PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a, b) = (z^6 + z^5 + z^3 + z^2, z)$

GUO-NIU HAN* AND YINING HU*

Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a, b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a, b) in $\mathbb{F}_2[x] \setminus \mathbb{F}_2$. In this paper we prove the conjecture for $(a, b) = (z^6 + z^5 + z^3 + z^2, z)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \ldots)$ of elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{\ddots + \cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a, b)-Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \ldots) = (a, b, b, a, b, a, a, b, \ldots)$$

that we shall denote

$$\mathrm{CF}(a, b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When* $(a,b) = (z^6+z^5+z^3+z^2, z)$*, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4 + p_3(z)y^3 + p_2(z)y^2 + p_1(z)y + p_0(z) = 0$$

where, for $\mathrm{CF}(a,b)$*,*

$$\begin{aligned}
p_0(z) &= z^{17}+z^{16}+z^{14}+z^{12}+z^{11}+z^{10}+z^9+z^6+z^5+z^4+z^3+z+1,\\
p_1(z) &= z^{23}+z^{21}+z^{19}+z^{18}+z^{15}+z^{13}+z^{12}+z^{10}+z^9+z^6+z^4+z^2,\\
p_2(z) &= z^{24}+z^{22}+z^{20}+z^{18}+z^8+z^4,\\
p_3(z) &= z^{23}+z^{21}+z^{19}+z^{18}+z^{15}+z^{13}+z^{12}+z^{10}+z^9+z^6+z^4+z^2,\\
p_4(z) &= z^{22}+z^{20}+z^{18}+z^{17}+z^{14}+z^{11}+z^{10}+z^9+z^8+z^6+z^5+z^4+z^3+z\\
&\quad,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$*,*

$$\begin{aligned}
p_0(z) &= z^{20}+z^{17}+z^{14}+z^{12}+z^{11}+z^{10}+z^9+z^8+z^6+z^5+z^3+z,\\
p_1(z) &= z^{23}+z^{21}+z^{19}+z^{18}+z^{15}+z^{13}+z^{12}+z^{10}+z^9+z^6+z^4+z^2,\\
p_2(z) &= z^{24}+z^{22}+z^{20}+z^{18}+z^8+z^4,\\
p_3(z) &= z^{23}+z^{21}+z^{19}+z^{18}+z^{15}+z^{13}+z^{12}+z^{10}+z^9+z^6+z^4+z^2,\\
p_4(z) &= z^{22}+z^{18}+z^{17}+z^{16}+z^{14}+z^{11}+z^{10}+z^9+z^6+z^5+z^3+z+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\ge 0}$ and $(Q_n(z))_{n\ge 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} \in \mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x) &= x^{\deg(t_{2^n-1})}\begin{pmatrix} t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix} \times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad \cdots \times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x) &= x^{\deg(\bar t_{2^n-1})}\begin{pmatrix} \bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix} \times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix} \bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad \cdots \times x^{\deg(\bar t_0)}\begin{pmatrix} \bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\ge 0$

$$\begin{aligned}
M_{n+1}(x) &= W_n(x)\cdot M_n(x),\\
W_{n+1}(x) &= M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(175, 175, 175, 175, 175)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z \left(\phi(W^o,0,0)(x,z),\ z^{12}\cdot\phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 630 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 630, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z\left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$ to order 770 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 770, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. □

Define

$$R^e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned}
M^e[:14u] = {} & M^e[:7u] + x^u M^e[:6u] + x^{2u} M^e[:5u] + x^u M^e[:7u] + x^{2u} M^e[:6u] \\
& + x^{3u} M^e[:5u] + x^{3u} M^e[:7u] + x^{4u} M^e[:6u] + x^{5u} M^e[:5u] \\
& + x^{4u} M^e[:7u] + x^{5u} M^e[:6u] + x^{6u} M^e[:5u] + x^{6u} M^e[:7u] \\
& + x^{7u} M^e[:6u] + x^{8u} M^e[:5u] + x^{7u} M^e[:7u] + x^{12u} M^e[:2u] \\
& + (x^{7u} + x^{8u} + x^{10u} + x^{11u} + x^{13u}) R^e, \\
W^e[:14u] = {} & W^e[:7u] + x^u W^e[:6u] + x^{2u} W^e[:5u] + x^u W^e[:7u] + x^{2u} W^e[:6u]
\end{aligned}$$

$$\begin{aligned}
&+ x^{3u}W^e[:5u] + x^{3u}W^e[:7u] + x^{4u}W^e[:6u] + x^{5u}W^e[:5u] \\
&+ x^{4u}W^e[:7u] + x^{5u}W^e[:6u] + x^{6u}W^e[:5u] + x^{6u}W^e[:7u] \\
&+ x^{7u}W^e[:6u] + x^{8u}W^e[:5u] + x^{7u}W^e[:7u] + x^{12u}W^e[:2u] \\
&+ (x^{7u} + x^{8u} + x^{10u} + x^{11u} + x^{13u})R^e.
\end{aligned}$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$\begin{aligned}
M^o[:14u] = {}& M^o[:7u] + x^u M^o[:6u] + x^{2u} M^o[:5u] + x^u M^o[:7u] + x^{2u} M^o[:6u] \\
&+ x^{3u} M^o[:5u] + x^{3u} M^o[:7u] + x^{4u} M^o[:6u] + x^{5u} M^o[:5u] \\
&+ x^{4u} M^o[:7u] + x^{5u} M^o[:6u] + x^{6u} M^o[:5u] + x^{6u} M^o[:7u] \\
&+ x^{7u} M^o[:6u] + x^{8u} M^o[:5u] + x^{7u} M^o[:7u] + x^{12u} M^o[:2u] \\
&+ (x^{7u} + x^{8u} + x^{10u} + x^{11u} + x^{13u})R^o, \\
W^o[:14u] = {}& W^o[:7u] + x^u W^o[:6u] + x^{2u} W^o[:5u] + x^u W^o[:7u] + x^{2u} W^o[:6u] \\
&+ x^{3u} W^o[:5u] + x^{3u} W^o[:7u] + x^{4u} W^o[:6u] + x^{5u} W^o[:5u] \\
&+ x^{4u} W^o[:7u] + x^{5u} W^o[:6u] + x^{6u} W^o[:5u] + x^{6u} W^o[:7u] \\
&+ x^{7u} W^o[:6u] + x^{8u} W^o[:5u] + x^{7u} W^o[:7u] + x^{12u} W^o[:2u] \\
&+ (x^{7u} + x^{8u} + x^{10u} + x^{11u} + x^{13u})R^o.
\end{aligned}$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$\begin{aligned}
T[:14u] = {}& T[:7u] + x^u T[:6u] + x^{2u} T[:5u] + x^u T[:7u] + x^{2u} T[:6u] + x^{3u} T[:5u] \\
&+ x^{3u} T[:7u] + x^{4u} T[:6u] + x^{5u} T[:5u] + x^{4u} T[:7u] + x^{5u} T[:6u] \\
&+ x^{6u} T[:5u] + x^{6u} T[:7u] + x^{7u} T[:6u] + x^{8u} T[:5u] + x^{7u} T[:7u] \\
&+ x^{12u} T[:2u].
\end{aligned}$$

The proof of the other 15 cases are similar. We break down the above identity into 7 parts:

$$\begin{aligned}
0 &= x^{2u}T[5u:6u] + x^u T[6u:7u] + T[7u:8u], \\
0 &= x^{8u}T[:u] + x^{3u}T[5u:6u] + T[8u:9u], \\
0 &= x^{8u}T[u:2u] + x^{3u}T[6u:7u] + T[9u:10u], \\
0 &= x^{8u}T[2u:3u] + x^{5u}T[5u:6u] + x^{4u}T[6u:7u] + T[10u:11u], \\
0 &= x^{8u}T[3u:4u] + x^{6u}T[5u:6u] + T[11u:12u], \\
0 &= x^{12u}T[:u] + x^{8u}T[4u:5u] + x^{6u}T[6u:7u] + T[12u:13u], \\
0 &= x^{12u}T[u:2u] + x^{7u}T[6u:7u] + T[13u:14u],
\end{aligned}$$

which can be rewritten as

$$0 = T[[101w]_2] + T[[110w]_2] + T[[111w]_2], \tag{2.7}$$

$$0 = T[[w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.8}$$

$$0 = T[[1w]_2] + T[[110w]_2] + T[[1001w]_2], \tag{2.9}$$

$$0 = T[[10w]_2] + T[[101w]_2] + T[[110w]_2] + T[[1010w]_2], \tag{2.10}$$
$$0 = T[[11w]_2] + T[[101w]_2] + T[[1011w]_2], \tag{2.11}$$
$$0 = T[[w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1100w]_2], \tag{2.12}$$
$$0 = T[[1w]_2] + T[[110w]_2] + T[[1101w]_2], \tag{2.13}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$0 = T[[101w]_2] + T[[110w]_2] + T[[111w]_2], \tag{2.14}$$
$$0 = T[[w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.15}$$
$$0 = T[[1w]_2] + T[[110w]_2] + T[[1001w]_2], \tag{2.16}$$
$$0 = T[[10w]_2] + T[[101w]_2] + T[[110w]_2] + T[[1010w]_2], \tag{2.17}$$
$$0 = T[[11w]_2] + T[[101w]_2] + T[[1011w]_2], \tag{2.18}$$
$$0 = T[[w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1100w]_2], \tag{2.19}$$
$$0 = T[[1w]_2] + T[[110w]_2] + T[[1101w]_2], \tag{2.20}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 866 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.20) can be written as

$$0 = \tau(A(s, 101)) + \tau(A(s, 110)) + \tau(A(s, 111)), \tag{2.21}$$
$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 101)) + \tau(A(s, 1000)), \tag{2.22}$$
$$0 = \tau(A(s, 1)) + \tau(A(s, 110)) + \tau(A(s, 1001)), \tag{2.23}$$
$$0 = \tau(A(s, 10)) + \tau(A(s, 101)) + \tau(A(s, 110)) + \tau(A(s, 1010)), \tag{2.24}$$
$$0 = \tau(A(s, 11)) + \tau(A(s, 101)) + \tau(A(s, 1011)), \tag{2.25}$$
$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 100)) + \tau(A(s, 110)) + \tau(A(s, 1100)), \tag{2.26}$$
$$0 = \tau(A(s, 1)) + \tau(A(s, 110)) + \tau(A(s, 1101)), \tag{2.27}$$

for all $s \in E_{2k}$, and

$$0 = \tau(A(s, 101)) + \tau(A(s, 110)) + \tau(A(s, 111)), \tag{2.28}$$
$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 101)) + \tau(A(s, 1000)), \tag{2.29}$$
$$0 = \tau(A(s, 1)) + \tau(A(s, 110)) + \tau(A(s, 1001)), \tag{2.30}$$
$$0 = \tau(A(s, 10)) + \tau(A(s, 101)) + \tau(A(s, 110)) + \tau(A(s, 1010)), \tag{2.31}$$
$$0 = \tau(A(s, 11)) + \tau(A(s, 101)) + \tau(A(s, 1011)), \tag{2.32}$$
$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 100)) + \tau(A(s, 110)) + \tau(A(s, 1100)), \tag{2.33}$$
$$0 = \tau(A(s, 1)) + \tau(A(s, 110)) + \tau(A(s, 1101)), \tag{2.34}$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{28}), E_{28}) = (A(i, 0^{16}), E_{16})$, so that we only have to verify that identities (2.21) through (2.34) hold for $2 \leq k \leq 14$, which turns out to be true. $\square$

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \geq 2$, and $u = 2^{2k-1}$,*

$$M_{2k} = M^e[:7u] + x^u M^e[:6u] + x^{2u} M^e[:5u] + x^{7u} R^e,$$
$$W_{2k} = W^e[:7u] + x^u W^e[:6u] + x^{2u} W^e[:5u] + x^{7u} R^e.$$

For all integer $k \geq 2$, and $u = 2^{2k}$,

$$M_{2k+1} = M^o[:7u] + x^u M^o[:6u] + x^{2u} M^o[:5u] + x^{7u} R^o,$$
$$W_{2k+1} = W^o[:7u] + x^u W^o[:6u] + x^{2u} W^o[:5u] + x^{7u} R^o.$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \geq 2$, we claim that

