723] | 803 | [570, 620] |
| 76 | [143, 144] | 258 | [156, 55] | 440 | [35, 582] | 622 | [724, 725] | 804 | [583, 129] |
| 77 | [145, 146] | 259 | [319, 370] | 441 | [582, 342] | 623 | [726, 727] | 805 | [588, 588] |
| 78 | [147, 148] | 260 | [162, 371] | 442 | [316, 583] | 624 | [728, 729] | 806 | [840, 179] |
| 79 | [32, 132] | 261 | [372, 373] | 443 | [158, 55] | 625 | [730, 731] | 807 | [641, 841] |
| 80 | [149, 150] | 262 | [315, 122] | 444 | [182, 584] | 626 | [435, 732] | 808 | [587, 584] |
| 81 | [126, 8] | 263 | [32, 374] | 445 | [585, 343] | 627 | [220, 733] | 809 | [619, 155] |
| 82 | [151, 134] | 264 | [375, 376] | 446 | [325, 323] | 628 | [734, 735] | 810 | [640, 328] |
| 83 | [35, 128] | 265 | [7, 315] | 447 | [586, 158] | 629 | [697, 736] | 811 | [612, 376] |
| 84 | [152, 119] | 266 | [33, 377] | 448 | [181, 587] | 630 | [737, 738] | 812 | [175, 334] |
| 85 | [31, 153] | 267 | [355, 145] | 449 | [588, 589] | 631 | [739, 740] | 813 | [177, 774] |
| 86 | [154, 155] | 268 | [378, 379] | 450 | [365, 574] | 632 | [741, 421] | 814 | [605, 842] |
| 87 | [153, 132] | 269 | [170, 380] | 451 | [590, 352] | 633 | [742, 743] | 815 | [55, 843] |
| 88 | [156, 157] | 270 | [165, 381] | 452 | [11, 136] | 634 | [490, 744] | 816 | [159, 130] |
| 89 | [158, 157] | 271 | [177, 359] | 453 | [188, 140] | 635 | [745, 542] | 817 | [171, 399] |
| 90 | [159, 156] | 272 | [189, 382] | 454 | [160, 591] | 636 | [746, 467] | 818 | [167, 364] |
| 91 | [159, 158] | 273 | [383, 384] | 455 | [174, 592] | 637 | [743, 747] | 819 | [595, 844] |
| 92 | [160, 125] | 274 | [385, 386] | 456 | [10, 332] | 638 | [748, 555] | 820 | [600, 411] |
| 93 | [161, 162] | 275 | [138, 185] | 457 | [169, 593] | 639 | [749, 17] | 821 | [589, 589] |
| 94 | [163, 164] | 276 | [188, 376] | 458 | [405, 171] | 640 | [750, 751] | 822 | [358, 618] |
| 95 | [165, 166] | 277 | [331, 122] | 459 | [60, 144] | 641 | [752, 753] | 823 | [625, 845] |
| 96 | [167, 168] | 278 | [141, 387] | 460 | [315, 124] | 642 | [754, 755] | 824 | [9, 352] |
| 97 | [169, 170] | 279 | [355, 352] | 461 | [154, 369] | 643 | [756, 757] | 825 | [46, 153] |
| 98 | [171, 172] | 280 | [388, 13] | 462 | [594, 188] | 644 | [758, 759] | 826 | [771, 394] |
| 99 | [151, 173] | 281 | [162, 137] | 463 | [595, 596] | 645 | [760, 532] | 827 | [363, 637] |
| 100 | [135, 174] | 282 | [389, 390] | 464 | [597, 166] | 646 | [761, 762] | 828 | [613, 846] |
| 101 | [175, 176] | 283 | [391, 172] | 465 | [598, 599] | 647 | [179, 380] | 829 | [378, 847] |
