PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a,b)=(z^3+z^2, z^4+z^3+z+1)$

GUO-NIU HAN* AND YINING HU*

Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a,b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a,b) in $\mathbb{F}_2[x]\setminus\mathbb{F}_2$. In this paper we prove the conjecture for $(a,b)=(z^3+z^2, z^4+z^3+z+1)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a}=(a_0(z),a_1(z),a_2(z),\ldots)$ of elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{a_2(z)+\cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{\ddots+\cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1}+c_{-2}z^{-2}+c_{-3}z^{-3}+\cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a,b)-Thue-Morse sequence

$$\mathbf{t}=(t_0(z),t_1(z),t_2(z),\ldots)=(a,b,b,a,b,a,a,b,\ldots)$$

that we shall denote

$$\mathrm{CF}(a,b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

Date: 9/04/2020 19:08:42 .

2010 *Mathematics Subject Classification.* 11B85, 11J70, 11B50, 11Y65, 05A15.

Key words and phrases. automatic sequence, continued fraction, Thue-Morse sequence.

* This paper was automatically generated from a computer program developed by the authors, with 20.6 minutes CPU time used.

Theorem 1.1. *When* $(a,b)=(z^3+z^2, z^4+z^3+z+1)$*, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

where, for $\mathrm{CF}(a,b)$*,*

$$\begin{aligned}
p_0(z)&=z^{17}+z^{15}+z^{14}+z^{10}+z^5+z^4+z^2+1,\\
p_1(z)&=z^{21}+z^{16}+z^{13}+z^{12}+z^9+z^7+z^5+z^4+z^3+z^2,\\
p_2(z)&=z^{24}+z^{20}+z^{18}+z^{14}+z^{12}+z^{10}+z^6+z^4,\\
p_3(z)&=z^{21}+z^{16}+z^{13}+z^{12}+z^9+z^7+z^5+z^4+z^3+z^2,\\
p_4(z)&=z^{18}+z^{17}+z^{15}+z^{12}+z^6+z^5+z^2,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$*,*

$$\begin{aligned}
p_0(z)&=z^{17}+z^{15}+z^{14}+z^{12}+z^{10}+z^8+z^5+z^4+z^2,\\
p_1(z)&=z^{21}+z^{16}+z^{13}+z^{12}+z^9+z^7+z^5+z^4+z^3+z^2,\\
p_2(z)&=z^{24}+z^{20}+z^{18}+z^{14}+z^{12}+z^{10}+z^6+z^4,\\
p_3(z)&=z^{21}+z^{16}+z^{13}+z^{12}+z^9+z^7+z^5+z^4+z^3+z^2,\\
p_4(z)&=z^{18}+z^{17}+z^{15}+z^8+z^6+z^5+z^2+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix}P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix}t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix}t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x)&=x^{\deg(t_{2^n-1})}\begin{pmatrix}t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix}t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(t_0)}\begin{pmatrix}t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x)&=x^{\deg(\bar t_{2^n-1})}\begin{pmatrix}\bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix}\bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(\bar t_0)}\begin{pmatrix}\bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\geq 0$

$$\begin{aligned}
M_{n+1}(x)&=W_n(x)\cdot M_n(x),\\
W_{n+1}(x)&=M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(175, 175, 175, 175, 175)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z\left(\phi(W^o,0,0)(x,z),\ z^{12}\cdot\phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 630 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 630, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z\left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ to order 770 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 770, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. □

Define

$$R^e = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned}M^e[:14u] = {}& M^e[:7u] + x^{2u}M^e[:5u] + x^{3u}M^e[:4u] + x^{5u}M^e[:2u] + x^{6u}M^e[:u]\\ &+ x^{2u}M^e[:7u] + x^{4u}M^e[:5u] + x^{5u}M^e[:4u] + x^{7u}M^e[:2u]\\ &+ x^{8u}M^e[:u] + x^{3u}M^e[:7u] + x^{5u}M^e[:5u] + x^{6u}M^e[:4u]\\ &+ x^{8u}M^e[:2u] + x^{9u}M^e[:u] + x^{4u}M^e[:7u] + x^{6u}M^e[:5u]\\ &+ x^{7u}M^e[:4u] + x^{9u}M^e[:2u] + x^{10u}M^e[:u] + x^{5u}M^e[:7u]\\ &+ x^{7u}M^e[:5u] + x^{8u}M^e[:4u] + x^{10u}M^e[:2u] + x^{11u}M^e[:u]\end{aligned}$$

$$\begin{aligned}
&+ x^{6u}M^e[:7u] + x^{8u}M^e[:5u] + x^{9u}M^e[:4u] + x^{11u}M^e[:2u]\\
&+ x^{12u}M^e[:u] + x^{7u}M^e[:7u] + x^{8u}M^e[:6u] + x^{10u}M^e[:4u]\\
&+ x^{11u}M^e[:3u] + x^{13u}M^e[:u]\\
&+ (x^{7u} + x^{9u} + x^{10u} + x^{11u} + x^{12u} + x^{13u})R^e,\\
W^e[:14u] = {}& W^e[:7u] + x^{2u}W^e[:5u] + x^{3u}W^e[:4u] + x^{5u}W^e[:2u] + x^{6u}W^e[:u]\\
&+ x^{2u}W^e[:7u] + x^{4u}W^e[:5u] + x^{5u}W^e[:4u] + x^{7u}W^e[:2u]\\
&+ x^{8u}W^e[:u] + x^{3u}W^e[:7u] + x^{5u}W^e[:5u] + x^{6u}W^e[:4u]\\
&+ x^{8u}W^e[:2u] + x^{9u}W^e[:u] + x^{4u}W^e[:7u] + x^{6u}W^e[:5u]\\
&+ x^{7u}W^e[:4u] + x^{9u}W^e[:2u] + x^{10u}W^e[:u] + x^{5u}W^e[:7u]\\
&+ x^{7u}W^e[:5u] + x^{8u}W^e[:4u] + x^{10u}W^e[:2u] + x^{11u}W^e[:u]\\
&+ x^{6u}W^e[:7u] + x^{8u}W^e[:5u] + x^{9u}W^e[:4u] + x^{11u}W^e[:2u]\\
&+ x^{12u}W^e[:u] + x^{7u}W^e[:7u] + x^{8u}W^e[:6u] + x^{10u}W^e[:4u]\\
&+ x^{11u}W^e[:3u] + x^{13u}W^e[:u]\\
&+ (x^{7u} + x^{9u} + x^{10u} + x^{11u} + x^{12u} + x^{13u})R^e.
\end{aligned}$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$\begin{aligned}
M^o[:14u] = {}& M^o[:7u] + x^{2u}M^o[:5u] + x^{3u}M^o[:4u] + x^{5u}M^o[:2u] + x^{6u}M^o[:u]\\
&+ x^{2u}M^o[:7u] + x^{4u}M^o[:5u] + x^{5u}M^o[:4u] + x^{7u}M^o[:2u]\\
&+ x^{8u}M^o[:u] + x^{3u}M^o[:7u] + x^{5u}M^o[:5u] + x^{6u}M^o[:4u]\\
&+ x^{8u}M^o[:2u] + x^{9u}M^o[:u] + x^{4u}M^o[:7u] + x^{6u}M^o[:5u]\\
&+ x^{7u}M^o[:4u] + x^{9u}M^o[:2u] + x^{10u}M^o[:u] + x^{5u}M^o[:7u]\\
&+ x^{7u}M^o[:5u] + x^{8u}M^o[:4u] + x^{10u}M^o[:2u] + x^{11u}M^o[:u]\\
&+ x^{6u}M^o[:7u] + x^{8u}M^o[:5u] + x^{9u}M^o[:4u] + x^{11u}M^o[:2u]\\
&+ x^{12u}M^o[:u] + x^{7u}M^o[:7u] + x^{8u}M^o[:6u] + x^{10u}M^o[:4u]\\
&+ x^{11u}M^o[:3u] + x^{13u}M^o[:u]\\
&+ (x^{7u} + x^{9u} + x^{10u} + x^{11u} + x^{12u} + x^{13u})R^o,\\
W^o[:14u] = {}& W^o[:7u] + x^{2u}W^o[:5u] + x^{3u}W^o[:4u] + x^{5u}W^o[:2u] + x^{6u}W^o[:u]\\
&+ x^{2u}W^o[:7u] + x^{4u}W^o[:5u] + x^{5u}W^o[:4u] + x^{7u}W^o[:2u]\\
&+ x^{8u}W^o[:u] + x^{3u}W^o[:7u] + x^{5u}W^o[:5u] + x^{6u}W^o[:4u]\\
&+ x^{8u}W^o[:2u] + x^{9u}W^o[:u] + x^{4u}W^o[:7u] + x^{6u}W^o[:5u]\\
&+ x^{7u}W^o[:4u] + x^{9u}W^o[:2u] + x^{10u}W^o[:u] + x^{5u}W^o[:7u]\\
&+ x^{7u}W^o[:5u] + x^{8u}W^o[:4u] + x^{10u}W^o[:2u] + x^{11u}W^o[:u]\\
&+ x^{6u}W^o[:7u] + x^{8u}W^o[:5u] + x^{9u}W^o[:4u] + x^{11u}W^o[:2u]\\
&+ x^{12u}W^o[:u] + x^{7u}W^o[:7u] + x^{8u}W^o[:6u] + x^{10u}W^o[:4u]\\
&+ x^{11u}W^o[:3u] + x^{13u}W^o[:u]\\
&+ (x^{7u} + x^{9u} + x^{10u} + x^{11u} + x^{12u} + x^{13u})R^o.
\end{aligned}$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M_{0,0}^o$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$\begin{aligned}T[:14u] = {} & T[:7u] + x^{2u}T[:5u] + x^{3u}T[:4u] + x^{5u}T[:2u] + x^{6u}T[:u] + x^{2u}T[:7u] \\ & + x^{4u}T[:5u] + x^{5u}T[:4u] + x^{7u}T[:2u] + x^{8u}T[:u] + x^{3u}T[:7u] \\ & + x^{5u}T[:5u] + x^{6u}T[:4u] + x^{8u}T[:2u] + x^{9u}T[:u] + x^{4u}T[:7u] \\ & + x^{6u}T[:5u] + x^{7u}T[:4u] + x^{9u}T[:2u] + x^{10u}T[:u] + x^{5u}T[:7u] \\ & + x^{7u}T[:5u] + x^{8u}T[:4u] + x^{10u}T[:2u] + x^{11u}T[:u] + x^{6u}T[:7u] \\ & + x^{8u}T[:5u] + x^{9u}T[:4u] + x^{11u}T[:2u] + x^{12u}T[:u] + x^{7u}T[:7u] \\ & + x^{8u}T[:6u] + x^{10u}T[:4u] + x^{11u}T[:3u] + x^{13u}T[:u].\end{aligned}$$

