PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a, b) = (z, z^6 + z^5 + z^3 + z^2 + z)$

GUO-NIU HAN* AND YINING HU*

Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a, b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a, b) in $\mathbb{F}_2[x] \setminus \mathbb{F}_2$. In this paper we prove the conjecture for $(a, b) = (z, z^6 + z^5 + z^3 + z^2 + z)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \ldots)$ of elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{\ddots + \cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a, b)-Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \ldots) = (a, b, b, a, b, a, a, b, \ldots)$$

that we shall denote

$$\mathrm{CF}(a, b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When $(a,b)=(z,z^6+z^5+z^3+z^2+z)$, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

where, for $\mathrm{CF}(a,b)$*,*

$$\begin{aligned}
p_0(z)&=z^{18}+z^{15}+z^{12}+z^9+z^8+z^7+z^5+z^4+1,\\
p_1(z)&=z^{21}+z^{19}+z^{17}+z^{16}+z^{13}+z^{10}+z^8+z^6,\\
p_2(z)&=z^{22}+z^{20}+z^{18}+z^{16}+z^6,\\
p_3(z)&=z^{21}+z^{19}+z^{17}+z^{16}+z^{13}+z^{10}+z^8+z^6,\\
p_4(z)&=z^{20}+z^{16}+z^{15}+z^{14}+z^{12}+z^{10}+z^9+z^8+z^7+z^6+z^5+z^4+1,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$*,*

$$\begin{aligned}
p_0(z)&=z^{15}+z^{14}+z^{12}+z^9+z^8+z^7+z^6+z^5+z^4+z^2+1,\\
p_1(z)&=z^{21}+z^{19}+z^{17}+z^{16}+z^{13}+z^{10}+z^8+z^6,\\
p_2(z)&=z^{22}+z^{20}+z^{18}+z^{16}+z^6,\\
p_3(z)&=z^{21}+z^{19}+z^{17}+z^{16}+z^{13}+z^{10}+z^8+z^6,\\
p_4(z)&=z^{20}+z^{18}+z^{16}+z^{15}+z^{12}+z^{10}+z^9+z^8+z^7+z^5+z^4+z^2+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix}P_n(z)&Q_n(z)\\P_{n-1}(z)&Q_{n-1}(z)\end{pmatrix}:=\begin{pmatrix}t_n(z)&1\\1&0\end{pmatrix}\begin{pmatrix}t_{n-1}(z)&1\\1&0\end{pmatrix}\cdots\begin{pmatrix}t_0(z)&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x)&=x^{\deg(t_{2^n-1})}\begin{pmatrix}t_{2^n-1}(1/x)&1\\1&0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix}t_{2^n-2}(1/x)&1\\1&0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(t_0)}\begin{pmatrix}t_0(1/x)&1\\1&0\end{pmatrix},\\
W_n(x)&=x^{\deg(\bar t_{2^n-1})}\begin{pmatrix}\bar t_{2^n-1}(1/x)&1\\1&0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix}\bar t_{2^n-2}(1/x)&1\\1&0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(\bar t_0)}\begin{pmatrix}\bar t_0(1/x)&1\\1&0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\geq 0$

$$\begin{aligned}
M_{n+1}(x)&=W_n(x)\cdot M_n(x),\\
W_{n+1}(x)&=M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(175, 175, 175, 175, 175)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0} M^o_{0,0}$ and $W^o_{0,1} M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z \left(\phi(W^o,0,0)(x,z),\ z^{12} \cdot \phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0} M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0} M^o_{0,0}$ to order 630 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 630, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1} M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z \left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$ to order 770 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 770, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. $\square$

Define

$$R^e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned} M^e[:14u] = {} & M^e[:7u] + x^u M^e[:6u] + x^{2u} M^e[:5u] + x^{5u} M^e[:2u] + x^u M^e[:7u] \\ & + x^{2u} M^e[:6u] + x^{3u} M^e[:5u] + x^{6u} M^e[:2u] + x^{3u} M^e[:7u] \\ & + x^{4u} M^e[:6u] + x^{5u} M^e[:5u] + x^{8u} M^e[:2u] + x^{4u} M^e[:7u] \\ & + x^{5u} M^e[:6u] + x^{6u} M^e[:5u] + x^{9u} M^e[:2u] + x^{5u} M^e[:7u] \\ & + x^{6u} M^e[:6u] + x^{7u} M^e[:5u] + x^{10u} M^e[:2u] + x^{6u} M^e[:7u] \\ & + x^{7u} M^e[:6u] + x^{8u} M^e[:5u] + x^{11u} M^e[:2u] + x^{7u} M^e[:7u] \end{aligned}$$