| 102 | [127, 124] | 284 | [392, 184] | 466 | [411, 54] | 648 | [32, 135] | 830 | [33, 572] |
| 103 | [177, 178] | 285 | [172, 171] | 467 | [600, 53] | 649 | [763, 373] | 831 | [39, 158] |
| 104 | [179, 180] | 286 | [184, 184] | 468 | [601, 184] | 650 | [126, 331] | 832 | [42, 126] |
| 105 | [181, 182] | 287 | [8, 124] | 469 | [147, 602] | 651 | [40, 349] | 833 | [848, 48] |
| 106 | [183, 184] | 288 | [122, 192] | 470 | [370, 154] | 652 | [582, 129] | 834 | [139, 849] |
| 107 | [146, 185] | 289 | [174, 368] | 471 | [174, 603] | 653 | [406, 394] | 835 | [332, 619] |
| 108 | [13, 150] | 290 | [393, 60] | 472 | [604, 162] | 654 | [764, 168] | 836 | [344, 850] |
| 109 | [8, 122] | 291 | [383, 394] | 473 | [605, 606] | 655 | [339, 123] | 837 | [43, 127] |
| 110 | [33, 186] | 292 | [157, 395] | 474 | [356, 191] | 656 | [765, 395] | 838 | [851, 852] |
| 111 | [187, 188] | 293 | [32, 346] | 475 | [46, 607] | 657 | [370, 766] | 839 | [853, 546] |
| 112 | [189, 190] | 294 | [147, 173] | 476 | [354, 45] | 658 | [44, 368] | 840 | [106, 477] |
| 113 | [51, 191] | 295 | [314, 150] | 477 | [608, 383] | 659 | [170, 180] | 841 | [854, 855] |
| 114 | [46, 131] | 296 | [363, 396] | 478 | [184, 183] | 660 | [329, 767] | 842 | [856, 857] |

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|---|---|---|---|---|---|---|---|---|---|
| 115 | [124, 192] | 297 | [170, 353] | 479 | [154, 609] | 661 | [346, 174] | 843 | [858, 859] |
| 116 | [40, 55] | 298 | [397, 398] | 480 | [43, 8] | 662 | [146, 144] | 844 | [860, 861] |
| 117 | [193, 75] | 299 | [399, 400] | 481 | [41, 119] | 663 | [372, 643] | 845 | [862, 863] |
| 118 | [194, 195] | 300 | [392, 183] | 482 | [397, 49] | 664 | [335, 610] | 846 | [864, 234] |
| 119 | [196, 197] | 301 | [401, 402] | 483 | [400, 404] | 665 | [768, 176] | 847 | [823, 865] |
| 120 | [198, 199] | 302 | [403, 180] | 484 | [372, 142] | 666 | [121, 769] | 848 | [800, 854] |
| 121 | [200, 201] | 303 | [171, 404] | 485 | [161, 575] | 667 | [612, 140] | 849 | [866, 867] |
| 122 | [16, 79] | 304 | [405, 400] | 486 | [12, 371] | 668 | [770, 771] | 850 | [819, 868] |
| 123 | [202, 203] | 305 | [37, 356] | 487 | [354, 603] | 669 | [404, 171] | 851 | [323, 597] |
| 124 | [204, 205] | 306 | [31, 131] | 488 | [610, 127] | 670 | [365, 772] | 852 | [36, 129] |
| 125 | [206, 207] | 307 | [406, 407] | 489 | [124, 123] | 671 | [769, 619] | 853 | [635, 403] |
| 126 | [208, 204] | 308 | [408, 191] | 490 | [611, 612] | 672 | [7, 127] | 854 | [793, 768] |