The proof of the other 15 cases are similar. We break down the above identity into 7 parts:

$$\begin{aligned}0 = {} & x^{6u}T[u:2u] + x^{5u}T[2u:3u] + x^{3u}T[4u:5u] + x^{2u}T[5u:6u] \\ & + T[7u:8u], \\ 0 = {} & x^{8u}T[:u] + x^{6u}T[2u:3u] + x^{5u}T[3u:4u] + x^{3u}T[5u:6u] \\ & + x^{2u}T[6u:7u] + T[8u:9u], \\ 0 = {} & x^{9u}T[:u] + x^{7u}T[2u:3u] + x^{6u}T[3u:4u] + x^{4u}T[5u:6u] \\ & + x^{3u}T[6u:7u] + T[9u:10u], \\ 0 = {} & x^{10u}T[:u] + x^{8u}T[2u:3u] + x^{7u}T[3u:4u] + x^{5u}T[5u:6u] \\ & + x^{4u}T[6u:7u] + T[10u:11u], \\ 0 = {} & x^{11u}T[:u] + x^{9u}T[2u:3u] + x^{8u}T[3u:4u] + x^{6u}T[5u:6u] \\ & + x^{5u}T[6u:7u] + T[11u:12u], \\ 0 = {} & x^{12u}T[:u] + x^{10u}T[2u:3u] + x^{9u}T[3u:4u] + x^{7u}T[5u:6u] \\ & + x^{6u}T[6u:7u] + T[12u:13u], \\ 0 = {} & x^{13u}T[:u] + x^{11u}T[2u:3u] + x^{10u}T[3u:4u] + x^{8u}T[5u:6u] \\ & + x^{7u}T[6u:7u] + T[13u:14u],\end{aligned}$$

which can be rewritten as

$$0 = T[[1w]_2] + T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[111w]_2], \tag{2.7}$$

$$\begin{aligned}0 = {} & T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2] \\ & + T[[1000w]_2],\end{aligned} \tag{2.8}$$

$$\begin{aligned}0 = {} & T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2] \\ & + T[[1001w]_2],\end{aligned} \tag{2.9}$$

$$\begin{aligned}0 = {} & T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2] \\ & + T[[1010w]_2],\end{aligned} \tag{2.10}$$

$$\begin{aligned}0 = {} & T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2] \\ & + T[[1011w]_2],\end{aligned} \tag{2.11}$$

$$\begin{aligned}0 = T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2]\\ + T[[1100w]_2],\end{aligned} \tag{2.12}$$

$$\begin{aligned}0 = T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2]\\ + T[[1101w]_2],\end{aligned} \tag{2.13}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$0 = T[[1w]_2] + T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[111w]_2], \tag{2.14}$$

$$\begin{aligned}0 = T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2]\\ + T[[1000w]_2],\end{aligned} \tag{2.15}$$

$$\begin{aligned}0 = T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2]\\ + T[[1001w]_2],\end{aligned} \tag{2.16}$$

$$\begin{aligned}0 = T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2]\\ + T[[1010w]_2],\end{aligned} \tag{2.17}$$

$$\begin{aligned}0 = T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2]\\ + T[[1011w]_2],\end{aligned} \tag{2.18}$$

$$\begin{aligned}0 = T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2]\\ + T[[1100w]_2],\end{aligned} \tag{2.19}$$

$$\begin{aligned}0 = T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2]\\ + T[[1101w]_2],\end{aligned} \tag{2.20}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 885 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.20) can be written as

$$\begin{aligned}0 = \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,101))\\ + \tau(A(s,111)),\end{aligned} \tag{2.21}$$

$$\begin{aligned}0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101))\\ + \tau(A(s,110)) + \tau(A(s,1000)),\end{aligned} \tag{2.22}$$

$$\begin{aligned}0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101))\\ + \tau(A(s,110)) + \tau(A(s,1001)),\end{aligned} \tag{2.23}$$

$$\begin{aligned}0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101))\\ + \tau(A(s,110)) + \tau(A(s,1010)),\end{aligned} \tag{2.24}$$

$$\begin{aligned}0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101))\\ + \tau(A(s,110)) + \tau(A(s,1011)),\end{aligned} \tag{2.25}$$

$$\begin{aligned}0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101))\\ + \tau(A(s,110)) + \tau(A(s,1100)),\end{aligned} \tag{2.26}$$

$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101))\\ &\quad + \tau(A(s,110)) + \tau(A(s,1101)),\end{aligned} \tag{2.27}$$

for all $s \in E_{2k}$, and

$$\begin{aligned}0 &= \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,101))\\ &\quad + \tau(A(s,111)),\end{aligned} \tag{2.28}$$

$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101))\\ &\quad + \tau(A(s,110)) + \tau(A(s,1000)),\end{aligned} \tag{2.29}$$

$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101))\\ &\quad + \tau(A(s,110)) + \tau(A(s,1001)),\end{aligned} \tag{2.30}$$

$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101))\\ &\quad + \tau(A(s,110)) + \tau(A(s,1010)),\end{aligned} \tag{2.31}$$

$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101))\\ &\quad + \tau(A(s,110)) + \tau(A(s,1011)),\end{aligned} \tag{2.32}$$

$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101))\\ &\quad + \tau(A(s,110)) + \tau(A(s,1100)),\end{aligned} \tag{2.33}$$

$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101))\\ &\quad + \tau(A(s,110)) + \tau(A(s,1101)),\end{aligned} \tag{2.34}$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{24}), E_{24}) = (A(i, 0^{20}), E_{20})$, so that we only have to verify that identities (2.21) through (2.34) hold for $2 \le k \le 12$, which turns out to be true. $\square$

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \ge 2$, and $u = 2^{2k-1}$,*

$$\begin{aligned}M_{2k} &= M^e[:7u] + x^{2u}M^e[:5u] + x^{3u}M^e[:4u] + x^{5u}M^e[:2u] + x^{6u}M^e[:u]\\ &\quad + x^{7u}R^e,\\ W_{2k} &= W^e[:7u] + x^{2u}W^e[:5u] + x^{3u}W^e[:4u] + x^{5u}W^e[:2u] + x^{6u}W^e[:u]\\ &\quad + x^{7u}R^e.\end{aligned}$$

For all integer $k \ge 2$, and $u = 2^{2k}$,

$$\begin{aligned}M_{2k+1} &= M^o[:7u] + x^{2u}M^o[:5u] + x^{3u}M^o[:4u] + x^{5u}M^o[:2u] + x^{6u}M^o[:u]\\ &\quad + x^{7u}R^o,\\ W_{2k+1} &= W^o[:7u] + x^{2u}W^o[:5u] + x^{3u}W^o[:4u] + x^{5u}W^o[:2u] + x^{6u}W^o[:u]\\ &\quad + x^{7u}R^o.\end{aligned}$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \ge 2$, we claim that

$$\begin{aligned}M_{2k} \wedge W_{2k} &\Rightarrow M_{2k+1} \wedge W_{2k+1},\\ M_{2k+1} \wedge W_{2k+1} &\Rightarrow M_{2k+2} \wedge W_{2k+2}.\end{aligned}$$

We give the proof of
$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$
the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to
$$\begin{aligned} W_{2k}M_{2k} = \big(&W^e[:7v] + x^{2v}W^e[:5v] + x^{3v}W^e[:4v] + x^{5v}W^e[:2v] + x^{6v}W^e[:v] \\ &+ x^{7u}R^e\big) \times \big(M^e[:7v] + x^{2v}M^e[:5v] + x^{3v}M^e[:4v] + x^{5v}M^e[:2v] \\ &+ x^{6v}M^e[:v] + x^{7u}R^e\big); \end{aligned} \tag{2.35}$$
Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{14v}R^o$. Therefore we only need to prove that their difference is $O(x^{14v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{14v}, to
$$\begin{aligned} W^e[:14v]M^e[:14v] + x^{4v}W^e[:10v]M^e[:10v] + x^{6v}W^e[:8v]M^e[:8v] \\ + x^{10v}W^e[:4v]M^e[:4v] + x^{12v}W^e[:2v]M^e[:2v]. \end{aligned}$$
For all $n \le 14$, replace the occurrences of $W^e[:n\cdot v]$ and $M^e[:n\cdot v]$ in the above expression by the reduction modulo $x^{n\cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define
$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n\cdot v : (n+1)\cdot v]/X^n, \\ b_n &:= M^e[n\cdot v : (n+1)\cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$
Using the notation introduced above, we can represent the expressions lhs (2.35) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{14}, b_1, ..., b_{14}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{14})$, which completes the proof. $\square$

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a,b)$; for $\mathrm{CF}(b,a)$, the proof is similar. By Lemma 2.3, we have For all $0 \le j,k \le 1$,
$$\begin{aligned} \lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k}, \\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k}, \\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k}, \\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}. \end{aligned}$$
Let $z = 1/x$. By the convergence theorem and identity (2.2),
$$\mathrm{CF}(a,b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$
By definition, $\phi(M^e,0,1)$ and $\phi(M^e,0,0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore
$$P(x,y) = \mathrm{Res}_t\left(\phi(M^e,0,1)(x,t), y^{12}\phi(M^e,0,1)(x,t/y)\right)$$
is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x,y)=q_4(x)y^4+q_3(x)y^3+q_2(x)y^2+q_1(x)y+q_0(x),$$

where

$$\begin{aligned}
q_0(x)&=x^{24}+x^{22}+x^{20}+x^{19}+x^{14}+x^{10}+x^9+x^7,\\
q_1(x)&=x^{22}+x^{21}+x^{20}+x^{19}+x^{17}+x^{15}+x^{12}+x^{11}+x^8+x^3,\\
q_2(x)&=x^{20}+x^{18}+x^{14}+x^{12}+x^{10}+x^6+x^4+1,\\
q_3(x)&=x^{22}+x^{21}+x^{20}+x^{19}+x^{17}+x^{15}+x^{12}+x^{11}+x^8+x^3,\\
q_4(x)&=x^{22}+x^{19}+x^{18}+x^{12}+x^9+x^7+x^6.
\end{aligned}$$

The polynomial $Q(x,y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and $R(x,y):=P(x,y)/Q(x,y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 216, and substitute it for y in $R(z,y)$, we get a series with valuation smaller than 216, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b)=f(1/z)$. □

3. Annexe

All 16 polynomials are of the form

$$p_0(x)+p_3(x)y^3+p_6(x)y^6+p_9(x)y^9+p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e,0,0)$:

$$\begin{aligned}
p_0(x)&=x^{132}+x^{126}+x^{124}+x^{120}+x^{116}+x^{114}+x^{110}+x^{108}+x^{106}+x^{102}\\
&\quad+x^{96}+x^{94}+x^{92}+x^{88}+x^{80}+x^{74}+x^{72}+x^{68}+x^{64}+x^{60}+x^{54}\\
&\quad+x^{52}+x^{50}+x^{46}+x^{40}+x^{38}+x^{36},\\
p_3(x)&=x^{126}+x^{124}+x^{120}+x^{116}+x^{114}+x^{112}+x^{100}+x^{92}+x^{88}+x^{78}\\
&\quad+x^{66}+x^{64}+x^{60}+x^{48}+x^{46}+x^{38}+x^{36}+x^{34}+x^{32}+x^{28}+x^{26}\\
&\quad+x^{20},\\
p_6(x)&=x^{126}+x^{120}+x^{118}+x^{110}+x^{108}+x^{106}+x^{102}+x^{100}+x^{94}+x^{92}\\
&\quad+x^{90}+x^{86}+x^{84}+x^{82}+x^{76}+x^{74}+x^{68}+x^{66}+x^{60}+x^{58}+x^{54}\\
&\quad+x^{50}+x^{48}+x^{46}+x^{40}+x^{28}+x^{26}+x^{24}+x^{20}+x^{14}+x^{12}+x^8\\
&\quad+x^6+1,\\
p_9(x)&=x^{128}+x^{124}+x^{122}+x^{120}+x^{118}+x^{106}+x^{104}+x^{100}+x^{98}+x^{86}\\
&\quad+x^{76}+x^{74}+x^{66}+x^{62}+x^{56}+x^{54}+x^{52}+x^{50}+x^{48}+x^{46}+x^{44}\\
&\quad+x^{42}+x^{40}+x^{36}+x^{34}+x^{32}+x^{20}+x^{16}+x^{12}+x^6,\\
p_{12}(x)&=x^{128}+x^{120}+x^{104}+x^{96}+x^{48}+x^{40}+x^{32}+x^{16}+x^8+1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0].