$$
\begin{aligned}
&+ x^{10u}M^e[:4u] + x^{11u}M^e[:3u] + x^{13u}M^e[:u]\\
&+ (x^{7u} + x^{8u} + x^{10u} + x^{11u} + x^{12u} + x^{13u})R^e,\\
W^e[:14u] = {}& W^e[:7u] + x^uW^e[:6u] + x^{2u}W^e[:5u] + x^{5u}W^e[:2u] + x^uW^e[:7u]\\
&+ x^{2u}W^e[:6u] + x^{3u}W^e[:5u] + x^{6u}W^e[:2u] + x^{3u}W^e[:7u]\\
&+ x^{4u}W^e[:6u] + x^{5u}W^e[:5u] + x^{8u}W^e[:2u] + x^{4u}W^e[:7u]\\
&+ x^{5u}W^e[:6u] + x^{6u}W^e[:5u] + x^{9u}W^e[:2u] + x^{5u}W^e[:7u]\\
&+ x^{6u}W^e[:6u] + x^{7u}W^e[:5u] + x^{10u}W^e[:2u] + x^{6u}W^e[:7u]\\
&+ x^{7u}W^e[:6u] + x^{8u}W^e[:5u] + x^{11u}W^e[:2u] + x^{7u}W^e[:7u]\\
&+ x^{10u}W^e[:4u] + x^{11u}W^e[:3u] + x^{13u}W^e[:u]\\
&+ (x^{7u} + x^{8u} + x^{10u} + x^{11u} + x^{12u} + x^{13u})R^e.
\end{aligned}
$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$
\begin{aligned}
M^o[:14u] = {}& M^o[:7u] + x^uM^o[:6u] + x^{2u}M^o[:5u] + x^{5u}M^o[:2u] + x^uM^o[:7u]\\
&+ x^{2u}M^o[:6u] + x^{3u}M^o[:5u] + x^{6u}M^o[:2u] + x^{3u}M^o[:7u]\\
&+ x^{4u}M^o[:6u] + x^{5u}M^o[:5u] + x^{8u}M^o[:2u] + x^{4u}M^o[:7u]\\
&+ x^{5u}M^o[:6u] + x^{6u}M^o[:5u] + x^{9u}M^o[:2u] + x^{5u}M^o[:7u]\\
&+ x^{6u}M^o[:6u] + x^{7u}M^o[:5u] + x^{10u}M^o[:2u] + x^{6u}M^o[:7u]\\
&+ x^{7u}M^o[:6u] + x^{8u}M^o[:5u] + x^{11u}M^o[:2u] + x^{7u}M^o[:7u]\\
&+ x^{10u}M^o[:4u] + x^{11u}M^o[:3u] + x^{13u}M^o[:u]\\
&+ (x^{7u} + x^{8u} + x^{10u} + x^{11u} + x^{12u} + x^{13u})R^o,\\
W^o[:14u] = {}& W^o[:7u] + x^uW^o[:6u] + x^{2u}W^o[:5u] + x^{5u}W^o[:2u] + x^uW^o[:7u]\\
&+ x^{2u}W^o[:6u] + x^{3u}W^o[:5u] + x^{6u}W^o[:2u] + x^{3u}W^o[:7u]\\
&+ x^{4u}W^o[:6u] + x^{5u}W^o[:5u] + x^{8u}W^o[:2u] + x^{4u}W^o[:7u]\\
&+ x^{5u}W^o[:6u] + x^{6u}W^o[:5u] + x^{9u}W^o[:2u] + x^{5u}W^o[:7u]\\
&+ x^{6u}W^o[:6u] + x^{7u}W^o[:5u] + x^{10u}W^o[:2u] + x^{6u}W^o[:7u]\\
&+ x^{7u}W^o[:6u] + x^{8u}W^o[:5u] + x^{11u}W^o[:2u] + x^{7u}W^o[:7u]\\
&+ x^{10u}W^o[:4u] + x^{11u}W^o[:3u] + x^{13u}W^o[:u]\\
&+ (x^{7u} + x^{8u} + x^{10u} + x^{11u} + x^{12u} + x^{13u})R^o.
\end{aligned}
$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$
\begin{aligned}
T[:14u] = {}& T[:7u] + x^uT[:6u] + x^{2u}T[:5u] + x^{5u}T[:2u] + x^uT[:7u] + x^{2u}T[:6u]\\
&+ x^{3u}T[:5u] + x^{6u}T[:2u] + x^{3u}T[:7u] + x^{4u}T[:6u] + x^{5u}T[:5u]\\
&+ x^{8u}T[:2u] + x^{4u}T[:7u] + x^{5u}T[:6u] + x^{6u}T[:5u] + x^{9u}T[:2u]\\
&+ x^{5u}T[:7u] + x^{6u}T[:6u] + x^{7u}T[:5u] + x^{10u}T[:2u] + x^{6u}T[:7u]\\
&+ x^{7u}T[:6u] + x^{8u}T[:5u] + x^{11u}T[:2u] + x^{7u}T[:7u] + x^{10u}T[:4u]
\end{aligned}
$$

$$+ x^{11u}T[:3u] + x^{13u}T[:u].$$

The proof of the other 15 cases are similar. We break down the above identity into 7 parts:

$$\begin{aligned}
0 &= x^{7u}T[:u] + x^{5u}T[2u:3u] + x^{2u}T[5u:6u] + x^{u}T[6u:7u] + T[7u:8u],\\
0 &= x^{7u}T[u:2u] + x^{6u}T[2u:3u] + x^{5u}T[3u:4u] + x^{3u}T[5u:6u]\\
&\quad + T[8u:9u],\\
0 &= x^{9u}T[:u] + x^{7u}T[2u:3u] + x^{6u}T[3u:4u] + x^{5u}T[4u:5u]\\
&\quad + x^{3u}T[6u:7u] + T[9u:10u],\\
0 &= x^{9u}T[u:2u] + x^{8u}T[2u:3u] + x^{7u}T[3u:4u] + x^{6u}T[4u:5u]\\
&\quad + x^{4u}T[6u:7u] + T[10u:11u],\\
0 &= x^{8u}T[3u:4u] + x^{7u}T[4u:5u] + x^{5u}T[6u:7u] + T[11u:12u],\\
0 &= x^{10u}T[2u:3u] + x^{8u}T[4u:5u] + x^{6u}T[6u:7u] + T[12u:13u],\\
0 &= x^{13u}T[:u] + x^{11u}T[2u:3u] + x^{10u}T[3u:4u] + x^{7u}T[6u:7u]\\
&\quad + T[13u:14u],
\end{aligned}$$

