PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a,b) = (z^6 + z^4 + z^3 + z^2, z + 1)$

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ABSTRACT. The continued fraction conjecture claims that the continued fraction defined by the (a,b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a,b) in $\mathbb{F}_2[x] \setminus \mathbb{F}_2$. In this paper we prove the conjecture for $(a,b) = (z^6 + z^4 + z^3 + z^2, z + 1)$.

1. INTRODUCTION

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \ldots)$ of elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{\ddots + \cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a,b)-Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \ldots) = (a, b, b, a, b, a, a, b, \ldots)$$

that we shall denote

$$\mathrm{CF}(a,b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When $(a,b)=(z^6+z^4+z^3+z^2, z+1)$, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

where, for $\mathrm{CF}(a,b)$*,*

$$\begin{aligned}
p_0(z) &= z^{15}+z^{14}+z^{12}+z^9+z^7+z^5+1,\\
p_1(z) &= z^{21}+z^{20}+z^{19}+z^{18}+z^{16}+z^{14}+z^{13}+z^{12}+z^{10}+z^4+z^3+z^2,\\
p_2(z) &= z^{22}+z^{18}+z^8+z^4,\\
p_3(z) &= z^{21}+z^{20}+z^{19}+z^{18}+z^{16}+z^{14}+z^{13}+z^{12}+z^{10}+z^4+z^3+z^2,\\
p_4(z) &= z^{20}+z^{18}+z^{15}+z^{14}+z^{12}+z^{10}+z^9+z^7+z^6+z^5+z^2,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$*,*

$$\begin{aligned}
p_0(z) &= z^{18}+z^{16}+z^{15}+z^{14}+z^{12}+z^9+z^7+z^6+z^5+z^4+z^2,\\
p_1(z) &= z^{21}+z^{20}+z^{19}+z^{18}+z^{16}+z^{14}+z^{13}+z^{12}+z^{10}+z^4+z^3+z^2,\\
p_2(z) &= z^{22}+z^{18}+z^8+z^4,\\
p_3(z) &= z^{21}+z^{20}+z^{19}+z^{18}+z^{16}+z^{14}+z^{13}+z^{12}+z^{10}+z^4+z^3+z^2,\\
p_4(z) &= z^{20}+z^{16}+z^{15}+z^{14}+z^{12}+z^{10}+z^9+z^7+z^5+z^4+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x) &= x^{\deg(t_{2^n-1})}\begin{pmatrix} t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x) &= x^{\deg(\bar t_{2^n-1})}\begin{pmatrix} \bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix} \bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(\bar t_0)}\begin{pmatrix} \bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\geq 0$

$$\begin{aligned}
M_{n+1}(x) &= W_n(x)\cdot M_n(x),\\
W_{n+1}(x) &= M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(175, 175, 175, 175, 175)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x, y) = \mathrm{Res}_z \left(\phi(W^o, 0, 0)(x, z),\ z^{12} \cdot \phi(M^o, 0, 0)(x, y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x, y)$. To prove that $\phi_0(x, y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x, y)$ of multiplicity m and that $Q(x, y) := P(x, y)/\phi_0(x, y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 630 and substitute it for y in $Q(x, y)$. We get a series of valuation less than 630, which proves that $Q(x, y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x, y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e, 0, 0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x, y) = \mathrm{Res}_z \left(\phi_0(x, z),\ \phi_1(x, y + z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e, 0, 0)$ is an irreducible factor of $S(x, y)$ of multiplicity μ, and that $Q(x, y) := S(x, y)/\phi(M^e, 0, 0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ to order 770 and substitute it for y in $Q(x, y)$. We get a series of valuation less than 770, and therefore $Q(x, y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e, 0, 0)$, we know that the two series are one and the same. □

Define

$$R^e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned} M^e[{:}\,14u] = {} & M^e[{:}\,7u] + x^u M^e[{:}\,6u] + x^{3u} M^e[{:}\,4u] + x^{4u} M^e[{:}\,3u] + x^{5u} M^e[{:}\,2u] \\ & + x^u M^e[{:}\,7u] + x^{2u} M^e[{:}\,6u] + x^{4u} M^e[{:}\,4u] + x^{5u} M^e[{:}\,3u] \\ & + x^{6u} M^e[{:}\,2u] + x^{2u} M^e[{:}\,7u] + x^{3u} M^e[{:}\,6u] + x^{5u} M^e[{:}\,4u] \\ & + x^{6u} M^e[{:}\,3u] + x^{7u} M^e[{:}\,2u] + x^{5u} M^e[{:}\,7u] + x^{6u} M^e[{:}\,6u] \\ & + x^{8u} M^e[{:}\,4u] + x^{9u} M^e[{:}\,3u] + x^{10u} M^e[{:}\,2u] + x^{6u} M^e[{:}\,7u] \\ & + x^{7u} M^e[{:}\,6u] + x^{9u} M^e[{:}\,4u] + x^{10u} M^e[{:}\,3u] + x^{11u} M^e[{:}\,2u] \end{aligned}$$

$$
\begin{aligned}
&+ x^{7u}M^e[:7u] + x^{8u}M^e[:6u] + x^{11u}M^e[:3u] + x^{9u}M^e[:5u]\\
&+ x^{13u}M^e[:u] + (x^{7u} + x^{8u} + x^{9u} + x^{12u} + x^{13u})R^e,\\
W^e[:14u] = {}& W^e[:7u] + x^{u}W^e[:6u] + x^{3u}W^e[:4u] + x^{4u}W^e[:3u] + x^{5u}W^e[:2u]\\
&+ x^{u}W^e[:7u] + x^{2u}W^e[:6u] + x^{4u}W^e[:4u] + x^{5u}W^e[:3u]\\
&+ x^{6u}W^e[:2u] + x^{2u}W^e[:7u] + x^{3u}W^e[:6u] + x^{5u}W^e[:4u]\\
&+ x^{6u}W^e[:3u] + x^{7u}W^e[:2u] + x^{5u}W^e[:7u] + x^{6u}W^e[:6u]\\
&+ x^{8u}W^e[:4u] + x^{9u}W^e[:3u] + x^{10u}W^e[:2u] + x^{6u}W^e[:7u]\\
&+ x^{7u}W^e[:6u] + x^{9u}W^e[:4u] + x^{10u}W^e[:3u] + x^{11u}W^e[:2u]\\
&+ x^{7u}W^e[:7u] + x^{8u}W^e[:6u] + x^{11u}W^e[:3u] + x^{9u}W^e[:5u]\\
&+ x^{13u}W^e[:u] + (x^{7u} + x^{8u} + x^{9u} + x^{12u} + x^{13u})R^e.
\end{aligned}
$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$
\begin{aligned}
M^o[:14u] = {}& M^o[:7u] + x^{u}M^o[:6u] + x^{3u}M^o[:4u] + x^{4u}M^o[:3u] + x^{5u}M^o[:2u]\\
&+ x^{u}M^o[:7u] + x^{2u}M^o[:6u] + x^{4u}M^o[:4u] + x^{5u}M^o[:3u]\\
&+ x^{6u}M^o[:2u] + x^{2u}M^o[:7u] + x^{3u}M^o[:6u] + x^{5u}M^o[:4u]\\
&+ x^{6u}M^o[:3u] + x^{7u}M^o[:2u] + x^{5u}M^o[:7u] + x^{6u}M^o[:6u]\\
&+ x^{8u}M^o[:4u] + x^{9u}M^o[:3u] + x^{10u}M^o[:2u] + x^{6u}M^o[:7u]\\
&+ x^{7u}M^o[:6u] + x^{9u}M^o[:4u] + x^{10u}M^o[:3u] + x^{11u}M^o[:2u]\\
&+ x^{7u}M^o[:7u] + x^{8u}M^o[:6u] + x^{11u}M^o[:3u] + x^{9u}M^o[:5u]\\
&+ x^{13u}M^o[:u] + (x^{7u} + x^{8u} + x^{9u} + x^{12u} + x^{13u})R^o,\\
W^o[:14u] = {}& W^o[:7u] + x^{u}W^o[:6u] + x^{3u}W^o[:4u] + x^{4u}W^o[:3u] + x^{5u}W^o[:2u]\\
&+ x^{u}W^o[:7u] + x^{2u}W^o[:6u] + x^{4u}W^o[:4u] + x^{5u}W^o[:3u]\\
&+ x^{6u}W^o[:2u] + x^{2u}W^o[:7u] + x^{3u}W^o[:6u] + x^{5u}W^o[:4u]\\
&+ x^{6u}W^o[:3u] + x^{7u}W^o[:2u] + x^{5u}W^o[:7u] + x^{6u}W^o[:6u]\\
&+ x^{8u}W^o[:4u] + x^{9u}W^o[:3u] + x^{10u}W^o[:2u] + x^{6u}W^o[:7u]\\
&+ x^{7u}W^o[:6u] + x^{9u}W^o[:4u] + x^{10u}W^o[:3u] + x^{11u}W^o[:2u]\\
&+ x^{7u}W^o[:7u] + x^{8u}W^o[:6u] + x^{11u}W^o[:3u] + x^{9u}W^o[:5u]\\
&+ x^{13u}W^o[:u] + (x^{7u} + x^{8u} + x^{9u} + x^{12u} + x^{13u})R^o.
\end{aligned}
$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$
\begin{aligned}
T[:14u] = {}& T[:7u] + x^{u}T[:6u] + x^{3u}T[:4u] + x^{4u}T[:3u] + x^{5u}T[:2u] + x^{u}T[:7u]\\
&+ x^{2u}T[:6u] + x^{4u}T[:4u] + x^{5u}T[:3u] + x^{6u}T[:2u] + x^{2u}T[:7u]\\
&+ x^{3u}T[:6u] + x^{5u}T[:4u] + x^{6u}T[:3u] + x^{7u}T[:2u] + x^{5u}T[:7u]\\
&+ x^{6u}T[:6u] + x^{8u}T[:4u] + x^{9u}T[:3u] + x^{10u}T[:2u] + x^{6u}T[:7u]\\
&+ x^{7u}T[:6u] + x^{9u}T[:4u] + x^{10u}T[:3u] + x^{11u}T[:2u] + x^{7u}T[:7u]
\end{aligned}
$$

$$+ x^{8u}T[:6u] + x^{11u}T[:3u] + x^{9u}T[:5u] + x^{13u}T[:u].$$

The proof of the other 15 cases are similar. We break down the above identity into 7 parts:

$$\begin{aligned}
0 &= x^{7u}T[:u] + x^{5u}T[2u:3u] + x^{4u}T[3u:4u] + x^{3u}T[4u:5u] \\
&\quad + x^{u}T[6u:7u] + T[7u:8u], \\
0 &= x^{7u}T[u:2u] + x^{6u}T[2u:3u] + x^{3u}T[5u:6u] + x^{2u}T[6u:7u] \\
&\quad + T[8u:9u], \\
0 &= x^{9u}T[:u] + x^{5u}T[4u:5u] + T[9u:10u], \\
0 &= x^{9u}T[u:2u] + x^{5u}T[5u:6u] + T[10u:11u], \\
0 &= x^{9u}T[2u:3u] + x^{5u}T[6u:7u] + T[11u:12u], \\
0 &= x^{10u}T[2u:3u] + x^{8u}T[4u:5u] + x^{6u}T[6u:7u] + T[12u:13u], \\
0 &= x^{13u}T[:u] + x^{11u}T[2u:3u] + x^{9u}T[4u:5u] + x^{8u}T[5u:6u] \\
&\quad + x^{7u}T[6u:7u] + T[13u:14u],
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
0 &= T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[110w]_2] \\
&\quad + T[[111w]_2], && (2.7) \\
0 &= T[[1w]_2] + T[[10w]_2] + T[[101w]_2] + T[[110w]_2] + T[[1000w]_2], && (2.8) \\
0 &= T[[w]_2] + T[[100w]_2] + T[[1001w]_2], && (2.9) \\
0 &= T[[1w]_2] + T[[101w]_2] + T[[1010w]_2], && (2.10) \\
0 &= T[[10w]_2] + T[[110w]_2] + T[[1011w]_2], && (2.11) \\
0 &= T[[10w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1100w]_2], && (2.12) \\
0 &= T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[110w]_2] \\
&\quad + T[[1101w]_2], && (2.13)
\end{aligned}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$\begin{aligned}
0 &= T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[110w]_2] \\
&\quad + T[[111w]_2], && (2.14) \\
0 &= T[[1w]_2] + T[[10w]_2] + T[[101w]_2] + T[[110w]_2] + T[[1000w]_2], && (2.15) \\
0 &= T[[w]_2] + T[[100w]_2] + T[[1001w]_2], && (2.16) \\
0 &= T[[1w]_2] + T[[101w]_2] + T[[1010w]_2], && (2.17) \\
0 &= T[[10w]_2] + T[[110w]_2] + T[[1011w]_2], && (2.18) \\
0 &= T[[10w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1100w]_2], && (2.19) \\
0 &= T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[110w]_2] \\
&\quad + T[[1101w]_2], && (2.20)
\end{aligned}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 824 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i,w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.20) can be written as