For $\phi(M^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{129} + x^{127} + x^{125} + x^{124} + x^{122} + x^{121} + x^{120} + x^{119} + x^{116} + x^{115} \\
&\quad + x^{114} + x^{112} + x^{110} + x^{107} + x^{105} + x^{104} + x^{103} + x^{100} + x^{98} + x^{97} \\
&\quad + x^{95} + x^{91} + x^{88} + x^{86} + x^{85} + x^{83} + x^{81} + x^{79} + x^{75} + x^{73} + x^{72} \\
&\quad + x^{70} + x^{66} + x^{62} + x^{61} + x^{60} + x^{59} + x^{52} + x^{51} + x^{46} + x^{45} + x^{44} \\
&\quad + x^{42} + x^{40} + x^{39}, \\
p_3(x) &= x^{125} + x^{124} + x^{123} + x^{120} + x^{118} + x^{115} + x^{113} + x^{112} + x^{110} + x^{109} \\
&\quad + x^{108} + x^{107} + x^{106} + x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{91} + x^{90} \\
&\quad + x^{89} + x^{87} + x^{85} + x^{84} + x^{82} + x^{79} + x^{77} + x^{76} + x^{75} + x^{73} + x^{69} \\
&\quad + x^{68} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{58} + x^{57} + x^{55} + x^{53} + x^{52} \\
&\quad + x^{50} + x^{47} + x^{41} + x^{40} + x^{38} + x^{34} + x^{32} + x^{31} + x^{30} + x^{27}, \\
p_6(x) &= x^{123} + x^{121} + x^{120} + x^{118} + x^{113} + x^{111} + x^{110} + x^{109} + x^{108} + x^{107} \\
&\quad + x^{106} + x^{105} + x^{104} + x^{102} + x^{101} + x^{99} + x^{98} + x^{96} + x^{91} + x^{88} \\
&\quad + x^{83} + x^{80} + x^{79} + x^{77} + x^{76} + x^{75} + x^{74} + x^{72} + x^{71} + x^{68} + x^{67} \\
&\quad + x^{64} + x^{63} + x^{60} + x^{57} + x^{54} + x^{53} + x^{50} + x^{43} + x^{41} + x^{40} + x^{39} \\
&\quad + x^{38} + x^{37} + x^{36} + x^{34} + x^{33} + x^{31} + x^{30} + x^{28} + x^{21} + x^{18}, \\
p_9(x) &= x^{121} + x^{120} + x^{118} + x^{114} + x^{113} + x^{110} + x^{109} + x^{107} + x^{105} + x^{104} \\
&\quad + x^{103} + x^{100} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} \\
&\quad + x^{87} + x^{79} + x^{78} + x^{77} + x^{75} + x^{69} + x^{68} + x^{66} + x^{63} + x^{59} + x^{58} \\
&\quad + x^{57} + x^{54} + x^{53} + x^{51} + x^{49} + x^{48} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} \\
&\quad + x^{41} + x^{40} + x^{37} + x^{35} + x^{34} + x^{33} + x^{31} + x^{29} + x^{28} + x^{26} + x^{22} \\
&\quad + x^{21} + x^{19} + x^{17} + x^{16} + x^{14} + x^{13} + x^{12} + x^{9}, \\
p_{12}(x) &= x^{119} + x^{116} + x^{111} + x^{108} + x^{107} + x^{104} + x^{99} + x^{96} + x^{39} + x^{36} \\
&\quad + x^{31} + x^{28} + x^{27} + x^{24} + x^{23} + x^{20} + x^{19} + x^{16} + x^{15} + x^{12} + x^{11} \\
&\quad + x^{8} + x^{3} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 0, 1, 0, 0, 1, 0].

For $\phi(M^e, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{129} + x^{127} + x^{125} + x^{124} + x^{122} + x^{121} + x^{120} + x^{119} + x^{116} + x^{115} \\
&\quad + x^{114} + x^{112} + x^{110} + x^{107} + x^{105} + x^{104} + x^{103} + x^{100} + x^{98} + x^{97} \\
&\quad + x^{95} + x^{91} + x^{88} + x^{86} + x^{85} + x^{83} + x^{81} + x^{79} + x^{75} + x^{73} + x^{72} \\
&\quad + x^{70} + x^{66} + x^{62} + x^{61} + x^{60} + x^{59} + x^{52} + x^{51} + x^{46} + x^{45} + x^{44} \\
&\quad + x^{42} + x^{40} + x^{39}, \\
p_3(x) &= x^{125} + x^{124} + x^{123} + x^{120} + x^{118} + x^{115} + x^{113} + x^{112} + x^{110} + x^{109} \\
&\quad + x^{108} + x^{107} + x^{106} + x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{91} + x^{90} \\
&\quad + x^{89} + x^{87} + x^{85} + x^{84} + x^{82} + x^{79} + x^{77} + x^{76} + x^{75} + x^{73} + x^{69}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{68} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{58} + x^{57} + x^{55} + x^{53} + x^{52}\\
&+ x^{50} + x^{47} + x^{41} + x^{40} + x^{38} + x^{34} + x^{32} + x^{31} + x^{30} + x^{27},\\
p_6(x) = {}& x^{123} + x^{121} + x^{120} + x^{118} + x^{113} + x^{111} + x^{110} + x^{109} + x^{108} + x^{107}\\
&+ x^{106} + x^{105} + x^{104} + x^{102} + x^{101} + x^{99} + x^{98} + x^{96} + x^{91} + x^{88}\\
&+ x^{83} + x^{80} + x^{79} + x^{77} + x^{76} + x^{75} + x^{74} + x^{72} + x^{71} + x^{68} + x^{67}\\
&+ x^{64} + x^{63} + x^{60} + x^{57} + x^{54} + x^{53} + x^{50} + x^{43} + x^{41} + x^{40} + x^{39}\\
&+ x^{38} + x^{37} + x^{36} + x^{34} + x^{33} + x^{31} + x^{30} + x^{28} + x^{21} + x^{18},\\
p_9(x) = {}& x^{121} + x^{120} + x^{118} + x^{114} + x^{113} + x^{110} + x^{109} + x^{107} + x^{105} + x^{104}\\
&+ x^{103} + x^{100} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90}\\
&+ x^{87} + x^{79} + x^{78} + x^{77} + x^{75} + x^{69} + x^{68} + x^{66} + x^{63} + x^{59} + x^{58}\\
&+ x^{57} + x^{54} + x^{53} + x^{51} + x^{49} + x^{48} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42}\\
&+ x^{41} + x^{40} + x^{37} + x^{35} + x^{34} + x^{33} + x^{31} + x^{29} + x^{28} + x^{26} + x^{22}\\
&+ x^{21} + x^{19} + x^{17} + x^{16} + x^{14} + x^{13} + x^{12} + x^{9},\\
p_{12}(x) = {}& x^{119} + x^{116} + x^{111} + x^{108} + x^{107} + x^{104} + x^{99} + x^{96} + x^{39} + x^{36}\\
&+ x^{31} + x^{28} + x^{27} + x^{24} + x^{23} + x^{20} + x^{19} + x^{16} + x^{15} + x^{12} + x^{11}\\
&+ x^{8} + x^{3} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 0, 1, 0, 0, 1, 0].

For $\phi(M^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) = {}& x^{126} + x^{122} + x^{120} + x^{114} + x^{110} + x^{106} + x^{96} + x^{94} + x^{90} + x^{88}\\
&+ x^{84} + x^{80} + x^{78} + x^{76} + x^{74} + x^{70} + x^{66} + x^{60} + x^{58} + x^{56} + x^{54}\\
&+ x^{52} + x^{50} + x^{46} + x^{42},\\
p_3(x) = {}& x^{114} + x^{112} + x^{104} + x^{100} + x^{98} + x^{96} + x^{94} + x^{90} + x^{88} + x^{86} + x^{84}\\
&+ x^{82} + x^{80} + x^{76} + x^{68} + x^{60} + x^{58} + x^{54} + x^{52} + x^{44} + x^{38} + x^{24}\\
&+ x^{20} + x^{18},\\
p_6(x) = {}& x^{114} + x^{112} + x^{110} + x^{106} + x^{100} + x^{98} + x^{92} + x^{90} + x^{86} + x^{84} + x^{82}\\
&+ x^{74} + x^{72} + x^{70} + x^{66} + x^{64} + x^{62} + x^{48} + x^{46} + x^{38} + x^{36} + x^{34}\\
&+ x^{26} + x^{20} + x^{18} + x^{12} + x^{8} + 1,\\
p_9(x) = {}& x^{110} + x^{106} + x^{102} + x^{98} + x^{96} + x^{94} + x^{88} + x^{82} + x^{78} + x^{76} + x^{72}\\
&+ x^{70} + x^{68} + x^{64} + x^{62} + x^{56} + x^{54} + x^{50} + x^{46} + x^{42} + x^{40} + x^{38}\\
&+ x^{36} + x^{34} + x^{32} + x^{30} + x^{26} + x^{22} + x^{20} + x^{18} + x^{16} + x^{6},\\
p_{12}(x) = {}& x^{110} + x^{104} + x^{102} + x^{96} + x^{30} + x^{24} + x^{22} + x^{16} + x^{14} + x^{8} + x^{6} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0].

For $\phi(W^e, 0, 0)$:

$$
\begin{aligned}
p_0(x) = {}& x^{126} + x^{122} + x^{120} + x^{118} + x^{112} + x^{108} + x^{104} + x^{102} + x^{94} + x^{92}\\
&+ x^{86} + x^{84} + x^{82} + x^{76} + x^{72} + x^{70} + x^{62} + x^{52} + x^{50} + x^{48} + x^{44}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{42} + x^{40} + x^{38} + x^{36},\\
p_3(x) &= x^{114} + x^{112} + x^{106} + x^{102} + x^{100} + x^{96} + x^{94} + x^{86} + x^{78} + x^{74} + x^{72}\\
&+ x^{68} + x^{64} + x^{62} + x^{60} + x^{58} + x^{56} + x^{52} + x^{48} + x^{30} + x^{28} + x^{20},\\
p_6(x) &= x^{114} + x^{112} + x^{110} + x^{104} + x^{100} + x^{98} + x^{96} + x^{88} + x^{78} + x^{76} + x^{70}\\
&+ x^{62} + x^{60} + x^{58} + x^{52} + x^{50} + x^{48} + x^{40} + x^{38} + x^{36} + x^{34} + x^{32}\\
&+ x^{30} + x^{28} + x^{24} + x^{18} + x^{8} + 1,\\
p_9(x) &= x^{110} + x^{106} + x^{102} + x^{98} + x^{96} + x^{94} + x^{88} + x^{82} + x^{78} + x^{76} + x^{72}\\
&+ x^{70} + x^{68} + x^{64} + x^{62} + x^{56} + x^{54} + x^{50} + x^{46} + x^{42} + x^{40} + x^{38}\\
&+ x^{36} + x^{34} + x^{32} + x^{30} + x^{26} + x^{22} + x^{20} + x^{18} + x^{16} + x^{6},\\
p_{12}(x) &= x^{110} + x^{104} + x^{102} + x^{96} + x^{30} + x^{24} + x^{22} + x^{16} + x^{14} + x^{8} + x^{6} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0].