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1} \wedge W_{2k+1},$$
$$M_{2k+1} \wedge W_{2k+1} \Rightarrow M_{2k+2} \wedge W_{2k+2}.$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$\begin{aligned} W_{2k} M_{2k} = \big(W^e[:7v] + x^v W^e[:6v] + x^{2v} W^e[:5v] + x^{7u} R^e\big) \times \big(M^e[:7v] \\ + x^v M^e[:6v] + x^{2v} M^e[:5v] + x^{7u} R^e\big); \end{aligned} \tag{2.35}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{14v} R^o$. Therefore we only need to prove that their difference is $O(x^{14v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{14v}, to

$$W^e[:14v] M^e[:14v] + x^{2v} W^e[:12v] M^e[:12v] + x^{4v} W^e[:10v] M^e[:10v].$$

For all $n \leq 14$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.35) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{14}, b_1, ..., b_{14}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{14})$, which completes the proof. $\square$

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \leq j, k \leq 1$,

$$\lim_{n \to \infty} M_{2n,j,k} = M^e_{j,k},$$

$$\lim_{n\to\infty} M_{2n+1,j,k} = M^o_{j,k},$$
$$\lim_{n\to\infty} W_{2n,j,k} = W^e_{j,k},$$
$$\lim_{n\to\infty} W_{2n+1,j,k} = W^o_{j,k}.$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a,b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x,y) = \mathrm{Res}_t\left(\phi(M^e,0,1)(x,t), y^{12}\phi(M^e,0,1)(x,t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x,y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned}
q_0(x) &= x^{24} + x^{23} + x^{21} + x^{20} + x^{19} + x^{18} + x^{15} + x^{14} + x^{13} + x^{12} + x^{10} + x^8 \\
&\quad + x^7, \\
q_1(x) &= x^{22} + x^{20} + x^{18} + x^{15} + x^{14} + x^{12} + x^{11} + x^9 + x^6 + x^5 + x^3 + x, \\
q_2(x) &= x^{20} + x^{16} + x^6 + x^4 + x^2 + 1, \\
q_3(x) &= x^{22} + x^{20} + x^{18} + x^{15} + x^{14} + x^{12} + x^{11} + x^9 + x^6 + x^5 + x^3 + x, \\
q_4(x) &= x^{23} + x^{21} + x^{20} + x^{19} + x^{18} + x^{16} + x^{15} + x^{14} + x^{13} + x^{10} + x^7 + x^6 \\
&\quad + x^4 + x^2.
\end{aligned}$$

The polynomial $Q(x,y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and $R(x,y) := P(x,y)/Q(x,y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 216, and substitute it for y in $R(z,y)$, we get a series with valuation smaller than 216, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b) = f(1/z)$. $\square$

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{114} + x^{112} + x^{110} + x^{108} + x^{106} + x^{104} + x^{96} + x^{94} + x^{92} + x^{88} \\
&\quad + x^{84} + x^{74} + x^{72} + x^{70} + x^{66} + x^{64} + x^{62} + x^{60} + x^{54} + x^{50} + x^{48} \\
&\quad + x^{46} + x^{44} + x^{38} + x^{34} + x^{30} + x^{26} + x^{22} + x^{16} + x^{14} + x^{12}, \\
p_3(x) &= x^{106} + x^{102} + x^{100} + x^{98} + x^{96} + x^{94} + x^{86} + x^{84} + x^{78} + x^{72} + x^{70}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{66} + x^{64} + x^{62} + x^{60} + x^{58} + x^{56} + x^{52} + x^{50} + x^{44} + x^{42} + x^{38}\\
&+ x^{22} + x^{20} + x^{18} + x^{14} + x^{12} + x^{8},\\
p_6(x) &= x^{106} + x^{94} + x^{92} + x^{90} + x^{86} + x^{80} + x^{74} + x^{72} + x^{68} + x^{62} + x^{54}\\
&+ x^{52} + x^{50} + x^{48} + x^{44} + x^{42} + x^{40} + x^{36} + x^{34} + x^{30} + x^{26} + x^{22}\\
&+ x^{20} + x^{18} + x^{16} + x^{14} + x^{12} + x^{10} + x^{8} + 1,\\
p_9(x) &= x^{104} + x^{102} + x^{100} + x^{96} + x^{94} + x^{92} + x^{86} + x^{84} + x^{80} + x^{78} + x^{74}\\
&+ x^{62} + x^{60} + x^{58} + x^{56} + x^{54} + x^{52} + x^{46} + x^{44} + x^{42} + x^{38} + x^{36}\\
&+ x^{34} + x^{30} + x^{28} + x^{20} + x^{16} + x^{12} + x^{8} + x^{2},\\
p_{12}(x) &= x^{104} + x^{102} + x^{98} + x^{96} + x^{88} + x^{86} + x^{82} + x^{80} + x^{56} + x^{54} + x^{50}\\
&+ x^{48} + x^{40} + x^{38} + x^{34} + x^{32} + x^{8} + x^{6} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0].

For $\phi(M^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{129} + x^{127} + x^{126} + x^{122} + x^{119} + x^{118} + x^{116} + x^{110} + x^{109} + x^{107}\\
&+ x^{105} + x^{103} + x^{100} + x^{99} + x^{96} + x^{93} + x^{92} + x^{91} + x^{89} + x^{88} + x^{87}\\
&+ x^{86} + x^{85} + x^{84} + x^{83} + x^{80} + x^{79} + x^{78} + x^{76} + x^{75} + x^{73} + x^{69}\\
&+ x^{66} + x^{64} + x^{63} + x^{61} + x^{60} + x^{57} + x^{55} + x^{53} + x^{51} + x^{47} + x^{46}\\
&+ x^{44} + x^{43} + x^{40} + x^{38} + x^{37} + x^{36} + x^{34} + x^{33} + x^{32} + x^{31} + x^{29}\\
&+ x^{27},\\
p_3(x) &= x^{122} + x^{121} + x^{120} + x^{116} + x^{114} + x^{113} + x^{108} + x^{107} + x^{105} + x^{104}\\
&+ x^{102} + x^{99} + x^{98} + x^{97} + x^{90} + x^{89} + x^{88} + x^{84} + x^{82} + x^{81} + x^{76}\\
&+ x^{75} + x^{74} + x^{70} + x^{68} + x^{67} + x^{60} + x^{59} + x^{58} + x^{54} + x^{52} + x^{51}\\
&+ x^{50} + x^{49} + x^{48} + x^{43} + x^{41} + x^{38} + x^{34} + x^{30} + x^{25} + x^{24} + x^{22}\\
&+ x^{20} + x^{18} + x^{16} + x^{14} + x^{11} + x^{10} + x^{9},\\
p_6(x) &= x^{120} + x^{119} + x^{117} + x^{116} + x^{106} + x^{105} + x^{103} + x^{101} + x^{99} + x^{97}\\
&+ x^{95} + x^{93} + x^{91} + x^{89} + x^{87} + x^{85} + x^{84} + x^{80} + x^{79} + x^{77} + x^{76}\\
&+ x^{70} + x^{69} + x^{67} + x^{65} + x^{64} + x^{62} + x^{61} + x^{59} + x^{58} + x^{56} + x^{55}\\
&+ x^{54} + x^{42} + x^{41} + x^{39} + x^{38} + x^{36} + x^{35} + x^{34} + x^{30} + x^{29} + x^{27}\\
&+ x^{26} + x^{24} + x^{23} + x^{21} + x^{19} + x^{17} + x^{16} + x^{10} + x^{9} + x^{7} + x^{6},\\
p_9(x) &= x^{118} + x^{117} + x^{116} + x^{114} + x^{113} + x^{110} + x^{109} + x^{108} + x^{104} + x^{102}\\
&+ x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{93} + x^{92} + x^{91} + x^{86} + x^{85} + x^{83}\\
&+ x^{82} + x^{80} + x^{77} + x^{74} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{66} + x^{65}\\
&+ x^{64} + x^{63} + x^{62} + x^{57} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{46}\\
&+ x^{42} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{34} + x^{33} + x^{32} + x^{31} + x^{29}\\
&+ x^{27} + x^{26} + x^{25} + x^{18} + x^{17} + x^{16} + x^{14} + x^{13} + x^{10} + x^{9} + x^{7} + x^{5}\\
&+ x^{4} + x^{3},
\end{aligned}
$$

$$\begin{aligned}
p_{12}(x) = {} & x^{116}+x^{115}+x^{113}+x^{111}+x^{109}+x^{108}+x^{104}+x^{103}+x^{101}+x^{100}\\
& +x^{96}+x^{95}+x^{93}+x^{92}+x^{88}+x^{87}+x^{85}+x^{83}+x^{81}+x^{80}+x^{68}\\
& +x^{67}+x^{65}+x^{63}+x^{61}+x^{60}+x^{56}+x^{55}+x^{53}+x^{52}+x^{48}+x^{47}\\
& +x^{45}+x^{44}+x^{40}+x^{39}+x^{37}+x^{35}+x^{33}+x^{32}+x^{20}+x^{19}+x^{17}\\
& +x^{15}+x^{13}+x^{12}+x^{8}+x^{7}+x^{5}+x^{3}+x+1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 1, 1, 1, 0].

For $\phi(M^e,1,0)$:

$$\begin{aligned}
p_{0}(x) = {} & x^{129}+x^{127}+x^{126}+x^{122}+x^{119}+x^{118}+x^{116}+x^{110}+x^{109}+x^{107}\\
& +x^{105}+x^{103}+x^{100}+x^{99}+x^{96}+x^{93}+x^{92}+x^{91}+x^{89}+x^{88}+x^{87}\\
& +x^{86}+x^{85}+x^{84}+x^{83}+x^{80}+x^{79}+x^{78}+x^{76}+x^{75}+x^{73}+x^{69}\\
& +x^{66}+x^{64}+x^{63}+x^{61}+x^{60}+x^{57}+x^{55}+x^{53}+x^{51}+x^{47}+x^{46}\\
& +x^{44}+x^{43}+x^{40}+x^{38}+x^{37}+x^{36}+x^{34}+x^{33}+x^{32}+x^{31}+x^{29}\\
& +x^{27},\\
p_{3}(x) = {} & x^{122}+x^{121}+x^{120}+x^{116}+x^{114}+x^{113}+x^{108}+x^{107}+x^{105}+x^{104}\\
& +x^{102}+x^{99}+x^{98}+x^{97}+x^{90}+x^{89}+x^{88}+x^{84}+x^{82}+x^{81}+x^{76}\\
& +x^{75}+x^{74}+x^{70}+x^{68}+x^{67}+x^{60}+x^{59}+x^{58}+x^{54}+x^{52}+x^{51}\\
& +x^{50}+x^{49}+x^{48}+x^{43}+x^{41}+x^{38}+x^{34}+x^{30}+x^{25}+x^{24}+x^{22}\\
& +x^{20}+x^{18}+x^{16}+x^{14}+x^{11}+x^{10}+x^{9},\\
p_{6}(x) = {} & x^{120}+x^{119}+x^{117}+x^{116}+x^{106}+x^{105}+x^{103}+x^{101}+x^{99}+x^{97}\\
& +x^{95}+x^{93}+x^{91}+x^{89}+x^{87}+x^{85}+x^{84}+x^{80}+x^{79}+x^{77}+x^{76}\\
& +x^{70}+x^{69}+x^{67}+x^{65}+x^{64}+x^{62}+x^{61}+x^{59}+x^{58}+x^{56}+x^{55}\\
& +x^{54}+x^{42}+x^{41}+x^{39}+x^{38}+x^{36}+x^{35}+x^{34}+x^{30}+x^{29}+x^{27}\\
& +x^{26}+x^{24}+x^{23}+x^{21}+x^{19}+x^{17}+x^{16}+x^{10}+x^{9}+x^{7}+x^{6},\\
p_{9}(x) = {} & x^{118}+x^{117}+x^{116}+x^{114}+x^{113}+x^{110}+x^{109}+x^{108}+x^{104}+x^{102}\\
& +x^{101}+x^{100}+x^{99}+x^{97}+x^{96}+x^{93}+x^{92}+x^{91}+x^{86}+x^{85}+x^{83}\\
& +x^{82}+x^{80}+x^{77}+x^{74}+x^{72}+x^{71}+x^{70}+x^{69}+x^{68}+x^{66}+x^{65}\\
& +x^{64}+x^{63}+x^{62}+x^{57}+x^{55}+x^{54}+x^{53}+x^{52}+x^{51}+x^{50}+x^{46}\\
& +x^{42}+x^{40}+x^{39}+x^{38}+x^{37}+x^{36}+x^{34}+x^{33}+x^{32}+x^{31}+x^{29}\\
& +x^{27}+x^{26}+x^{25}+x^{18}+x^{17}+x^{16}+x^{14}+x^{13}+x^{10}+x^{9}+x^{7}+x^{5}\\
& +x^{4}+x^{3},
\end{aligned}$$

$$\begin{aligned}
p_{12}(x) = {} & x^{116}+x^{115}+x^{113}+x^{111}+x^{109}+x^{108}+x^{104}+x^{103}+x^{101}+x^{100}\\
& +x^{96}+x^{95}+x^{93}+x^{92}+x^{88}+x^{87}+x^{85}+x^{83}+x^{81}+x^{80}+x^{68}\\
& +x^{67}+x^{65}+x^{63}+x^{61}+x^{60}+x^{56}+x^{55}+x^{53}+x^{52}+x^{48}+x^{47}\\
& +x^{45}+x^{44}+x^{40}+x^{39}+x^{37}+x^{35}+x^{33}+x^{32}+x^{20}+x^{19}+x^{17}\\
& +x^{15}+x^{13}+x^{12}+x^{8}+x^{7}+x^{5}+x^{3}+x+1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 1, 1, 1, 0].