| 127 | [209, 210] | 309 | [363, 168] | 491 | [613, 614] | 673 | [401, 773] | 855 | [768, 334] |
| 128 | [211, 212] | 310 | [409, 410] | 492 | [324, 381] | 674 | [771, 384] | 856 | [765, 869] |
| 129 | [213, 214] | 311 | [411, 341] | 493 | [151, 148] | 675 | [405, 391] | 857 | [347, 584] |
| 130 | [215, 216] | 312 | [138, 144] | 494 | [615, 333] | 676 | [638, 404] | 858 | [13, 870] |
| 131 | [217, 218] | 313 | [40, 157] | 495 | [375, 140] | 677 | [404, 400] | 859 | [135, 44] |
| 132 | [62, 202] | 314 | [412, 413] | 496 | [347, 616] | 678 | [41, 578] | 860 | [322, 871] |
| 133 | [219, 220] | 315 | [414, 415] | 497 | [617, 618] | 679 | [118, 408] | 861 | [593, 180] |
| 134 | [221, 222] | 316 | [416, 417] | 498 | [315, 160] | 680 | [128, 342] | 862 | [579, 872] |
| 135 | [223, 224] | 317 | [418, 274] | 499 | [320, 34] | 681 | [36, 585] | 863 | [587, 348] |
| 136 | [225, 226] | 318 | [419, 420] | 500 | [318, 10] | 682 | [583, 342] | 864 | [51, 873] |
| 137 | [227, 228] | 319 | [421, 422] | 501 | [619, 369] | 683 | [358, 774] | 865 | [129, 343] |
| 138 | [229, 67] | 320 | [423, 211] | 502 | [36, 342] | 684 | [31, 607] | 866 | [350, 874] |
| 139 | [230, 231] | 321 | [3, 108] | 503 | [578, 38] | 685 | [598, 2] | 867 | [367, 332] |
| 140 | [232, 233] | 322 | [424, 425] | 504 | [118, 356] | 686 | [775, 776] | 868 | [157, 56] |
| 141 | [234, 96] | 323 | [426, 427] | 505 | [316, 128] | 687 | [627, 384] | 869 | [875, 876] |
| 142 | [235, 236] | 324 | [428, 429] | 506 | [620, 165] | 688 | [385, 777] | 870 | [877, 81] |
| 143 | [237, 238] | 325 | [430, 431] | 507 | [621, 622] | 689 | [332, 154] | 871 | [878, 879] |
| 144 | [115, 239] | 326 | [432, 276] | 508 | [393, 146] | 690 | [329, 386] | 872 | [880, 881] |
| 145 | [240, 241] | 327 | [433, 434] | 509 | [53, 50] | 691 | [778, 131] | 873 | [882, 883] |
| 146 | [242, 243] | 328 | [104, 435] | 510 | [177, 618] | 692 | [611, 139] | 874 | [814, 884] |
| 147 | [244, 245] | 329 | [436, 437] | 511 | [146, 320] | 693 | [575, 137] | 875 | [575, 885] |
| 148 | [246, 247] | 330 | [438, 439] | 512 | [354, 368] | 694 | [172, 391] | 876 | [327, 154] |
| 149 | [248, 249] | 331 | [440, 441] | 513 | [117, 578] | 695 | [779, 593] | 877 | [361, 886] |
| 150 | [250, 69] | 332 | [442, 443] | 514 | [406, 384] | 696 | [183, 601] | 878 | [188, 887] |
| 151 | [251, 252] | 333 | [444, 445] | 515 | [129, 580] | 697 | [588, 629] | 879 | [346, 44] |
| 152 | [30, 253] | 334 | [268, 77] | 516 | [623, 624] | 698 | [766, 369] | 880 | [314, 888] |