which can be rewritten as

$$0 = T[[w]_2] + T[[10w]_2] + T[[101w]_2] + T[[110w]_2] + T[[111w]_2], \tag{2.7}$$

$$0 = T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.8}$$

$$\begin{aligned}0 = {}& T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[110w]_2]\\ &+ T[[1001w]_2],\end{aligned} \tag{2.9}$$

$$\begin{aligned}0 = {}& T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[110w]_2]\\ &+ T[[1010w]_2],\end{aligned} \tag{2.10}$$

$$0 = T[[11w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1011w]_2], \tag{2.11}$$

$$0 = T[[10w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1100w]_2], \tag{2.12}$$

$$0 = T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[110w]_2] + T[[1101w]_2], \tag{2.13}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$0 = T[[w]_2] + T[[10w]_2] + T[[101w]_2] + T[[110w]_2] + T[[111w]_2], \tag{2.14}$$

$$0 = T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.15}$$

$$\begin{aligned}0 = {}& T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[110w]_2]\\ &+ T[[1001w]_2],\end{aligned} \tag{2.16}$$

$$\begin{aligned}0 = {}& T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[110w]_2]\\ &+ T[[1010w]_2],\end{aligned} \tag{2.17}$$

$$0 = T[[11w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1011w]_2], \tag{2.18}$$

$$0 = T[[10w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1100w]_2], \tag{2.19}$$

$$0 = T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[110w]_2] + T[[1101w]_2], \tag{2.20}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 1177 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.20) can be written as

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,101)) + \tau(A(s,110)) \\ &\quad + \tau(A(s,111)), \end{aligned} \tag{2.21}$$

$$\begin{aligned} 0 &= \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101)) \\ &\quad + \tau(A(s,1000)), \end{aligned} \tag{2.22}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ &\quad + \tau(A(s,110)) + \tau(A(s,1001)), \end{aligned} \tag{2.23}$$

$$\begin{aligned} 0 &= \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ &\quad + \tau(A(s,110)) + \tau(A(s,1010)), \end{aligned} \tag{2.24}$$

$$0 = \tau(A(s,11)) + \tau(A(s,100)) + \tau(A(s,110)) + \tau(A(s,1011)), \tag{2.25}$$

$$0 = \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,110)) + \tau(A(s,1100)), \tag{2.26}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,110)) \\ &\quad + \tau(A(s,1101)), \end{aligned} \tag{2.27}$$

for all $s \in E_{2k}$, and

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,101)) + \tau(A(s,110)) \\ &\quad + \tau(A(s,111)), \end{aligned} \tag{2.28}$$

$$\begin{aligned} 0 &= \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101)) \\ &\quad + \tau(A(s,1000)), \end{aligned} \tag{2.29}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ &\quad + \tau(A(s,110)) + \tau(A(s,1001)), \end{aligned} \tag{2.30}$$

$$\begin{aligned} 0 &= \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ &\quad + \tau(A(s,110)) + \tau(A(s,1010)), \end{aligned} \tag{2.31}$$

$$0 = \tau(A(s,11)) + \tau(A(s,100)) + \tau(A(s,110)) + \tau(A(s,1011)), \tag{2.32}$$

$$0 = \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,110)) + \tau(A(s,1100)), \tag{2.33}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,110)) \\ &\quad + \tau(A(s,1101)), \end{aligned} \tag{2.34}$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{28}), E_{28}) = (A(i, 0^{16}), E_{16})$, so that we only have to verify that identities (2.21) through (2.34) hold for $2 \leq k \leq 14$, which turns out to be true. $\square$

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \geq 2$, and $u = 2^{2k-1}$,*

$$\begin{aligned} M_{2k} &= M^e[:7u] + x^u M^e[:6u] + x^{2u} M^e[:5u] + x^{5u} M^e[:2u] + x^{7u} R^e, \\ W_{2k} &= W^e[:7u] + x^u W^e[:6u] + x^{2u} W^e[:5u] + x^{5u} W^e[:2u] + x^{7u} R^e. \end{aligned}$$

For all integer $k \geq 2$, and $u = 2^{2k}$,

$$M_{2k+1} = M^o[:7u] + x^u M^o[:6u] + x^{2u} M^o[:5u] + x^{5u} M^o[:2u] + x^{7u} R^o,$$
$$W_{2k+1} = W^o[:7u] + x^u W^o[:6u] + x^{2u} W^o[:5u] + x^{5u} W^o[:2u] + x^{7u} R^o.$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \geq 2$, we claim that

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1} \wedge W_{2k+1},$$
$$M_{2k+1} \wedge W_{2k+1} \Rightarrow M_{2k+2} \wedge W_{2k+2}.$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$W_{2k}M_{2k} = \big(W^e[:7v] + x^v W^e[:6v] + x^{2v} W^e[:5v] + x^{5v} W^e[:2v] + x^{7u} R^e\big) \times \big(M^e[:7v] + x^v M^e[:6v] + x^{2v} M^e[:5v] + x^{5v} M^e[:2v] + x^{7u} R^e\big); \tag{2.35}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{14v}R^o$. Therefore we only need to prove that their difference is $O(x^{14v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{14v}, to

$$W^e[:14v]M^e[:14v] + x^{2v}W^e[:12v]M^e[:12v] + x^{4v}W^e[:10v]M^e[:10v] + x^{10v}W^e[:4v]M^e[:4v].$$