For $\phi(W^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{129} + x^{125} + x^{120} + x^{119} + x^{118} + x^{117} + x^{116} + x^{113} + x^{112} + x^{111}\\
&+ x^{107} + x^{106} + x^{105} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96}\\
&+ x^{95} + x^{94} + x^{91} + x^{88} + x^{87} + x^{85} + x^{82} + x^{78} + x^{76} + x^{75} + x^{72}\\
&+ x^{69} + x^{68} + x^{65} + x^{64} + x^{61} + x^{58} + x^{56} + x^{54} + x^{53} + x^{51} + x^{48}\\
&+ x^{47} + x^{45} + x^{42} + x^{40} + x^{39},\\
p_3(x) &= x^{125} + x^{124} + x^{123} + x^{120} + x^{118} + x^{115} + x^{113} + x^{112} + x^{110} + x^{109}\\
&+ x^{108} + x^{107} + x^{106} + x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{91} + x^{90}\\
&+ x^{89} + x^{87} + x^{85} + x^{84} + x^{82} + x^{79} + x^{77} + x^{76} + x^{75} + x^{73} + x^{69}\\
&+ x^{68} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{58} + x^{57} + x^{55} + x^{53} + x^{52}\\
&+ x^{50} + x^{47} + x^{41} + x^{40} + x^{38} + x^{34} + x^{32} + x^{31} + x^{30} + x^{27},\\
p_6(x) &= x^{123} + x^{121} + x^{120} + x^{118} + x^{113} + x^{111} + x^{110} + x^{109} + x^{108} + x^{107}\\
&+ x^{106} + x^{105} + x^{104} + x^{102} + x^{101} + x^{99} + x^{98} + x^{96} + x^{91} + x^{88}\\
&+ x^{83} + x^{80} + x^{79} + x^{77} + x^{76} + x^{75} + x^{74} + x^{72} + x^{71} + x^{68} + x^{67}\\
&+ x^{64} + x^{63} + x^{60} + x^{57} + x^{54} + x^{53} + x^{50} + x^{43} + x^{41} + x^{40} + x^{39}\\
&+ x^{38} + x^{37} + x^{36} + x^{34} + x^{33} + x^{31} + x^{30} + x^{28} + x^{21} + x^{18},\\
p_9(x) &= x^{121} + x^{120} + x^{118} + x^{114} + x^{113} + x^{110} + x^{109} + x^{107} + x^{105} + x^{104}\\
&+ x^{103} + x^{100} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90}\\
&+ x^{87} + x^{79} + x^{78} + x^{77} + x^{75} + x^{69} + x^{68} + x^{66} + x^{63} + x^{59} + x^{58}\\
&+ x^{57} + x^{54} + x^{53} + x^{51} + x^{49} + x^{48} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42}\\
&+ x^{41} + x^{40} + x^{37} + x^{35} + x^{34} + x^{33} + x^{31} + x^{29} + x^{28} + x^{26} + x^{22}\\
&+ x^{21} + x^{19} + x^{17} + x^{16} + x^{14} + x^{13} + x^{12} + x^{9},\\
p_{12}(x) &= x^{119} + x^{116} + x^{111} + x^{108} + x^{107} + x^{104} + x^{99} + x^{96} + x^{39} + x^{36}\\
&+ x^{31} + x^{28} + x^{27} + x^{24} + x^{23} + x^{20} + x^{19} + x^{16} + x^{15} + x^{12} + x^{11}
\end{aligned}
$$

$$+ x^{8} + x^{3} + 1,$$

and the initial terms are [0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 1, 0, 1, 0, 0].

For $\phi(W^e, 1, 0)$:

$$\begin{aligned}
p_0(x) = {} & x^{129} + x^{125} + x^{120} + x^{119} + x^{118} + x^{117} + x^{116} + x^{113} + x^{112} + x^{111} \\
& + x^{107} + x^{106} + x^{105} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} \\
& + x^{95} + x^{94} + x^{91} + x^{88} + x^{87} + x^{85} + x^{82} + x^{78} + x^{76} + x^{75} + x^{72} \\
& + x^{69} + x^{68} + x^{65} + x^{64} + x^{61} + x^{58} + x^{56} + x^{54} + x^{53} + x^{51} + x^{48} \\
& + x^{47} + x^{45} + x^{42} + x^{40} + x^{39}, \\
p_3(x) = {} & x^{125} + x^{124} + x^{123} + x^{120} + x^{118} + x^{115} + x^{113} + x^{112} + x^{110} + x^{109} \\
& + x^{108} + x^{107} + x^{106} + x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{91} + x^{90} \\
& + x^{89} + x^{87} + x^{85} + x^{84} + x^{82} + x^{79} + x^{77} + x^{76} + x^{75} + x^{73} + x^{69} \\
& + x^{68} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{58} + x^{57} + x^{55} + x^{53} + x^{52} \\
& + x^{50} + x^{47} + x^{41} + x^{40} + x^{38} + x^{34} + x^{32} + x^{31} + x^{30} + x^{27}, \\
p_6(x) = {} & x^{123} + x^{121} + x^{120} + x^{118} + x^{113} + x^{111} + x^{110} + x^{109} + x^{108} + x^{107} \\
& + x^{106} + x^{105} + x^{104} + x^{102} + x^{101} + x^{99} + x^{98} + x^{96} + x^{91} + x^{88} \\
& + x^{83} + x^{80} + x^{79} + x^{77} + x^{76} + x^{75} + x^{74} + x^{72} + x^{71} + x^{68} + x^{67} \\
& + x^{64} + x^{63} + x^{60} + x^{57} + x^{54} + x^{53} + x^{50} + x^{43} + x^{41} + x^{40} + x^{39} \\
& + x^{38} + x^{37} + x^{36} + x^{34} + x^{33} + x^{31} + x^{30} + x^{28} + x^{21} + x^{18}, \\
p_9(x) = {} & x^{121} + x^{120} + x^{118} + x^{114} + x^{113} + x^{110} + x^{109} + x^{107} + x^{105} + x^{104} \\
& + x^{103} + x^{100} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} \\
& + x^{87} + x^{79} + x^{78} + x^{77} + x^{75} + x^{69} + x^{68} + x^{66} + x^{63} + x^{59} + x^{58} \\
& + x^{57} + x^{54} + x^{53} + x^{51} + x^{49} + x^{48} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} \\
& + x^{41} + x^{40} + x^{37} + x^{35} + x^{34} + x^{33} + x^{31} + x^{29} + x^{28} + x^{26} + x^{22} \\
& + x^{21} + x^{19} + x^{17} + x^{16} + x^{14} + x^{13} + x^{12} + x^{9}, \\
p_{12}(x) = {} & x^{119} + x^{116} + x^{111} + x^{108} + x^{107} + x^{104} + x^{99} + x^{96} + x^{39} + x^{36} \\
& + x^{31} + x^{28} + x^{27} + x^{24} + x^{23} + x^{20} + x^{19} + x^{16} + x^{15} + x^{12} + x^{11} \\
& + x^{8} + x^{3} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 1, 0, 1, 0, 0].

For $\phi(W^e, 1, 1)$:

$$\begin{aligned}
p_0(x) = {} & x^{132} + x^{128} + x^{126} + x^{124} + x^{120} + x^{118} + x^{114} + x^{106} + x^{104} + x^{100} \\
& + x^{98} + x^{96} + x^{94} + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{76} + x^{72} + x^{70} \\
& + x^{62} + x^{60} + x^{58} + x^{52} + x^{50} + x^{48} + x^{46} + x^{42}, \\
p_3(x) = {} & x^{126} + x^{122} + x^{118} + x^{116} + x^{102} + x^{100} + x^{96} + x^{94} + x^{92} + x^{84} \\
& + x^{82} + x^{80} + x^{78} + x^{74} + x^{70} + x^{66} + x^{62} + x^{60} + x^{56} + x^{48} + x^{40} \\
& + x^{32} + x^{30} + x^{26} + x^{20} + x^{18},
\end{aligned}$$

$$\begin{aligned}
p_6(x) &= x^{126} + x^{124} + x^{122} + x^{120} + x^{116} + x^{114} + x^{112} + x^{110} + x^{108} + x^{106} \\
&\quad + x^{104} + x^{100} + x^{90} + x^{88} + x^{86} + x^{84} + x^{80} + x^{72} + x^{70} + x^{66} + x^{62} \\
&\quad + x^{58} + x^{56} + x^{54} + x^{52} + x^{50} + x^{40} + x^{38} + x^{34} + x^{32} + x^{30} + x^{26} \\
&\quad + x^{24} + x^{18} + x^{14} + x^{8} + x^{6} + 1, \\
p_9(x) &= x^{128} + x^{124} + x^{122} + x^{120} + x^{118} + x^{106} + x^{104} + x^{100} + x^{98} + x^{86} \\
&\quad + x^{76} + x^{74} + x^{66} + x^{62} + x^{56} + x^{54} + x^{52} + x^{50} + x^{48} + x^{46} + x^{44} \\
&\quad + x^{42} + x^{40} + x^{36} + x^{34} + x^{32} + x^{20} + x^{16} + x^{12} + x^{6}, \\
p_{12}(x) &= x^{128} + x^{120} + x^{104} + x^{96} + x^{48} + x^{40} + x^{32} + x^{16} + x^{8} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0].

For $\phi(M^o, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{129} + x^{127} + x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117} \\
&\quad + x^{116} + x^{115} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{107} + x^{103} \\
&\quad + x^{97} + x^{96} + x^{95} + x^{93} + x^{89} + x^{88} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} \\
&\quad + x^{81} + x^{80} + x^{79} + x^{77} + x^{69} + x^{59} + x^{57} + x^{54} + x^{52} + x^{50} + x^{49} \\
&\quad + x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{36}, \\
p_3(x) &= x^{127} + x^{125} + x^{124} + x^{119} + x^{118} + x^{113} + x^{108} + x^{106} + x^{104} + x^{103} \\
&\quad + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{90} + x^{88} \\
&\quad + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} \\
&\quad + x^{70} + x^{68} + x^{66} + x^{65} + x^{64} + x^{63} + x^{61} + x^{59} + x^{58} + x^{57} + x^{54} \\
&\quad + x^{51} + x^{50} + x^{49} + x^{46} + x^{41} + x^{36} + x^{35} + x^{33} + x^{31} + x^{29} + x^{27} \\
&\quad + x^{26} + x^{24}, \\
p_6(x) &= x^{127} + x^{125} + x^{124} + x^{119} + x^{118} + x^{117} + x^{115} + x^{112} + x^{111} + x^{110} \\
&\quad + x^{109} + x^{108} + x^{107} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{95} \\
&\quad + x^{94} + x^{93} + x^{91} + x^{87} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} \\
&\quad + x^{78} + x^{77} + x^{75} + x^{74} + x^{73} + x^{72} + x^{70} + x^{69} + x^{65} + x^{64} + x^{62} \\
&\quad + x^{56} + x^{52} + x^{51} + x^{48} + x^{47} + x^{46} + x^{45} + x^{42} + x^{41} + x^{40} + x^{39} \\
&\quad + x^{37} + x^{36} + x^{35} + x^{32} + x^{31} + x^{30} + x^{26} + x^{24} + x^{21} + x^{20} + x^{17} \\
&\quad + x^{16} + x^{14} + x^{11} + x^{8} + x^{3} + 1, \\
p_9(x) &= x^{127} + x^{124} + x^{123} + x^{120} + x^{119} + x^{116} + x^{111} + x^{108} + x^{47} + x^{44} \\
&\quad + x^{43} + x^{40} + x^{39} + x^{36} + x^{27} + x^{24} + x^{23} + x^{20} + x^{15} + x^{12}, \\
p_{12}(x) &= x^{127} + x^{124} + x^{123} + x^{120} + x^{107} + x^{104} + x^{99} + x^{96} + x^{47} + x^{44} \\
&\quad + x^{43} + x^{40} + x^{31} + x^{28} + x^{19} + x^{16} + x^{11} + x^{8} + x^{3} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1].