| 153 | [254, 255] | 335 | [446, 447] | 517 | [607, 132] | 699 | [156, 765] | 881 | [370, 619] |
| 154 | [256, 257] | 336 | [448, 449] | 518 | [128, 579] | 700 | [620, 324] | 882 | [175, 889] |
| 155 | [243, 258] | 337 | [450, 451] | 519 | [326, 165] | 701 | [355, 393] | 883 | [374, 174] |
| 156 | [259, 260] | 338 | [452, 453] | 520 | [185, 377] | 702 | [641, 402] | 884 | [585, 580] |
| 157 | [261, 262] | 339 | [454, 215] | 521 | [625, 626] | 703 | [780, 645] | 885 | [890, 891] |
| 158 | [263, 264] | 340 | [455, 456] | 522 | [408, 52] | 704 | [53, 341] | 886 | [828, 892] |

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|---|---|---|---|---|---|---|---|---|---|
| 159 | [265, 261] | 341 | [457, 458] | 523 | [608, 627] | 705 | [766, 155] | 887 | [893, 894] |
| 160 | [266, 3] | 342 | [253, 459] | 524 | [628, 629] | 706 | [781, 390] | 888 | [895, 896] |
| 161 | [267, 268] | 343 | [197, 460] | 525 | [630, 334] | 707 | [119, 51] | 889 | [897, 672] |
| 162 | [269, 270] | 344 | [461, 462] | 526 | [60, 185] | 708 | [321, 165] | 890 | [621, 898] |
| 163 | [271, 272] | 345 | [463, 464] | 527 | [165, 631] | 709 | [9, 393] | 891 | [10, 327] |
| 164 | [28, 273] | 346 | [91, 88] | 528 | [390, 584] | 710 | [9, 59] | 892 | [349, 395] |
| 165 | [274, 275] | 347 | [465, 466] | 529 | [337, 2] | 711 | [347, 782] | 893 | [623, 899] |
| 166 | [276, 277] | 348 | [467, 468] | 530 | [593, 353] | 712 | [324, 166] | 894 | [610, 8] |
| 167 | [278, 279] | 349 | [469, 470] | 531 | [172, 400] | 713 | [337, 599] | 895 | [57, 900] |
| 168 | [280, 281] | 350 | [471, 472] | 532 | [182, 348] | 714 | [783, 410] | 896 | [121, 332] |
| 169 | [98, 282] | 351 | [473, 474] | 533 | [632, 364] | 715 | [318, 121] | 897 | [47, 901] |
| 170 | [245, 225] | 352 | [475, 476] | 534 | [143, 33] | 716 | [784, 379] | 898 | [566, 902] |
| 171 | [283, 284] | 353 | [477, 478] | 535 | [335, 7] | 717 | [770, 406] | 899 | [310, 903] |
| 172 | [285, 286] | 354 | [479, 480] | 536 | [633, 362] | 718 | [638, 172] | 900 | [272, 904] |
| 173 | [287, 288] | 355 | [481, 242] | 537 | [391, 405] | 719 | [147, 134] | 901 | [759, 905] |
| 174 | [289, 290] | 356 | [82, 287] | 538 | [598, 634] | 720 | [187, 375] | 902 | [579, 343] |
| 175 | [291, 292] | 357 | [482, 483] | 539 | [390, 348] | 721 | [139, 376] | 903 | [55, 395] |
| 176 | [24, 293] | 358 | [484, 85] | 540 | [392, 392] | 722 | [607, 346] | 904 | [342, 580] |
| 177 | [294, 208] | 359 | [485, 486] | 541 | [171, 405] | 723 | [785, 612] | 905 | [765, 56] |