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ &\quad + \tau(A(s,110)) + \tau(A(s,111)), \end{aligned} \tag{2.21}$$

$$\begin{aligned} 0 &= \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,101)) + \tau(A(s,110)) \\ &\quad + \tau(A(s,1000)), \end{aligned} \tag{2.22}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,100)) + \tau(A(s,1001)), \tag{2.23}$$

$$0 = \tau(A(s,1)) + \tau(A(s,101)) + \tau(A(s,1010)), \tag{2.24}$$

$$0 = \tau(A(s,10)) + \tau(A(s,110)) + \tau(A(s,1011)), \tag{2.25}$$

$$0 = \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,110)) + \tau(A(s,1100)), \tag{2.26}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,101)) \\ &\quad + \tau(A(s,110)) + \tau(A(s,1101)), \end{aligned} \tag{2.27}$$

for all $s \in E_{2k}$, and

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ &\quad + \tau(A(s,110)) + \tau(A(s,111)), \end{aligned} \tag{2.28}$$

$$\begin{aligned} 0 &= \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,101)) + \tau(A(s,110)) \\ &\quad + \tau(A(s,1000)), \end{aligned} \tag{2.29}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,100)) + \tau(A(s,1001)), \tag{2.30}$$

$$0 = \tau(A(s,1)) + \tau(A(s,101)) + \tau(A(s,1010)), \tag{2.31}$$

$$0 = \tau(A(s,10)) + \tau(A(s,110)) + \tau(A(s,1011)), \tag{2.32}$$

$$0 = \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,110)) + \tau(A(s,1100)), \tag{2.33}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,101)) \\ &\quad + \tau(A(s,110)) + \tau(A(s,1101)), \end{aligned} \tag{2.34}$$

for $s = A(i, 0^{2k})$. We find that $(A(i,0^{32}), E_{32}) = (A(i,0^{20}), E_{20})$, so that we only have to verify that identities (2.21) through (2.34) hold for $2 \leq k \leq 16$, which turns out to be true. □

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \geq 2$, and $u = 2^{2k-1}$,*

$$\begin{aligned} M_{2k} &= M^e[:7u] + x^u M^e[:6u] + x^{3u} M^e[:4u] + x^{4u} M^e[:3u] + x^{5u} M^e[:2u] \\ &\quad + x^{7u} R^e, \\ W_{2k} &= W^e[:7u] + x^u W^e[:6u] + x^{3u} W^e[:4u] + x^{4u} W^e[:3u] + x^{5u} W^e[:2u] \\ &\quad + x^{7u} R^e. \end{aligned}$$

For all integer $k \geq 2$, and $u = 2^{2k}$,

$$\begin{aligned} M_{2k+1} &= M^o[:7u] + x^u M^o[:6u] + x^{3u} M^o[:4u] + x^{4u} M^o[:3u] + x^{5u} M^o[:2u] \\ &\quad + x^{7u} R^o, \end{aligned}$$

$$\begin{aligned}W_{2k+1} = W^o[:7u] + x^u W^o[:6u] + x^{3u} W^o[:4u] + x^{4u} W^o[:3u] + x^{5u} W^o[:2u] \\ + x^{7u} R^o.\end{aligned}$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \geq 2$, we claim that

$$\begin{aligned}M_{2k} \wedge W_{2k} &\Rightarrow M_{2k+1} \wedge W_{2k+1}, \\ M_{2k+1} \wedge W_{2k+1} &\Rightarrow M_{2k+2} \wedge W_{2k+2}.\end{aligned}$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$\begin{aligned}W_{2k} M_{2k} = \big(&W^e[:7v] + x^v W^e[:6v] + x^{3v} W^e[:4v] + x^{4v} W^e[:3v] \\ &+ x^{5v} W^e[:2v] + x^{7u} R^e\big) \times \big(M^e[:7v] + x^v M^e[:6v] + x^{3v} M^e[:4v] \\ &+ x^{4v} M^e[:3v] + x^{5v} M^e[:2v] + x^{7u} R^e\big);\end{aligned} \tag{2.35}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{14v} R^o$. Therefore we only need to prove that their difference is $O(x^{14v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{14v}, to

$$\begin{aligned}W^e[:14v] M^e[:14v] + x^{2v} W^e[:12v] M^e[:12v] + x^{6v} W^e[:8v] M^e[:8v] \\ + x^{8v} W^e[:6v] M^e[:6v] + x^{10v} W^e[:4v] M^e[:4v].\end{aligned}$$

For all $n \leq 14$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$\begin{aligned}X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v] / X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v] / X^n, \\ c &:= R^e.\end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.35) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{14}, b_1, ..., b_{14}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{14})$, which completes the proof. $\square$

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \leq j, k \leq 1$,

$$\begin{aligned}\lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k}, \\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k}, \\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k}, \\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}.\end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a,b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x,y) = \mathrm{Res}_t\left(\phi(M^e,0,1)(x,t), y^{12}\phi(M^e,0,1)(x,t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x,y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned}
q_0(x) &= x^{22} + x^{17} + x^{15} + x^{13} + x^{10} + x^8 + x^7,\\
q_1(x) &= x^{20} + x^{19} + x^{18} + x^{12} + x^{10} + x^9 + x^8 + x^6 + x^4 + x^3 + x^2 + x,\\
q_2(x) &= x^{18} + x^{14} + x^4 + 1,\\
q_3(x) &= x^{20} + x^{19} + x^{18} + x^{12} + x^{10} + x^9 + x^8 + x^6 + x^4 + x^3 + x^2 + x,\\
q_4(x) &= x^{20} + x^{17} + x^{16} + x^{15} + x^{13} + x^{12} + x^{10} + x^8 + x^7 + x^4 + x^2.
\end{aligned}$$

The polynomial $Q(x,y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and $R(x,y) := P(x,y)/Q(x,y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 216, and substitute it for y in $R(z,y)$, we get a series with valuation smaller than 216, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b) = f(1/z)$. □

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e, 0, 0)$:

$$\begin{aligned}
p_0(x) = {} & x^{102} + x^{98} + x^{94} + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{80} + x^{76} + x^{74}\\
& + x^{72} + x^{70} + x^{66} + x^{64} + x^{60} + x^{54} + x^{46} + x^{42} + x^{26} + x^{16} + x^{14}\\
& + x^{12},\\
p_3(x) = {} & x^{96} + x^{94} + x^{92} + x^{90} + x^{88} + x^{84} + x^{82} + x^{78} + x^{74} + x^{72} + x^{64}\\
& + x^{62} + x^{58} + x^{52} + x^{44} + x^{42} + x^{40} + x^{32} + x^{28} + x^{18} + x^{16} + x^{14}\\
& + x^{12} + x^8,\\
p_6(x) = {} & x^{96} + x^{92} + x^{88} + x^{82} + x^{72} + x^{68} + x^{62} + x^{60} + x^{58} + x^{56} + x^{52}\\
& + x^{48} + x^{44} + x^{42} + x^{38} + x^{36} + x^{34} + x^{22} + x^{20} + x^{16} + x^{14} + x^{12}\\
& + x^8 + 1,
\end{aligned}$$

$$\begin{aligned}
p_9(x) &= x^{98} + x^{94} + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{80} + x^{72} + x^{70} + x^{60} \\
&\quad + x^{56} + x^{52} + x^{44} + x^{42} + x^{40} + x^{36} + x^{30} + x^{28} + x^{26} + x^{22} + x^{18} \\
&\quad + x^{16} + x^{12} + x^{10} + x^{6} + x^{4} + x^{2}, \\
p_{12}(x) &= x^{98} + x^{92} + x^{88} + x^{84} + x^{80} + x^{76} + x^{72} + x^{68} + x^{64} + x^{60} + x^{56} \\
&\quad + x^{52} + x^{50} + x^{48} + x^{18} + x^{12} + x^{8} + x^{4} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0].

For $\phi(M^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{117} + x^{115} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{105} + x^{103} + x^{99} \\
&\quad + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{87} + x^{86} + x^{82} + x^{81} + x^{80} + x^{79} \\
&\quad + x^{78} + x^{76} + x^{75} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{62} + x^{60} + x^{58} \\
&\quad + x^{56} + x^{51} + x^{49} + x^{48} + x^{43} + x^{42} + x^{38} + x^{37} + x^{36} + x^{34} + x^{33} \\
&\quad + x^{32} + x^{31} + x^{29} + x^{27}, \\
p_3(x) &= x^{113} + x^{112} + x^{111} + x^{110} + x^{108} + x^{106} + x^{104} + x^{102} + x^{100} + x^{99} \\
&\quad + x^{96} + x^{95} + x^{94} + x^{92} + x^{90} + x^{89} + x^{87} + x^{83} + x^{82} + x^{79} + x^{75} \\
&\quad + x^{74} + x^{71} + x^{67} + x^{66} + x^{63} + x^{59} + x^{58} + x^{57} + x^{56} + x^{54} + x^{52} \\
&\quad + x^{51} + x^{49} + x^{41} + x^{40} + x^{39} + x^{38} + x^{36} + x^{34} + x^{32} + x^{30} + x^{28} \\
&\quad + x^{27} + x^{24} + x^{23} + x^{22} + x^{20} + x^{18} + x^{16} + x^{14} + x^{12} + x^{11} + x^{9}, \\
p_6(x) &= x^{111} + x^{108} + x^{107} + x^{106} + x^{105} + x^{102} + x^{101} + x^{100} + x^{93} + x^{90} \\
&\quad + x^{88} + x^{86} + x^{85} + x^{84} + x^{81} + x^{79} + x^{78} + x^{77} + x^{74} + x^{72} + x^{70} \\
&\quad + x^{68} + x^{67} + x^{66} + x^{63} + x^{60} + x^{58} + x^{57} + x^{56} + x^{45} + x^{43} + x^{42} \\
&\quad + x^{41} + x^{39} + x^{38} + x^{37} + x^{35} + x^{34} + x^{31} + x^{29} + x^{28} + x^{25} + x^{22} \\
&\quad + x^{21} + x^{20} + x^{17} + x^{14} + x^{12} + x^{10} + x^{8} + x^{7} + x^{6}, \\
p_9(x) &= x^{109} + x^{108} + x^{105} + x^{103} + x^{102} + x^{99} + x^{95} + x^{94} + x^{93} + x^{92} + x^{90} \\
&\quad + x^{89} + x^{87} + x^{85} + x^{84} + x^{83} + x^{82} + x^{80} + x^{78} + x^{77} + x^{74} + x^{70} \\
&\quad + x^{67} + x^{65} + x^{64} + x^{60} + x^{59} + x^{58} + x^{56} + x^{54} + x^{52} + x^{51} + x^{49} \\
&\quad + x^{47} + x^{46} + x^{45} + x^{44} + x^{42} + x^{41} + x^{38} + x^{35} + x^{33} + x^{32} + x^{31} \\
&\quad + x^{30} + x^{29} + x^{27} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{18} + x^{17} + x^{16} \\
&\quad + x^{15} + x^{13} + x^{7} + x^{6} + x^{3}, \\
p_{12}(x) &= x^{107} + x^{105} + x^{104} + x^{103} + x^{101} + x^{100} + x^{95} + x^{93} + x^{92} + x^{91} \\
&\quad + x^{89} + x^{88} + x^{87} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{79} + x^{77} + x^{76} \\
&\quad + x^{75} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{61} \\
&\quad + x^{60} + x^{51} + x^{49} + x^{48} + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{15} \\
&\quad + x^{13} + x^{12} + x^{3} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1].