For all $n \leq 14$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$X := x^v,$$
$$a_n := W^e[n \cdot v : (n+1) \cdot v]/X^n,$$
$$b_n := M^e[n \cdot v : (n+1) \cdot v]/X^n,$$
$$c := R^e.$$

Using the notation introduced above, we can represent the expressions lhs (2.35) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{14}, b_1, ..., b_{14}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{14})$, which completes the proof. $\square$

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \leq j, k \leq 1$,

$$\lim_{n\to\infty} M_{2n,j,k} = M^e_{j,k},$$
$$\lim_{n\to\infty} M_{2n+1,j,k} = M^o_{j,k},$$
$$\lim_{n\to\infty} W_{2n,j,k} = W^e_{j,k},$$
$$\lim_{n\to\infty} W_{2n+1,j,k} = W^o_{j,k}.$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a, b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x, y) = \mathrm{Res}_t\left(\phi(M^e, 0, 1)(x, t), y^{12}\phi(M^e, 0, 1)(x, t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x, y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned}
q_0(x) &= x^{22} + x^{18} + x^{17} + x^{15} + x^{14} + x^{13} + x^{10} + x^7 + x^4,\\
q_1(x) &= x^{16} + x^{14} + x^{12} + x^9 + x^6 + x^5 + x^3 + x,\\
q_2(x) &= x^{16} + x^6 + x^4 + x^2 + 1,\\
q_3(x) &= x^{16} + x^{14} + x^{12} + x^9 + x^6 + x^5 + x^3 + x,\\
q_4(x) &= x^{22} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{10} + x^8 + x^7 + x^6\\
&\quad + x^2.
\end{aligned}$$

The polynomial $Q(x, y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x, y)$ of multiplicity m and $R(x, y) := P(x, y)/Q(x, y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 216, and substitute it for y in $R(z, y)$, we get a series with valuation smaller than 216, which proves that $R(z, y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z, y)$ is the minimal polynomial of $\mathrm{CF}(a, b) = f(1/z)$. □

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{132} + x^{124} + x^{122} + x^{120} + x^{118} + x^{114} + x^{108} + x^{106} + x^{102} + x^{100}\\
&\quad + x^{98} + x^{96} + x^{94} + x^{92} + x^{90} + x^{78} + x^{74} + x^{72} + x^{68} + x^{66} + x^{62}\\
&\quad + x^{58} + x^{56} + x^{54} + x^{50} + x^{48} + x^{46} + x^{44} + x^{40} + x^{36} + x^{34} + x^{24}\\
&\quad + x^{22} + x^{20} + x^{12},\\
p_3(x) &= x^{108} + x^{104} + x^{100} + x^{98} + x^{94} + x^{92} + x^{84} + x^{80} + x^{78} + x^{74} + x^{68}\\
&\quad + x^{66} + x^{62} + x^{58} + x^{56} + x^{54} + x^{48} + x^{32} + x^{28} + x^{24} + x^{22} + x^{18}\\
&\quad + x^{16} + x^8,\\
p_6(x) &= x^{104} + x^{100} + x^{92} + x^{84} + x^{80} + x^{78} + x^{72} + x^{68} + x^{64} + x^{62} + x^{60}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{56} + x^{52} + x^{50} + x^{48} + x^{46} + x^{44} + x^{42} + x^{38} + x^{30} + x^{24} + x^{22}\\
&+ x^{20} + x^{18} + x^{16} + x^{12} + x^{8} + x^{4} + x^{2} + 1,\\
p_9(x) &= x^{92} + x^{82} + x^{80} + x^{74} + x^{72} + x^{70} + x^{68} + x^{66} + x^{52} + x^{44} + x^{42}\\
&+ x^{40} + x^{36} + x^{30} + x^{26} + x^{22} + x^{20} + x^{18} + x^{14} + x^{12} + x^{8} + x^{6} + x^{4}\\
&+ x^{2},\\
p_{12}(x) &= x^{88} + x^{80} + x^{64} + x^{48} + x^{24} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0].