| 178 | [111, 295] | 360 | [487, 488] | 542 | [635, 179] | 724 | [644, 786] | | |
| 179 | [296, 297] | 361 | [489, 490] | 543 | [143, 185] | 725 | [38, 52] | | |
| 180 | [298, 299] | 362 | [491, 492] | 544 | [615, 630] | 726 | [360, 609] | | |
| 181 | [300, 105] | 363 | [493, 114] | 545 | [149, 14] | 727 | [787, 12] | | |

Output function $n \mapsto \tau(n)$:

| $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 130 | 1 | 260 | 0 | 390 | 1 | 520 | 0 | 650 | 0 | 780 | 0 |
| 1 | 0 | 131 | 1 | 261 | 0 | 391 | 0 | 521 | 1 | 651 | 1 | 781 | 0 |
| 2 | 0 | 132 | 0 | 262 | 1 | 392 | 0 | 522 | 0 | 652 | 1 | 782 | 1 |
| 3 | 0 | 133 | 0 | 263 | 1 | 393 | 0 | 523 | 1 | 653 | 1 | 783 | 0 |
| 4 | 0 | 134 | 1 | 264 | 1 | 394 | 0 | 524 | 1 | 654 | 0 | 784 | 0 |
| 5 | 0 | 135 | 0 | 265 | 0 | 395 | 0 | 525 | 1 | 655 | 0 | 785 | 0 |
| 6 | 0 | 136 | 0 | 266 | 0 | 396 | 1 | 526 | 0 | 656 | 0 | 786 | 0 |
| 7 | 0 | 137 | 1 | 267 | 0 | 397 | 1 | 527 | 1 | 657 | 1 | 787 | 0 |
| 8 | 1 | 138 | 0 | 268 | 0 | 398 | 0 | 528 | 1 | 658 | 1 | 788 | 1 |
| 9 | 0 | 139 | 1 | 269 | 0 | 399 | 1 | 529 | 1 | 659 | 0 | 789 | 0 |
| 10 | 1 | 140 | 1 | 270 | 1 | 400 | 0 | 530 | 0 | 660 | 1 | 790 | 0 |
| 11 | 0 | 141 | 0 | 271 | 0 | 401 | 1 | 531 | 1 | 661 | 0 | 791 | 0 |
| 12 | 0 | 142 | 0 | 272 | 0 | 402 | 1 | 532 | 1 | 662 | 0 | 792 | 0 |
| 13 | 0 | 143 | 0 | 273 | 1 | 403 | 0 | 533 | 0 | 663 | 0 | 793 | 1 |
| 14 | 0 | 144 | 0 | 274 | 1 | 404 | 1 | 534 | 0 | 664 | 1 | 794 | 1 |
| 15 | 0 | 145 | 0 | 275 | 0 | 405 | 1 | 535 | 1 | 665 | 1 | 795 | 1 |
| 16 | 0 | 146 | 0 | 276 | 1 | 406 | 1 | 536 | 0 | 666 | 1 | 796 | 0 |
| 17 | 1 | 147 | 0 | 277 | 1 | 407 | 0 | 537 | 0 | 667 | 1 | 797 | 0 |

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 18 | 0 | 148 | 1 | 278 | 0 | 408 | 0 | 538 | 1 | 668 | 1 | 798 | 1 |
| 19 | 0 | 149 | 0 | 279 | 0 | 409 | 0 | 539 | 1 | 669 | 1 | 799 | 1 |
| 20 | 0 | 150 | 0 | 280 | 1 | 410 | 1 | 540 | 0 | 670 | 1 | 800 | 0 |
| 21 | 1 | 151 | 0 | 281 | 0 | 411 | 0 | 541 | 0 | 671 | 1 | 801 | 1 |
| 22 | 1 | 152 | 0 | 282 | 0 | 412 | 0 | 542 | 0 | 672 | 0 | 802 | 1 |
| 23 | 0 | 153 | 1 | 283 | 0 | 413 | 0 | 543 | 0 | 673 | 1 | 803 | 1 |
| 24 | 0 | 154 | 1 | 284 | 0 | 414 | 1 | 544 | 1 | 674 | 1 | 804 | 1 |
| 25 | 0 | 155 | 0 | 285 | 1 | 415 | 0 | 545 | 0 | 675 | 1 | 805 | 1 |
| 26 | 0 | 156 | 1 | 286 | 0 | 416 | 1 | 546 | 1 | 676 | 0 | 806 | 0 |
| 27 | 0 | 157 | 0 | 287 | 1 | 417 | 0 | 547 | 0 | 677 | 1 | 807 | 1 |
| 28 | 0 | 158 | 1 | 288 | 0 | 418 | 0 | 548 | 0 | 678 | 0 | 808 | 1 |