For $\phi(M^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{144} + x^{142} + x^{140} + x^{136} + x^{132} + x^{126} + x^{124} + x^{122} + x^{120} + x^{114} \\
&\quad + x^{112} + x^{108} + x^{98} + x^{88} + x^{86} + x^{84} + x^{82} + x^{80} + x^{68} + x^{66} + x^{64} \\
&\quad + x^{56} + x^{46} + x^{44} + x^{42}, \\
p_3(x) &= x^{132} + x^{130} + x^{120} + x^{118} + x^{112} + x^{106} + x^{104} + x^{98} + x^{96} + x^{92} \\
&\quad + x^{90} + x^{88} + x^{82} + x^{80} + x^{74} + x^{68} + x^{62} + x^{60} + x^{58} + x^{52} + x^{50} \\
&\quad + x^{48} + x^{46} + x^{42} + x^{40} + x^{34} + x^{30} + x^{24} + x^{22} + x^{20} + x^{8} + x^{6}, \\
p_6(x) &= x^{132} + x^{130} + x^{128} + x^{126} + x^{122} + x^{106} + x^{98} + x^{96} + x^{94} + x^{90} \\
&\quad + x^{86} + x^{76} + x^{70} + x^{68} + x^{64} + x^{62} + x^{60} + x^{54} + x^{52} + x^{50} + x^{48} \\
&\quad + x^{46} + x^{42} + x^{38} + x^{36} + x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{24} + x^{12} \\
&\quad + x^{8} + x^{6} + x^{2} + 1, \\
p_9(x) &= x^{128} + x^{124} + x^{120} + x^{116} + x^{110} + x^{104} + x^{102} + x^{100} + x^{98} + x^{94} \\
&\quad + x^{92} + x^{90} + x^{88} + x^{84} + x^{80} + x^{74} + x^{70} + x^{66} + x^{56} + x^{54} + x^{52} \\
&\quad + x^{48} + x^{44} + x^{42} + x^{40} + x^{32} + x^{30} + x^{26} + x^{20} + x^{18} + x^{14} + x^{12} \\
&\quad + x^{8} + x^{6} + x^{4} + x^{2}, \\
p_{12}(x) &= x^{128} + x^{120} + x^{112} + x^{96} + x^{88} + x^{72} + x^{64} + x^{48} + x^{40} + x^{24} + x^{8} + 1 \\
&\quad ,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0].

For $\phi(W^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{144} + x^{142} + x^{140} + x^{128} + x^{120} + x^{118} + x^{116} + x^{114} + x^{110} + x^{106} \\
&\quad + x^{98} + x^{92} + x^{90} + x^{82} + x^{78} + x^{76} + x^{74} + x^{70} + x^{64} + x^{52} + x^{44} \\
&\quad + x^{42} + x^{36} + x^{34} + x^{32} + x^{24} + x^{22} + x^{20} + x^{12}, \\
p_3(x) &= x^{132} + x^{130} + x^{124} + x^{118} + x^{110} + x^{108} + x^{106} + x^{104} + x^{100} + x^{96} \\
&\quad + x^{92} + x^{90} + x^{86} + x^{84} + x^{78} + x^{74} + x^{70} + x^{64} + x^{62} + x^{58} + x^{54} \\
&\quad + x^{50} + x^{40} + x^{36} + x^{32} + x^{28} + x^{26} + x^{24} + x^{22} + x^{18} + x^{16} + x^{8}, \\
p_6(x) &= x^{132} + x^{130} + x^{128} + x^{126} + x^{124} + x^{122} + x^{112} + x^{110} + x^{108} + x^{106} \\
&\quad + x^{100} + x^{96} + x^{94} + x^{92} + x^{90} + x^{82} + x^{80} + x^{78} + x^{68} + x^{64} + x^{62} \\
&\quad + x^{56} + x^{50} + x^{38} + x^{36} + x^{32} + x^{20} + x^{18} + x^{16} + x^{12} + x^{8} + x^{4} + x^{2} \\
&\quad + 1, \\
p_9(x) &= x^{128} + x^{124} + x^{120} + x^{116} + x^{110} + x^{104} + x^{102} + x^{100} + x^{98} + x^{94} \\
&\quad + x^{92} + x^{90} + x^{88} + x^{84} + x^{80} + x^{74} + x^{70} + x^{66} + x^{56} + x^{54} + x^{52} \\
&\quad + x^{48} + x^{44} + x^{42} + x^{40} + x^{32} + x^{30} + x^{26} + x^{20} + x^{18} + x^{14} + x^{12} \\
&\quad + x^{8} + x^{6} + x^{4} + x^{2}, \\
p_{12}(x) &= x^{128} + x^{120} + x^{112} + x^{96} + x^{88} + x^{72} + x^{64} + x^{48} + x^{40} + x^{24} + x^{8} + 1 \\
&\quad ,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0].

For $\phi(W^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{129} + x^{127} + x^{125} + x^{124} + x^{123} + x^{122} + x^{119} + x^{117} + x^{110} + x^{104} \\
&+ x^{100} + x^{98} + x^{97} + x^{96} + x^{93} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{76} \\
&+ x^{74} + x^{73} + x^{71} + x^{69} + x^{67} + x^{64} + x^{62} + x^{61} + x^{60} + x^{57} + x^{56} \\
&+ x^{55} + x^{52} + x^{50} + x^{44} + x^{43} + x^{40} + x^{34} + x^{32} + x^{31} + x^{29} + x^{26} \\
&+ x^{24} + x^{23} + x^{19} + x^{18}, \\
p_3(x) &= x^{122} + x^{121} + x^{120} + x^{116} + x^{114} + x^{113} + x^{108} + x^{107} + x^{105} + x^{104} \\
&+ x^{102} + x^{99} + x^{98} + x^{97} + x^{90} + x^{89} + x^{88} + x^{84} + x^{82} + x^{81} + x^{76} \\
&+ x^{75} + x^{74} + x^{70} + x^{68} + x^{67} + x^{60} + x^{59} + x^{58} + x^{54} + x^{52} + x^{51} \\
&+ x^{50} + x^{49} + x^{48} + x^{43} + x^{41} + x^{38} + x^{34} + x^{30} + x^{25} + x^{24} + x^{22} \\
&+ x^{20} + x^{18} + x^{16} + x^{14} + x^{11} + x^{10} + x^{9}, \\
p_6(x) &= x^{120} + x^{119} + x^{117} + x^{116} + x^{106} + x^{105} + x^{103} + x^{101} + x^{99} + x^{97} \\
&+ x^{95} + x^{93} + x^{91} + x^{89} + x^{87} + x^{85} + x^{84} + x^{80} + x^{79} + x^{77} + x^{76} \\
&+ x^{70} + x^{69} + x^{67} + x^{65} + x^{64} + x^{62} + x^{61} + x^{59} + x^{58} + x^{56} + x^{55} \\
&+ x^{54} + x^{42} + x^{41} + x^{39} + x^{38} + x^{36} + x^{35} + x^{34} + x^{30} + x^{29} + x^{27} \\
&+ x^{26} + x^{24} + x^{23} + x^{21} + x^{19} + x^{17} + x^{16} + x^{10} + x^{9} + x^{7} + x^{6}, \\
p_9(x) &= x^{118} + x^{117} + x^{116} + x^{114} + x^{113} + x^{110} + x^{109} + x^{108} + x^{104} + x^{102} \\
&+ x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{93} + x^{92} + x^{91} + x^{86} + x^{85} + x^{83} \\
&+ x^{82} + x^{80} + x^{77} + x^{74} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{66} + x^{65} \\
&+ x^{64} + x^{63} + x^{62} + x^{57} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{46} \\
&+ x^{42} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{34} + x^{33} + x^{32} + x^{31} + x^{29} \\
&+ x^{27} + x^{26} + x^{25} + x^{18} + x^{17} + x^{16} + x^{14} + x^{13} + x^{10} + x^{9} + x^{7} + x^{5} \\
&+ x^{4} + x^{3}, \\
p_{12}(x) &= x^{116} + x^{115} + x^{113} + x^{111} + x^{109} + x^{108} + x^{104} + x^{103} + x^{101} + x^{100} \\
&+ x^{96} + x^{95} + x^{93} + x^{92} + x^{88} + x^{87} + x^{85} + x^{83} + x^{81} + x^{80} + x^{68} \\
&+ x^{67} + x^{65} + x^{63} + x^{61} + x^{60} + x^{56} + x^{55} + x^{53} + x^{52} + x^{48} + x^{47} \\
&+ x^{45} + x^{44} + x^{40} + x^{39} + x^{37} + x^{35} + x^{33} + x^{32} + x^{20} + x^{19} + x^{17} \\
&+ x^{15} + x^{13} + x^{12} + x^{8} + x^{7} + x^{5} + x^{3} + x + 1,
\end{aligned}$$

and the initial terms are [0, 1, 0, 0, 0, 1, 0, 0, 1, 1, 1, 0, 1, 0, 0, 0].

For $\phi(W^e, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{129} + x^{127} + x^{125} + x^{124} + x^{123} + x^{122} + x^{119} + x^{117} + x^{110} + x^{104} \\
&+ x^{100} + x^{98} + x^{97} + x^{96} + x^{93} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{76} \\
&+ x^{74} + x^{73} + x^{71} + x^{69} + x^{67} + x^{64} + x^{62} + x^{61} + x^{60} + x^{57} + x^{56} \\
&+ x^{55} + x^{52} + x^{50} + x^{44} + x^{43} + x^{40} + x^{34} + x^{32} + x^{31} + x^{29} + x^{26}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{24} + x^{23} + x^{19} + x^{18},\\
p_3(x) &= x^{122} + x^{121} + x^{120} + x^{116} + x^{114} + x^{113} + x^{108} + x^{107} + x^{105} + x^{104}\\
&+ x^{102} + x^{99} + x^{98} + x^{97} + x^{90} + x^{89} + x^{88} + x^{84} + x^{82} + x^{81} + x^{76}\\
&+ x^{75} + x^{74} + x^{70} + x^{68} + x^{67} + x^{60} + x^{59} + x^{58} + x^{54} + x^{52} + x^{51}\\
&+ x^{50} + x^{49} + x^{48} + x^{43} + x^{41} + x^{38} + x^{34} + x^{30} + x^{25} + x^{24} + x^{22}\\
&+ x^{20} + x^{18} + x^{16} + x^{14} + x^{11} + x^{10} + x^{9},\\
p_6(x) &= x^{120} + x^{119} + x^{117} + x^{116} + x^{106} + x^{105} + x^{103} + x^{101} + x^{99} + x^{97}\\
&+ x^{95} + x^{93} + x^{91} + x^{89} + x^{87} + x^{85} + x^{84} + x^{80} + x^{79} + x^{77} + x^{76}\\
&+ x^{70} + x^{69} + x^{67} + x^{65} + x^{64} + x^{62} + x^{61} + x^{59} + x^{58} + x^{56} + x^{55}\\
&+ x^{54} + x^{42} + x^{41} + x^{39} + x^{38} + x^{36} + x^{35} + x^{34} + x^{30} + x^{29} + x^{27}\\
&+ x^{26} + x^{24} + x^{23} + x^{21} + x^{19} + x^{17} + x^{16} + x^{10} + x^{9} + x^{7} + x^{6},\\
p_9(x) &= x^{118} + x^{117} + x^{116} + x^{114} + x^{113} + x^{110} + x^{109} + x^{108} + x^{104} + x^{102}\\
&+ x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{93} + x^{92} + x^{91} + x^{86} + x^{85} + x^{83}\\
&+ x^{82} + x^{80} + x^{77} + x^{74} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{66} + x^{65}\\
&+ x^{64} + x^{63} + x^{62} + x^{57} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{46}\\
&+ x^{42} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{34} + x^{33} + x^{32} + x^{31} + x^{29}\\
&+ x^{27} + x^{26} + x^{25} + x^{18} + x^{17} + x^{16} + x^{14} + x^{13} + x^{10} + x^{9} + x^{7} + x^{5}\\
&+ x^{4} + x^{3},\\
p_{12}(x) &= x^{116} + x^{115} + x^{113} + x^{111} + x^{109} + x^{108} + x^{104} + x^{103} + x^{101} + x^{100}\\
&+ x^{96} + x^{95} + x^{93} + x^{92} + x^{88} + x^{87} + x^{85} + x^{83} + x^{81} + x^{80} + x^{68}\\
&+ x^{67} + x^{65} + x^{63} + x^{61} + x^{60} + x^{56} + x^{55} + x^{53} + x^{52} + x^{48} + x^{47}\\
&+ x^{45} + x^{44} + x^{40} + x^{39} + x^{37} + x^{35} + x^{33} + x^{32} + x^{20} + x^{19} + x^{17}\\
&+ x^{15} + x^{13} + x^{12} + x^{8} + x^{7} + x^{5} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 0, 0, 0, 1, 0, 0, 1, 1, 1, 0, 1, 0, 0, 0].