For $\phi(M^e, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{117} + x^{115} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{105} + x^{103} + x^{99} \\
&+ x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{87} + x^{86} + x^{82} + x^{81} + x^{80} + x^{79} \\
&+ x^{78} + x^{76} + x^{75} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{62} + x^{60} + x^{58} \\
&+ x^{56} + x^{51} + x^{49} + x^{48} + x^{43} + x^{42} + x^{38} + x^{37} + x^{36} + x^{34} + x^{33} \\
&+ x^{32} + x^{31} + x^{29} + x^{27}, \\
p_3(x) &= x^{113} + x^{112} + x^{111} + x^{110} + x^{108} + x^{106} + x^{104} + x^{102} + x^{100} + x^{99} \\
&+ x^{96} + x^{95} + x^{94} + x^{92} + x^{90} + x^{89} + x^{87} + x^{83} + x^{82} + x^{79} + x^{75} \\
&+ x^{74} + x^{71} + x^{67} + x^{66} + x^{63} + x^{59} + x^{58} + x^{57} + x^{56} + x^{54} + x^{52} \\
&+ x^{51} + x^{49} + x^{41} + x^{40} + x^{39} + x^{38} + x^{36} + x^{34} + x^{32} + x^{30} + x^{28} \\
&+ x^{27} + x^{24} + x^{23} + x^{22} + x^{20} + x^{18} + x^{16} + x^{14} + x^{12} + x^{11} + x^{9}, \\
p_6(x) &= x^{111} + x^{108} + x^{107} + x^{106} + x^{105} + x^{102} + x^{101} + x^{100} + x^{93} + x^{90} \\
&+ x^{88} + x^{86} + x^{85} + x^{84} + x^{81} + x^{79} + x^{78} + x^{77} + x^{74} + x^{72} + x^{70} \\
&+ x^{68} + x^{67} + x^{66} + x^{63} + x^{60} + x^{58} + x^{57} + x^{56} + x^{45} + x^{43} + x^{42} \\
&+ x^{41} + x^{39} + x^{38} + x^{37} + x^{35} + x^{34} + x^{31} + x^{29} + x^{28} + x^{25} + x^{22} \\
&+ x^{21} + x^{20} + x^{17} + x^{14} + x^{12} + x^{10} + x^{8} + x^{7} + x^{6}, \\
p_9(x) &= x^{109} + x^{108} + x^{105} + x^{103} + x^{102} + x^{99} + x^{95} + x^{94} + x^{93} + x^{92} + x^{90} \\
&+ x^{89} + x^{87} + x^{85} + x^{84} + x^{83} + x^{82} + x^{80} + x^{78} + x^{77} + x^{74} + x^{70} \\
&+ x^{67} + x^{65} + x^{64} + x^{60} + x^{59} + x^{58} + x^{56} + x^{54} + x^{52} + x^{51} + x^{49} \\
&+ x^{47} + x^{46} + x^{45} + x^{44} + x^{42} + x^{41} + x^{38} + x^{35} + x^{33} + x^{32} + x^{31} \\
&+ x^{30} + x^{29} + x^{27} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{18} + x^{17} + x^{16} \\
&+ x^{15} + x^{13} + x^{7} + x^{6} + x^{3}, \\
p_{12}(x) &= x^{107} + x^{105} + x^{104} + x^{103} + x^{101} + x^{100} + x^{95} + x^{93} + x^{92} + x^{91} \\
&+ x^{89} + x^{88} + x^{87} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{79} + x^{77} + x^{76} \\
&+ x^{75} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{61} \\
&+ x^{60} + x^{51} + x^{49} + x^{48} + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{15} \\
&+ x^{13} + x^{12} + x^{3} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1].

For $\phi(M^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{132} + x^{122} + x^{118} + x^{114} + x^{112} + x^{108} + x^{98} + x^{96} + x^{90} + x^{88} \\
&+ x^{80} + x^{78} + x^{70} + x^{68} + x^{66} + x^{60} + x^{56} + x^{52} + x^{46} + x^{44} + x^{42}, \\
p_3(x) &= x^{120} + x^{118} + x^{112} + x^{110} + x^{98} + x^{96} + x^{92} + x^{90} + x^{80} + x^{70} + x^{62} \\
&+ x^{54} + x^{52} + x^{50} + x^{48} + x^{40} + x^{38} + x^{28} + x^{22} + x^{20} + x^{10} + x^{6}, \\
p_6(x) &= x^{120} + x^{118} + x^{116} + x^{112} + x^{110} + x^{106} + x^{104} + x^{98} + x^{94} + x^{90} \\
&+ x^{86} + x^{82} + x^{80} + x^{78} + x^{76} + x^{68} + x^{66} + x^{64} + x^{62} + x^{60} + x^{58}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{56} + x^{54} + x^{52} + x^{36} + x^{30} + x^{28} + x^{26} + x^{14} + x^{12} + x^{8} + x^{6} + x^{2}\\
&+ 1,\\
p_9(x) &= x^{116} + x^{110} + x^{108} + x^{104} + x^{100} + x^{94} + x^{90} + x^{84} + x^{78} + x^{76} + x^{72}\\
&+ x^{68} + x^{62} + x^{60} + x^{58} + x^{54} + x^{52} + x^{48} + x^{44} + x^{38} + x^{34} + x^{28}\\
&+ x^{22} + x^{20} + x^{16} + x^{12} + x^{6} + x^{2},\\
p_{12}(x) &= x^{116} + x^{112} + x^{108} + x^{104} + x^{92} + x^{88} + x^{84} + x^{80} + x^{76} + x^{72} + x^{52}\\
&+ x^{48} + x^{36} + x^{32} + x^{28} + x^{24} + x^{20} + x^{16} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0].

For $\phi(W^e, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{132} + x^{124} + x^{122} + x^{118} + x^{116} + x^{114} + x^{106} + x^{104} + x^{102} + x^{100}\\
&+ x^{96} + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{76} + x^{74} + x^{72} + x^{70} + x^{66}\\
&+ x^{64} + x^{62} + x^{60} + x^{58} + x^{56} + x^{54} + x^{52} + x^{50} + x^{48} + x^{46} + x^{44}\\
&+ x^{34} + x^{24} + x^{22} + x^{20} + x^{12},\\
p_3(x) &= x^{120} + x^{118} + x^{108} + x^{104} + x^{102} + x^{96} + x^{92} + x^{90} + x^{86} + x^{84} + x^{80}\\
&+ x^{76} + x^{72} + x^{70} + x^{68} + x^{66} + x^{64} + x^{56} + x^{50} + x^{46} + x^{42} + x^{40}\\
&+ x^{38} + x^{30} + x^{22} + x^{16} + x^{12} + x^{8},\\
p_6(x) &= x^{120} + x^{118} + x^{116} + x^{106} + x^{102} + x^{96} + x^{94} + x^{92} + x^{90} + x^{82} + x^{78}\\
&+ x^{76} + x^{72} + x^{68} + x^{58} + x^{52} + x^{48} + x^{46} + x^{42} + x^{40} + x^{28} + x^{26}\\
&+ x^{24} + x^{16} + x^{14} + x^{12} + x^{10} + x^{4} + x^{2} + 1,\\
p_9(x) &= x^{116} + x^{110} + x^{108} + x^{104} + x^{100} + x^{94} + x^{90} + x^{84} + x^{78} + x^{76} + x^{72}\\
&+ x^{68} + x^{62} + x^{60} + x^{58} + x^{54} + x^{52} + x^{48} + x^{44} + x^{38} + x^{34} + x^{28}\\
&+ x^{22} + x^{20} + x^{16} + x^{12} + x^{6} + x^{2},\\
p_{12}(x) &= x^{116} + x^{112} + x^{108} + x^{104} + x^{92} + x^{88} + x^{84} + x^{80} + x^{76} + x^{72} + x^{52}\\
&+ x^{48} + x^{36} + x^{32} + x^{28} + x^{24} + x^{20} + x^{16} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{117} + x^{111} + x^{109} + x^{108} + x^{107} + x^{106} + x^{105} + x^{104} + x^{102} + x^{99}\\
&+ x^{97} + x^{96} + x^{95} + x^{94} + x^{83} + x^{82} + x^{79} + x^{78} + x^{77} + x^{74} + x^{73}\\
&+ x^{68} + x^{65} + x^{64} + x^{63} + x^{62} + x^{60} + x^{59} + x^{58} + x^{56} + x^{54} + x^{52}\\
&+ x^{50} + x^{46} + x^{44} + x^{39} + x^{37} + x^{34} + x^{32} + x^{27} + x^{22} + x^{21} + x^{20}\\
&+ x^{19} + x^{18},\\
p_3(x) &= x^{113} + x^{112} + x^{111} + x^{110} + x^{108} + x^{106} + x^{104} + x^{102} + x^{100} + x^{99}\\
&+ x^{96} + x^{95} + x^{94} + x^{92} + x^{90} + x^{89} + x^{87} + x^{83} + x^{82} + x^{79} + x^{75}\\
&+ x^{74} + x^{71} + x^{67} + x^{66} + x^{63} + x^{59} + x^{58} + x^{57} + x^{56} + x^{54} + x^{52}\\
&+ x^{51} + x^{49} + x^{41} + x^{40} + x^{39} + x^{38} + x^{36} + x^{34} + x^{32} + x^{30} + x^{28}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{27} + x^{24} + x^{23} + x^{22} + x^{20} + x^{18} + x^{16} + x^{14} + x^{12} + x^{11} + x^{9},\\
p_6(x) &= x^{111} + x^{108} + x^{107} + x^{106} + x^{105} + x^{102} + x^{101} + x^{100} + x^{93} + x^{90}\\
&+ x^{88} + x^{86} + x^{85} + x^{84} + x^{81} + x^{79} + x^{78} + x^{77} + x^{74} + x^{72} + x^{70}\\
&+ x^{68} + x^{67} + x^{66} + x^{63} + x^{60} + x^{58} + x^{57} + x^{56} + x^{45} + x^{43} + x^{42}\\
&+ x^{41} + x^{39} + x^{38} + x^{37} + x^{35} + x^{34} + x^{31} + x^{29} + x^{28} + x^{25} + x^{22}\\
&+ x^{21} + x^{20} + x^{17} + x^{14} + x^{12} + x^{10} + x^{8} + x^{7} + x^{6},\\
p_9(x) &= x^{109} + x^{108} + x^{105} + x^{103} + x^{102} + x^{99} + x^{95} + x^{94} + x^{93} + x^{92} + x^{90}\\
&+ x^{89} + x^{87} + x^{85} + x^{84} + x^{83} + x^{82} + x^{80} + x^{78} + x^{77} + x^{74} + x^{70}\\
&+ x^{67} + x^{65} + x^{64} + x^{60} + x^{59} + x^{58} + x^{56} + x^{54} + x^{52} + x^{51} + x^{49}\\
&+ x^{47} + x^{46} + x^{45} + x^{44} + x^{42} + x^{41} + x^{38} + x^{35} + x^{33} + x^{32} + x^{31}\\
&+ x^{30} + x^{29} + x^{27} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{18} + x^{17} + x^{16}\\
&+ x^{15} + x^{13} + x^{7} + x^{6} + x^{3},\\
p_{12}(x) &= x^{107} + x^{105} + x^{104} + x^{103} + x^{101} + x^{100} + x^{95} + x^{93} + x^{92} + x^{91}\\
&+ x^{89} + x^{88} + x^{87} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{79} + x^{77} + x^{76}\\
&+ x^{75} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{61}\\
&+ x^{60} + x^{51} + x^{49} + x^{48} + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{15}\\
&+ x^{13} + x^{12} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 1, 1, 1, 0, 0, 0, 1, 1, 1, 0, 0, 0, 1, 0].