| 29 | 0 | 159 | 0 | 289 | 1 | 419 | 1 | 549 | 1 | 679 | 0 | 809 | 1 |
| 30 | 0 | 160 | 0 | 290 | 0 | 420 | 0 | 550 | 1 | 680 | 1 | 810 | 1 |
| 31 | 0 | 161 | 0 | 291 | 1 | 421 | 1 | 551 | 1 | 681 | 1 | 811 | 1 |
| 32 | 1 | 162 | 0 | 292 | 0 | 422 | 0 | 552 | 0 | 682 | 1 | 812 | 1 |
| 33 | 0 | 163 | 0 | 293 | 1 | 423 | 0 | 553 | 1 | 683 | 0 | 813 | 0 |
| 34 | 1 | 164 | 0 | 294 | 0 | 424 | 1 | 554 | 0 | 684 | 0 | 814 | 1 |
| 35 | 1 | 165 | 1 | 295 | 0 | 425 | 0 | 555 | 1 | 685 | 1 | 815 | 0 |
| 36 | 1 | 166 | 1 | 296 | 0 | 426 | 0 | 556 | 1 | 686 | 0 | 816 | 0 |
| 37 | 0 | 167 | 0 | 297 | 0 | 427 | 0 | 557 | 1 | 687 | 1 | 817 | 0 |
| 38 | 0 | 168 | 1 | 298 | 1 | 428 | 1 | 558 | 0 | 688 | 1 | 818 | 0 |
| 39 | 0 | 169 | 0 | 299 | 1 | 429 | 1 | 559 | 1 | 689 | 1 | 819 | 1 |
| 40 | 1 | 170 | 0 | 300 | 0 | 430 | 1 | 560 | 0 | 690 | 1 | 820 | 1 |
| 41 | 0 | 171 | 0 | 301 | 1 | 431 | 0 | 561 | 0 | 691 | 0 | 821 | 1 |
| 42 | 1 | 172 | 1 | 302 | 0 | 432 | 0 | 562 | 0 | 692 | 0 | 822 | 0 |
| 43 | 0 | 173 | 1 | 303 | 0 | 433 | 1 | 563 | 0 | 693 | 0 | 823 | 1 |
| 44 | 1 | 174 | 1 | 304 | 1 | 434 | 0 | 564 | 1 | 694 | 1 | 824 | 0 |
| 45 | 1 | 175 | 1 | 305 | 0 | 435 | 1 | 565 | 0 | 695 | 0 | 825 | 0 |
| 46 | 0 | 176 | 0 | 306 | 0 | 436 | 1 | 566 | 0 | 696 | 0 | 826 | 1 |
| 47 | 0 | 177 | 0 | 307 | 1 | 437 | 0 | 567 | 1 | 697 | 1 | 827 | 0 |
| 48 | 0 | 178 | 0 | 308 | 0 | 438 | 0 | 568 | 0 | 698 | 1 | 828 | 1 |
| 49 | 0 | 179 | 0 | 309 | 0 | 439 | 1 | 569 | 0 | 699 | 1 | 829 | 0 |
| 50 | 0 | 180 | 1 | 310 | 0 | 440 | 1 | 570 | 1 | 700 | 0 | 830 | 0 |
| 51 | 0 | 181 | 0 | 311 | 0 | 441 | 1 | 571 | 0 | 701 | 0 | 831 | 0 |
| 52 | 1 | 182 | 1 | 312 | 0 | 442 | 1 | 572 | 1 | 702 | 1 | 832 | 1 |
| 53 | 0 | 183 | 0 | 313 | 1 | 443 | 1 | 573 | 1 | 703 | 0 | 833 | 0 |
| 54 | 0 | 184 | 0 | 314 | 0 | 444 | 1 | 574 | 1 | 704 | 0 | 834 | 1 |
| 55 | 0 | 185 | 0 | 315 | 1 | 445 | 1 | 575 | 0 | 705 | 1 | 835 | 1 |
| 56 | 0 | 186 | 1 | 316 | 1 | 446 | 1 | 576 | 1 | 706 | 0 | 836 | 1 |
| 57 | 0 | 187 | 0 | 317 | 0 | 447 | 0 | 577 | 0 | 707 | 0 | 837 | 0 |
| 58 | 0 | 188 | 1 | 318 | 1 | 448 | 0 | 578 | 0 | 708 | 0 | 838 | 0 |
| 59 | 0 | 189 | 0 | 319 | 1 | 449 | 1 | 579 | 1 | 709 | 0 | 839 | 0 |
| 60 | 0 | 190 | 1 | 320 | 0 | 450 | 1 | 580 | 0 | 710 | 0 | 840 | 0 |
| 61 | 0 | 191 | 1 | 321 | 0 | 451 | 0 | 581 | 0 | 711 | 1 | 841 | 1 |

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 62 | 0 | 192 | 0 | 322 | 1 | 452 | 0 | 582 | 1 | 712 | 1 | 842 | 0 |
| 63 | 1 | 193 | 0 | 323 | 0 | 453 | 1 | 583 | 1 | 713 | 1 | 843 | 0 |