For $\phi(W^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{114} + x^{112} + x^{110} + x^{104} + x^{98} + x^{94} + x^{92} + x^{86} + x^{80} + x^{74} + x^{72}\\
&+ x^{68} + x^{66} + x^{60} + x^{58} + x^{54} + x^{50} + x^{38} + x^{36} + x^{34} + x^{32} + x^{26}\\
&+ x^{24},\\
p_3(x) &= x^{106} + x^{102} + x^{94} + x^{88} + x^{86} + x^{84} + x^{80} + x^{78} + x^{76} + x^{74} + x^{64}\\
&+ x^{60} + x^{58} + x^{56} + x^{54} + x^{52} + x^{42} + x^{40} + x^{36} + x^{34} + x^{30} + x^{26}\\
&+ x^{22} + x^{18} + x^{14} + x^{6},\\
p_6(x) &= x^{106} + x^{100} + x^{98} + x^{92} + x^{88} + x^{82} + x^{78} + x^{76} + x^{74} + x^{72} + x^{70}\\
&+ x^{64} + x^{60} + x^{58} + x^{54} + x^{46} + x^{36} + x^{32} + x^{30} + x^{28} + x^{24} + x^{16}\\
&+ x^{4} + 1,\\
p_9(x) &= x^{104} + x^{102} + x^{100} + x^{96} + x^{94} + x^{92} + x^{86} + x^{84} + x^{80} + x^{78} + x^{74}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{62} + x^{60} + x^{58} + x^{56} + x^{54} + x^{52} + x^{46} + x^{44} + x^{42} + x^{38} + x^{36}\\
&+ x^{34} + x^{30} + x^{28} + x^{20} + x^{16} + x^{12} + x^{8} + x^{2},\\
p_{12}(x) &= x^{104} + x^{102} + x^{98} + x^{96} + x^{88} + x^{86} + x^{82} + x^{80} + x^{56} + x^{54} + x^{50}\\
&+ x^{48} + x^{40} + x^{38} + x^{34} + x^{32} + x^{8} + x^{6} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0].

For $\phi(M^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{129} + x^{127} + x^{126} + x^{125} + x^{124} + x^{122} + x^{121} + x^{119} + x^{116} + x^{113}\\
&+ x^{112} + x^{111} + x^{110} + x^{107} + x^{105} + x^{102} + x^{99} + x^{97} + x^{92} + x^{90}\\
&+ x^{87} + x^{86} + x^{84} + x^{83} + x^{79} + x^{78} + x^{77} + x^{72} + x^{62} + x^{61} + x^{60}\\
&+ x^{59} + x^{54} + x^{53} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{44} + x^{43} + x^{42}\\
&+ x^{41} + x^{39} + x^{38} + x^{36} + x^{33} + x^{29} + x^{28} + x^{25} + x^{23} + x^{20} + x^{19}\\
&+ x^{18} + x^{17} + x^{16} + x^{14} + x^{13} + x^{12},\\
p_3(x) &= x^{124} + x^{121} + x^{120} + x^{119} + x^{114} + x^{113} + x^{112} + x^{110} + x^{108} + x^{107}\\
&+ x^{106} + x^{104} + x^{103} + x^{100} + x^{99} + x^{98} + x^{96} + x^{95} + x^{93} + x^{92}\\
&+ x^{91} + x^{90} + x^{89} + x^{85} + x^{84} + x^{80} + x^{79} + x^{76} + x^{73} + x^{70} + x^{69}\\
&+ x^{67} + x^{66} + x^{62} + x^{59} + x^{58} + x^{57} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50}\\
&+ x^{49} + x^{48} + x^{47} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{38}\\
&+ x^{37} + x^{36} + x^{35} + x^{32} + x^{29} + x^{27} + x^{25} + x^{23} + x^{22} + x^{20} + x^{19}\\
&+ x^{18} + x^{13} + x^{12} + x^{10} + x^{9} + x^{8},\\
p_6(x) &= x^{124} + x^{121} + x^{120} + x^{119} + x^{115} + x^{114} + x^{113} + x^{110} + x^{109} + x^{108}\\
&+ x^{107} + x^{103} + x^{102} + x^{101} + x^{98} + x^{96} + x^{94} + x^{91} + x^{88} + x^{87}\\
&+ x^{86} + x^{84} + x^{83} + x^{82} + x^{79} + x^{78} + x^{77} + x^{76} + x^{74} + x^{71} + x^{67}\\
&+ x^{65} + x^{64} + x^{63} + x^{61} + x^{60} + x^{58} + x^{55} + x^{52} + x^{51} + x^{50} + x^{47}\\
&+ x^{46} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39} + x^{34} + x^{32} + x^{31} + x^{30} + x^{29}\\
&+ x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{22} + x^{21} + x^{19} + x^{17} + x^{16} + x^{14}\\
&+ x^{10} + x^{8} + x^{7} + x^{6} + x^{4} + x^{3} + x + 1,\\
p_9(x) &= x^{124} + x^{123} + x^{121} + x^{119} + x^{117} + x^{115} + x^{113} + x^{112} + x^{100} + x^{99}\\
&+ x^{97} + x^{96} + x^{92} + x^{91} + x^{89} + x^{87} + x^{85} + x^{84} + x^{76} + x^{75} + x^{73}\\
&+ x^{71} + x^{69} + x^{67} + x^{65} + x^{64} + x^{52} + x^{51} + x^{49} + x^{48} + x^{44} + x^{43}\\
&+ x^{41} + x^{39} + x^{37} + x^{36} + x^{28} + x^{27} + x^{25} + x^{23} + x^{21} + x^{19} + x^{17}\\
&+ x^{16} + x^{12} + x^{11} + x^{9} + x^{7} + x^{5} + x^{4},\\
p_{12}(x) &= x^{124} + x^{123} + x^{121} + x^{119} + x^{117} + x^{116} + x^{112} + x^{111} + x^{109} + x^{108}\\
&+ x^{104} + x^{103} + x^{101} + x^{99} + x^{97} + x^{95} + x^{93} + x^{91} + x^{89} + x^{88} + x^{84}\\
&+ x^{83} + x^{81} + x^{80} + x^{76} + x^{75} + x^{73} + x^{71} + x^{69} + x^{68} + x^{64} + x^{63}\\
&+ x^{61} + x^{60} + x^{56} + x^{55} + x^{53} + x^{51} + x^{49} + x^{47} + x^{45} + x^{43} + x^{41}
\end{aligned}
$$

$$+ x^{40} + x^{36} + x^{35} + x^{33} + x^{32} + x^{28} + x^{27} + x^{25} + x^{23} + x^{21} + x^{20} + x^{16} + x^{15} + x^{13} + x^{11} + x^{9} + x^{8} + x^{4} + x^{3} + x + 1,$$

and the initial terms are [1, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 1, 1, 1, 1, 1].

For $\phi(M^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{144} + x^{142} + x^{139} + x^{137} + x^{134} + x^{133} + x^{132} + x^{130} + x^{128} + x^{120} \\
&+ x^{118} + x^{116} + x^{113} + x^{110} + x^{109} + x^{107} + x^{104} + x^{101} + x^{99} + x^{95} \\
&+ x^{92} + x^{90} + x^{89} + x^{88} + x^{87} + x^{85} + x^{84} + x^{83} + x^{82} + x^{79} + x^{78} \\
&+ x^{74} + x^{73} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{64} + x^{63} + x^{59} + x^{57} \\
&+ x^{54} + x^{53} + x^{49} + x^{48} + x^{46} + x^{41} + x^{39} + x^{37} + x^{35} + x^{34} + x^{33} \\
&+ x^{31} + x^{29} + x^{28} + x^{27}, \\
p_3(x) &= x^{130} + x^{128} + x^{124} + x^{123} + x^{120} + x^{118} + x^{117} + x^{115} + x^{114} + x^{112} \\
&+ x^{111} + x^{109} + x^{108} + x^{105} + x^{104} + x^{103} + x^{102} + x^{98} + x^{97} + x^{96} \\
&+ x^{92} + x^{91} + x^{88} + x^{86} + x^{85} + x^{83} + x^{79} + x^{77} + x^{75} + x^{73} + x^{71} \\
&+ x^{69} + x^{67} + x^{65} + x^{63} + x^{61} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{53} \\
&+ x^{52} + x^{49} + x^{48} + x^{47} + x^{46} + x^{41} + x^{34} + x^{32} + x^{28} + x^{27} + x^{26} \\
&+ x^{22} + x^{21} + x^{20} + x^{16} + x^{15} + x^{14} + x^{9}, \\
p_6(x) &= x^{132} + x^{128} + x^{120} + x^{118} + x^{116} + x^{106} + x^{96} + x^{92} + x^{88} + x^{84} \\
&+ x^{82} + x^{80} + x^{74} + x^{68} + x^{66} + x^{64} + x^{62} + x^{58} + x^{56} + x^{50} + x^{48} \\
&+ x^{46} + x^{34} + x^{30} + x^{28} + x^{26} + x^{24} + x^{22} + x^{18} + x^{16} + x^{10} + x^{6}, \\
p_9(x) &= x^{134} + x^{132} + x^{130} + x^{127} + x^{123} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117} \\
&+ x^{115} + x^{114} + x^{111} + x^{110} + x^{109} + x^{106} + x^{103} + x^{101} + x^{100} \\
&+ x^{94} + x^{93} + x^{91} + x^{90} + x^{89} + x^{88} + x^{86} + x^{84} + x^{83} + x^{82} + x^{79} \\
&+ x^{75} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{67} + x^{66} + x^{63} + x^{62} + x^{61} \\
&+ x^{58} + x^{55} + x^{53} + x^{52} + x^{46} + x^{45} + x^{43} + x^{42} + x^{41} + x^{40} + x^{38} \\
&+ x^{36} + x^{35} + x^{34} + x^{31} + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{19} \\
&+ x^{14} + x^{13} + x^{11} + x^{10} + x^{9} + x^{8} + x^{3}, \\
p_{12}(x) &= x^{136} + x^{128} + x^{120} + x^{112} + x^{108} + x^{104} + x^{100} + x^{96} + x^{92} + x^{84} \\
&+ x^{72} + x^{64} + x^{60} + x^{56} + x^{52} + x^{48} + x^{44} + x^{36} + x^{12} + x^{8} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 0, 1, 1, 1, 1].