For $\phi(M^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{117} + x^{116} + x^{115} + x^{110} + x^{109} + x^{105} + x^{103} + x^{100} + x^{99} + x^{98}\\
&+ x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{85} + x^{83} + x^{82} + x^{80} + x^{79}\\
&+ x^{77} + x^{76} + x^{75} + x^{74} + x^{71} + x^{69} + x^{68} + x^{66} + x^{64} + x^{63} + x^{62}\\
&+ x^{60} + x^{59} + x^{57} + x^{55} + x^{52} + x^{49} + x^{46} + x^{45} + x^{44} + x^{41} + x^{37}\\
&+ x^{36} + x^{35} + x^{33} + x^{32} + x^{31} + x^{27} + x^{26} + x^{24} + x^{19} + x^{18},\\
p_3(x) &= x^{85} + x^{84} + x^{82} + x^{81} + x^{75} + x^{74} + x^{73} + x^{69} + x^{68} + x^{67} + x^{59}\\
&+ x^{58} + x^{57} + x^{53} + x^{52} + x^{51} + x^{45} + x^{44} + x^{43} + x^{37} + x^{36} + x^{34}\\
&+ x^{33} + x^{29} + x^{28} + x^{26} + x^{25} + x^{11} + x^{10} + x^{9},\\
p_6(x) &= x^{81} + x^{80} + x^{79} + x^{77} + x^{76} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{67}\\
&+ x^{66} + x^{65} + x^{60} + x^{58} + x^{55} + x^{51} + x^{46} + x^{45} + x^{42} + x^{38} + x^{36}\\
&+ x^{35} + x^{34} + x^{31} + x^{30} + x^{29} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{19}\\
&+ x^{17} + x^{16} + x^{11} + x^{10} + x^{9} + x^{7} + x^{6},\\
p_9(x) &= x^{77} + x^{76} + x^{74} + x^{72} + x^{70} + x^{69} + x^{67} + x^{66} + x^{64} + x^{62} + x^{61}\\
&+ x^{57} + x^{56} + x^{54} + x^{53} + x^{51} + x^{50} + x^{48} + x^{47} + x^{39} + x^{38} + x^{37}\\
&+ x^{33} + x^{32} + x^{31} + x^{29} + x^{28} + x^{27} + x^{17} + x^{16} + x^{14} + x^{13} + x^{9}\\
&+ x^{8} + x^{7} + x^{5} + x^{4} + x^{3},\\
p_{12}(x) &= x^{73} + x^{72} + x^{71} + x^{67} + x^{63} + x^{61} + x^{60} + x^{49} + x^{48} + x^{47} + x^{45}\\
&+ x^{44} + x^{41} + x^{40} + x^{39} + x^{35} + x^{33} + x^{32} + x^{25} + x^{24} + x^{23} + x^{19}\\
&+ x^{15} + x^{13} + x^{12} + x^{9} + x^{8} + x^{7} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 0, 0, 0, 1, 0, 0, 1, 1, 1, 0, 1, 1, 0, 1].

For $\phi(M^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{117} + x^{116} + x^{115} + x^{110} + x^{109} + x^{105} + x^{103} + x^{100} + x^{99} + x^{98}\\
&+ x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{85} + x^{83} + x^{82} + x^{80} + x^{79}\\
&+ x^{77} + x^{76} + x^{75} + x^{74} + x^{71} + x^{69} + x^{68} + x^{66} + x^{64} + x^{63} + x^{62}\\
&+ x^{60} + x^{59} + x^{57} + x^{55} + x^{52} + x^{49} + x^{46} + x^{45} + x^{44} + x^{41} + x^{37}\\
&+ x^{36} + x^{35} + x^{33} + x^{32} + x^{31} + x^{27} + x^{26} + x^{24} + x^{19} + x^{18},\\
p_3(x) &= x^{85} + x^{84} + x^{82} + x^{81} + x^{75} + x^{74} + x^{73} + x^{69} + x^{68} + x^{67} + x^{59}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{58} + x^{57} + x^{53} + x^{52} + x^{51} + x^{45} + x^{44} + x^{43} + x^{37} + x^{36} + x^{34}\\
&+ x^{33} + x^{29} + x^{28} + x^{26} + x^{25} + x^{11} + x^{10} + x^{9},\\
p_6(x) &= x^{81} + x^{80} + x^{79} + x^{77} + x^{76} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{67}\\
&+ x^{66} + x^{65} + x^{60} + x^{58} + x^{55} + x^{51} + x^{46} + x^{45} + x^{42} + x^{38} + x^{36}\\
&+ x^{35} + x^{34} + x^{31} + x^{30} + x^{29} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{19}\\
&+ x^{17} + x^{16} + x^{11} + x^{10} + x^{9} + x^{7} + x^{6},\\
p_9(x) &= x^{77} + x^{76} + x^{74} + x^{72} + x^{70} + x^{69} + x^{67} + x^{66} + x^{64} + x^{62} + x^{61}\\
&+ x^{57} + x^{56} + x^{54} + x^{53} + x^{51} + x^{50} + x^{48} + x^{47} + x^{39} + x^{38} + x^{37}\\
&+ x^{33} + x^{32} + x^{31} + x^{29} + x^{28} + x^{27} + x^{17} + x^{16} + x^{14} + x^{13} + x^{9}\\
&+ x^{8} + x^{7} + x^{5} + x^{4} + x^{3},\\
p_{12}(x) &= x^{73} + x^{72} + x^{71} + x^{67} + x^{63} + x^{61} + x^{60} + x^{49} + x^{48} + x^{47} + x^{45}\\
&+ x^{44} + x^{41} + x^{40} + x^{39} + x^{35} + x^{33} + x^{32} + x^{25} + x^{24} + x^{23} + x^{19}\\
&+ x^{15} + x^{13} + x^{12} + x^{9} + x^{8} + x^{7} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 0, 0, 0, 1, 0, 0, 1, 1, 1, 0, 1, 1, 0, 1].