| 64 | 0 | 194 | 0 | 324 | 1 | 454 | 0 | 584 | 1 | 714 | 0 | 844 | 1 |
| 65 | 0 | 195 | 0 | 325 | 1 | 455 | 1 | 585 | 1 | 715 | 1 | 845 | 1 |
| 66 | 0 | 196 | 0 | 326 | 0 | 456 | 1 | 586 | 0 | 716 | 0 | 846 | 0 |
| 67 | 1 | 197 | 0 | 327 | 1 | 457 | 0 | 587 | 1 | 717 | 1 | 847 | 1 |
| 68 | 1 | 198 | 1 | 328 | 0 | 458 | 1 | 588 | 1 | 718 | 0 | 848 | 0 |
| 69 | 1 | 199 | 0 | 329 | 1 | 459 | 0 | 589 | 1 | 719 | 0 | 849 | 1 |
| 70 | 0 | 200 | 1 | 330 | 0 | 460 | 1 | 590 | 0 | 720 | 0 | 850 | 1 |
| 71 | 1 | 201 | 0 | 331 | 1 | 461 | 1 | 591 | 0 | 721 | 1 | 851 | 0 |
| 72 | 0 | 202 | 0 | 332 | 1 | 462 | 0 | 592 | 1 | 722 | 1 | 852 | 1 |
| 73 | 0 | 203 | 0 | 333 | 1 | 463 | 1 | 593 | 0 | 723 | 0 | 853 | 0 |
| 74 | 1 | 204 | 0 | 334 | 0 | 464 | 1 | 594 | 0 | 724 | 0 | 854 | 1 |
| 75 | 0 | 205 | 0 | 335 | 1 | 465 | 1 | 595 | 1 | 725 | 0 | 855 | 1 |
| 76 | 0 | 206 | 0 | 336 | 0 | 466 | 0 | 596 | 1 | 726 | 1 | 856 | 0 |
| 77 | 0 | 207 | 0 | 337 | 1 | 467 | 1 | 597 | 1 | 727 | 0 | 857 | 1 |
| 78 | 0 | 208 | 0 | 338 | 0 | 468 | 0 | 598 | 1 | 728 | 0 | 858 | 0 |
| 79 | 1 | 209 | 1 | 339 | 0 | 469 | 0 | 599 | 0 | 729 | 1 | 859 | 0 |
| 80 | 0 | 210 | 1 | 340 | 1 | 470 | 1 | 600 | 1 | 730 | 1 | 860 | 1 |
| 81 | 0 | 211 | 1 | 341 | 0 | 471 | 1 | 601 | 0 | 731 | 0 | 861 | 0 |
| 82 | 0 | 212 | 0 | 342 | 1 | 472 | 0 | 602 | 1 | 732 | 0 | 862 | 1 |
| 83 | 1 | 213 | 1 | 343 | 0 | 473 | 1 | 603 | 1 | 733 | 1 | 863 | 1 |
| 84 | 0 | 214 | 1 | 344 | 1 | 474 | 0 | 604 | 0 | 734 | 1 | 864 | 0 |
| 85 | 0 | 215 | 1 | 345 | 1 | 475 | 0 | 605 | 1 | 735 | 1 | 865 | 1 |
| 86 | 1 | 216 | 1 | 346 | 0 | 476 | 1 | 606 | 0 | 736 | 0 | 866 | 1 |
| 87 | 1 | 217 | 1 | 347 | 1 | 477 | 1 | 607 | 1 | 737 | 1 | 867 | 1 |
| 88 | 1 | 218 | 0 | 348 | 1 | 478 | 0 | 608 | 1 | 738 | 0 | 868 | 0 |
| 89 | 1 | 219 | 0 | 349 | 0 | 479 | 1 | 609 | 0 | 739 | 1 | 869 | 0 |
| 90 | 0 | 220 | 1 | 350 | 1 | 480 | 0 | 610 | 0 | 740 | 1 | 870 | 0 |
| 91 | 0 | 221 | 1 | 351 | 1 | 481 | 0 | 611 | 0 | 741 | 0 | 871 | 1 |
| 92 | 0 | 222 | 1 | 352 | 0 | 482 | 1 | 612 | 1 | 742 | 0 | 872 | 0 |
| 93 | 0 | 223 | 0 | 353 | 1 | 483 | 0 | 613 | 1 | 743 | 1 | 873 | 1 |
| 94 | 0 | 224 | 0 | 354 | 1 | 484 | 0 | 614 | 0 | 744 | 1 | 874 | 1 |
| 95 | 1 | 225 | 0 | 355 | 0 | 485 | 0 | 615 | 1 | 745 | 0 | 875 | 0 |
| 96 | 0 | 226 | 1 | 356 | 0 | 486 | 0 | 616 | 1 | 746 | 0 | 876 | 1 |
| 97 | 0 | 227 | 1 | 357 | 1 | 487 | 1 | 617 | 0 | 747 | 1 | 877 | 0 |
| 98 | 0 | 228 | 1 | 358 | 0 | 488 | 0 | 618 | 0 | 748 | 0 | 878 | 1 |