For $\phi(M^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{122} + x^{119} + x^{117} + x^{116} + x^{114} + x^{112} + x^{109} + x^{108} + x^{105} + x^{103} \\
&+ x^{101} + x^{98} + x^{97} + x^{96} + x^{93} + x^{92} + x^{89} + x^{88} + x^{87} + x^{83} + x^{81} \\
&+ x^{80} + x^{78} + x^{77} + x^{75} + x^{74} + x^{73} + x^{72} + x^{70} + x^{65} + x^{63} + x^{59} \\
&+ x^{56} + x^{53} + x^{51} + x^{49} + x^{48} + x^{47} + x^{45} + x^{42} + x^{41} + x^{40} + x^{39} \\
&+ x^{34} + x^{33} + x^{25} + x^{22} + x^{21} + x^{20} + x^{18},
\end{aligned}$$

$$\begin{aligned}
p_3(x) &= x^{114} + x^{112} + x^{110} + x^{107} + x^{106} + x^{104} + x^{103} + x^{102} + x^{101} + x^{97}\\
&\quad + x^{82} + x^{80} + x^{78} + x^{75} + x^{74} + x^{72} + x^{71} + x^{70} + x^{69} + x^{66} + x^{65}\\
&\quad + x^{64} + x^{62} + x^{59} + x^{58} + x^{56} + x^{55} + x^{54} + x^{53} + x^{50} + x^{49} + x^{48}\\
&\quad + x^{46} + x^{43} + x^{39} + x^{37} + x^{35} + x^{33} + x^{31} + x^{29} + x^{27} + x^{25} + x^{23}\\
&\quad + x^{21} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{9},\\
p_6(x) &= x^{116} + x^{102} + x^{98} + x^{94} + x^{90} + x^{86} + x^{82} + x^{80} + x^{78} + x^{74} + x^{72}\\
&\quad + x^{70} + x^{68} + x^{64} + x^{58} + x^{52} + x^{50} + x^{48} + x^{46} + x^{44} + x^{42} + x^{40}\\
&\quad + x^{36} + x^{34} + x^{26} + x^{20} + x^{16} + x^{6},\\
p_9(x) &= x^{118} + x^{116} + x^{112} + x^{111} + x^{108} + x^{106} + x^{105} + x^{104} + x^{103} + x^{102}\\
&\quad + x^{100} + x^{99} + x^{96} + x^{95} + x^{92} + x^{90} + x^{89} + x^{88} + x^{87} + x^{83} + x^{70}\\
&\quad + x^{68} + x^{64} + x^{63} + x^{60} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{52} + x^{51}\\
&\quad + x^{48} + x^{47} + x^{44} + x^{42} + x^{41} + x^{40} + x^{39} + x^{35} + x^{22} + x^{20} + x^{16}\\
&\quad + x^{15} + x^{12} + x^{10} + x^{9} + x^{8} + x^{7} + x^{3},\\
p_{12}(x) &= x^{120} + x^{116} + x^{108} + x^{100} + x^{96} + x^{92} + x^{88} + x^{80} + x^{72} + x^{68} + x^{60}\\
&\quad + x^{52} + x^{48} + x^{44} + x^{40} + x^{32} + x^{24} + x^{20} + x^{12} + x^{8} + 1,
\end{aligned}$$

and the initial terms are [0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0].

For $\phi(M^o, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{129} + x^{127} + x^{123} + x^{120} + x^{116} + x^{115} + x^{113} + x^{111} + x^{109} + x^{103}\\
&\quad + x^{102} + x^{101} + x^{100} + x^{99} + x^{97} + x^{94} + x^{91} + x^{90} + x^{89} + x^{87} + x^{86}\\
&\quad + x^{85} + x^{82} + x^{81} + x^{80} + x^{79} + x^{76} + x^{75} + x^{74} + x^{72} + x^{71} + x^{70}\\
&\quad + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{62} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54}\\
&\quad + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{45} + x^{42} + x^{41} + x^{39}\\
&\quad + x^{37} + x^{36} + x^{33},\\
p_3(x) &= x^{124} + x^{121} + x^{117} + x^{115} + x^{114} + x^{110} + x^{108} + x^{107} + x^{105} + x^{103}\\
&\quad + x^{100} + x^{99} + x^{97} + x^{95} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87}\\
&\quad + x^{84} + x^{83} + x^{81} + x^{79} + x^{76} + x^{74} + x^{71} + x^{70} + x^{62} + x^{59} + x^{55}\\
&\quad + x^{54} + x^{52} + x^{49} + x^{44} + x^{41} + x^{38} + x^{36} + x^{33} + x^{32} + x^{31} + x^{24}\\
&\quad + x^{23} + x^{22} + x^{20} + x^{19} + x^{18} + x^{13} + x^{12},\\
p_6(x) &= x^{124} + x^{121} + x^{117} + x^{114} + x^{111} + x^{110} + x^{108} + x^{106} + x^{104} + x^{102}\\
&\quad + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{89} + x^{88} + x^{86} + x^{85}\\
&\quad + x^{83} + x^{82} + x^{79} + x^{78} + x^{77} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70}\\
&\quad + x^{69} + x^{64} + x^{62} + x^{61} + x^{59} + x^{55} + x^{54} + x^{53} + x^{52} + x^{50} + x^{49}\\
&\quad + x^{48} + x^{47} + x^{46} + x^{41} + x^{39} + x^{38} + x^{37} + x^{36} + x^{33} + x^{31} + x^{27}\\
&\quad + x^{24} + x^{23} + x^{22} + x^{20} + x^{19} + x^{17} + x^{16} + x^{14} + x^{9} + x^{7} + x^{5} + x^{3}\\
&\quad + x + 1,
\end{aligned}$$

$$\begin{aligned}
p_9(x) &= x^{124} + x^{123} + x^{121} + x^{119} + x^{117} + x^{115} + x^{113} + x^{112} + x^{100} + x^{99} \\
&\quad + x^{97} + x^{96} + x^{92} + x^{91} + x^{89} + x^{87} + x^{85} + x^{84} + x^{76} + x^{75} + x^{73} \\
&\quad + x^{71} + x^{69} + x^{67} + x^{65} + x^{64} + x^{52} + x^{51} + x^{49} + x^{48} + x^{44} + x^{43} \\
&\quad + x^{41} + x^{39} + x^{37} + x^{36} + x^{28} + x^{27} + x^{25} + x^{23} + x^{21} + x^{19} + x^{17} \\
&\quad + x^{16} + x^{12} + x^{11} + x^{9} + x^{7} + x^{5} + x^{4}, \\
p_{12}(x) &= x^{124} + x^{123} + x^{121} + x^{119} + x^{117} + x^{116} + x^{112} + x^{111} + x^{109} + x^{108} \\
&\quad + x^{104} + x^{103} + x^{101} + x^{99} + x^{97} + x^{95} + x^{93} + x^{91} + x^{89} + x^{88} + x^{84} \\
&\quad + x^{83} + x^{81} + x^{80} + x^{76} + x^{75} + x^{73} + x^{71} + x^{69} + x^{68} + x^{64} + x^{63} \\
&\quad + x^{61} + x^{60} + x^{56} + x^{55} + x^{53} + x^{51} + x^{49} + x^{47} + x^{45} + x^{43} + x^{41} \\
&\quad + x^{40} + x^{36} + x^{35} + x^{33} + x^{32} + x^{28} + x^{27} + x^{25} + x^{23} + x^{21} + x^{20} \\
&\quad + x^{16} + x^{15} + x^{13} + x^{11} + x^{9} + x^{8} + x^{4} + x^{3} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 0, 1, 0, 1].

For $\phi(W^o, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{129} + x^{127} + x^{126} + x^{125} + x^{124} + x^{122} + x^{121} + x^{119} + x^{116} + x^{113} \\
&\quad + x^{112} + x^{111} + x^{110} + x^{107} + x^{105} + x^{102} + x^{99} + x^{97} + x^{92} + x^{90} \\
&\quad + x^{87} + x^{86} + x^{84} + x^{83} + x^{79} + x^{78} + x^{77} + x^{72} + x^{62} + x^{61} + x^{60} \\
&\quad + x^{59} + x^{54} + x^{53} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{44} + x^{43} + x^{42} \\
&\quad + x^{41} + x^{39} + x^{38} + x^{36} + x^{33} + x^{29} + x^{28} + x^{25} + x^{23} + x^{20} + x^{19} \\
&\quad + x^{18} + x^{17} + x^{16} + x^{14} + x^{13} + x^{12}, \\
p_3(x) &= x^{124} + x^{121} + x^{120} + x^{119} + x^{114} + x^{113} + x^{112} + x^{110} + x^{108} + x^{107} \\
&\quad + x^{106} + x^{104} + x^{103} + x^{100} + x^{99} + x^{98} + x^{96} + x^{95} + x^{93} + x^{92} \\
&\quad + x^{91} + x^{90} + x^{89} + x^{85} + x^{84} + x^{80} + x^{79} + x^{76} + x^{73} + x^{70} + x^{69} \\
&\quad + x^{67} + x^{66} + x^{62} + x^{59} + x^{58} + x^{57} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} \\
&\quad + x^{49} + x^{48} + x^{47} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{38} \\
&\quad + x^{37} + x^{36} + x^{35} + x^{32} + x^{29} + x^{27} + x^{25} + x^{23} + x^{22} + x^{20} + x^{19} \\
&\quad + x^{18} + x^{13} + x^{12} + x^{10} + x^{9} + x^{8}, \\
p_6(x) &= x^{124} + x^{121} + x^{120} + x^{119} + x^{115} + x^{114} + x^{113} + x^{110} + x^{109} + x^{108} \\
&\quad + x^{107} + x^{103} + x^{102} + x^{101} + x^{98} + x^{96} + x^{94} + x^{91} + x^{88} + x^{87} \\
&\quad + x^{86} + x^{84} + x^{83} + x^{82} + x^{79} + x^{78} + x^{77} + x^{76} + x^{74} + x^{71} + x^{67} \\
&\quad + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} + x^{58} + x^{55} + x^{52} + x^{51} + x^{50} + x^{47} \\
&\quad + x^{46} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39} + x^{34} + x^{32} + x^{31} + x^{30} + x^{29} \\
&\quad + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{22} + x^{21} + x^{19} + x^{17} + x^{16} + x^{14} \\
&\quad + x^{10} + x^{8} + x^{7} + x^{6} + x^{4} + x^{3} + x + 1, \\
p_9(x) &= x^{124} + x^{123} + x^{121} + x^{119} + x^{117} + x^{115} + x^{113} + x^{112} + x^{100} + x^{99} \\
&\quad + x^{97} + x^{96} + x^{92} + x^{91} + x^{89} + x^{87} + x^{85} + x^{84} + x^{76} + x^{75} + x^{73}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{71} + x^{69} + x^{67} + x^{65} + x^{64} + x^{52} + x^{51} + x^{49} + x^{48} + x^{44} + x^{43}\\
&+ x^{41} + x^{39} + x^{37} + x^{36} + x^{28} + x^{27} + x^{25} + x^{23} + x^{21} + x^{19} + x^{17}\\
&+ x^{16} + x^{12} + x^{11} + x^{9} + x^{7} + x^{5} + x^{4},\\
p_{12}(x) = &\; x^{124} + x^{123} + x^{121} + x^{119} + x^{117} + x^{116} + x^{112} + x^{111} + x^{109} + x^{108}\\
&+ x^{104} + x^{103} + x^{101} + x^{99} + x^{97} + x^{95} + x^{93} + x^{91} + x^{89} + x^{88} + x^{84}\\
&+ x^{83} + x^{81} + x^{80} + x^{76} + x^{75} + x^{73} + x^{71} + x^{69} + x^{68} + x^{64} + x^{63}\\
&+ x^{61} + x^{60} + x^{56} + x^{55} + x^{53} + x^{51} + x^{49} + x^{47} + x^{45} + x^{43} + x^{41}\\
&+ x^{40} + x^{36} + x^{35} + x^{33} + x^{32} + x^{28} + x^{27} + x^{25} + x^{23} + x^{21} + x^{20}\\
&+ x^{16} + x^{15} + x^{13} + x^{11} + x^{9} + x^{8} + x^{4} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 1, 1, 1, 1, 1].