For $\phi(M^o, 0, 1)$:

$$p_0(x) = x^{130} + x^{128} + x^{126} + x^{124} + x^{122} + x^{121} + x^{120} + x^{117} + x^{116} + x^{114}$$

$$
\begin{aligned}
&+ x^{109} + x^{108} + x^{107} + x^{103} + x^{101} + x^{100} + x^{98} + x^{96} + x^{95} + x^{94}\\
&+ x^{93} + x^{92} + x^{91} + x^{89} + x^{86} + x^{85} + x^{84} + x^{83} + x^{79} + x^{78} + x^{77}\\
&+ x^{74} + x^{72} + x^{70} + x^{69} + x^{67} + x^{66} + x^{64} + x^{61} + x^{59} + x^{58} + x^{57}\\
&+ x^{54} + x^{53} + x^{52} + x^{51} + x^{47} + x^{42} + x^{41} + x^{40} + x^{39},\\
p_3(x) &= x^{116} + x^{115} + x^{114} + x^{113} + x^{112} + x^{109} + x^{108} + x^{106} + x^{105} + x^{104}\\
&+ x^{101} + x^{100} + x^{98} + x^{97} + x^{96} + x^{93} + x^{91} + x^{84} + x^{83} + x^{82} + x^{81}\\
&+ x^{80} + x^{77} + x^{76} + x^{74} + x^{73} + x^{72} + x^{69} + x^{67} + x^{52} + x^{51} + x^{50}\\
&+ x^{49} + x^{48} + x^{45} + x^{44} + x^{42} + x^{41} + x^{40} + x^{37} + x^{36} + x^{34} + x^{33}\\
&+ x^{32} + x^{29} + x^{27},\\
p_6(x) &= x^{118} + x^{108} + x^{104} + x^{100} + x^{98} + x^{94} + x^{92} + x^{90} + x^{88} + x^{78} + x^{76}\\
&+ x^{72} + x^{66} + x^{64} + x^{58} + x^{56} + x^{54} + x^{48} + x^{46} + x^{44} + x^{42} + x^{40}\\
&+ x^{36} + x^{32} + x^{30} + x^{26} + x^{24} + x^{22} + x^{20} + x^{18},\\
p_9(x) &= x^{120} + x^{119} + x^{115} + x^{114} + x^{113} + x^{112} + x^{111} + x^{110} + x^{107} + x^{105}\\
&+ x^{40} + x^{39} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{27} + x^{25} + x^{24}\\
&+ x^{23} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{11} + x^{9},\\
p_{12}(x) &= x^{122} + x^{120} + x^{102} + x^{100} + x^{98} + x^{96} + x^{42} + x^{40} + x^{26} + x^{24} + x^{22}\\
&+ x^{20} + x^{18} + x^{16} + x^{6} + x^{4} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0].

For $\phi(M^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{132} + x^{130} + x^{127} + x^{125} + x^{124} + x^{120} + x^{117} + x^{116} + x^{115} + x^{114}\\
&+ x^{113} + x^{112} + x^{108} + x^{107} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101}\\
&+ x^{99} + x^{98} + x^{97} + x^{95} + x^{94} + x^{92} + x^{91} + x^{87} + x^{86} + x^{84} + x^{82}\\
&+ x^{81} + x^{80} + x^{79} + x^{78} + x^{72} + x^{71} + x^{67} + x^{65} + x^{63} + x^{61} + x^{60}\\
&+ x^{59} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{48} + x^{47} + x^{46}\\
&+ x^{43} + x^{40} + x^{39},\\
p_3(x) &= x^{130} + x^{129} + x^{125} + x^{124} + x^{123} + x^{122} + x^{121} + x^{119} + x^{118} + x^{116}\\
&+ x^{115} + x^{114} + x^{112} + x^{110} + x^{108} + x^{105} + x^{104} + x^{102} + x^{101}\\
&+ x^{99} + x^{98} + x^{96} + x^{95} + x^{93} + x^{92} + x^{90} + x^{89} + x^{87} + x^{86} + x^{84}\\
&+ x^{83} + x^{81} + x^{80} + x^{78} + x^{77} + x^{75} + x^{73} + x^{72} + x^{71} + x^{67} + x^{66}\\
&+ x^{65} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{55} + x^{54} + x^{52} + x^{51} + x^{50}\\
&+ x^{48} + x^{46} + x^{44} + x^{41} + x^{40} + x^{38} + x^{37} + x^{35} + x^{33} + x^{32} + x^{31}\\
&+ x^{27},\\
p_6(x) &= x^{132} + x^{130} + x^{122} + x^{116} + x^{114} + x^{106} + x^{104} + x^{102} + x^{100} + x^{98}\\
&+ x^{96} + x^{92} + x^{86} + x^{84} + x^{74} + x^{68} + x^{66} + x^{64} + x^{62} + x^{60} + x^{54}\\
&+ x^{52} + x^{48} + x^{46} + x^{42} + x^{38} + x^{36} + x^{34} + x^{28} + x^{18},
\end{aligned}
$$

$$\begin{aligned}
p_9(x) &= x^{134} + x^{133} + x^{132} + x^{131} + x^{129} + x^{128} + x^{123} + x^{120} + x^{119} + x^{116} \\
&\quad + x^{111} + x^{110} + x^{109} + x^{105} + x^{54} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} \\
&\quad + x^{43} + x^{40} + x^{39} + x^{38} + x^{37} + x^{35} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} \\
&\quad + x^{27} + x^{25} + x^{24} + x^{23} + x^{20} + x^{15} + x^{14} + x^{13} + x^{9}, \\
p_{12}(x) &= x^{136} + x^{132} + x^{124} + x^{120} + x^{116} + x^{108} + x^{104} + x^{96} + x^{56} + x^{52} \\
&\quad + x^{44} + x^{20} + x^{16} + x^{12} + x^{8} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0].

For $\phi(M^o, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{129} + x^{121} + x^{116} + x^{115} + x^{113} + x^{112} + x^{111} + x^{109} + x^{104} + x^{101} \\
&\quad + x^{99} + x^{96} + x^{95} + x^{93} + x^{90} + x^{89} + x^{87} + x^{86} + x^{82} + x^{80} + x^{79} \\
&\quad + x^{76} + x^{74} + x^{70} + x^{62} + x^{58} + x^{57} + x^{56} + x^{53} + x^{50} + x^{48} + x^{47} \\
&\quad + x^{46} + x^{45} + x^{42}, \\
p_3(x) &= x^{127} + x^{125} + x^{124} + x^{123} + x^{121} + x^{120} + x^{116} + x^{115} + x^{113} + x^{112} \\
&\quad + x^{111} + x^{106} + x^{104} + x^{101} + x^{95} + x^{93} + x^{92} + x^{90} + x^{89} + x^{88} \\
&\quad + x^{87} + x^{86} + x^{84} + x^{83} + x^{82} + x^{81} + x^{78} + x^{74} + x^{71} + x^{68} + x^{67} \\
&\quad + x^{66} + x^{65} + x^{63} + x^{61} + x^{58} + x^{56} + x^{55} + x^{52} + x^{50} + x^{49} + x^{48} \\
&\quad + x^{47} + x^{46} + x^{44} + x^{41} + x^{39} + x^{32} + x^{31} + x^{30} + x^{29} + x^{26}, \\
p_6(x) &= x^{127} + x^{125} + x^{124} + x^{123} + x^{121} + x^{120} + x^{117} + x^{116} + x^{115} + x^{113} \\
&\quad + x^{112} + x^{106} + x^{105} + x^{98} + x^{96} + x^{95} + x^{94} + x^{93} + x^{91} + x^{85} + x^{83} \\
&\quad + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{73} + x^{70} + x^{69} + x^{67} + x^{65} + x^{63} \\
&\quad + x^{62} + x^{60} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{50} + x^{46} + x^{45} + x^{44} \\
&\quad + x^{43} + x^{42} + x^{39} + x^{38} + x^{36} + x^{34} + x^{33} + x^{28} + x^{26} + x^{24} + x^{23} \\
&\quad + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{12} + x^{11} + x^{8} + x^{3} + 1, \\
p_9(x) &= x^{127} + x^{124} + x^{123} + x^{120} + x^{119} + x^{116} + x^{111} + x^{108} + x^{47} + x^{44} \\
&\quad + x^{43} + x^{40} + x^{39} + x^{36} + x^{27} + x^{24} + x^{23} + x^{20} + x^{15} + x^{12}, \\
p_{12}(x) &= x^{127} + x^{124} + x^{123} + x^{120} + x^{107} + x^{104} + x^{99} + x^{96} + x^{47} + x^{44} \\
&\quad + x^{43} + x^{40} + x^{31} + x^{28} + x^{19} + x^{16} + x^{11} + x^{8} + x^{3} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 1, 1, 1, 0, 1].

For $\phi(W^o, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{129} + x^{127} + x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117} \\
&\quad + x^{116} + x^{115} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{107} + x^{103} \\
&\quad + x^{97} + x^{96} + x^{95} + x^{93} + x^{89} + x^{88} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} \\
&\quad + x^{81} + x^{80} + x^{79} + x^{77} + x^{69} + x^{59} + x^{57} + x^{54} + x^{52} + x^{50} + x^{49} \\
&\quad + x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{36}, \\
p_3(x) &= x^{127} + x^{125} + x^{124} + x^{119} + x^{118} + x^{113} + x^{108} + x^{106} + x^{104} + x^{103}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{90} + x^{88}\\
&+ x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71}\\
&+ x^{70} + x^{68} + x^{66} + x^{65} + x^{64} + x^{63} + x^{61} + x^{59} + x^{58} + x^{57} + x^{54}\\
&+ x^{51} + x^{50} + x^{49} + x^{46} + x^{41} + x^{36} + x^{35} + x^{33} + x^{31} + x^{29} + x^{27}\\
&+ x^{26} + x^{24},\\
p_6(x) = {}& x^{127} + x^{125} + x^{124} + x^{119} + x^{118} + x^{117} + x^{115} + x^{112} + x^{111} + x^{110}\\
&+ x^{109} + x^{108} + x^{107} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{95}\\
&+ x^{94} + x^{93} + x^{91} + x^{87} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79}\\
&+ x^{78} + x^{77} + x^{75} + x^{74} + x^{73} + x^{72} + x^{70} + x^{69} + x^{65} + x^{64} + x^{62}\\
&+ x^{56} + x^{52} + x^{51} + x^{48} + x^{47} + x^{46} + x^{45} + x^{42} + x^{41} + x^{40} + x^{39}\\
&+ x^{37} + x^{36} + x^{35} + x^{32} + x^{31} + x^{30} + x^{26} + x^{24} + x^{21} + x^{20} + x^{17}\\
&+ x^{16} + x^{14} + x^{11} + x^{8} + x^{3} + 1,\\
p_9(x) = {}& x^{127} + x^{124} + x^{123} + x^{120} + x^{119} + x^{116} + x^{111} + x^{108} + x^{47} + x^{44}\\
&+ x^{43} + x^{40} + x^{39} + x^{36} + x^{27} + x^{24} + x^{23} + x^{20} + x^{15} + x^{12},\\
p_{12}(x) = {}& x^{127} + x^{124} + x^{123} + x^{120} + x^{107} + x^{104} + x^{99} + x^{96} + x^{47} + x^{44}\\
&+ x^{43} + x^{40} + x^{31} + x^{28} + x^{19} + x^{16} + x^{11} + x^{8} + x^{3} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1].

For $\phi(W^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) = {}& x^{132} + x^{130} + x^{127} + x^{125} + x^{124} + x^{120} + x^{117} + x^{116} + x^{115} + x^{114}\\
&+ x^{113} + x^{112} + x^{108} + x^{107} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101}\\
&+ x^{99} + x^{98} + x^{97} + x^{95} + x^{94} + x^{92} + x^{91} + x^{87} + x^{86} + x^{84} + x^{82}\\
&+ x^{81} + x^{80} + x^{79} + x^{78} + x^{72} + x^{71} + x^{67} + x^{65} + x^{63} + x^{61} + x^{60}\\
&+ x^{59} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{48} + x^{47} + x^{46}\\
&+ x^{43} + x^{40} + x^{39},\\
p_3(x) = {}& x^{130} + x^{129} + x^{125} + x^{124} + x^{123} + x^{122} + x^{121} + x^{119} + x^{118} + x^{116}\\
&+ x^{115} + x^{114} + x^{112} + x^{110} + x^{108} + x^{105} + x^{104} + x^{102} + x^{101}\\
&+ x^{99} + x^{98} + x^{96} + x^{95} + x^{93} + x^{92} + x^{90} + x^{89} + x^{87} + x^{86} + x^{84}\\
&+ x^{83} + x^{81} + x^{80} + x^{78} + x^{77} + x^{75} + x^{73} + x^{72} + x^{71} + x^{67} + x^{66}\\
&+ x^{65} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{55} + x^{54} + x^{52} + x^{51} + x^{50}\\
&+ x^{48} + x^{46} + x^{44} + x^{41} + x^{40} + x^{38} + x^{37} + x^{35} + x^{33} + x^{32} + x^{31}\\
&+ x^{27},\\
p_6(x) = {}& x^{132} + x^{130} + x^{122} + x^{116} + x^{114} + x^{106} + x^{104} + x^{102} + x^{100} + x^{98}\\
&+ x^{96} + x^{92} + x^{86} + x^{84} + x^{74} + x^{68} + x^{66} + x^{64} + x^{62} + x^{60} + x^{54}\\
&+ x^{52} + x^{48} + x^{46} + x^{42} + x^{38} + x^{36} + x^{34} + x^{28} + x^{18},\\
p_9(x) = {}& x^{134} + x^{133} + x^{132} + x^{131} + x^{129} + x^{128} + x^{123} + x^{120} + x^{119} + x^{116}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{111} + x^{110} + x^{109} + x^{105} + x^{54} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48}\\
&+ x^{43} + x^{40} + x^{39} + x^{38} + x^{37} + x^{35} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29}\\
&+ x^{27} + x^{25} + x^{24} + x^{23} + x^{20} + x^{15} + x^{14} + x^{13} + x^{9},\\
p_{12}(x) &= x^{136} + x^{132} + x^{124} + x^{120} + x^{116} + x^{108} + x^{104} + x^{96} + x^{56} + x^{52}\\
&+ x^{44} + x^{20} + x^{16} + x^{12} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0].