For $\phi(M^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{102} + x^{100} + x^{98} + x^{94} + x^{90} + x^{84} + x^{82} + x^{80} + x^{78} + x^{76} + x^{74}\\
&+ x^{70} + x^{68} + x^{62} + x^{60} + x^{56} + x^{50} + x^{46} + x^{38} + x^{36} + x^{32} + x^{26}\\
&+ x^{24},\\
p_3(x) &= x^{78} + x^{76} + x^{72} + x^{64} + x^{54} + x^{52} + x^{44} + x^{32} + x^{30} + x^{28} + x^{24}\\
&+ x^{22} + x^{20} + x^{16} + x^{14} + x^{6},\\
p_6(x) &= x^{74} + x^{72} + x^{68} + x^{66} + x^{62} + x^{58} + x^{48} + x^{46} + x^{44} + x^{42} + x^{40}\\
&+ x^{38} + x^{36} + x^{34} + x^{30} + x^{28} + x^{26} + x^{18} + x^{12} + x^{10} + x^{4} + 1,\\
p_9(x) &= x^{62} + x^{60} + x^{58} + x^{52} + x^{50} + x^{38} + x^{36} + x^{34} + x^{32} + x^{26} + x^{24}\\
&+ x^{22} + x^{18} + x^{14} + x^{8} + x^{2},\\
p_{12}(x) &= x^{58} + x^{56} + x^{54} + x^{50} + x^{48} + x^{42} + x^{40} + x^{38} + x^{34} + x^{32} + x^{10} + x^{8}\\
&+ x^{6} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{102} + x^{100} + x^{96} + x^{94} + x^{92} + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{72}\\
&+ x^{66} + x^{64} + x^{60} + x^{54} + x^{48} + x^{44} + x^{42} + x^{16} + x^{14} + x^{12},\\
p_3(x) &= x^{78} + x^{76} + x^{74} + x^{68} + x^{60} + x^{56} + x^{54} + x^{50} + x^{46} + x^{40} + x^{38}\\
&+ x^{34} + x^{28} + x^{26} + x^{22} + x^{14} + x^{12} + x^{8},\\
p_6(x) &= x^{74} + x^{72} + x^{70} + x^{66} + x^{64} + x^{60} + x^{58} + x^{56} + x^{54} + x^{52} + x^{46}\\
&+ x^{44} + x^{38} + x^{34} + x^{20} + x^{16} + x^{8} + 1,\\
p_9(x) &= x^{62} + x^{60} + x^{58} + x^{52} + x^{50} + x^{38} + x^{36} + x^{34} + x^{32} + x^{26} + x^{24}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{22} + x^{18} + x^{14} + x^{8} + x^{2},\\
p_{12}(x) &= x^{58} + x^{56} + x^{54} + x^{50} + x^{48} + x^{42} + x^{40} + x^{38} + x^{34} + x^{32} + x^{10} + x^{8}\\
&+ x^{6} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{117} + x^{116} + x^{115} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{107} + x^{104}\\
&+ x^{102} + x^{97} + x^{96} + x^{95} + x^{92} + x^{89} + x^{86} + x^{85} + x^{84} + x^{80} + x^{79}\\
&+ x^{78} + x^{77} + x^{76} + x^{74} + x^{69} + x^{68} + x^{66} + x^{63} + x^{62} + x^{60} + x^{58}\\
&+ x^{56} + x^{54} + x^{53} + x^{52} + x^{49} + x^{47} + x^{46} + x^{44} + x^{43} + x^{41} + x^{40}\\
&+ x^{39} + x^{38} + x^{37} + x^{36} + x^{34} + x^{33} + x^{32} + x^{31} + x^{29} + x^{27},\\
p_3(x) &= x^{85} + x^{84} + x^{82} + x^{81} + x^{75} + x^{74} + x^{73} + x^{69} + x^{68} + x^{67} + x^{59}\\
&+ x^{58} + x^{57} + x^{53} + x^{52} + x^{51} + x^{45} + x^{44} + x^{43} + x^{37} + x^{36} + x^{34}\\
&+ x^{33} + x^{29} + x^{28} + x^{26} + x^{25} + x^{11} + x^{10} + x^{9},\\
p_6(x) &= x^{81} + x^{80} + x^{79} + x^{77} + x^{76} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{67}\\
&+ x^{66} + x^{65} + x^{60} + x^{58} + x^{55} + x^{51} + x^{46} + x^{45} + x^{42} + x^{38} + x^{36}\\
&+ x^{35} + x^{34} + x^{31} + x^{30} + x^{29} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{19}\\
&+ x^{17} + x^{16} + x^{11} + x^{10} + x^{9} + x^{7} + x^{6},\\
p_9(x) &= x^{77} + x^{76} + x^{74} + x^{72} + x^{70} + x^{69} + x^{67} + x^{66} + x^{64} + x^{62} + x^{61}\\
&+ x^{57} + x^{56} + x^{54} + x^{53} + x^{51} + x^{50} + x^{48} + x^{47} + x^{39} + x^{38} + x^{37}\\
&+ x^{33} + x^{32} + x^{31} + x^{29} + x^{28} + x^{27} + x^{17} + x^{16} + x^{14} + x^{13} + x^{9}\\
&+ x^{8} + x^{7} + x^{5} + x^{4} + x^{3},\\
p_{12}(x) &= x^{73} + x^{72} + x^{71} + x^{67} + x^{63} + x^{61} + x^{60} + x^{49} + x^{48} + x^{47} + x^{45}\\
&+ x^{44} + x^{41} + x^{40} + x^{39} + x^{35} + x^{33} + x^{32} + x^{25} + x^{24} + x^{23} + x^{19}\\
&+ x^{15} + x^{13} + x^{12} + x^{9} + x^{8} + x^{7} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 1, 1, 1, 1].