For $\phi(W^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) = &\; x^{122} + x^{119} + x^{117} + x^{116} + x^{114} + x^{112} + x^{109} + x^{108} + x^{105} + x^{103}\\
&+ x^{101} + x^{98} + x^{97} + x^{96} + x^{93} + x^{92} + x^{89} + x^{88} + x^{87} + x^{83} + x^{81}\\
&+ x^{80} + x^{78} + x^{77} + x^{75} + x^{74} + x^{73} + x^{72} + x^{70} + x^{65} + x^{63} + x^{59}\\
&+ x^{56} + x^{53} + x^{51} + x^{49} + x^{48} + x^{47} + x^{45} + x^{42} + x^{41} + x^{40} + x^{39}\\
&+ x^{34} + x^{33} + x^{25} + x^{22} + x^{21} + x^{20} + x^{18},\\
p_3(x) = &\; x^{114} + x^{112} + x^{110} + x^{107} + x^{106} + x^{104} + x^{103} + x^{102} + x^{101} + x^{97}\\
&+ x^{82} + x^{80} + x^{78} + x^{75} + x^{74} + x^{72} + x^{71} + x^{70} + x^{69} + x^{66} + x^{65}\\
&+ x^{64} + x^{62} + x^{59} + x^{58} + x^{56} + x^{55} + x^{54} + x^{53} + x^{50} + x^{49} + x^{48}\\
&+ x^{46} + x^{43} + x^{39} + x^{37} + x^{35} + x^{33} + x^{31} + x^{29} + x^{27} + x^{25} + x^{23}\\
&+ x^{21} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{9},\\
p_6(x) = &\; x^{116} + x^{102} + x^{98} + x^{94} + x^{90} + x^{86} + x^{82} + x^{80} + x^{78} + x^{74} + x^{72}\\
&+ x^{70} + x^{68} + x^{64} + x^{58} + x^{52} + x^{50} + x^{48} + x^{46} + x^{44} + x^{42} + x^{40}\\
&+ x^{36} + x^{34} + x^{26} + x^{20} + x^{16} + x^{6},\\
p_9(x) = &\; x^{118} + x^{116} + x^{112} + x^{111} + x^{108} + x^{106} + x^{105} + x^{104} + x^{103} + x^{102}\\
&+ x^{100} + x^{99} + x^{96} + x^{95} + x^{92} + x^{90} + x^{89} + x^{88} + x^{87} + x^{83} + x^{70}\\
&+ x^{68} + x^{64} + x^{63} + x^{60} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{52} + x^{51}\\
&+ x^{48} + x^{47} + x^{44} + x^{42} + x^{41} + x^{40} + x^{39} + x^{35} + x^{22} + x^{20} + x^{16}\\
&+ x^{15} + x^{12} + x^{10} + x^{9} + x^{8} + x^{7} + x^{3},\\
p_{12}(x) = &\; x^{120} + x^{116} + x^{108} + x^{100} + x^{96} + x^{92} + x^{88} + x^{80} + x^{72} + x^{68} + x^{60}\\
&+ x^{52} + x^{48} + x^{44} + x^{40} + x^{32} + x^{24} + x^{20} + x^{12} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0].

For $\phi(W^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) = &\; x^{144} + x^{142} + x^{139} + x^{137} + x^{134} + x^{133} + x^{132} + x^{130} + x^{128} + x^{120}\\
&+ x^{118} + x^{116} + x^{113} + x^{110} + x^{109} + x^{107} + x^{104} + x^{101} + x^{99} + x^{95}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{92} + x^{90} + x^{89} + x^{88} + x^{87} + x^{85} + x^{84} + x^{83} + x^{82} + x^{79} + x^{78} \\
&+ x^{74} + x^{73} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{64} + x^{63} + x^{59} + x^{57} \\
&+ x^{54} + x^{53} + x^{49} + x^{48} + x^{46} + x^{41} + x^{39} + x^{37} + x^{35} + x^{34} + x^{33} \\
&+ x^{31} + x^{29} + x^{28} + x^{27}, \\
p_3(x) &= x^{130} + x^{128} + x^{124} + x^{123} + x^{120} + x^{118} + x^{117} + x^{115} + x^{114} + x^{112} \\
&+ x^{111} + x^{109} + x^{108} + x^{105} + x^{104} + x^{103} + x^{102} + x^{98} + x^{97} + x^{96} \\
&+ x^{92} + x^{91} + x^{88} + x^{86} + x^{85} + x^{83} + x^{79} + x^{77} + x^{75} + x^{73} + x^{71} \\
&+ x^{69} + x^{67} + x^{65} + x^{63} + x^{61} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{53} \\
&+ x^{52} + x^{49} + x^{48} + x^{47} + x^{46} + x^{41} + x^{34} + x^{32} + x^{28} + x^{27} + x^{26} \\
&+ x^{22} + x^{21} + x^{20} + x^{16} + x^{15} + x^{14} + x^{9}, \\
p_6(x) &= x^{132} + x^{128} + x^{120} + x^{118} + x^{116} + x^{106} + x^{96} + x^{92} + x^{88} + x^{84} \\
&+ x^{82} + x^{80} + x^{74} + x^{68} + x^{66} + x^{64} + x^{62} + x^{58} + x^{56} + x^{50} + x^{48} \\
&+ x^{46} + x^{34} + x^{30} + x^{28} + x^{26} + x^{24} + x^{22} + x^{18} + x^{16} + x^{10} + x^{6}, \\
p_9(x) &= x^{134} + x^{132} + x^{130} + x^{127} + x^{123} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117} \\
&+ x^{115} + x^{114} + x^{111} + x^{110} + x^{109} + x^{106} + x^{103} + x^{101} + x^{100} \\
&+ x^{94} + x^{93} + x^{91} + x^{90} + x^{89} + x^{88} + x^{86} + x^{84} + x^{83} + x^{82} + x^{79} \\
&+ x^{75} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{67} + x^{66} + x^{63} + x^{62} + x^{61} \\
&+ x^{58} + x^{55} + x^{53} + x^{52} + x^{46} + x^{45} + x^{43} + x^{42} + x^{41} + x^{40} + x^{38} \\
&+ x^{36} + x^{35} + x^{34} + x^{31} + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{19} \\
&+ x^{14} + x^{13} + x^{11} + x^{10} + x^{9} + x^{8} + x^{3}, \\
p_{12}(x) &= x^{136} + x^{128} + x^{120} + x^{112} + x^{108} + x^{104} + x^{100} + x^{96} + x^{92} + x^{84} \\
&+ x^{72} + x^{64} + x^{60} + x^{56} + x^{52} + x^{48} + x^{44} + x^{36} + x^{12} + x^{8} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 0, 1, 1, 1, 1].

For $\phi(W^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{129} + x^{127} + x^{123} + x^{120} + x^{116} + x^{115} + x^{113} + x^{111} + x^{109} + x^{103} \\
&+ x^{102} + x^{101} + x^{100} + x^{99} + x^{97} + x^{94} + x^{91} + x^{90} + x^{89} + x^{87} + x^{86} \\
&+ x^{85} + x^{82} + x^{81} + x^{80} + x^{79} + x^{76} + x^{75} + x^{74} + x^{72} + x^{71} + x^{70} \\
&+ x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{62} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} \\
&+ x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{45} + x^{42} + x^{41} + x^{39} \\
&+ x^{37} + x^{36} + x^{33}, \\
p_3(x) &= x^{124} + x^{121} + x^{117} + x^{115} + x^{114} + x^{110} + x^{108} + x^{107} + x^{105} + x^{103} \\
&+ x^{100} + x^{99} + x^{97} + x^{95} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} \\
&+ x^{84} + x^{83} + x^{81} + x^{79} + x^{76} + x^{74} + x^{71} + x^{70} + x^{62} + x^{59} + x^{55} \\
&+ x^{54} + x^{52} + x^{49} + x^{44} + x^{41} + x^{38} + x^{36} + x^{33} + x^{32} + x^{31} + x^{24} \\
&+ x^{23} + x^{22} + x^{20} + x^{19} + x^{18} + x^{13} + x^{12},
\end{aligned}
$$

$$\begin{aligned}
p_6(x) = {} & x^{124} + x^{121} + x^{117} + x^{114} + x^{111} + x^{110} + x^{108} + x^{106} + x^{104} + x^{102} \\
& + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{89} + x^{88} + x^{86} + x^{85} \\
& + x^{83} + x^{82} + x^{79} + x^{78} + x^{77} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} \\
& + x^{69} + x^{64} + x^{62} + x^{61} + x^{59} + x^{55} + x^{54} + x^{53} + x^{52} + x^{50} + x^{49} \\
& + x^{48} + x^{47} + x^{46} + x^{41} + x^{39} + x^{38} + x^{37} + x^{36} + x^{33} + x^{31} + x^{27} \\
& + x^{24} + x^{23} + x^{22} + x^{20} + x^{19} + x^{17} + x^{16} + x^{14} + x^{9} + x^{7} + x^{5} + x^{3} \\
& + x + 1, \\
p_9(x) = {} & x^{124} + x^{123} + x^{121} + x^{119} + x^{117} + x^{115} + x^{113} + x^{112} + x^{100} + x^{99} \\
& + x^{97} + x^{96} + x^{92} + x^{91} + x^{89} + x^{87} + x^{85} + x^{84} + x^{76} + x^{75} + x^{73} \\
& + x^{71} + x^{69} + x^{67} + x^{65} + x^{64} + x^{52} + x^{51} + x^{49} + x^{48} + x^{44} + x^{43} \\
& + x^{41} + x^{39} + x^{37} + x^{36} + x^{28} + x^{27} + x^{25} + x^{23} + x^{21} + x^{19} + x^{17} \\
& + x^{16} + x^{12} + x^{11} + x^{9} + x^{7} + x^{5} + x^{4}, \\
p_{12}(x) = {} & x^{124} + x^{123} + x^{121} + x^{119} + x^{117} + x^{116} + x^{112} + x^{111} + x^{109} + x^{108} \\
& + x^{104} + x^{103} + x^{101} + x^{99} + x^{97} + x^{95} + x^{93} + x^{91} + x^{89} + x^{88} + x^{84} \\
& + x^{83} + x^{81} + x^{80} + x^{76} + x^{75} + x^{73} + x^{71} + x^{69} + x^{68} + x^{64} + x^{63} \\
& + x^{61} + x^{60} + x^{56} + x^{55} + x^{53} + x^{51} + x^{49} + x^{47} + x^{45} + x^{43} + x^{41} \\
& + x^{40} + x^{36} + x^{35} + x^{33} + x^{32} + x^{28} + x^{27} + x^{25} + x^{23} + x^{21} + x^{20} \\
& + x^{16} + x^{15} + x^{13} + x^{11} + x^{9} + x^{8} + x^{4} + x^{3} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 0, 1, 0, 1].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	174	[302, 303]	348	[171, 199]	522	[457, 150]	696	[479, 783]
1	[3, 4]	175	[304, 305]	349	[444, 60]	523	[662, 198]	697	[784, 785]
2	[5, 6]	176	[306, 307]	350	[172, 200]	524	[689, 418]	698	[786, 787]
3	[7, 8]	177	[308, 309]	351	[467, 468]	525	[196, 194]	699	[788, 278]
4	[9, 10]	178	[310, 311]	352	[469, 470]	526	[690, 446]	700	[250, 789]
5	[11, 12]	179	[312, 313]	353	[471, 472]	527	[691, 149]	701	[790, 791]
6	[13, 14]	180	[314, 247]	354	[246, 473]	528	[426, 194]	702	[792, 471]
7	[15, 16]	181	[315, 316]	355	[474, 475]	529	[388, 433]	703	[793, 794]
8	[17, 18]	182	[317, 318]	356	[476, 477]	530	[682, 441]	704	[795, 796]
9	[19, 20]	183	[319, 320]	357	[478, 479]	531	[692, 52]	705	[797, 798]
10	[21, 22]	184	[321, 322]	358	[480, 481]	532	[388, 154]	706	[799, 779]
11	[23, 24]	185	[323, 324]	359	[482, 483]	533	[693, 694]	707	[800, 801]
12	[25, 26]	186	[325, 326]	360	[484, 485]	534	[194, 407]	708	[802, 511]
13	[27, 28]	187	[327, 328]	361	[117, 486]	535	[189, 178]	709	[297, 803]
14	[29, 30]	188	[329, 330]	362	[487, 488]	536	[351, 695]	710	[804, 805]
15	[31, 32]	189	[331, 332]	363	[489, 490]	537	[410, 696]	711	[806, 807]