For $\phi(W^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{130} + x^{128} + x^{126} + x^{124} + x^{122} + x^{121} + x^{120} + x^{117} + x^{116} + x^{114}\\
&+ x^{109} + x^{108} + x^{107} + x^{103} + x^{101} + x^{100} + x^{98} + x^{96} + x^{95} + x^{94}\\
&+ x^{93} + x^{92} + x^{91} + x^{89} + x^{86} + x^{85} + x^{84} + x^{83} + x^{79} + x^{78} + x^{77}\\
&+ x^{74} + x^{72} + x^{70} + x^{69} + x^{67} + x^{66} + x^{64} + x^{61} + x^{59} + x^{58} + x^{57}\\
&+ x^{54} + x^{53} + x^{52} + x^{51} + x^{47} + x^{42} + x^{41} + x^{40} + x^{39},\\
p_3(x) &= x^{116} + x^{115} + x^{114} + x^{113} + x^{112} + x^{109} + x^{108} + x^{106} + x^{105} + x^{104}\\
&+ x^{101} + x^{100} + x^{98} + x^{97} + x^{96} + x^{93} + x^{91} + x^{84} + x^{83} + x^{82} + x^{81}\\
&+ x^{80} + x^{77} + x^{76} + x^{74} + x^{73} + x^{72} + x^{69} + x^{67} + x^{52} + x^{51} + x^{50}\\
&+ x^{49} + x^{48} + x^{45} + x^{44} + x^{42} + x^{41} + x^{40} + x^{37} + x^{36} + x^{34} + x^{33}\\
&+ x^{32} + x^{29} + x^{27},\\
p_6(x) &= x^{118} + x^{108} + x^{104} + x^{100} + x^{98} + x^{94} + x^{92} + x^{90} + x^{88} + x^{78} + x^{76}\\
&+ x^{72} + x^{66} + x^{64} + x^{58} + x^{56} + x^{54} + x^{48} + x^{46} + x^{44} + x^{42} + x^{40}\\
&+ x^{36} + x^{32} + x^{30} + x^{26} + x^{24} + x^{22} + x^{20} + x^{18},\\
p_9(x) &= x^{120} + x^{119} + x^{115} + x^{114} + x^{113} + x^{112} + x^{111} + x^{110} + x^{107} + x^{105}\\
&+ x^{40} + x^{39} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{27} + x^{25} + x^{24}\\
&+ x^{23} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{11} + x^{9},\\
p_{12}(x) &= x^{122} + x^{120} + x^{102} + x^{100} + x^{98} + x^{96} + x^{42} + x^{40} + x^{26} + x^{24} + x^{22}\\
&+ x^{20} + x^{18} + x^{16} + x^{6} + x^{4} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0].

For $\phi(W^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{129} + x^{121} + x^{116} + x^{115} + x^{113} + x^{112} + x^{111} + x^{109} + x^{104} + x^{101}\\
&+ x^{99} + x^{96} + x^{95} + x^{93} + x^{90} + x^{89} + x^{87} + x^{86} + x^{82} + x^{80} + x^{79}\\
&+ x^{76} + x^{74} + x^{70} + x^{62} + x^{58} + x^{57} + x^{56} + x^{53} + x^{50} + x^{48} + x^{47}\\
&+ x^{46} + x^{45} + x^{42},\\
p_3(x) &= x^{127} + x^{125} + x^{124} + x^{123} + x^{121} + x^{120} + x^{116} + x^{115} + x^{113} + x^{112}\\
&+ x^{111} + x^{106} + x^{104} + x^{101} + x^{95} + x^{93} + x^{92} + x^{90} + x^{89} + x^{88}\\
&+ x^{87} + x^{86} + x^{84} + x^{83} + x^{82} + x^{81} + x^{78} + x^{74} + x^{71} + x^{68} + x^{67}\\
&+ x^{66} + x^{65} + x^{63} + x^{61} + x^{58} + x^{56} + x^{55} + x^{52} + x^{50} + x^{49} + x^{48}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{47}+x^{46}+x^{44}+x^{41}+x^{39}+x^{32}+x^{31}+x^{30}+x^{29}+x^{26},\\
p_6(x)=\;&x^{127}+x^{125}+x^{124}+x^{123}+x^{121}+x^{120}+x^{117}+x^{116}+x^{115}+x^{113}\\
&+x^{112}+x^{106}+x^{105}+x^{98}+x^{96}+x^{95}+x^{94}+x^{93}+x^{91}+x^{85}+x^{83}\\
&+x^{82}+x^{81}+x^{80}+x^{79}+x^{78}+x^{73}+x^{70}+x^{69}+x^{67}+x^{65}+x^{63}\\
&+x^{62}+x^{60}+x^{57}+x^{56}+x^{55}+x^{54}+x^{53}+x^{50}+x^{46}+x^{45}+x^{44}\\
&+x^{43}+x^{42}+x^{39}+x^{38}+x^{36}+x^{34}+x^{33}+x^{28}+x^{26}+x^{24}+x^{23}\\
&+x^{18}+x^{17}+x^{16}+x^{15}+x^{14}+x^{12}+x^{11}+x^{8}+x^{3}+1,\\
p_9(x)=\;&x^{127}+x^{124}+x^{123}+x^{120}+x^{119}+x^{116}+x^{111}+x^{108}+x^{47}+x^{44}\\
&+x^{43}+x^{40}+x^{39}+x^{36}+x^{27}+x^{24}+x^{23}+x^{20}+x^{15}+x^{12},\\
p_{12}(x)=\;&x^{127}+x^{124}+x^{123}+x^{120}+x^{107}+x^{104}+x^{99}+x^{96}+x^{47}+x^{44}\\
&+x^{43}+x^{40}+x^{31}+x^{28}+x^{19}+x^{16}+x^{11}+x^{8}+x^{3}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 1, 1, 1, 0, 1].

Below is the transition function and output function of an 2-automaton that generates $T=M^o_{0,0}$:

Transition function $(n,j)\mapsto\delta(n,j)$ $(\Lambda(n):=[\delta(n,0),\delta(n,1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	178	[303, 304]	356	[477, 478]	534	[400, 634]	712	[11, 801]
1	[3, 4]	179	[305, 217]	357	[479, 71]	535	[626, 376]	713	[183, 639]
2	[5, 6]	180	[306, 307]	358	[480, 300]	536	[180, 125]	714	[346, 651]
3	[7, 8]	181	[222, 308]	359	[249, 481]	537	[401, 45]	715	[628, 643]
4	[9, 10]	182	[309, 310]	360	[482, 483]	538	[429, 630]	716	[156, 150]
5	[11, 12]	183	[311, 277]	361	[231, 484]	539	[133, 378]	717	[802, 792]
6	[13, 14]	184	[312, 313]	362	[485, 486]	540	[438, 149]	718	[803, 610]
7	[15, 16]	185	[88, 255]	363	[261, 487]	541	[594, 413]	719	[804, 805]
8	[17, 18]	186	[314, 315]	364	[88, 488]	542	[118, 650]	720	[670, 355]
9	[19, 20]	187	[316, 317]	365	[489, 490]	543	[616, 651]	721	[421, 371]
10	[21, 22]	188	[318, 319]	366	[491, 281]	544	[652, 653]	722	[673, 663]
11	[23, 24]	189	[81, 320]	367	[492, 493]	545	[654, 351]	723	[641, 640]
12	[25, 26]	190	[321, 322]	368	[245, 482]	546	[145, 160]	724	[350, 623]
13	[27, 28]	191	[323, 324]	369	[494, 324]	547	[655, 656]	725	[646, 164]
14	[29, 30]	192	[233, 325]	370	[77, 495]	548	[657, 405]	726	[662, 49]
15	[31, 32]	193	[326, 327]	371	[496, 497]	549	[373, 658]	727	[369, 403]
16	[33, 34]	194	[328, 329]	372	[498, 499]	550	[180, 338]	728	[631, 792]
17	[35, 36]	195	[330, 331]	373	[500, 237]	551	[402, 119]	729	[604, 348]
18	[37, 38]	196	[332, 333]	374	[501, 200]	552	[35, 42]	730	[605, 408]
19	[39, 40]	197	[334, 334]	375	[502, 503]	553	[192, 194]	731	[332, 799]
20	[41, 42]	198	[335, 336]	376	[504, 505]	554	[604, 406]	732	[802, 417]
21	[43, 40]	199	[337, 338]	377	[273, 230]	555	[632, 659]	733	[614, 806]
22	[33, 44]	200	[339, 340]	378	[506, 507]	556	[402, 144]	734	[129, 643]
23	[45, 46]	201	[134, 152]	379	[21, 508]	557	[188, 172]	735	[185, 162]
24	[47, 48]	202	[341, 115]	380	[214, 111]	558	[599, 409]	736	[406, 603]