For $\phi(W^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{117} + x^{111} + x^{109} + x^{108} + x^{107} + x^{106} + x^{105} + x^{104} + x^{102} + x^{99}\\
&+ x^{97} + x^{96} + x^{95} + x^{94} + x^{83} + x^{82} + x^{79} + x^{78} + x^{77} + x^{74} + x^{73}\\
&+ x^{68} + x^{65} + x^{64} + x^{63} + x^{62} + x^{60} + x^{59} + x^{58} + x^{56} + x^{54} + x^{52}\\
&+ x^{50} + x^{46} + x^{44} + x^{39} + x^{37} + x^{34} + x^{32} + x^{27} + x^{22} + x^{21} + x^{20}\\
&+ x^{19} + x^{18},\\
p_3(x) &= x^{113} + x^{112} + x^{111} + x^{110} + x^{108} + x^{106} + x^{104} + x^{102} + x^{100} + x^{99}\\
&+ x^{96} + x^{95} + x^{94} + x^{92} + x^{90} + x^{89} + x^{87} + x^{83} + x^{82} + x^{79} + x^{75}\\
&+ x^{74} + x^{71} + x^{67} + x^{66} + x^{63} + x^{59} + x^{58} + x^{57} + x^{56} + x^{54} + x^{52}\\
&+ x^{51} + x^{49} + x^{41} + x^{40} + x^{39} + x^{38} + x^{36} + x^{34} + x^{32} + x^{30} + x^{28}\\
&+ x^{27} + x^{24} + x^{23} + x^{22} + x^{20} + x^{18} + x^{16} + x^{14} + x^{12} + x^{11} + x^{9},\\
p_6(x) &= x^{111} + x^{108} + x^{107} + x^{106} + x^{105} + x^{102} + x^{101} + x^{100} + x^{93} + x^{90}\\
&+ x^{88} + x^{86} + x^{85} + x^{84} + x^{81} + x^{79} + x^{78} + x^{77} + x^{74} + x^{72} + x^{70}\\
&+ x^{68} + x^{67} + x^{66} + x^{63} + x^{60} + x^{58} + x^{57} + x^{56} + x^{45} + x^{43} + x^{42}\\
&+ x^{41} + x^{39} + x^{38} + x^{37} + x^{35} + x^{34} + x^{31} + x^{29} + x^{28} + x^{25} + x^{22}\\
&+ x^{21} + x^{20} + x^{17} + x^{14} + x^{12} + x^{10} + x^{8} + x^{7} + x^{6},\\
p_9(x) &= x^{109} + x^{108} + x^{105} + x^{103} + x^{102} + x^{99} + x^{95} + x^{94} + x^{93} + x^{92} + x^{90}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{89}+x^{87}+x^{85}+x^{84}+x^{83}+x^{82}+x^{80}+x^{78}+x^{77}+x^{74}+x^{70}\\
&+x^{67}+x^{65}+x^{64}+x^{60}+x^{59}+x^{58}+x^{56}+x^{54}+x^{52}+x^{51}+x^{49}\\
&+x^{47}+x^{46}+x^{45}+x^{44}+x^{42}+x^{41}+x^{38}+x^{35}+x^{33}+x^{32}+x^{31}\\
&+x^{30}+x^{29}+x^{27}+x^{25}+x^{24}+x^{23}+x^{22}+x^{21}+x^{18}+x^{17}+x^{16}\\
&+x^{15}+x^{13}+x^{7}+x^{6}+x^{3},\\
p_{12}(x)&=x^{107}+x^{105}+x^{104}+x^{103}+x^{101}+x^{100}+x^{95}+x^{93}+x^{92}+x^{91}\\
&+x^{89}+x^{88}+x^{87}+x^{85}+x^{84}+x^{83}+x^{81}+x^{80}+x^{79}+x^{77}+x^{76}\\
&+x^{75}+x^{73}+x^{72}+x^{71}+x^{69}+x^{68}+x^{67}+x^{65}+x^{64}+x^{63}+x^{61}\\
&+x^{60}+x^{51}+x^{49}+x^{48}+x^{27}+x^{25}+x^{24}+x^{23}+x^{21}+x^{20}+x^{15}\\
&+x^{13}+x^{12}+x^{3}+x+1,
\end{aligned}
$$

and the initial terms are [0, 1, 1, 1, 1, 0, 0, 0, 1, 1, 1, 0, 0, 0, 1, 0].

For $\phi(W^e,1,1)$:

$$
\begin{aligned}
p_0(x)&=x^{102}+x^{94}+x^{92}+x^{90}+x^{88}+x^{86}+x^{82}+x^{78}+x^{74}+x^{58}+x^{56}\\
&+x^{52}+x^{48}+x^{46}+x^{44}+x^{38}+x^{36}+x^{30}+x^{28}+x^{26}+x^{24},\\
p_3(x)&=x^{96}+x^{90}+x^{88}+x^{86}+x^{84}+x^{80}+x^{78}+x^{72}+x^{64}+x^{60}+x^{56}\\
&+x^{50}+x^{46}+x^{44}+x^{38}+x^{36}+x^{26}+x^{22}+x^{18}+x^{16}+x^{8}+x^{6},\\
p_6(x)&=x^{96}+x^{94}+x^{90}+x^{88}+x^{84}+x^{80}+x^{78}+x^{68}+x^{58}+x^{52}+x^{50}\\
&+x^{40}+x^{38}+x^{34}+x^{26}+x^{24}+x^{22}+x^{16}+x^{10}+x^{8}+x^{4}+1,\\
p_9(x)&=x^{98}+x^{94}+x^{90}+x^{88}+x^{86}+x^{84}+x^{82}+x^{80}+x^{72}+x^{70}+x^{60}\\
&+x^{56}+x^{52}+x^{44}+x^{42}+x^{40}+x^{36}+x^{30}+x^{28}+x^{26}+x^{22}+x^{18}\\
&+x^{16}+x^{12}+x^{10}+x^{6}+x^{4}+x^{2},\\
p_{12}(x)&=x^{98}+x^{92}+x^{88}+x^{84}+x^{80}+x^{76}+x^{72}+x^{68}+x^{64}+x^{60}+x^{56}\\
&+x^{52}+x^{50}+x^{48}+x^{18}+x^{12}+x^{8}+x^{4}+x^{2}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0].

For $\phi(M^o,0,0)$:

$$
\begin{aligned}
p_0(x)&=x^{117}+x^{115}+x^{112}+x^{110}+x^{109}+x^{107}+x^{106}+x^{105}+x^{101}+x^{99}\\
&+x^{98}+x^{94}+x^{93}+x^{91}+x^{90}+x^{87}+x^{85}+x^{82}+x^{78}+x^{77}+x^{76}\\
&+x^{71}+x^{70}+x^{67}+x^{66}+x^{61}+x^{60}+x^{59}+x^{57}+x^{56}+x^{54}+x^{50}\\
&+x^{49}+x^{48}+x^{46}+x^{45}+x^{44}+x^{41}+x^{40}+x^{38}+x^{37}+x^{36}+x^{35}\\
&+x^{29}+x^{28}+x^{27}+x^{20}+x^{19}+x^{18}+x^{16}+x^{14}+x^{13}+x^{12},\\
p_3(x)&=x^{115}+x^{112}+x^{111}+x^{109}+x^{108}+x^{106}+x^{105}+x^{104}+x^{103}+x^{102}\\
&+x^{99}+x^{98}+x^{96}+x^{94}+x^{91}+x^{89}+x^{84}+x^{83}+x^{80}+x^{79}+x^{78}\\
&+x^{75}+x^{70}+x^{69}+x^{63}+x^{61}+x^{60}+x^{57}+x^{56}+x^{55}+x^{49}+x^{46}\\
&+x^{45}+x^{44}+x^{39}+x^{38}+x^{34}+x^{32}+x^{31}+x^{28}+x^{24}+x^{23}+x^{22}\\
&+x^{21}+x^{20}+x^{19}+x^{17}+x^{14}+x^{13}+x^{10}+x^{9}+x^{8},
\end{aligned}
$$

$$
\begin{aligned}
p_6(x) = {} & x^{115} + x^{112} + x^{111} + x^{109} + x^{108} + x^{106} + x^{104} + x^{102} + x^{100} + x^{96} \\
& + x^{95} + x^{94} + x^{92} + x^{91} + x^{89} + x^{88} + x^{87} + x^{86} + x^{84} + x^{81} + x^{79} \\
& + x^{78} + x^{76} + x^{74} + x^{71} + x^{69} + x^{67} + x^{66} + x^{63} + x^{62} + x^{61} + x^{59} \\
& + x^{58} + x^{57} + x^{54} + x^{53} + x^{52} + x^{50} + x^{46} + x^{45} + x^{44} + x^{42} + x^{41} \\
& + x^{40} + x^{39} + x^{34} + x^{33} + x^{29} + x^{28} + x^{26} + x^{24} + x^{23} + x^{21} + x^{19} \\
& + x^{18} + x^{17} + x^{16} + x^{12} + x^{10} + x^{9} + x^{6} + x^{5} + x^{4} + x^{3} + x + 1, \\
p_9(x) = {} & x^{115} + x^{113} + x^{112} + x^{99} + x^{97} + x^{96} + x^{95} + x^{93} + x^{92} + x^{91} + x^{89} \\
& + x^{88} + x^{87} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{79} + x^{77} + x^{76} + x^{75} \\
& + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{63} + x^{61} + x^{60} + x^{59} + x^{57} + x^{56} \\
& + x^{55} + x^{53} + x^{52} + x^{35} + x^{33} + x^{32} + x^{15} + x^{13} + x^{12} + x^{11} + x^{9} \\
& + x^{8} + x^{7} + x^{5} + x^{4}, \\
p_{12}(x) = {} & x^{115} + x^{113} + x^{112} + x^{107} + x^{105} + x^{104} + x^{103} + x^{101} + x^{100} + x^{99} \\
& + x^{97} + x^{96} + x^{67} + x^{65} + x^{64} + x^{59} + x^{57} + x^{56} + x^{55} + x^{53} + x^{52} \\
& + x^{51} + x^{49} + x^{48} + x^{35} + x^{33} + x^{32} + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} \\
& + x^{20} + x^{11} + x^{9} + x^{8} + x^{7} + x^{5} + x^{4} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 1, 0, 1, 0, 1, 1, 1, 1, 0, 0, 0, 1, 0, 0].