16	[33, 34]	190	[333, 334]	364	[491, 492]	538	[183, 379]	712	[592, 808]
17	[9, 35]	191	[334, 335]	365	[493, 494]	539	[364, 185]	713	[809, 810]
18	[36, 37]	192	[336, 337]	366	[495, 496]	540	[151, 697]	714	[811, 812]
19	[38, 39]	193	[338, 339]	367	[497, 498]	541	[661, 169]	715	[813, 503]
20	[40, 41]	194	[340, 341]	368	[499, 500]	542	[428, 140]	716	[814, 815]
21	[42, 43]	195	[342, 343]	369	[501, 502]	543	[118, 677]	717	[816, 817]
22	[44, 45]	196	[344, 345]	370	[503, 504]	544	[686, 698]	718	[818, 819]
23	[46, 47]	197	[335, 346]	371	[505, 506]	545	[669, 45]	719	[820, 821]
24	[48, 49]	198	[347, 5]	372	[103, 507]	546	[699, 464]	720	[822, 823]
25	[32, 50]	199	[348, 349]	373	[508, 509]	547	[142, 354]	721	[824, 17]
26	[51, 52]	200	[350, 116]	374	[510, 484]	548	[144, 700]	722	[265, 825]
27	[53, 54]	201	[351, 45]	375	[511, 512]	549	[186, 401]	723	[826, 82]
28	[55, 56]	202	[131, 121]	376	[513, 514]	550	[367, 701]	724	[276, 827]
29	[57, 58]	203	[46, 352]	377	[515, 516]	551	[410, 670]	725	[828, 829]
30	[59, 60]	204	[60, 173]	378	[299, 517]	552	[661, 129]	726	[648, 830]
31	[61, 62]	205	[353, 354]	379	[518, 205]	553	[370, 14]	727	[704, 682]
32	[63, 64]	206	[51, 355]	380	[519, 315]	554	[195, 163]	728	[134, 685]
33	[65, 66]	207	[356, 357]	381	[520, 521]	555	[702, 459]	729	[706, 461]
34	[67, 68]	208	[159, 167]	382	[522, 523]	556	[367, 703]	730	[716, 713]
35	[69, 70]	209	[59, 172]	383	[524, 525]	557	[165, 369]	731	[430, 179]
36	[71, 72]	210	[49, 191]	384	[526, 527]	558	[404, 386]	732	[190, 190]
37	[73, 74]	211	[358, 178]	385	[517, 487]	559	[461, 130]	733	[134, 51]
38	[75, 76]	212	[359, 360]	386	[318, 528]	560	[704, 32]	734	[831, 398]
39	[77, 63]	213	[361, 167]	387	[529, 530]	561	[396, 705]	735	[832, 176]
40	[78, 79]	214	[362, 176]	388	[531, 532]	562	[706, 169]	736	[138, 466]
41	[80, 81]	215	[161, 363]	389	[533, 534]	563	[424, 354]	737	[725, 833]
42	[82, 83]	216	[364, 365]	390	[535, 536]	564	[155, 375]	738	[404, 374]
43	[84, 85]	217	[156, 366]	391	[537, 538]	565	[35, 677]	739	[440, 834]
44	[86, 87]	218	[367, 368]	392	[539, 540]	566	[408, 707]	740	[661, 461]
45	[88, 89]	219	[359, 365]	393	[343, 541]	567	[165, 426]	741	[158, 674]
46	[90, 91]	220	[369, 164]	394	[542, 543]	568	[708, 405]	742	[835, 32]
47	[92, 93]	221	[370, 55]	395	[544, 545]	569	[709, 432]	743	[196, 406]
48	[94, 95]	222	[371, 372]	396	[546, 338]	570	[672, 710]	744	[685, 355]
49	[49, 49]	223	[139, 373]	397	[547, 548]	571	[430, 41]	745	[668, 836]
50	[96, 97]	224	[374, 375]	398	[301, 549]	572	[32, 12]	746	[404, 155]
51	[98, 99]	225	[376, 176]	399	[550, 551]	573	[711, 355]	747	[725, 409]
52	[100, 101]	226	[377, 162]	400	[552, 553]	574	[377, 363]	748	[693, 445]
53	[102, 103]	227	[357, 137]	401	[554, 555]	575	[383, 352]	749	[837, 363]
54	[104, 105]	228	[170, 374]	402	[556, 557]	576	[59, 145]	750	[119, 438]
55	[106, 107]	229	[378, 370]	403	[558, 559]	577	[699, 361]	751	[53, 175]
56	[108, 109]	230	[165, 196]	404	[560, 467]	578	[663, 400]	752	[11, 441]
57	[110, 111]	231	[378, 379]	405	[561, 562]	579	[687, 187]	753	[837, 392]
58	[112, 113]	232	[380, 180]	406	[289, 563]	580	[46, 187]	754	[682, 12]
59	[114, 115]	233	[381, 382]	407	[564, 565]	581	[119, 394]	755	[361, 160]

60	[116, 117]	234	[383, 187]	408	[566, 567]	582	[427, 712]	756	[838, 36]
61	[9, 118]	235	[384, 385]	409	[568, 569]	583	[148, 696]	757	[169, 420]
62	[119, 120]	236	[386, 387]	410	[570, 571]	584	[39, 198]	758	[452, 139]
63	[57, 121]	237	[388, 389]	411	[477, 572]	585	[184, 365]	759	[44, 695]
64	[122, 59]	238	[119, 56]	412	[573, 574]	586	[194, 671]	760	[171, 145]
65	[123, 124]	239	[390, 391]	413	[575, 576]	587	[402, 713]	761	[381, 462]
66	[125, 126]	240	[377, 392]	414	[577, 578]	588	[714, 365]	762	[145, 200]
67	[127, 128]	241	[393, 121]	415	[579, 580]	589	[163, 406]	763	[39, 447]
68	[129, 130]	242	[48, 191]	416	[581, 582]	590	[385, 700]	764	[721, 361]
69	[131, 132]	243	[192, 394]	417	[583, 584]	591	[715, 183]	765	[686, 121]
70	[42, 133]	244	[395, 52]	418	[585, 586]	592	[463, 41]	766	[667, 834]
71	[134, 135]	245	[396, 54]	419	[587, 588]	593	[177, 188]	767	[374, 387]
72	[136, 137]	246	[32, 397]	420	[589, 590]	594	[673, 694]	768	[161, 460]
73	[138, 34]	247	[393, 58]	421	[591, 592]	595	[156, 432]	769	[155, 387]
74	[139, 140]	248	[60, 200]	422	[320, 593]	596	[669, 462]	770	[715, 664]
75	[141, 142]	249	[380, 398]	423	[594, 595]	597	[190, 49]	771	[692, 355]
76	[143, 144]	250	[399, 400]	424	[596, 597]	598	[9, 415]	772	[33, 466]
77	[145, 122]	251	[42, 178]	425	[598, 508]	599	[684, 135]	773	[465, 55]
78	[146, 147]	252	[383, 401]	426	[599, 220]	600	[131, 698]	774	[668, 701]
79	[148, 149]	253	[402, 403]	427	[600, 601]	601	[686, 58]	775	[693, 400]
80	[150, 151]	254	[195, 369]	428	[602, 603]	602	[669, 695]	776	[358, 43]
81	[152, 142]	255	[404, 405]	429	[604, 605]	603	[145, 173]	777	[146, 710]
82	[153, 154]	256	[406, 407]	430	[606, 607]	604	[57, 698]	778	[714, 185]
83	[155, 156]	257	[408, 409]	431	[345, 608]	605	[189, 43]	779	[405, 156]
84	[157, 158]	258	[410, 149]	432	[609, 610]	606	[716, 717]	780	[152, 353]
85	[159, 160]	259	[150, 411]	433	[611, 612]	607	[673, 400]	781	[157, 661]
86	[161, 162]	260	[379, 14]	434	[486, 613]	608	[718, 682]	782	[456, 370]
87	[163, 164]	261	[8, 169]	435	[614, 615]	609	[374, 709]	783	[675, 353]
88	[165, 163]	262	[412, 363]	436	[616, 314]	610	[135, 355]	784	[196, 676]
89	[166, 167]	263	[413, 354]	437	[617, 539]	611	[430, 128]	785	[705, 176]
90	[168, 154]	264	[414, 415]	438	[618, 619]	612	[411, 719]	786	[660, 158]
91	[158, 169]	265	[142, 416]	439	[620, 621]	613	[171, 60]	787	[458, 181]
92	[170, 155]	266	[123, 417]	440	[622, 623]	614	[691, 670]	788	[425, 428]
93	[166, 160]	267	[138, 418]	441	[62, 624]	615	[387, 366]	789	[398, 181]
94	[171, 172]	268	[419, 149]	442	[625, 618]	616	[436, 450]	790	[839, 136]
95	[173, 59]	269	[129, 420]	443	[626, 627]	617	[690, 39]	791	[375, 366]
96	[174, 140]	270	[421, 422]	444	[628, 629]	618	[682, 50]	792	[145, 436]
97	[175, 176]	271	[36, 137]	445	[630, 631]	619	[144, 712]	793	[708, 386]
98	[177, 178]	272	[138, 423]	446	[632, 579]	620	[714, 360]	794	[137, 719]
99	[177, 133]	273	[424, 416]	447	[76, 547]	621	[398, 459]	795	[384, 427]
100	[40, 179]	274	[425, 139]	448	[633, 634]	622	[711, 52]	796	[413, 726]
101	[180, 181]	275	[384, 422]	449	[635, 520]	623	[440, 435]	797	[177, 43]
102	[182, 183]	276	[426, 164]	450	[636, 350]	624	[118, 683]	798	[131, 58]
103	[184, 185]	277	[190, 48]	451	[637, 585]	625	[421, 144]	799	[666, 705]

104	[186, 187]	278	[143, 427]	452	[638, 639]	626	[429, 439]	800	[839, 36]
105	[42, 188]	279	[428, 373]	453	[640, 268]	627	[388, 126]	801	[37, 697]
106	[189, 188]	280	[429, 391]	454	[641, 642]	628	[436, 444]	802	[425, 434]
107	[190, 191]	281	[33, 418]	455	[643, 644]	629	[200, 59]	803	[192, 438]
108	[192, 120]	282	[430, 431]	456	[629, 29]	630	[357, 151]	804	[840, 8]
109	[175, 193]	283	[387, 432]	457	[645, 524]	631	[718, 32]	805	[37, 719]
110	[153, 126]	284	[396, 376]	458	[646, 233]	632	[48, 190]	806	[393, 132]
111	[163, 194]	285	[136, 37]	459	[647, 648]	633	[362, 193]	807	[381, 695]
112	[195, 196]	286	[125, 433]	460	[649, 650]	634	[667, 435]	808	[183, 370]
113	[197, 198]	287	[434, 140]	461	[651, 652]	635	[720, 413]	809	[841, 426]
114	[172, 173]	288	[141, 424]	462	[653, 654]	636	[59, 199]	810	[137, 697]
115	[191, 49]	289	[371, 435]	463	[655, 656]	637	[721, 464]	811	[402, 717]
116	[199, 173]	290	[60, 436]	464	[349, 304]	638	[666, 175]	812	[148, 670]
117	[200, 171]	291	[437, 51]	465	[657, 658]	639	[424, 722]	813	[665, 680]
118	[201, 202]	292	[55, 438]	466	[527, 659]	640	[182, 664]	814	[716, 842]
119	[203, 204]	293	[390, 439]	467	[168, 126]	641	[723, 8]	815	[708, 155]
120	[205, 206]	294	[440, 372]	468	[357, 37]	642	[169, 130]	816	[838, 136]
121	[207, 208]	295	[32, 441]	469	[660, 661]	643	[127, 431]	817	[676, 671]
122	[209, 210]	296	[442, 192]	470	[662, 447]	644	[164, 407]	818	[843, 415]
123	[22, 211]	297	[443, 435]	471	[358, 188]	645	[452, 434]	819	[14, 56]
124	[212, 213]	298	[191, 191]	472	[436, 59]	646	[48, 48]	820	[155, 709]
125	[214, 215]	299	[444, 145]	473	[54, 193]	647	[688, 395]	821	[422, 700]
126	[216, 217]	300	[122, 444]	474	[663, 445]	648	[174, 373]	822	[453, 118]
127	[218, 219]	301	[173, 171]	475	[664, 370]	649	[693, 724]	823	[413, 722]
128	[220, 221]	302	[383, 47]	476	[665, 32]	650	[674, 130]	824	[704, 11]
129	[222, 223]	303	[122, 171]	477	[443, 372]	651	[125, 389]	825	[415, 677]
130	[224, 225]	304	[381, 45]	478	[666, 376]	652	[413, 416]	826	[720, 424]
131	[226, 227]	305	[44, 382]	479	[369, 194]	653	[449, 155]	827	[160, 434]
132	[228, 229]	306	[399, 445]	480	[667, 455]	654	[702, 181]	828	[672, 844]
133	[230, 231]	307	[446, 447]	481	[196, 164]	655	[725, 454]	829	[708, 374]
134	[232, 100]	308	[448, 418]	482	[668, 368]	656	[127, 41]	830	[395, 355]
135	[233, 234]	309	[150, 37]	483	[127, 179]	657	[684, 395]	831	[845, 344]
136	[235, 236]	310	[449, 374]	484	[405, 387]	658	[424, 726]	832	[252, 535]
137	[237, 238]	311	[197, 447]	485	[379, 55]	659	[411, 697]	833	[846, 847]
138	[239, 240]	312	[158, 129]	486	[358, 133]	660	[727, 308]	834	[848, 849]
139	[241, 242]	313	[160, 139]	487	[669, 382]	661	[728, 729]	835	[850, 96]
140	[243, 244]	314	[450, 60]	488	[351, 462]	662	[95, 201]	836	[851, 852]
141	[245, 246]	315	[451, 144]	489	[419, 670]	663	[730, 731]	837	[853, 854]
142	[247, 248]	316	[14, 438]	490	[164, 671]	664	[732, 106]	838	[855, 90]
143	[249, 250]	317	[452, 174]	491	[672, 147]	665	[733, 25]	839	[856, 262]
144	[251, 252]	318	[453, 35]	492	[673, 445]	666	[734, 614]	840	[857, 749]
145	[210, 114]	319	[60, 122]	493	[661, 674]	667	[735, 736]	841	[858, 753]
146	[253, 254]	320	[351, 382]	494	[675, 142]	668	[737, 738]	842	[859, 860]
147	[255, 256]	321	[408, 454]	495	[163, 676]	669	[739, 740]	843	[861, 216]