25	[49, 8]	203	[151, 135]	381	[509, 506]	559	[660, 661]	737	[642, 659]
26	[50, 51]	204	[342, 144]	382	[510, 511]	560	[662, 156]	738	[785, 618]
27	[52, 53]	205	[343, 344]	383	[512, 106]	561	[609, 663]	739	[807, 431]
28	[54, 55]	206	[339, 57]	384	[62, 513]	562	[653, 664]	740	[423, 625]
29	[56, 57]	207	[345, 131]	385	[472, 514]	563	[665, 408]	741	[435, 185]
30	[58, 59]	208	[338, 126]	386	[515, 516]	564	[666, 667]	742	[666, 808]
31	[60, 61]	209	[346, 347]	387	[517, 518]	565	[620, 409]	743	[809, 801]
32	[62, 63]	210	[348, 349]	388	[519, 520]	566	[619, 668]	744	[810, 169]
33	[64, 65]	211	[350, 351]	389	[521, 522]	567	[177, 635]	745	[811, 362]
34	[66, 67]	212	[191, 352]	390	[510, 523]	568	[669, 156]	746	[179, 649]
35	[68, 69]	213	[353, 173]	391	[220, 524]	569	[670, 400]	747	[622, 651]
36	[70, 71]	214	[354, 355]	392	[525, 253]	570	[633, 439]	748	[418, 406]
37	[72, 73]	215	[356, 192]	393	[526, 527]	571	[42, 189]	749	[812, 675]
38	[74, 75]	216	[357, 44]	394	[318, 528]	572	[163, 647]	750	[670, 139]
39	[76, 77]	217	[358, 154]	395	[529, 530]	573	[9, 131]	751	[353, 192]
40	[78, 79]	218	[359, 360]	396	[531, 532]	574	[364, 671]	752	[379, 432]
41	[80, 81]	219	[357, 34]	397	[282, 451]	575	[168, 672]	753	[414, 50]
42	[82, 65]	220	[159, 159]	398	[533, 534]	576	[673, 610]	754	[794, 653]
43	[83, 84]	221	[162, 146]	399	[535, 476]	577	[348, 603]	755	[813, 408]
44	[85, 86]	222	[161, 45]	400	[536, 315]	578	[674, 675]	756	[797, 814]
45	[87, 88]	223	[159, 55]	401	[537, 278]	579	[154, 176]	757	[152, 172]
46	[89, 28]	224	[361, 362]	402	[538, 539]	580	[39, 633]	758	[803, 663]
47	[90, 91]	225	[363, 131]	403	[540, 541]	581	[395, 193]	759	[815, 347]
48	[92, 93]	226	[364, 365]	404	[542, 543]	582	[343, 668]	760	[783, 610]
49	[94, 95]	227	[366, 367]	405	[544, 545]	583	[160, 401]	761	[640, 640]
50	[16, 96]	228	[368, 37]	406	[498, 546]	584	[660, 676]	762	[407, 606]
51	[97, 98]	229	[369, 144]	407	[547, 79]	585	[648, 180]	763	[358, 373]
52	[99, 100]	230	[40, 370]	408	[548, 549]	586	[375, 627]	764	[816, 664]
53	[101, 102]	231	[171, 371]	409	[550, 539]	587	[126, 431]	765	[817, 351]
54	[54, 54]	232	[359, 38]	410	[519, 551]	588	[31, 188]	766	[128, 160]
55	[103, 104]	233	[116, 331]	411	[552, 553]	589	[117, 174]	767	[818, 808]
56	[105, 18]	234	[372, 373]	412	[554, 555]	590	[677, 678]	768	[136, 819]
57	[106, 107]	235	[374, 34]	413	[556, 557]	591	[679, 680]	769	[433, 370]
58	[108, 109]	236	[375, 376]	414	[558, 557]	592	[509, 681]	770	[152, 820]
59	[110, 111]	237	[377, 378]	415	[559, 560]	593	[94, 682]	771	[821, 189]
60	[112, 113]	238	[379, 124]	416	[73, 561]	594	[683, 106]	772	[641, 641]
61	[114, 115]	239	[380, 381]	417	[562, 563]	595	[512, 684]	773	[674, 796]
62	[116, 117]	240	[128, 146]	418	[467, 80]	596	[253, 290]	774	[822, 58]
63	[118, 119]	241	[335, 48]	419	[564, 297]	597	[685, 686]	775	[784, 413]
64	[120, 121]	242	[382, 157]	420	[203, 565]	598	[687, 88]	776	[787, 823]
65	[59, 122]	243	[383, 122]	421	[566, 270]	599	[688, 689]	777	[800, 405]
66	[123, 124]	244	[331, 174]	422	[567, 568]	600	[690, 691]	778	[621, 805]
67	[125, 126]	245	[384, 385]	423	[569, 533]	601	[692, 693]	779	[349, 418]
68	[127, 128]	246	[342, 119]	424	[458, 570]	602	[514, 548]	780	[812, 796]

69	[129, 130]	247	[185, 128]	425	[230, 571]	603	[197, 267]	781	[751, 552]
70	[131, 132]	248	[386, 387]	426	[572, 573]	604	[677, 694]	782	[824, 825]
71	[133, 51]	249	[143, 113]	427	[574, 575]	605	[695, 696]	783	[768, 826]
72	[134, 135]	250	[372, 154]	428	[462, 96]	606	[697, 698]	784	[286, 275]
73	[136, 137]	251	[388, 180]	429	[16, 576]	607	[699, 700]	785	[827, 4]
74	[138, 139]	252	[132, 158]	430	[577, 578]	608	[701, 702]	786	[828, 829]
75	[135, 140]	253	[45, 160]	431	[218, 579]	609	[325, 703]	787	[731, 830]
76	[112, 141]	254	[162, 160]	432	[580, 509]	610	[704, 705]	788	[831, 832]
77	[142, 139]	255	[159, 54]	433	[477, 581]	611	[473, 706]	789	[246, 777]
78	[143, 141]	256	[161, 145]	434	[582, 243]	612	[707, 266]	790	[833, 443]
79	[118, 144]	257	[361, 389]	435	[583, 280]	613	[464, 708]	791	[248, 240]
80	[145, 146]	258	[390, 124]	436	[584, 585]	614	[514, 329]	792	[834, 835]
81	[147, 148]	259	[391, 365]	437	[586, 587]	615	[495, 709]	793	[80, 699]
82	[149, 150]	260	[392, 393]	438	[588, 558]	616	[710, 711]	794	[321, 454]
83	[151, 152]	261	[356, 173]	439	[513, 589]	617	[198, 467]	795	[836, 246]
84	[153, 154]	262	[369, 119]	440	[590, 409]	618	[712, 713]	796	[732, 837]
85	[155, 156]	263	[358, 175]	441	[332, 591]	619	[229, 714]	797	[463, 838]
86	[157, 158]	264	[394, 385]	442	[592, 412]	620	[467, 715]	798	[839, 530]
87	[54, 159]	265	[138, 355]	443	[377, 51]	621	[716, 457]	799	[840, 841]
88	[160, 161]	266	[395, 137]	444	[593, 131]	622	[63, 717]	800	[842, 697]
89	[162, 46]	267	[185, 145]	445	[594, 381]	623	[718, 719]	801	[843, 312]
90	[163, 164]	268	[396, 148]	446	[595, 178]	624	[720, 506]	802	[844, 845]
91	[165, 124]	269	[131, 157]	447	[363, 10]	625	[502, 553]	803	[846, 718]
92	[166, 167]	270	[397, 398]	448	[596, 597]	626	[721, 722]	804	[847, 206]
93	[168, 169]	271	[170, 359]	449	[598, 367]	627	[723, 724]	805	[715, 699]
94	[170, 37]	272	[35, 115]	450	[599, 191]	628	[725, 726]	806	[848, 849]
95	[171, 137]	273	[399, 333]	451	[40, 439]	629	[727, 728]	807	[721, 533]
96	[32, 172]	274	[354, 400]	452	[357, 600]	630	[729, 730]	808	[329, 549]
97	[173, 174]	275	[37, 360]	453	[409, 352]	631	[731, 732]	809	[103, 850]
98	[175, 176]	276	[46, 401]	454	[146, 185]	632	[733, 63]	810	[851, 278]
99	[177, 178]	277	[45, 146]	455	[386, 601]	633	[201, 696]	811	[273, 852]
100	[179, 180]	278	[401, 162]	456	[592, 121]	634	[734, 735]	812	[853, 481]
101	[181, 182]	279	[160, 185]	457	[126, 398]	635	[736, 737]	813	[854, 581]
102	[183, 184]	280	[46, 161]	458	[33, 600]	636	[588, 556]	814	[855, 856]
103	[161, 162]	281	[401, 145]	459	[602, 405]	637	[583, 491]	815	[266, 759]
104	[46, 185]	282	[402, 403]	460	[603, 604]	638	[738, 54]	816	[699, 320]
105	[186, 187]	283	[404, 405]	461	[605, 606]	639	[739, 740]	817	[857, 204]
106	[188, 140]	284	[406, 349]	462	[607, 154]	640	[247, 741]	818	[441, 296]
107	[36, 189]	285	[407, 408]	463	[374, 608]	641	[240, 735]	819	[858, 859]
108	[190, 191]	286	[190, 409]	464	[609, 610]	642	[742, 319]	820	[860, 769]
109	[192, 174]	287	[47, 336]	465	[611, 611]	643	[743, 744]	821	[861, 587]
110	[171, 193]	288	[410, 411]	466	[612, 613]	644	[745, 746]	822	[213, 110]
111	[173, 194]	289	[56, 340]	467	[334, 55]	645	[727, 109]	823	[830, 837]
112	[72, 74]	290	[52, 130]	468	[435, 161]	646	[219, 747]	824	[862, 611]

113	[195, 196]	291	[381, 57]	469	[614, 615]	647	[748, 749]	825	[863, 796]
114	[197, 198]	292	[8, 50]	470	[616, 347]	648	[306, 750]	826	[657, 782]
115	[199, 200]	293	[114, 42]	471	[115, 617]	649	[751, 502]	827	[864, 676]
116	[201, 202]	294	[120, 412]	472	[384, 433]	650	[752, 753]	828	[166, 865]
117	[203, 204]	295	[413, 57]	473	[55, 334]	651	[754, 755]	829	[366, 866]
118	[205, 206]	296	[113, 38]	474	[147, 618]	652	[688, 301]	830	[117, 194]
119	[207, 208]	297	[136, 193]	475	[432, 594]	653	[480, 688]	831	[141, 867]
120	[105, 209]	298	[152, 140]	476	[413, 340]	654	[756, 441]	832	[373, 176]
121	[210, 211]	299	[149, 414]	477	[330, 117]	655	[63, 589]	833	[156, 414]
122	[109, 212]	300	[415, 336]	478	[142, 355]	656	[757, 719]	834	[664, 794]
123	[213, 214]	301	[127, 162]	479	[619, 344]	657	[696, 758]	835	[868, 613]
124	[215, 216]	302	[416, 417]	480	[435, 127]	658	[268, 80]	836	[818, 667]
125	[217, 218]	303	[418, 348]	481	[437, 34]	659	[759, 709]	837	[821, 617]
126	[219, 218]	304	[419, 351]	482	[620, 621]	660	[556, 760]	838	[135, 172]
127	[220, 221]	305	[368, 359]	483	[385, 439]	661	[761, 762]	839	[337, 125]
128	[222, 223]	306	[420, 188]	484	[622, 347]	662	[215, 763]	840	[155, 149]
129	[224, 225]	307	[421, 137]	485	[349, 604]	663	[764, 765]	841	[338, 625]
130	[226, 227]	308	[422, 148]	486	[419, 623]	664	[68, 766]	842	[395, 819]
131	[228, 229]	309	[410, 423]	487	[190, 621]	665	[767, 514]	843	[822, 428]
132	[20, 230]	310	[424, 425]	488	[422, 618]	666	[768, 769]	844	[437, 608]
133	[231, 232]	311	[426, 427]	489	[49, 624]	667	[770, 771]	845	[815, 651]
134	[233, 234]	312	[424, 122]	490	[625, 398]	668	[772, 773]	846	[342, 650]
135	[83, 235]	313	[428, 59]	491	[426, 601]	669	[580, 720]	847	[382, 132]
136	[236, 237]	314	[429, 430]	492	[158, 122]	670	[741, 734]	848	[117, 869]
137	[238, 239]	315	[397, 431]	493	[624, 50]	671	[774, 29]	849	[804, 352]
138	[240, 241]	316	[123, 432]	494	[626, 627]	672	[686, 775]	850	[426, 387]
139	[242, 243]	317	[132, 59]	495	[141, 38]	673	[441, 470]	851	[644, 2]
140	[234, 244]	318	[394, 433]	496	[388, 337]	674	[776, 73]	852	[404, 782]
141	[245, 246]	319	[434, 333]	497	[125, 625]	675	[777, 778]	853	[655, 814]
142	[247, 248]	320	[146, 161]	498	[128, 435]	676	[779, 780]	854	[870, 806]
143	[249, 250]	321	[145, 435]	499	[628, 130]	677	[146, 127]	855	[433, 871]
144	[251, 252]	322	[436, 387]	500	[124, 594]	678	[396, 618]	856	[139, 634]
145	[253, 254]	323	[432, 380]	501	[629, 630]	679	[781, 180]	857	[872, 823]
146	[255, 256]	324	[59, 425]	502	[186, 333]	680	[414, 377]	858	[345, 382]
147	[257, 258]	325	[437, 44]	503	[631, 417]	681	[602, 782]	859	[157, 59]
148	[259, 260]	326	[438, 156]	504	[603, 418]	682	[354, 139]	860	[138, 790]
149	[261, 262]	327	[150, 377]	505	[632, 613]	683	[41, 36]	861	[124, 380]
150	[263, 97]	328	[153, 373]	506	[633, 370]	684	[783, 663]	862	[741, 240]
151	[264, 265]	329	[433, 439]	507	[355, 634]	685	[636, 784]	863	[873, 196]
152	[76, 266]	330	[78, 440]	508	[607, 373]	686	[50, 378]	864	[452, 874]
153	[267, 268]	331	[60, 441]	509	[421, 193]	687	[785, 786]	865	[875, 876]
154	[269, 270]	332	[442, 443]	510	[420, 32]	688	[127, 145]	866	[876, 877]
155	[271, 272]	333	[444, 445]	511	[399, 591]	689	[415, 48]	867	[878, 830]
156	[19, 273]	334	[23, 103]	512	[399, 187]	690	[593, 382]	868	[879, 235]