For $\phi(M^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) = {} & x^{132} + x^{128} + x^{124} + x^{123} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117} + x^{114} \\
& + x^{113} + x^{112} + x^{103} + x^{102} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{89} \\
& + x^{87} + x^{86} + x^{85} + x^{82} + x^{81} + x^{78} + x^{76} + x^{75} + x^{73} + x^{71} + x^{69} \\
& + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{59} + x^{55} + x^{51} + x^{50} + x^{49} + x^{47} \\
& + x^{43} + x^{42} + x^{39} + x^{38} + x^{37} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} + x^{29} \\
& + x^{28} + x^{27}, \\
p_3(x) = {} & x^{118} + x^{117} + x^{116} + x^{109} + x^{107} + x^{106} + x^{103} + x^{100} + x^{98} + x^{96} \\
& + x^{95} + x^{94} + x^{93} + x^{90} + x^{89} + x^{88} + x^{81} + x^{79} + x^{77} + x^{76} + x^{75} \\
& + x^{72} + x^{65} + x^{63} + x^{62} + x^{59} + x^{56} + x^{54} + x^{52} + x^{51} + x^{50} + x^{49} \\
& + x^{46} + x^{45} + x^{44} + x^{37} + x^{35} + x^{34} + x^{31} + x^{28} + x^{26} + x^{24} + x^{23} \\
& + x^{20} + x^{18} + x^{17} + x^{16} + x^{14} + x^{12} + x^{11} + x^{10} + x^{9}, \\
p_6(x) = {} & x^{120} + x^{118} + x^{114} + x^{110} + x^{108} + x^{104} + x^{98} + x^{96} + x^{94} + x^{92} \\
& + x^{90} + x^{80} + x^{78} + x^{76} + x^{72} + x^{70} + x^{66} + x^{64} + x^{62} + x^{58} + x^{56} \\
& + x^{54} + x^{50} + x^{40} + x^{28} + x^{16} + x^{8} + x^{6}, \\
p_9(x) = {} & x^{122} + x^{121} + x^{119} + x^{117} + x^{116} + x^{114} + x^{111} + x^{110} + x^{109} + x^{105} \\
& + x^{103} + x^{101} + x^{97} + x^{96} + x^{95} + x^{93} + x^{92} + x^{91} + x^{89} + x^{88} + x^{87} \\
& + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{79} + x^{77} + x^{76} + x^{75} + x^{74} + x^{72} \\
& + x^{67} + x^{66} + x^{65} + x^{64} + x^{62} + x^{60} + x^{59} + x^{56} + x^{52} + x^{51} + x^{42}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{41}+x^{39}+x^{37}+x^{36}+x^{34}+x^{31}+x^{30}+x^{29}+x^{26}+x^{20}+x^{18}\\
&+x^{17}+x^{16}+x^{14}+x^{12}+x^{11}+x^{8}+x^{4}+x^{3},\\
p_{12}(x) &= x^{124}+x^{96}+x^{76}+x^{48}+x^{44}+x^{28}+x^{16}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0].

For $\phi(M^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{108}+x^{106}+x^{104}+x^{103}+x^{102}+x^{101}+x^{100}+x^{96}+x^{95}+x^{93}\\
&+x^{92}+x^{90}+x^{87}+x^{86}+x^{83}+x^{81}+x^{79}+x^{77}+x^{75}+x^{74}+x^{71}\\
&+x^{70}+x^{68}+x^{66}+x^{61}+x^{57}+x^{56}+x^{54}+x^{52}+x^{51}+x^{50}+x^{49}\\
&+x^{46}+x^{44}+x^{43}+x^{42}+x^{41}+x^{38}+x^{37}+x^{36}+x^{35}+x^{33}+x^{32}\\
&+x^{29}+x^{28}+x^{26}+x^{22}+x^{21}+x^{18},\\
p_3(x) &= x^{106}+x^{105}+x^{104}+x^{98}+x^{96}+x^{95}+x^{93}+x^{92}+x^{91}+x^{90}+x^{87}\\
&+x^{85}+x^{84}+x^{83}+x^{80}+x^{73}+x^{71}+x^{69}+x^{68}+x^{67}+x^{64}+x^{57}\\
&+x^{55}+x^{53}+x^{52}+x^{51}+x^{50}+x^{49}+x^{34}+x^{33}+x^{32}+x^{26}+x^{24}\\
&+x^{23}+x^{21}+x^{20}+x^{19}+x^{18}+x^{15}+x^{13}+x^{12}+x^{11}+x^{10}+x^{9},\\
p_6(x) &= x^{108}+x^{106}+x^{102}+x^{100}+x^{90}+x^{88}+x^{86}+x^{84}+x^{78}+x^{74}+x^{72}\\
&+x^{70}+x^{68}+x^{66}+x^{60}+x^{58}+x^{56}+x^{42}+x^{38}+x^{34}+x^{28}+x^{22}\\
&+x^{20}+x^{14}+x^{12}+x^{10}+x^{8}+x^{6},\\
p_9(x) &= x^{110}+x^{109}+x^{107}+x^{105}+x^{104}+x^{101}+x^{97}+x^{96}+x^{95}+x^{93}\\
&+x^{92}+x^{91}+x^{89}+x^{88}+x^{87}+x^{85}+x^{84}+x^{83}+x^{81}+x^{80}+x^{79}\\
&+x^{77}+x^{76}+x^{75}+x^{73}+x^{72}+x^{71}+x^{69}+x^{68}+x^{67}+x^{65}+x^{64}\\
&+x^{63}+x^{62}+x^{60}+x^{55}+x^{52}+x^{51}+x^{30}+x^{29}+x^{27}+x^{25}+x^{24}\\
&+x^{21}+x^{17}+x^{16}+x^{15}+x^{14}+x^{12}+x^{7}+x^{4}+x^{3},\\
p_{12}(x) &= x^{112}+x^{104}+x^{100}+x^{96}+x^{64}+x^{56}+x^{52}+x^{48}+x^{32}+x^{24}+x^{20}\\
&+x^{8}+x^{4}+1,
\end{aligned}
$$

and the initial terms are [0, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 1, 1, 0].

For $\phi(M^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{117}+x^{113}+x^{111}+x^{109}+x^{107}+x^{104}+x^{101}+x^{99}+x^{91}+x^{89}\\
&+x^{88}+x^{87}+x^{84}+x^{83}+x^{80}+x^{79}+x^{78}+x^{77}+x^{75}+x^{73}+x^{72}\\
&+x^{70}+x^{66}+x^{65}+x^{62}+x^{60}+x^{59}+x^{57}+x^{56}+x^{55}+x^{53}+x^{51}\\
&+x^{50}+x^{46}+x^{43}+x^{42}+x^{36}+x^{35}+x^{33},\\
p_3(x) &= x^{115}+x^{112}+x^{109}+x^{105}+x^{103}+x^{100}+x^{99}+x^{97}+x^{95}+x^{94}\\
&+x^{92}+x^{91}+x^{90}+x^{89}+x^{88}+x^{87}+x^{86}+x^{84}+x^{83}+x^{81}+x^{80}\\
&+x^{75}+x^{73}+x^{71}+x^{69}+x^{65}+x^{62}+x^{61}+x^{60}+x^{56}+x^{55}+x^{54}\\
&+x^{52}+x^{50}+x^{48}+x^{46}+x^{45}+x^{44}+x^{38}+x^{36}+x^{31}+x^{30}+x^{26}\\
&+x^{25}+x^{22}+x^{21}+x^{19}+x^{18}+x^{17}+x^{16}+x^{13}+x^{12},
\end{aligned}
$$

$$\begin{aligned}
p_6(x) = {} & x^{115} + x^{112} + x^{109} + x^{103} + x^{100} + x^{99} + x^{97} + x^{94} + x^{90} + x^{88} + x^{87} \\
& + x^{85} + x^{77} + x^{76} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} \\
& + x^{62} + x^{61} + x^{60} + x^{57} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} \\
& + x^{46} + x^{44} + x^{41} + x^{38} + x^{37} + x^{35} + x^{33} + x^{29} + x^{28} + x^{26} + x^{25} \\
& + x^{24} + x^{23} + x^{22} + x^{20} + x^{18} + x^{17} + x^{14} + x^{9} + x^{8} + x^{3} + x + 1, \\
p_9(x) = {} & x^{115} + x^{113} + x^{112} + x^{99} + x^{97} + x^{96} + x^{95} + x^{93} + x^{92} + x^{91} + x^{89} \\
& + x^{88} + x^{87} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{79} + x^{77} + x^{76} + x^{75} \\
& + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{63} + x^{61} + x^{60} + x^{59} + x^{57} + x^{56} \\
& + x^{55} + x^{53} + x^{52} + x^{35} + x^{33} + x^{32} + x^{15} + x^{13} + x^{12} + x^{11} + x^{9} \\
& + x^{8} + x^{7} + x^{5} + x^{4}, \\
p_{12}(x) = {} & x^{115} + x^{113} + x^{112} + x^{107} + x^{105} + x^{104} + x^{103} + x^{101} + x^{100} + x^{99} \\
& + x^{97} + x^{96} + x^{67} + x^{65} + x^{64} + x^{59} + x^{57} + x^{56} + x^{55} + x^{53} + x^{52} \\
& + x^{51} + x^{49} + x^{48} + x^{35} + x^{33} + x^{32} + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} \\
& + x^{20} + x^{11} + x^{9} + x^{8} + x^{7} + x^{5} + x^{4} + x^{3} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 1, 1, 0, 0].

For $\phi(W^o, 0, 0)$:

$$\begin{aligned}
p_0(x) = {} & x^{117} + x^{115} + x^{112} + x^{110} + x^{109} + x^{107} + x^{106} + x^{105} + x^{101} + x^{99} \\
& + x^{98} + x^{94} + x^{93} + x^{91} + x^{90} + x^{87} + x^{85} + x^{82} + x^{78} + x^{77} + x^{76} \\
& + x^{71} + x^{70} + x^{67} + x^{66} + x^{61} + x^{60} + x^{59} + x^{57} + x^{56} + x^{54} + x^{50} \\
& + x^{49} + x^{48} + x^{46} + x^{45} + x^{44} + x^{41} + x^{40} + x^{38} + x^{37} + x^{36} + x^{35} \\
& + x^{29} + x^{28} + x^{27} + x^{20} + x^{19} + x^{18} + x^{16} + x^{14} + x^{13} + x^{12}, \\
p_3(x) = {} & x^{115} + x^{112} + x^{111} + x^{109} + x^{108} + x^{106} + x^{105} + x^{104} + x^{103} + x^{102} \\
& + x^{99} + x^{98} + x^{96} + x^{94} + x^{91} + x^{89} + x^{84} + x^{83} + x^{80} + x^{79} + x^{78} \\
& + x^{75} + x^{70} + x^{69} + x^{63} + x^{61} + x^{60} + x^{57} + x^{56} + x^{55} + x^{49} + x^{46} \\
& + x^{45} + x^{44} + x^{39} + x^{38} + x^{34} + x^{32} + x^{31} + x^{28} + x^{24} + x^{23} + x^{22} \\
& + x^{21} + x^{20} + x^{19} + x^{17} + x^{14} + x^{13} + x^{10} + x^{9} + x^{8}, \\
p_6(x) = {} & x^{115} + x^{112} + x^{111} + x^{109} + x^{108} + x^{106} + x^{104} + x^{102} + x^{100} + x^{96} \\
& + x^{95} + x^{94} + x^{92} + x^{91} + x^{89} + x^{88} + x^{87} + x^{86} + x^{84} + x^{81} + x^{79} \\
& + x^{78} + x^{76} + x^{74} + x^{71} + x^{69} + x^{67} + x^{66} + x^{63} + x^{62} + x^{61} + x^{59} \\
& + x^{58} + x^{57} + x^{54} + x^{53} + x^{52} + x^{50} + x^{46} + x^{45} + x^{44} + x^{42} + x^{41} \\
& + x^{40} + x^{39} + x^{34} + x^{33} + x^{29} + x^{28} + x^{26} + x^{24} + x^{23} + x^{21} + x^{19} \\
& + x^{18} + x^{17} + x^{16} + x^{12} + x^{10} + x^{9} + x^{6} + x^{5} + x^{4} + x^{3} + x + 1, \\
p_9(x) = {} & x^{115} + x^{113} + x^{112} + x^{99} + x^{97} + x^{96} + x^{95} + x^{93} + x^{92} + x^{91} + x^{89} \\
& + x^{88} + x^{87} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{79} + x^{77} + x^{76} + x^{75} \\
& + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{63} + x^{61} + x^{60} + x^{59} + x^{57} + x^{56}
\end{aligned}$$

$$+ x^{55} + x^{53} + x^{52} + x^{35} + x^{33} + x^{32} + x^{15} + x^{13} + x^{12} + x^{11} + x^{9} + x^{8} + x^{7} + x^{5} + x^{4},$$

$$p_{12}(x) = x^{115} + x^{113} + x^{112} + x^{107} + x^{105} + x^{104} + x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{67} + x^{65} + x^{64} + x^{59} + x^{57} + x^{56} + x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{35} + x^{33} + x^{32} + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{11} + x^{9} + x^{8} + x^{7} + x^{5} + x^{4} + x^{3} + x + 1,$$

and the initial terms are [1, 0, 1, 0, 1, 0, 1, 1, 1, 1, 0, 0, 0, 1, 0, 0].