148	[257, 258]	322	[359, 185]	496	[10, 677]	670	[741, 742]	844	[862, 863]
149	[259, 260]	323	[357, 411]	497	[146, 678]	671	[743, 744]	845	[720, 353]
150	[4, 261]	324	[167, 139]	498	[195, 426]	672	[745, 746]	846	[679, 369]
151	[262, 263]	325	[440, 455]	499	[679, 426]	673	[747, 748]	847	[406, 671]
152	[264, 265]	326	[150, 137]	500	[461, 420]	674	[749, 750]	848	[673, 724]
153	[266, 267]	327	[356, 150]	501	[421, 385]	675	[751, 752]	849	[151, 719]
154	[268, 269]	328	[456, 379]	502	[8, 461]	676	[753, 754]	850	[864, 175]
155	[270, 271]	329	[457, 357]	503	[451, 427]	677	[578, 755]	851	[723, 706]
156	[272, 273]	330	[458, 459]	504	[680, 12]	678	[756, 757]	852	[676, 407]
157	[274, 222]	331	[377, 460]	505	[681, 417]	679	[758, 110]	853	[832, 193]
158	[275, 276]	332	[158, 461]	506	[161, 392]	680	[759, 760]	854	[125, 154]
159	[277, 241]	333	[122, 450]	507	[674, 420]	681	[580, 761]	855	[465, 192]
160	[278, 279]	334	[199, 200]	508	[682, 397]	682	[346, 762]	856	[141, 413]
161	[280, 281]	335	[450, 145]	509	[415, 683]	683	[20, 763]	857	[442, 119]
162	[282, 283]	336	[44, 462]	510	[684, 685]	684	[764, 489]	858	[665, 682]
163	[284, 285]	337	[436, 171]	511	[412, 392]	685	[549, 765]	859	[840, 706]
164	[286, 287]	338	[463, 179]	512	[434, 373]	686	[766, 767]	860	[375, 432]
165	[288, 289]	339	[464, 160]	513	[686, 132]	687	[768, 769]	861	[831, 180]
166	[290, 104]	340	[33, 423]	514	[46, 401]	688	[770, 474]	862	[841, 369]
167	[291, 292]	341	[192, 56]	515	[681, 124]	689	[771, 772]	863	[709, 366]
168	[293, 294]	342	[465, 119]	516	[667, 372]	690	[773, 4]	864	[865, 306]
169	[16, 295]	343	[448, 466]	517	[687, 401]	691	[774, 775]	865	[38, 446]
170	[296, 297]	344	[453, 10]	518	[688, 51]	692	[776, 776]		
171	[298, 299]	345	[386, 156]	519	[442, 14]	693	[777, 778]		
172	[300, 298]	346	[189, 133]	520	[689, 466]	694	[779, 780]		
173	[115, 301]	347	[437, 395]	521	[374, 156]	695	[781, 782]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	124	1	248	1	372	0	496	0	620	0	744	1
1	0	125	1	249	0	373	1	497	0	621	1	745	1
2	1	126	1	250	0	374	1	498	1	622	1	746	1
3	0	127	0	251	0	375	1	499	1	623	1	747	1
4	0	128	0	252	0	376	0	500	1	624	1	748	0
5	1	129	1	253	0	377	1	501	0	625	0	749	0
6	1	130	1	254	1	378	1	502	0	626	0	750	1
7	0	131	1	255	1	379	0	503	1	627	0	751	1
8	0	132	0	256	1	380	0	504	0	628	1	752	1
9	0	133	0	257	0	381	0	505	1	629	0	753	0
10	0	134	0	258	1	382	1	506	0	630	1	754	1
11	1	135	0	259	0	383	0	507	0	631	1	755	0
12	0	136	1	260	0	384	1	508	1	632	1	756	1
13	1	137	0	261	0	385	0	509	0	633	1	757	0
14	0	138	1	262	1	386	1	510	1	634	0	758	1

15	0	139	1	263	0	387	0	511	1	635	1	759	0
16	0	140	0	264	0	388	0	512	0	636	0	760	1
17	0	141	0	265	1	389	0	513	0	637	1	761	0
18	0	142	1	266	0	390	1	514	1	638	1	762	0
19	0	143	0	267	1	391	1	515	1	639	1	763	0
20	0	144	0	268	0	392	1	516	0	640	1	764	1
21	0	145	0	269	1	393	1	517	0	641	1	765	0
22	0	146	0	270	0	394	1	518	0	642	0	766	0
23	1	147	1	271	0	395	0	519	0	643	0	767	1
24	1	148	0	272	1	396	0	520	0	644	1	768	0
25	0	149	0	273	1	397	1	521	1	645	1	769	0
26	1	150	0	274	0	398	1	522	1	646	1	770	0
27	1	151	1	275	1	399	0	523	1	647	0	771	0
28	1	152	0	276	1	400	0	524	0	648	0	772	0
29	0	153	0	277	0	401	1	525	1	649	0	773	1
30	0	154	0	278	0	402	0	526	1	650	0	774	1
31	0	155	0	279	1	403	1	527	1	651	1	775	0
32	0	156	1	280	0	404	1	528	1	652	0	776	0
33	0	157	0	281	0	405	0	529	0	653	1	777	0
34	0	158	1	282	1	406	1	530	1	654	0	778	0
35	1	159	0	283	0	407	0	531	0	655	1	779	0
36	0	160	0	284	0	408	0	532	0	656	0	780	0
37	1	161	0	285	1	409	0	533	0	657	1	781	0
38	0	162	1	286	1	410	1	534	0	658	1	782	0
39	0	163	0	287	0	411	0	535	1	659	0	783	1
40	0	164	1	288	0	412	1	536	1	660	1	784	1
41	0	165	0	289	1	413	0	537	1	661	0	785	1
42	0	166	1	290	1	414	0	538	1	662	1	786	1
43	0	167	1	291	1	415	0	539	1	663	1	787	1
44	0	168	1	292	1	416	1	540	1	664	0	788	0
45	0	169	0	293	1	417	0	541	0	665	0	789	1
46	1	170	0	294	1	418	0	542	1	666	1	790	0
47	0	171	1	295	0	419	0	543	1	667	0	791	1
48	1	172	0	296	0	420	0	544	0	668	1	792	0
49	0	173	1	297	0	421	0	545	1	669	1	793	0
50	0	174	0	298	1	422	1	546	0	670	1	794	0
51	1	175	0	299	1	423	1	547	1	671	1	795	1
52	0	176	0	300	0	424	1	548	0	672	1	796	0
53	1	177	1	301	1	425	0	549	1	673	1	797	1
54	1	178	1	302	0	426	1	550	0	674	0	798	1
55	1	179	1	303	0	427	1	551	1	675	1	799	1
56	0	180	0	304	0	428	1	552	0	676	0	800	0
57	0	181	1	305	0	429	0	553	1	677	1	801	1
58	1	182	1	306	0	430	1	554	1	678	1	802	0

59	0	183	1	307	1	431	1	555	0	679	1	803	0
60	1	184	0	308	1	432	1	556	0	680	0	804	0
61	0	185	1	309	0	433	1	557	0	681	1	805	1
62	1	186	1	310	1	434	0	558	1	682	1	806	1
63	0	187	0	311	0	435	1	559	1	683	0	807	0
64	0	188	1	312	1	436	1	560	1	684	1	808	1
65	0	189	1	313	0	437	1	561	0	685	1	809	0
66	1	190	0	314	0	438	1	562	1	686	0	810	0
67	0	191	1	315	1	439	0	563	1	687	0	811	0
68	1	192	0	316	0	440	1	564	0	688	0	812	0
69	1	193	1	317	1	441	1	565	1	689	0	813	0
70	0	194	0	318	1	442	0	566	0	690	1	814	1
71	0	195	1	319	1	443	0	567	0	691	1	815	0
72	1	196	1	320	1	444	1	568	0	692	0	816	1
73	1	197	0	321	0	445	1	569	0	693	0	817	0
74	1	198	1	322	1	446	1	570	1	694	0	818	1
75	0	199	1	323	1	447	0	571	1	695	0	819	0
76	0	200	0	324	1	448	1	572	0	696	0	820	0
77	0	201	1	325	1	449	1	573	1	697	1	821	1
78	0	202	1	326	0	450	0	574	1	698	1	822	1
79	0	203	1	327	0	451	1	575	0	699	0	823	0
80	0	204	1	328	0	452	1	576	0	700	0	824	1
81	0	205	0	329	1	453	1	577	0	701	0	825	0
82	0	206	1	330	1	454	1	578	1	702	0	826	1
83	0	207	0	331	1	455	0	579	0	703	0	827	0
84	0	208	0	332	1	456	0	580	1	704	1	828	1
85	0	209	0	333	0	457	1	581	1	705	1	829	0
86	0	210	0	334	1	458	1	582	1	706	1	830	0
87	0	211	0	335	0	459	0	583	0	707	0	831	1
88	0	212	1	336	0	460	0	584	0	708	0	832	0
89	1	213	0	337	1	461	1	585	0	709	0	833	1
90	1	214	1	338	1	462	1	586	0	710	0	834	1
91	1	215	0	339	1	463	1	587	0	711	1	835	0
92	0	216	1	340	0	464	1	588	0	712	1	836	1
93	1	217	1	341	0	465	1	589	0	713	0	837	0
94	1	218	0	342	1	466	1	590	0	714	0	838	1
95	1	219	1	343	1	467	1	591	0	715	0	839	0
96	0	220	0	344	1	468	1	592	1	716	1	840	0
97	0	221	1	345	1	469	1	593	1	717	1	841	0
98	1	222	1	346	1	470	1	594	1	718	1	842	0
99	1	223	1	347	1	471	0	595	1	719	0	843	1
100	0	224	1	348	1	472	1	596	1	720	1	844	0
101	0	225	0	349	1	473	1	597	0	721	1	845	1
102	1	226	1	350	0	474	1	598	0	722	1	846	1

103	0	227	1	351	1	475	0	599	1	723	1	847	1
104	1	228	0	352	1	476	0	600	1	724	1	848	1
105	0	229	1	353	0	477	0	601	0	725	1	849	1
106	1	230	0	354	0	478	1	602	1	726	0	850	0
107	0	231	1	355	1	479	0	603	0	727	1	851	1
108	0	232	0	356	0	480	0	604	0	728	0	852	0
109	0	233	0	357	1	481	1	605	1	729	1	853	0
110	0	234	0	358	0	482	1	606	1	730	1	854	1
111	0	235	1	359	1	483	0	607	1	731	1	855	1
112	1	236	1	360	0	484	0	608	1	732	0	856	0
113	0	237	0	361	0	485	0	609	1	733	0	857	0
114	0	238	1	362	1	486	0	610	0	734	1	858	0
115	1	239	1	363	0	487	1	611	1	735	0	859	0
116	1	240	1	364	1	488	1	612	0	736	1	860	1
117	0	241	1	365	0	489	0	613	1	737	1	861	1
118	1	242	1	366	0	490	1	614	1	738	1	862	0
119	1	243	0	367	0	491	1	615	0	739	1	863	0
120	0	244	0	368	1	492	1	616	1	740	0	864	0
121	0	245	0	369	0	493	0	617	1	741	1	865	0
122	0	246	0	370	1	494	1	618	1	742	0		
123	0	247	1	371	1	495	0	619	0	743	1		

Université de Strasbourg, CNRS, IRMA UMR 7501, F-67000 Strasbourg, France
Email address: guoniu.han@unistra.fr

School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan, PR China
Email address: huyining@hust.edu.cn