| 99 | 0 | 229 | 0 | 359 | 0 | 489 | 0 | 619 | 1 | 749 | 0 | 879 | 0 |
| 100 | 0 | 230 | 1 | 360 | 1 | 490 | 0 | 620 | 0 | 750 | 1 | 880 | 0 |
| 101 | 1 | 231 | 0 | 361 | 0 | 491 | 1 | 621 | 1 | 751 | 0 | 881 | 1 |
| 102 | 1 | 232 | 1 | 362 | 1 | 492 | 1 | 622 | 0 | 752 | 1 | 882 | 1 |
| 103 | 0 | 233 | 0 | 363 | 0 | 493 | 0 | 623 | 1 | 753 | 0 | 883 | 0 |
| 104 | 0 | 234 | 0 | 364 | 1 | 494 | 1 | 624 | 0 | 754 | 1 | 884 | 1 |
| 105 | 0 | 235 | 0 | 365 | 1 | 495 | 1 | 625 | 1 | 755 | 0 | 885 | 1 |

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 106 | 0 | 236 | 0 | 366 | 1 | 496 | 1 | 626 | 1 | 756 | 0 | 886 | 1 |
| 107 | 0 | 237 | 0 | 367 | 1 | 497 | 0 | 627 | 1 | 757 | 1 | 887 | 1 |
| 108 | 0 | 238 | 0 | 368 | 1 | 498 | 1 | 628 | 1 | 758 | 0 | 888 | 0 |
| 109 | 1 | 239 | 1 | 369 | 0 | 499 | 0 | 629 | 1 | 759 | 0 | 889 | 0 |
| 110 | 0 | 240 | 0 | 370 | 1 | 500 | 1 | 630 | 1 | 760 | 0 | 890 | 1 |
| 111 | 0 | 241 | 0 | 371 | 1 | 501 | 1 | 631 | 1 | 761 | 0 | 891 | 1 |
| 112 | 0 | 242 | 0 | 372 | 0 | 502 | 1 | 632 | 0 | 762 | 0 | 892 | 0 |
| 113 | 0 | 243 | 0 | 373 | 0 | 503 | 0 | 633 | 0 | 763 | 0 | 893 | 1 |
| 114 | 0 | 244 | 0 | 374 | 0 | 504 | 0 | 634 | 0 | 764 | 0 | 894 | 0 |
| 115 | 0 | 245 | 0 | 375 | 1 | 505 | 1 | 635 | 0 | 765 | 0 | 895 | 0 |
| 116 | 1 | 246 | 1 | 376 | 1 | 506 | 0 | 636 | 0 | 766 | 1 | 896 | 1 |
| 117 | 0 | 247 | 0 | 377 | 1 | 507 | 1 | 637 | 1 | 767 | 0 | 897 | 0 |
| 118 | 0 | 248 | 0 | 378 | 0 | 508 | 0 | 638 | 0 | 768 | 1 | 898 | 0 |
| 119 | 0 | 249 | 1 | 379 | 1 | 509 | 0 | 639 | 0 | 769 | 1 | 899 | 0 |
| 120 | 1 | 250 | 0 | 380 | 1 | 510 | 0 | 640 | 1 | 770 | 1 | 900 | 0 |
| 121 | 1 | 251 | 0 | 381 | 1 | 511 | 0 | 641 | 1 | 771 | 1 | 901 | 0 |
| 122 | 0 | 252 | 0 | 382 | 1 | 512 | 1 | 642 | 1 | 772 | 1 | 902 | 1 |
| 123 | 0 | 253 | 1 | 383 | 1 | 513 | 0 | 643 | 0 | 773 | 1 | 903 | 0 |
| 124 | 0 | 254 | 1 | 384 | 0 | 514 | 1 | 644 | 0 | 774 | 0 | 904 | 1 |
| 125 | 0 | 255 | 1 | 385 | 1 | 515 | 1 | 645 | 0 | 775 | 0 | 905 | 0 |
| 126 | 0 | 256 | 1 | 386 | 0 | 516 | 1 | 646 | 0 | 776 | 0 | | |
| 127 | 1 | 257 | 0 | 387 | 0 | 517 | 1 | 647 | 0 | 777 | 0 | | |
| 128 | 1 | 258 | 1 | 388 | 1 | 518 | 1 | 648 | 1 | 778 | 0 | | |
| 129 | 1 | 259 | 1 | 389 | 0 | 519 | 0 | 649 | 0 | 779 | 0 | | |

Université de Strasbourg, CNRS, IRMA UMR 7501, F-67000 Strasbourg, France
*Email address*: `guoniu.han@unistra.fr`

School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan, PR China
*Email address*: `huyining@hust.edu.cn`