For $\phi(W^o, 0, 1)$:

$$p_0(x) = x^{108} + x^{106} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{96} + x^{95} + x^{93} + x^{92} + x^{90} + x^{87} + x^{86} + x^{83} + x^{81} + x^{79} + x^{77} + x^{75} + x^{74} + x^{71} + x^{70} + x^{68} + x^{66} + x^{61} + x^{57} + x^{56} + x^{54} + x^{52} + x^{51} + x^{50} + x^{49} + x^{46} + x^{44} + x^{43} + x^{42} + x^{41} + x^{38} + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{29} + x^{28} + x^{26} + x^{22} + x^{21} + x^{18},$$

$$p_3(x) = x^{106} + x^{105} + x^{104} + x^{98} + x^{96} + x^{95} + x^{93} + x^{92} + x^{91} + x^{90} + x^{87} + x^{85} + x^{84} + x^{83} + x^{80} + x^{73} + x^{71} + x^{69} + x^{68} + x^{67} + x^{64} + x^{57} + x^{55} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{34} + x^{33} + x^{32} + x^{26} + x^{24} + x^{23} + x^{21} + x^{20} + x^{19} + x^{18} + x^{15} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9},$$

$$p_6(x) = x^{108} + x^{106} + x^{102} + x^{100} + x^{90} + x^{88} + x^{86} + x^{84} + x^{78} + x^{74} + x^{72} + x^{70} + x^{68} + x^{66} + x^{60} + x^{58} + x^{56} + x^{42} + x^{38} + x^{34} + x^{28} + x^{22} + x^{20} + x^{14} + x^{12} + x^{10} + x^{8} + x^{6},$$

$$p_9(x) = x^{110} + x^{109} + x^{107} + x^{105} + x^{104} + x^{101} + x^{97} + x^{96} + x^{95} + x^{93} + x^{92} + x^{91} + x^{89} + x^{88} + x^{87} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{79} + x^{77} + x^{76} + x^{75} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{62} + x^{60} + x^{55} + x^{52} + x^{51} + x^{30} + x^{29} + x^{27} + x^{25} + x^{24} + x^{21} + x^{17} + x^{16} + x^{15} + x^{14} + x^{12} + x^{7} + x^{4} + x^{3},$$

$$p_{12}(x) = x^{112} + x^{104} + x^{100} + x^{96} + x^{64} + x^{56} + x^{52} + x^{48} + x^{32} + x^{24} + x^{20} + x^{8} + x^{4} + 1,$$

and the initial terms are [0, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 1, 1, 0].

For $\phi(W^o, 1, 0)$:

$$p_0(x) = x^{132} + x^{128} + x^{124} + x^{123} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117} + x^{114} + x^{113} + x^{112} + x^{103} + x^{102} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{89} + x^{87} + x^{86} + x^{85} + x^{82} + x^{81} + x^{78} + x^{76} + x^{75} + x^{73} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{59} + x^{55} + x^{51} + x^{50} + x^{49} + x^{47} + x^{43} + x^{42} + x^{39} + x^{38} + x^{37} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} + x^{29} + x^{28} + x^{27},$$

$$\begin{aligned}
p_3(x) &= x^{118} + x^{117} + x^{116} + x^{109} + x^{107} + x^{106} + x^{103} + x^{100} + x^{98} + x^{96} \\
&\quad + x^{95} + x^{94} + x^{93} + x^{90} + x^{89} + x^{88} + x^{81} + x^{79} + x^{77} + x^{76} + x^{75} \\
&\quad + x^{72} + x^{65} + x^{63} + x^{62} + x^{59} + x^{56} + x^{54} + x^{52} + x^{51} + x^{50} + x^{49} \\
&\quad + x^{46} + x^{45} + x^{44} + x^{37} + x^{35} + x^{34} + x^{31} + x^{28} + x^{26} + x^{24} + x^{23} \\
&\quad + x^{20} + x^{18} + x^{17} + x^{16} + x^{14} + x^{12} + x^{11} + x^{10} + x^{9}, \\
p_6(x) &= x^{120} + x^{118} + x^{114} + x^{110} + x^{108} + x^{104} + x^{98} + x^{96} + x^{94} + x^{92} \\
&\quad + x^{90} + x^{80} + x^{78} + x^{76} + x^{72} + x^{70} + x^{66} + x^{64} + x^{62} + x^{58} + x^{56} \\
&\quad + x^{54} + x^{50} + x^{40} + x^{28} + x^{16} + x^{8} + x^{6}, \\
p_9(x) &= x^{122} + x^{121} + x^{119} + x^{117} + x^{116} + x^{114} + x^{111} + x^{110} + x^{109} + x^{105} \\
&\quad + x^{103} + x^{101} + x^{97} + x^{96} + x^{95} + x^{93} + x^{92} + x^{91} + x^{89} + x^{88} + x^{87} \\
&\quad + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{79} + x^{77} + x^{76} + x^{75} + x^{74} + x^{72} \\
&\quad + x^{67} + x^{66} + x^{65} + x^{64} + x^{62} + x^{60} + x^{59} + x^{56} + x^{52} + x^{51} + x^{42} \\
&\quad + x^{41} + x^{39} + x^{37} + x^{36} + x^{34} + x^{31} + x^{30} + x^{29} + x^{26} + x^{20} + x^{18} \\
&\quad + x^{17} + x^{16} + x^{14} + x^{12} + x^{11} + x^{8} + x^{4} + x^{3}, \\
p_{12}(x) &= x^{124} + x^{96} + x^{76} + x^{48} + x^{44} + x^{28} + x^{16} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0].

For $\phi(W^o, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{117} + x^{113} + x^{111} + x^{109} + x^{107} + x^{104} + x^{101} + x^{99} + x^{91} + x^{89} \\
&\quad + x^{88} + x^{87} + x^{84} + x^{83} + x^{80} + x^{79} + x^{78} + x^{77} + x^{75} + x^{73} + x^{72} \\
&\quad + x^{70} + x^{66} + x^{65} + x^{62} + x^{60} + x^{59} + x^{57} + x^{56} + x^{55} + x^{53} + x^{51} \\
&\quad + x^{50} + x^{46} + x^{43} + x^{42} + x^{36} + x^{35} + x^{33}, \\
p_3(x) &= x^{115} + x^{112} + x^{109} + x^{105} + x^{103} + x^{100} + x^{99} + x^{97} + x^{95} + x^{94} \\
&\quad + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{84} + x^{83} + x^{81} + x^{80} \\
&\quad + x^{75} + x^{73} + x^{71} + x^{69} + x^{65} + x^{62} + x^{61} + x^{60} + x^{56} + x^{55} + x^{54} \\
&\quad + x^{52} + x^{50} + x^{48} + x^{46} + x^{45} + x^{44} + x^{38} + x^{36} + x^{31} + x^{30} + x^{26} \\
&\quad + x^{25} + x^{22} + x^{21} + x^{19} + x^{18} + x^{17} + x^{16} + x^{13} + x^{12}, \\
p_6(x) &= x^{115} + x^{112} + x^{109} + x^{103} + x^{100} + x^{99} + x^{97} + x^{94} + x^{90} + x^{88} + x^{87} \\
&\quad + x^{85} + x^{77} + x^{76} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} \\
&\quad + x^{62} + x^{61} + x^{60} + x^{57} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} \\
&\quad + x^{46} + x^{44} + x^{41} + x^{38} + x^{37} + x^{35} + x^{33} + x^{29} + x^{28} + x^{26} + x^{25} \\
&\quad + x^{24} + x^{23} + x^{22} + x^{20} + x^{18} + x^{17} + x^{14} + x^{9} + x^{8} + x^{3} + x + 1, \\
p_9(x) &= x^{115} + x^{113} + x^{112} + x^{99} + x^{97} + x^{96} + x^{95} + x^{93} + x^{92} + x^{91} + x^{89} \\
&\quad + x^{88} + x^{87} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{79} + x^{77} + x^{76} + x^{75} \\
&\quad + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{63} + x^{61} + x^{60} + x^{59} + x^{57} + x^{56} \\
&\quad + x^{55} + x^{53} + x^{52} + x^{35} + x^{33} + x^{32} + x^{15} + x^{13} + x^{12} + x^{11} + x^{9}
\end{aligned}$$

$$
\begin{aligned}
&+x^8+x^7+x^5+x^4,\\
p_{12}(x) = {}&x^{115}+x^{113}+x^{112}+x^{107}+x^{105}+x^{104}+x^{103}+x^{101}+x^{100}+x^{99}\\
&+x^{97}+x^{96}+x^{67}+x^{65}+x^{64}+x^{59}+x^{57}+x^{56}+x^{55}+x^{53}+x^{52}\\
&+x^{51}+x^{49}+x^{48}+x^{35}+x^{33}+x^{32}+x^{27}+x^{25}+x^{24}+x^{23}+x^{21}\\
&+x^{20}+x^{11}+x^{9}+x^{8}+x^{7}+x^{5}+x^{4}+x^{3}+x+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 1, 1, 0, 0].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	165	[276, 277]	330	[473, 236]	495	[32, 395]	660	[736, 272]
1	[3, 4]	166	[278, 279]	331	[209, 474]	496	[632, 166]	661	[737, 114]
2	[5, 6]	167	[280, 281]	332	[475, 476]	497	[633, 620]	662	[738, 562]
3	[7, 8]	168	[282, 283]	333	[247, 298]	498	[390, 634]	663	[81, 513]
4	[9, 10]	169	[284, 112]	334	[263, 477]	499	[635, 380]	664	[739, 740]
5	[11, 12]	170	[285, 286]	335	[478, 479]	500	[631, 384]	665	[466, 243]
6	[13, 14]	171	[287, 92]	336	[480, 481]	501	[407, 137]	666	[274, 571]
7	[15, 16]	172	[288, 289]	337	[482, 483]	502	[612, 404]	667	[300, 26]
8	[17, 18]	173	[290, 291]	338	[484, 485]	503	[340, 636]	668	[741, 713]
9	[19, 20]	174	[292, 293]	339	[486, 204]	504	[41, 135]	669	[742, 743]
10	[21, 22]	175	[291, 103]	340	[487, 488]	505	[181, 365]	670	[205, 713]
11	[23, 24]	176	[294, 295]	341	[198, 262]	506	[621, 394]	671	[744, 241]
12	[25, 26]	177	[101, 293]	342	[463, 28]	507	[395, 349]	672	[745, 503]
13	[27, 28]	178	[296, 290]	343	[72, 104]	508	[57, 637]	673	[746, 573]
14	[29, 30]	179	[297, 298]	344	[489, 17]	509	[638, 639]	674	[747, 461]
15	[31, 32]	180	[299, 300]	345	[490, 491]	510	[151, 640]	675	[748, 532]
16	[33, 34]	181	[301, 63]	346	[492, 493]	511	[47, 415]	676	[749, 207]
17	[35, 36]	182	[302, 303]	347	[286, 494]	512	[370, 641]	677	[750, 468]
18	[37, 38]	183	[64, 304]	348	[495, 496]	513	[642, 643]	678	[236, 24]
19	[39, 40]	184	[305, 92]	349	[66, 106]	514	[644, 185]	679	[751, 519]
20	[41, 42]	185	[306, 307]	350	[497, 498]	515	[645, 602]	680	[752, 701]
21	[43, 44]	186	[308, 279]	351	[499, 268]	516	[446, 424]	681	[753, 740]
22	[45, 46]	187	[309, 304]	352	[500, 283]	517	[33, 646]	682	[754, 77]
23	[47, 48]	188	[245, 89]	353	[501, 228]	518	[344, 55]	683	[755, 500]
24	[49, 50]	189	[310, 286]	354	[502, 229]	519	[647, 154]	684	[756, 731]
25	[51, 52]	190	[311, 312]	355	[503, 277]	520	[620, 43]	685	[757, 211]
26	[53, 54]	191	[313, 234]	356	[504, 73]	521	[607, 403]	686	[277, 488]
27	[55, 56]	192	[314, 267]	357	[505, 232]	522	[193, 646]	687	[758, 526]
28	[57, 58]	193	[315, 316]	358	[506, 268]	523	[192, 47]	688	[79, 114]
29	[59, 60]	194	[186, 149]	359	[496, 507]	524	[648, 649]	689	[606, 438]
30	[61, 62]	195	[317, 318]	360	[508, 509]	525	[650, 351]	690	[345, 611]
31	[63, 64]	196	[319, 320]	361	[510, 511]	526	[173, 432]	691	[635, 370]