For $\phi(W^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{117} + x^{116} + x^{115} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{107} + x^{104}\\
&+ x^{102} + x^{97} + x^{96} + x^{95} + x^{92} + x^{89} + x^{86} + x^{85} + x^{84} + x^{80} + x^{79}\\
&+ x^{78} + x^{77} + x^{76} + x^{74} + x^{69} + x^{68} + x^{66} + x^{63} + x^{62} + x^{60} + x^{58}\\
&+ x^{56} + x^{54} + x^{53} + x^{52} + x^{49} + x^{47} + x^{46} + x^{44} + x^{43} + x^{41} + x^{40}\\
&+ x^{39} + x^{38} + x^{37} + x^{36} + x^{34} + x^{33} + x^{32} + x^{31} + x^{29} + x^{27},\\
p_3(x) &= x^{85} + x^{84} + x^{82} + x^{81} + x^{75} + x^{74} + x^{73} + x^{69} + x^{68} + x^{67} + x^{59}\\
&+ x^{58} + x^{57} + x^{53} + x^{52} + x^{51} + x^{45} + x^{44} + x^{43} + x^{37} + x^{36} + x^{34}\\
&+ x^{33} + x^{29} + x^{28} + x^{26} + x^{25} + x^{11} + x^{10} + x^{9},\\
p_6(x) &= x^{81} + x^{80} + x^{79} + x^{77} + x^{76} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{67}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{66} + x^{65} + x^{60} + x^{58} + x^{55} + x^{51} + x^{46} + x^{45} + x^{42} + x^{38} + x^{36} \\
&+ x^{35} + x^{34} + x^{31} + x^{30} + x^{29} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{19} \\
&+ x^{17} + x^{16} + x^{11} + x^{10} + x^{9} + x^{7} + x^{6}, \\
p_9(x) = {}& x^{77} + x^{76} + x^{74} + x^{72} + x^{70} + x^{69} + x^{67} + x^{66} + x^{64} + x^{62} + x^{61} \\
&+ x^{57} + x^{56} + x^{54} + x^{53} + x^{51} + x^{50} + x^{48} + x^{47} + x^{39} + x^{38} + x^{37} \\
&+ x^{33} + x^{32} + x^{31} + x^{29} + x^{28} + x^{27} + x^{17} + x^{16} + x^{14} + x^{13} + x^{9} \\
&+ x^{8} + x^{7} + x^{5} + x^{4} + x^{3}, \\
p_{12}(x) = {}& x^{73} + x^{72} + x^{71} + x^{67} + x^{63} + x^{61} + x^{60} + x^{49} + x^{48} + x^{47} + x^{45} \\
&+ x^{44} + x^{41} + x^{40} + x^{39} + x^{35} + x^{33} + x^{32} + x^{25} + x^{24} + x^{23} + x^{19} \\
&+ x^{15} + x^{13} + x^{12} + x^{9} + x^{8} + x^{7} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 1, 1, 1, 1].

For $\phi(W^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) = {}& x^{132} + x^{128} + x^{122} + x^{118} + x^{116} + x^{112} + x^{110} + x^{108} + x^{104} + x^{102} \\
&+ x^{98} + x^{92} + x^{90} + x^{88} + x^{84} + x^{82} + x^{72} + x^{66} + x^{60} + x^{52} + x^{46} \\
&+ x^{44} + x^{42}, \\
p_3(x) = {}& x^{108} + x^{98} + x^{96} + x^{94} + x^{92} + x^{90} + x^{88} + x^{82} + x^{74} + x^{70} + x^{68} \\
&+ x^{66} + x^{64} + x^{62} + x^{60} + x^{58} + x^{56} + x^{54} + x^{52} + x^{50} + x^{38} + x^{24} \\
&+ x^{22} + x^{20} + x^{8} + x^{6}, \\
p_6(x) = {}& x^{104} + x^{96} + x^{92} + x^{90} + x^{82} + x^{76} + x^{72} + x^{70} + x^{64} + x^{62} + x^{46} \\
&+ x^{44} + x^{42} + x^{36} + x^{30} + x^{28} + x^{22} + x^{12} + x^{8} + x^{6} + x^{2} + 1, \\
p_9(x) = {}& x^{92} + x^{82} + x^{80} + x^{74} + x^{72} + x^{70} + x^{68} + x^{66} + x^{52} + x^{44} + x^{42} \\
&+ x^{40} + x^{36} + x^{30} + x^{26} + x^{22} + x^{20} + x^{18} + x^{14} + x^{12} + x^{8} + x^{6} + x^{4} \\
&+ x^{2}, \\
p_{12}(x) = {}& x^{88} + x^{80} + x^{64} + x^{48} + x^{24} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0].

For $\phi(M^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) = {}& x^{117} + x^{116} + x^{114} + x^{112} + x^{111} + x^{108} + x^{104} + x^{102} + x^{101} + x^{99} \\
&+ x^{98} + x^{94} + x^{93} + x^{89} + x^{84} + x^{83} + x^{80} + x^{79} + x^{78} + x^{76} + x^{74} \\
&+ x^{71} + x^{66} + x^{63} + x^{62} + x^{58} + x^{55} + x^{51} + x^{50} + x^{48} + x^{47} + x^{46} \\
&+ x^{43} + x^{42} + x^{41} + x^{37} + x^{33} + x^{28} + x^{23} + x^{20} + x^{18} + x^{16} + x^{14} \\
&+ x^{13} + x^{12}, \\
p_3(x) = {}& x^{93} + x^{92} + x^{90} + x^{88} + x^{86} + x^{81} + x^{80} + x^{79} + x^{77} + x^{75} + x^{74} \\
&+ x^{73} + x^{72} + x^{69} + x^{64} + x^{63} + x^{61} + x^{60} + x^{59} + x^{55} + x^{54} + x^{51} \\
&+ x^{50} + x^{48} + x^{43} + x^{42} + x^{41} + x^{38} + x^{31} + x^{23} + x^{22} + x^{20} + x^{18} \\
&+ x^{15} + x^{12} + x^{10} + x^{9} + x^{8},
\end{aligned}
$$