157	[274, 232]	335	[446, 447]	513	[590, 621]	691	[150, 50]	869	[880, 881]
158	[229, 275]	336	[448, 449]	514	[374, 44]	692	[181, 597]	870	[542, 881]
159	[276, 277]	337	[305, 219]	515	[595, 635]	693	[13, 393]	871	[882, 838]
160	[278, 253]	338	[450, 451]	516	[390, 636]	694	[334, 54]	872	[73, 75]
161	[254, 279]	339	[452, 97]	517	[637, 167]	695	[787, 788]	873	[870, 615]
162	[280, 281]	340	[275, 453]	518	[638, 639]	696	[434, 591]	874	[416, 792]
163	[282, 283]	341	[454, 455]	519	[43, 633]	697	[789, 417]	875	[636, 883]
164	[284, 285]	342	[456, 457]	520	[599, 621]	698	[790, 791]	876	[625, 431]
165	[228, 286]	343	[458, 459]	521	[640, 641]	699	[55, 55]	877	[883, 413]
166	[255, 287]	344	[460, 461]	522	[642, 613]	700	[436, 601]	878	[114, 821]
167	[288, 289]	345	[15, 462]	523	[41, 115]	701	[781, 337]	879	[872, 788]
168	[290, 279]	346	[463, 464]	524	[52, 643]	702	[380, 413]	880	[590, 804]
169	[291, 292]	347	[465, 466]	525	[644, 48]	703	[789, 792]	881	[113, 360]
170	[195, 293]	348	[267, 467]	526	[645, 378]	704	[793, 794]	882	[153, 798]
171	[294, 295]	349	[454, 468]	527	[411, 625]	705	[795, 796]	883	[884, 570]
172	[61, 296]	350	[469, 325]	528	[341, 42]	706	[55, 54]	884	[607, 175]
173	[264, 297]	351	[470, 471]	529	[629, 430]	707	[797, 656]		
174	[202, 298]	352	[322, 468]	530	[383, 425]	708	[798, 658]		
175	[299, 295]	353	[472, 328]	531	[646, 647]	709	[139, 791]		
176	[300, 301]	354	[473, 474]	532	[648, 649]	710	[434, 799]		
177	[110, 302]	355	[475, 476]	533	[32, 140]	711	[800, 782]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	127	1	254	1	381	1	508	1	635	1	762	0
1	0	128	1	255	1	382	1	509	1	636	0	763	0
2	0	129	0	256	1	383	1	510	1	637	0	764	1
3	0	130	0	257	0	384	0	511	1	638	1	765	0
4	0	131	0	258	1	385	0	512	1	639	1	766	1
5	0	132	0	259	1	386	1	513	1	640	0	767	1
6	0	133	0	260	1	387	0	514	1	641	1	768	0
7	0	134	0	261	1	388	1	515	1	642	0	769	1
8	1	135	1	262	1	389	0	516	1	643	1	770	0
9	0	136	0	263	0	390	1	517	0	644	1	771	1
10	1	137	1	264	1	391	1	518	1	645	1	772	1
11	0	138	1	265	1	392	1	519	1	646	1	773	1
12	1	139	1	266	1	393	1	520	1	647	0	774	0
13	0	140	1	267	0	394	1	521	0	648	1	775	0
14	0	141	0	268	1	395	1	522	0	649	0	776	1
15	0	142	0	269	0	396	1	523	0	650	1	777	1
16	0	143	1	270	0	397	0	524	0	651	0	778	0
17	1	144	1	271	1	398	0	525	1	652	1	779	1
18	0	145	0	272	1	399	1	526	1	653	0	780	0
19	0	146	1	273	1	400	1	527	1	654	1	781	0

20	0	147	0	274	1	401	0	528	1	655	0	782	0
21	1	148	1	275	0	402	0	529	1	656	0	783	0
22	0	149	1	276	1	403	0	530	1	657	0	784	0
23	0	150	0	277	0	404	0	531	1	658	1	785	1
24	0	151	1	278	0	405	1	532	1	659	1	786	1
25	1	152	0	279	0	406	1	533	0	660	0	787	1
26	0	153	0	280	1	407	0	534	1	661	0	788	0
27	0	154	0	281	0	408	0	535	1	662	1	789	1
28	0	155	1	282	0	409	1	536	1	663	1	790	0
29	0	156	0	283	0	410	1	537	0	664	1	791	1
30	0	157	1	284	1	411	1	538	0	665	1	792	1
31	0	158	1	285	0	412	1	539	0	666	0	793	0
32	0	159	1	286	0	413	0	540	0	667	0	794	0
33	0	160	0	287	0	414	1	541	0	668	1	795	1
34	0	161	1	288	1	415	0	542	0	669	0	796	0
35	1	162	1	289	0	416	0	543	0	670	0	797	1
36	0	163	0	290	0	417	0	544	1	671	0	798	0
37	0	164	1	291	1	418	0	545	1	672	0	799	1
38	1	165	0	292	1	419	0	546	0	673	1	800	1
39	0	166	1	293	0	420	1	547	0	674	1	801	0
40	1	167	1	294	0	421	1	548	0	675	1	802	0
41	0	168	0	295	0	422	0	549	1	676	1	803	1
42	1	169	1	296	1	423	0	550	1	677	1	804	1
43	1	170	1	297	0	424	0	551	0	678	1	805	1
44	1	171	0	298	0	425	1	552	1	679	0	806	1
45	0	172	0	299	1	426	0	553	0	680	1	807	1
46	1	173	1	300	0	427	0	554	1	681	1	808	1
47	0	174	1	301	1	428	1	555	1	682	1	809	1
48	1	175	1	302	0	429	0	556	0	683	0	810	1
49	1	176	0	303	0	430	0	557	1	684	0	811	1
50	0	177	0	304	0	431	1	558	1	685	0	812	0
51	1	178	0	305	0	432	0	559	0	686	0	813	0
52	0	179	0	306	1	433	1	560	1	687	1	814	1
53	1	180	1	307	1	434	0	561	0	688	1	815	1
54	0	181	1	308	0	435	0	562	0	689	0	816	1
55	1	182	1	309	1	436	0	563	1	690	1	817	0
56	0	183	0	310	0	437	0	564	0	691	0	818	1
57	1	184	0	311	0	438	0	565	0	692	1	819	0
58	0	185	0	312	0	439	1	566	1	693	0	820	1
59	0	186	0	313	1	440	1	567	0	694	0	821	1
60	0	187	0	314	0	441	1	568	0	695	1	822	0
61	0	188	1	315	0	442	1	569	0	696	0	823	1
62	0	189	0	316	0	443	1	570	0	697	1	824	0
63	0	190	0	317	0	444	1	571	1	698	0	825	0

64	0	191	0	318	1	445	0	572	0	699	1	826	0
65	0	192	0	319	0	446	1	573	0	700	0	827	1
66	0	193	0	320	1	447	1	574	0	701	0	828	1
67	0	194	0	321	0	448	0	575	0	702	1	829	0
68	1	195	1	322	0	449	1	576	1	703	1	830	1
69	0	196	1	323	0	450	1	577	0	704	0	831	0
70	0	197	0	324	0	451	1	578	1	705	1	832	1
71	0	198	1	325	0	452	1	579	0	706	1	833	0
72	0	199	0	326	0	453	1	580	0	707	1	834	1
73	0	200	1	327	0	454	1	581	1	708	0	835	0
74	1	201	0	328	0	455	1	582	0	709	1	836	1
75	1	202	1	329	1	456	1	583	0	710	0	837	1
76	0	203	1	330	1	457	1	584	0	711	1	838	1
77	0	204	1	331	0	458	0	585	1	712	0	839	0
78	1	205	0	332	1	459	1	586	0	713	0	840	1
79	0	206	1	333	1	460	0	587	1	714	1	841	1
80	0	207	0	334	0	461	1	588	0	715	1	842	1
81	0	208	1	335	1	462	1	589	1	716	0	843	0
82	1	209	1	336	0	463	1	590	1	717	0	844	0
83	1	210	0	337	0	464	0	591	0	718	1	845	1
84	0	211	1	338	1	465	1	592	1	719	1	846	1
85	1	212	0	339	1	466	1	593	1	720	0	847	1
86	1	213	0	340	0	467	0	594	0	721	1	848	1
87	0	214	1	341	1	468	0	595	1	722	1	849	1
88	0	215	1	342	1	469	1	596	0	723	1	850	0
89	1	216	1	343	0	470	0	597	0	724	1	851	1
90	0	217	0	344	0	471	0	598	1	725	1	852	0
91	0	218	1	345	0	472	0	599	1	726	1	853	0
92	1	219	1	346	1	473	1	600	1	727	1	854	0
93	0	220	1	347	1	474	0	601	1	728	1	855	1
94	1	221	1	348	0	475	0	602	1	729	1	856	1
95	0	222	1	349	1	476	0	603	0	730	1	857	0
96	0	223	1	350	1	477	1	604	1	731	1	858	0
97	1	224	0	351	0	478	0	605	1	732	0	859	1
98	1	225	1	352	0	479	1	606	1	733	1	860	1
99	0	226	0	353	0	480	0	607	1	734	0	861	1
100	0	227	0	354	1	481	0	608	0	735	0	862	0
101	1	228	0	355	0	482	0	609	0	736	1	863	0
102	0	229	1	356	1	483	0	610	0	737	0	864	1
103	1	230	1	357	1	484	0	611	1	738	1	865	0
104	1	231	0	358	0	485	1	612	1	739	1	866	0
105	0	232	1	359	1	486	0	613	0	740	0	867	0
106	1	233	0	360	0	487	0	614	1	741	0	868	0
107	0	234	1	361	0	488	0	615	0	742	0	869	1

108	0	235	1	362	1	489	1	616	0	743	1	870	0
109	0	236	0	363	1	490	0	617	1	744	1	871	0
110	0	237	1	364	0	491	0	618	0	745	1	872	0
111	1	238	1	365	1	492	1	619	1	746	0	873	0
112	0	239	1	366	0	493	0	620	0	747	0	874	0
113	1	240	1	367	1	494	1	621	0	748	0	875	0
114	0	241	1	368	0	495	0	622	0	749	0	876	0
115	0	242	1	369	1	496	1	623	1	750	0	877	1
116	0	243	1	370	0	497	0	624	0	751	0	878	0
117	1	244	0	371	1	498	1	625	0	752	1	879	0
118	0	245	0	372	1	499	1	626	1	753	1	880	1
119	0	246	1	373	1	500	1	627	1	754	0	881	1
120	0	247	0	374	1	501	1	628	1	755	0	882	0
121	0	248	1	375	0	502	0	629	1	756	1	883	1
122	0	249	1	376	0	503	1	630	1	757	0	884	1
123	0	250	1	377	1	504	0	631	1	758	1		
124	1	251	1	378	0	505	1	632	1	759	1		
125	0	252	0	379	1	506	0	633	0	760	0		
126	1	253	0	380	1	507	0	634	0	761	0		

Université de Strasbourg, CNRS, IRMA UMR 7501, F-67000 Strasbourg, France
Email address: guoniu.han@unistra.fr

School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan, PR China
Email address: huyining@hust.edu.cn