32	[65, 66]	197	[52, 174]	362	[512, 513]	527	[606, 388]	692	[319, 682]
33	[67, 68]	198	[321, 175]	363	[514, 216]	528	[355, 44]	693	[323, 182]
34	[69, 70]	199	[322, 323]	364	[515, 516]	529	[651, 167]	694	[663, 759]
35	[71, 72]	200	[324, 325]	365	[498, 305]	530	[652, 334]	695	[688, 154]
36	[73, 74]	201	[326, 159]	366	[76, 465]	531	[127, 609]	696	[760, 192]
37	[75, 76]	202	[327, 328]	367	[517, 518]	532	[129, 132]	697	[761, 148]
38	[77, 22]	203	[329, 330]	368	[519, 222]	533	[653, 328]	698	[45, 762]
39	[78, 79]	204	[185, 46]	369	[520, 521]	534	[42, 136]	699	[686, 373]
40	[80, 81]	205	[331, 332]	370	[522, 481]	535	[43, 371]	700	[435, 419]
41	[82, 83]	206	[333, 334]	371	[85, 523]	536	[423, 654]	701	[763, 764]
42	[84, 85]	207	[335, 156]	372	[524, 525]	537	[384, 655]	702	[335, 765]
43	[86, 87]	208	[336, 337]	373	[526, 300]	538	[420, 161]	703	[176, 51]
44	[88, 89]	209	[338, 339]	374	[468, 527]	539	[432, 400]	704	[352, 13]
45	[90, 91]	210	[340, 341]	375	[516, 458]	540	[656, 447]	705	[357, 130]
46	[92, 80]	211	[342, 318]	376	[528, 529]	541	[657, 604]	706	[766, 767]
47	[93, 94]	212	[343, 344]	377	[201, 473]	542	[124, 2]	707	[61, 451]
48	[95, 30]	213	[345, 346]	378	[530, 230]	543	[425, 120]	708	[414, 158]
49	[96, 97]	214	[190, 347]	379	[531, 532]	544	[431, 408]	709	[49, 138]
50	[98, 93]	215	[348, 336]	380	[533, 534]	545	[656, 440]	710	[184, 45]
51	[99, 54]	216	[183, 349]	381	[521, 535]	546	[658, 659]	711	[453, 397]
52	[100, 101]	217	[350, 120]	382	[536, 194]	547	[190, 325]	712	[166, 768]
53	[102, 103]	218	[351, 352]	383	[87, 256]	548	[171, 602]	713	[769, 445]
54	[54, 54]	219	[160, 353]	384	[537, 466]	549	[136, 141]	714	[170, 357]
55	[104, 87]	220	[354, 355]	385	[538, 303]	550	[128, 131]	715	[401, 176]
56	[105, 106]	221	[356, 357]	386	[539, 262]	551	[650, 364]	716	[761, 186]
57	[16, 107]	222	[358, 326]	387	[264, 201]	552	[434, 176]	717	[183, 396]
58	[108, 109]	223	[336, 359]	388	[540, 220]	553	[401, 321]	718	[770, 336]
59	[110, 111]	224	[360, 361]	389	[541, 218]	554	[162, 187]	719	[771, 772]
60	[112, 113]	225	[362, 123]	390	[542, 470]	555	[327, 660]	720	[187, 601]
61	[30, 63]	226	[189, 324]	391	[543, 493]	556	[661, 387]	721	[42, 140]
62	[114, 115]	227	[363, 60]	392	[70, 531]	557	[662, 412]	722	[661, 606]
63	[60, 116]	228	[364, 352]	393	[544, 108]	558	[663, 664]	723	[773, 431]
64	[117, 118]	229	[323, 365]	394	[545, 91]	559	[356, 35]	724	[378, 625]
65	[59, 119]	230	[150, 366]	395	[546, 274]	560	[665, 42]	725	[628, 404]
66	[120, 121]	231	[120, 367]	396	[547, 18]	561	[666, 667]	726	[774, 628]
67	[122, 123]	232	[368, 342]	397	[548, 549]	562	[668, 669]	727	[775, 134]
68	[124, 125]	233	[369, 370]	398	[550, 214]	563	[616, 667]	728	[38, 776]
69	[126, 127]	234	[355, 371]	399	[551, 97]	564	[648, 442]	729	[777, 191]
70	[128, 129]	235	[372, 373]	400	[552, 553]	565	[422, 359]	730	[630, 778]
71	[35, 130]	236	[159, 374]	401	[553, 101]	566	[670, 671]	731	[680, 779]
72	[131, 132]	237	[352, 375]	402	[554, 548]	567	[179, 659]	732	[426, 367]
73	[133, 134]	238	[39, 376]	403	[555, 289]	568	[672, 354]	733	[775, 444]
74	[135, 136]	239	[377, 329]	404	[556, 113]	569	[632, 651]	734	[780, 779]
75	[126, 137]	240	[378, 379]	405	[557, 558]	570	[10, 673]	735	[31, 441]

76	[138, 139]	241	[380, 381]	406	[559, 489]	571	[674, 414]	736	[781, 190]
77	[140, 141]	242	[192, 382]	407	[560, 538]	572	[146, 675]	737	[782, 392]
78	[142, 143]	243	[383, 12]	408	[561, 195]	573	[426, 121]	738	[394, 783]
79	[144, 145]	244	[358, 158]	409	[476, 562]	574	[615, 676]	739	[117, 417]
80	[146, 147]	245	[384, 385]	410	[563, 564]	575	[191, 677]	740	[10, 374]
81	[148, 149]	246	[333, 386]	411	[565, 566]	576	[429, 406]	741	[784, 785]
82	[37, 150]	247	[375, 14]	412	[567, 6]	577	[678, 640]	742	[408, 675]
83	[151, 152]	248	[387, 388]	413	[568, 540]	578	[168, 435]	743	[172, 603]
84	[153, 131]	249	[389, 342]	414	[569, 516]	579	[677, 47]	744	[165, 56]
85	[154, 62]	250	[390, 391]	415	[267, 505]	580	[679, 320]	745	[168, 418]
86	[11, 155]	251	[392, 330]	416	[570, 571]	581	[158, 10]	746	[760, 677]
87	[132, 156]	252	[393, 394]	417	[258, 258]	582	[680, 681]	747	[627, 612]
88	[137, 157]	253	[395, 396]	418	[572, 223]	583	[350, 426]	748	[604, 655]
89	[158, 159]	254	[394, 397]	419	[283, 573]	584	[682, 368]	749	[678, 152]
90	[160, 161]	255	[342, 398]	420	[574, 491]	585	[321, 51]	750	[781, 324]
91	[162, 163]	256	[399, 59]	421	[575, 531]	586	[364, 452]	751	[674, 358]
92	[164, 148]	257	[400, 178]	422	[576, 534]	587	[683, 351]	752	[186, 786]
93	[165, 8]	258	[178, 401]	423	[577, 247]	588	[684, 391]	753	[448, 636]
94	[166, 167]	259	[402, 403]	424	[273, 578]	589	[411, 685]	754	[770, 422]
95	[50, 168]	260	[127, 157]	425	[239, 202]	590	[657, 384]	755	[437, 153]
96	[169, 59]	261	[404, 38]	426	[573, 579]	591	[320, 389]	756	[787, 664]
97	[170, 35]	262	[174, 177]	427	[580, 314]	592	[322, 181]	757	[416, 118]
98	[171, 172]	263	[175, 173]	428	[581, 246]	593	[138, 168]	758	[433, 434]
99	[173, 174]	264	[357, 36]	429	[582, 225]	594	[686, 687]	759	[788, 789]
100	[175, 52]	265	[405, 406]	430	[583, 584]	595	[647, 61]	760	[790, 93]
101	[176, 175]	266	[407, 127]	431	[493, 66]	596	[389, 317]	761	[494, 521]
102	[177, 178]	267	[320, 368]	432	[103, 553]	597	[653, 660]	762	[791, 281]
103	[51, 173]	268	[408, 147]	433	[293, 295]	598	[688, 61]	763	[509, 79]
104	[179, 180]	269	[409, 410]	434	[585, 291]	599	[684, 634]	764	[792, 711]
105	[181, 182]	270	[411, 412]	435	[234, 479]	600	[372, 687]	765	[793, 465]
106	[119, 116]	271	[413, 387]	436	[586, 527]	601	[689, 523]	766	[794, 560]
107	[32, 183]	272	[414, 326]	437	[587, 217]	602	[690, 107]	767	[109, 204]
108	[184, 185]	273	[324, 347]	438	[588, 74]	603	[691, 483]	768	[795, 80]
109	[55, 8]	274	[382, 415]	439	[589, 68]	604	[527, 312]	769	[579, 503]
110	[164, 186]	275	[392, 378]	440	[590, 584]	605	[692, 112]	770	[220, 280]
111	[187, 188]	276	[416, 417]	441	[272, 248]	606	[693, 579]	771	[796, 239]
112	[189, 190]	277	[418, 419]	442	[113, 214]	607	[518, 108]	772	[507, 743]
113	[191, 192]	278	[420, 353]	443	[591, 248]	608	[232, 83]	773	[797, 104]
114	[193, 34]	279	[421, 55]	444	[592, 70]	609	[213, 518]	774	[798, 500]
115	[145, 163]	280	[348, 422]	445	[593, 273]	610	[694, 270]	775	[799, 558]
116	[111, 194]	281	[423, 424]	446	[594, 195]	611	[695, 696]	776	[800, 264]
117	[195, 196]	282	[425, 426]	447	[595, 489]	612	[697, 463]	777	[801, 217]
118	[197, 198]	283	[153, 129]	448	[230, 525]	613	[698, 238]	778	[802, 511]
119	[199, 76]	284	[427, 191]	449	[596, 594]	614	[699, 700]	779	[803, 789]

120	[200, 201]	285	[428, 421]	450	[584, 245]	615	[241, 701]	780	[743, 238]
121	[202, 203]	286	[429, 430]	451	[597, 494]	616	[702, 196]	781	[474, 73]
122	[204, 205]	287	[421, 165]	452	[598, 29]	617	[703, 263]	782	[804, 64]
123	[206, 207]	288	[405, 430]	453	[599, 307]	618	[704, 473]	783	[805, 240]
124	[208, 209]	289	[431, 146]	454	[485, 498]	619	[705, 578]	784	[562, 285]
125	[210, 211]	290	[432, 177]	455	[529, 209]	620	[696, 85]	785	[483, 711]
126	[212, 213]	291	[433, 401]	456	[600, 564]	621	[74, 548]	786	[806, 549]
127	[214, 72]	292	[174, 400]	457	[163, 601]	622	[706, 285]	787	[807, 205]
128	[215, 216]	293	[52, 432]	458	[602, 603]	623	[549, 529]	788	[658, 180]
129	[217, 218]	294	[53, 53]	459	[604, 385]	624	[312, 246]	789	[38, 366]
130	[219, 220]	295	[400, 433]	460	[605, 50]	625	[707, 249]	790	[808, 166]
131	[221, 222]	296	[178, 434]	461	[413, 606]	626	[708, 505]	791	[644, 45]
132	[216, 223]	297	[435, 436]	462	[607, 57]	627	[709, 75]	792	[446, 654]
133	[224, 225]	298	[437, 128]	463	[330, 379]	628	[710, 77]	793	[363, 119]
134	[226, 227]	299	[54, 53]	464	[608, 162]	629	[711, 509]	794	[809, 810]
135	[228, 229]	300	[434, 321]	465	[137, 609]	630	[712, 566]	795	[402, 57]
136	[83, 230]	301	[387, 438]	466	[139, 435]	631	[713, 213]	796	[811, 812]
137	[231, 232]	302	[439, 440]	467	[610, 611]	632	[203, 496]	797	[119, 648]
138	[233, 234]	303	[441, 183]	468	[612, 37]	633	[261, 594]	798	[813, 414]
139	[97, 29]	304	[116, 442]	469	[613, 614]	634	[714, 696]	799	[362, 814]
140	[235, 236]	305	[443, 146]	470	[615, 614]	635	[303, 252]	800	[682, 389]
141	[229, 237]	306	[133, 444]	471	[454, 323]	636	[715, 477]	801	[399, 363]
142	[238, 239]	307	[445, 446]	472	[351, 452]	637	[716, 535]	802	[652, 386]
143	[240, 241]	308	[439, 447]	473	[616, 617]	638	[513, 485]	803	[383, 155]
144	[242, 243]	309	[448, 341]	474	[618, 32]	639	[717, 81]	804	[129, 335]
145	[244, 245]	310	[449, 450]	475	[140, 619]	640	[26, 198]	805	[369, 380]
146	[246, 247]	311	[154, 451]	476	[403, 58]	641	[718, 109]	806	[808, 651]
147	[248, 249]	312	[452, 375]	477	[177, 433]	642	[719, 474]	807	[602, 815]
148	[250, 251]	313	[393, 453]	478	[136, 619]	643	[720, 565]	808	[816, 280]
149	[252, 240]	314	[454, 181]	479	[370, 381]	644	[721, 691]	809	[566, 568]
150	[253, 254]	315	[455, 456]	480	[610, 346]	645	[115, 691]	810	[535, 476]
151	[255, 256]	316	[455, 361]	481	[620, 355]	646	[722, 227]	811	[817, 568]
152	[257, 258]	317	[457, 458]	482	[621, 453]	647	[723, 540]	812	[281, 565]
153	[259, 111]	318	[218, 459]	483	[163, 188]	648	[724, 94]	813	[818, 519]
154	[260, 261]	319	[460, 461]	484	[622, 623]	649	[725, 459]	814	[819, 700]
155	[262, 263]	320	[462, 463]	485	[624, 443]	650	[726, 75]	815	[820, 507]
156	[222, 264]	321	[295, 290]	486	[330, 625]	651	[727, 251]	816	[445, 423]
157	[265, 203]	322	[464, 202]	487	[13, 626]	652	[728, 729]	817	[821, 40]
158	[266, 267]	323	[459, 261]	488	[627, 628]	653	[730, 731]	818	[822, 128]
159	[268, 254]	324	[465, 466]	489	[628, 37]	654	[732, 237]	819	[666, 617]
160	[269, 270]	325	[467, 115]	490	[629, 125]	655	[733, 305]	820	[645, 172]
161	[271, 272]	326	[461, 468]	491	[630, 410]	656	[734, 316]	821	[701, 560]
162	[227, 273]	327	[469, 470]	492	[631, 604]	657	[735, 538]	822	[823, 314]
163	[243, 274]	328	[471, 228]	493	[50, 139]	658	[304, 729]	823	[777, 760]

164	[275, 252]	329	[472, 17]	494	[450, 446]	659	[477, 526]	

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	118	0	236	0	354	1	472	1	590	0	708	1
1	0	119	1	237	1	355	0	473	1	591	0	709	0
2	1	120	0	238	1	356	0	474	1	592	1	710	0
3	0	121	1	239	0	357	1	475	0	593	0	711	1
4	1	122	0	240	0	358	0	476	1	594	1	712	0
5	1	123	1	241	0	359	1	477	1	595	1	713	1
6	0	124	0	242	0	360	0	478	1	596	0	714	1
7	0	125	0	243	0	361	1	479	1	597	0	715	1
8	0	126	0	244	0	362	1	480	0	598	0	716	1
9	1	127	1	245	1	363	1	481	0	599	1	717	0
10	1	128	0	246	1	364	0	482	0	600	0	718	1
11	1	129	1	247	1	365	0	483	0	601	0	719	0
12	1	130	1	248	1	366	0	484	1	602	1	720	1
13	0	131	0	249	0	367	0	485	0	603	1	721	1
14	0	132	0	250	0	368	1	486	1	604	0	722	1
15	0	133	0	251	0	369	0	487	0	605	1	723	1
16	0	134	0	252	1	370	1	488	0	606	0	724	0
17	0	135	0	253	1	371	1	489	0	607	0	725	0
18	0	136	1	254	0	372	0	490	1	608	1	726	1
19	1	137	0	255	1	373	1	491	0	609	1	727	1
20	0	138	0	256	1	374	1	492	1	610	0	728	0
21	1	139	1	257	0	375	1	493	1	611	0	729	1
22	1	140	0	258	1	376	0	494	1	612	1	730	0
23	1	141	0	259	1	377	0	495	0	613	1	731	1
24	0	142	1	260	1	378	0	496	1	614	1	732	1
25	1	143	0	261	1	379	1	497	1	615	0	733	1
26	1	144	0	262	0	380	0	498	0	616	1	734	0
27	0	145	0	263	0	381	0	499	1	617	1	735	0
28	0	146	1	264	1	382	0	500	1	618	1	736	1
29	0	147	1	265	0	383	0	501	1	619	1	737	1
30	0	148	0	266	1	384	1	502	1	620	0	738	0
31	0	149	1	267	0	385	0	503	0	621	0	739	0
32	0	150	1	268	0	386	1	504	0	622	1	740	1
33	0	151	1	269	1	387	1	505	1	623	1	741	0
34	0	152	0	270	1	388	0	506	0	624	0	742	0
35	0	153	1	271	0	389	0	507	1	625	0	743	0
36	0	154	1	272	1	390	0	508	0	626	1	744	1
37	0	155	0	273	0	391	0	509	0	627	0	745	0
38	0	156	0	274	0	392	0	510	1	628	0	746	0
39	1	157	0	275	0	393	1	511	1	629	1	747	0

40	1	158	1	276	1	394	0	512	1	630	0	748	0
41	0	159	0	277	1	395	1	513	0	631	1	749	0
42	1	160	1	278	0	396	1	514	1	632	1	750	1
43	1	161	0	279	1	397	1	515	0	633	1	751	0
44	0	162	1	280	0	398	0	516	1	634	1	752	1
45	1	163	0	281	0	399	1	517	0	635	1	753	1
46	0	164	0	282	0	400	0	518	0	636	1	754	1
47	1	165	1	283	1	401	1	519	1	637	1	755	0
48	1	166	0	284	0	402	1	520	0	638	0	756	1
49	0	167	0	285	1	403	1	521	0	639	0	757	1
50	1	168	0	286	1	404	1	522	1	640	1	758	0
51	1	169	0	287	1	405	0	523	0	641	1	759	1
52	0	170	1	288	0	406	0	524	0	642	0	760	0
53	1	171	1	289	1	407	1	525	1	643	1	761	1
54	0	172	0	290	1	408	0	526	1	644	1	762	1
55	0	173	1	291	0	409	1	527	0	645	0	763	0
56	1	174	0	292	0	410	1	528	0	646	1	764	1
57	0	175	0	293	0	411	1	529	1	647	1	765	1
58	0	176	1	294	1	412	0	530	0	648	0	766	1
59	0	177	1	295	0	413	0	531	1	649	0	767	0
60	0	178	1	296	1	414	1	532	1	650	1	768	1
61	0	179	0	297	0	415	0	533	0	651	1	769	1
62	1	180	0	298	0	416	1	534	1	652	0	770	1
63	0	181	1	299	0	417	1	535	1	653	0	771	0
64	0	182	1	300	0	418	1	536	0	654	1	772	1
65	0	183	0	301	1	419	1	537	1	655	1	773	1
66	0	184	0	302	1	420	0	538	0	656	0	774	1
67	0	185	0	303	1	421	1	539	1	657	0	775	1
68	0	186	1	304	1	422	1	540	0	658	1	776	1
69	0	187	1	305	0	423	0	541	0	659	1	777	1
70	0	188	1	306	0	424	0	542	0	660	1	778	0
71	0	189	0	307	0	425	0	543	0	661	1	779	0
72	0	190	1	308	1	426	1	544	1	662	0	780	0
73	0	191	1	309	1	427	0	545	0	663	0	781	1
74	0	192	0	310	0	428	1	546	1	664	0	782	1
75	0	193	1	311	1	429	1	547	1	665	1	783	0
76	0	194	1	312	0	430	1	548	1	666	0	784	0
77	0	195	0	313	1	431	1	549	1	667	0	785	0
78	1	196	1	314	0	432	1	550	0	668	0	786	0
79	0	197	0	315	1	433	0	551	1	669	0	787	1
80	1	198	0	316	1	434	0	552	0	670	1	788	1
81	0	199	1	317	0	435	0	553	1	671	1	789	0
82	0	200	0	318	1	436	0	554	1	672	0	790	0
83	1	201	0	319	1	437	0	555	1	673	0	791	1

84	1	202	1	320	0	438	1	556	1	674	0	792	1
85	1	203	1	321	0	439	1	557	0	675	0	793	1
86	1	204	0	322	1	440	0	558	0	676	0	794	1
87	0	205	1	323	0	441	1	559	0	677	1	795	1
88	0	206	1	324	0	442	1	560	1	678	0	796	0
89	1	207	1	325	0	443	0	561	0	679	0	797	1
90	1	208	0	326	0	444	1	562	0	680	1	798	1
91	1	209	1	327	1	445	0	563	1	681	1	799	1
92	0	210	0	328	0	446	1	564	0	682	1	800	1
93	1	211	1	329	1	447	1	565	1	683	0	801	1
94	0	212	0	330	1	448	1	566	1	684	1	802	0
95	1	213	1	331	1	449	0	567	0	685	1	803	0
96	0	214	1	332	0	450	1	568	0	686	1	804	1
97	1	215	0	333	1	451	0	569	1	687	0	805	0
98	1	216	0	334	0	452	0	570	1	688	0	806	0
99	1	217	1	335	1	453	1	571	0	689	0	807	1
100	0	218	1	336	0	454	0	572	1	690	1	808	0
101	1	219	1	337	0	455	1	573	1	691	1	809	1
102	1	220	1	338	1	456	0	574	0	692	1	810	1
103	1	221	0	339	1	457	0	575	1	693	0	811	0
104	0	222	0	340	0	458	1	576	1	694	0	812	0
105	1	223	0	341	0	459	0	577	0	695	0	813	1
106	1	224	0	342	1	460	1	578	0	696	0	814	0
107	0	225	1	343	0	461	0	579	1	697	1	815	0
108	0	226	0	344	0	462	0	580	0	698	1	816	0
109	0	227	1	345	1	463	1	581	1	699	1	817	0
110	0	228	0	346	1	464	1	582	1	700	0	818	1
111	1	229	0	347	1	465	0	583	1	701	0	819	0
112	0	230	1	348	0	466	1	584	1	702	1	820	0
113	1	231	0	349	0	467	0	585	0	703	1	821	0
114	1	232	1	350	1	468	1	586	0	704	1	822	1
115	0	233	0	351	1	469	1	587	0	705	1	823	1
116	1	234	0	352	1	470	0	588	1	706	1		
117	0	235	0	353	1	471	0	589	1	707	0		

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