$$\begin{aligned}
p_6(x) &= x^{85} + x^{84} + x^{82} + x^{80} + x^{78} + x^{68} + x^{64} + x^{63} + x^{61} + x^{60} + x^{57}\\
&\quad + x^{56} + x^{54} + x^{53} + x^{51} + x^{50} + x^{48} + x^{45} + x^{44} + x^{42} + x^{37} + x^{34}\\
&\quad + x^{32} + x^{30} + x^{29} + x^{27} + x^{26} + x^{22} + x^{20} + x^{16} + x^{14} + x^{13} + x^{11}\\
&\quad + x^{10} + x^{8} + x^{7} + x^{6} + x^{5} + x^{4} + x^{3} + x + 1,\\
p_9(x) &= x^{73} + x^{72} + x^{71} + x^{67} + x^{65} + x^{64} + x^{61} + x^{60} + x^{59} + x^{57} + x^{56}\\
&\quad + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{45} + x^{44} + x^{43} + x^{39} + x^{37} + x^{36}\\
&\quad + x^{25} + x^{24} + x^{23} + x^{19} + x^{17} + x^{16} + x^{13} + x^{12} + x^{11} + x^{7} + x^{5} + x^{4},\\
p_{12}(x) &= x^{65} + x^{64} + x^{63} + x^{59} + x^{57} + x^{56} + x^{53} + x^{52} + x^{51} + x^{47} + x^{43}\\
&\quad + x^{41} + x^{40} + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{17} + x^{16} + x^{15} + x^{11}\\
&\quad + x^{9} + x^{8} + x^{5} + x^{4} + x^{3} + x + 1,
\end{aligned}$$

and the initial terms are [1, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 1, 1].

For $\phi(M^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{112} + x^{110} + x^{106} + x^{105} + x^{103} + x^{101} + x^{98} + x^{96} + x^{95} + x^{92}\\
&\quad + x^{90} + x^{87} + x^{86} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{76} + x^{75} + x^{74}\\
&\quad + x^{73} + x^{72} + x^{70} + x^{68} + x^{67} + x^{66} + x^{63} + x^{62} + x^{61} + x^{60} + x^{57}\\
&\quad + x^{56} + x^{54} + x^{50} + x^{49} + x^{48} + x^{45} + x^{44} + x^{40} + x^{36} + x^{32} + x^{31}\\
&\quad + x^{26} + x^{25} + x^{22} + x^{21} + x^{20} + x^{18},\\
p_3(x) &= x^{84} + x^{82} + x^{78} + x^{77} + x^{74} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{66}\\
&\quad + x^{65} + x^{62} + x^{61} + x^{58} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{50} + x^{49}\\
&\quad + x^{46} + x^{45} + x^{44} + x^{40} + x^{39} + x^{36} + x^{33} + x^{32} + x^{31} + x^{28} + x^{25}\\
&\quad + x^{24} + x^{23} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{9},\\
p_6(x) &= x^{76} + x^{68} + x^{62} + x^{58} + x^{56} + x^{52} + x^{48} + x^{44} + x^{42} + x^{40} + x^{36}\\
&\quad + x^{34} + x^{26} + x^{20} + x^{16} + x^{6},\\
p_9(x) &= x^{68} + x^{66} + x^{64} + x^{61} + x^{60} + x^{58} + x^{57} + x^{56} + x^{55} + x^{52} + x^{51}\\
&\quad + x^{50} + x^{48} + x^{45} + x^{44} + x^{42} + x^{41} + x^{40} + x^{39} + x^{35} + x^{20} + x^{18}\\
&\quad + x^{16} + x^{13} + x^{12} + x^{10} + x^{9} + x^{8} + x^{7} + x^{3},\\
p_{12}(x) &= x^{60} + x^{56} + x^{48} + x^{44} + x^{40} + x^{32} + x^{12} + x^{8} + 1,
\end{aligned}$$

and the initial terms are [0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0].

For $\phi(M^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{132} + x^{130} + x^{128} + x^{125} + x^{123} + x^{120} + x^{119} + x^{116} + x^{113} + x^{112}\\
&\quad + x^{111} + x^{110} + x^{108} + x^{105} + x^{104} + x^{102} + x^{101} + x^{99} + x^{93} + x^{90}\\
&\quad + x^{89} + x^{88} + x^{87} + x^{85} + x^{84} + x^{81} + x^{78} + x^{76} + x^{73} + x^{72} + x^{70}\\
&\quad + x^{66} + x^{62} + x^{61} + x^{56} + x^{54} + x^{53} + x^{50} + x^{49} + x^{47} + x^{46} + x^{42}\\
&\quad + x^{41} + x^{40} + x^{39} + x^{37} + x^{35} + x^{33} + x^{32} + x^{31} + x^{29} + x^{28} + x^{27},\\
p_3(x) &= x^{104} + x^{102} + x^{100} + x^{97} + x^{96} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{88}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{87} + x^{84} + x^{83} + x^{82} + x^{80} + x^{78} + x^{77} + x^{74} + x^{73} + x^{70} + x^{68}\\
&+ x^{67} + x^{66} + x^{65} + x^{61} + x^{60} + x^{58} + x^{56} + x^{53} + x^{49} + x^{47} + x^{46}\\
&+ x^{45} + x^{44} + x^{43} + x^{41} + x^{38} + x^{37} + x^{35} + x^{33} + x^{32} + x^{28} + x^{26}\\
&+ x^{25} + x^{24} + x^{23} + x^{22} + x^{20} + x^{19} + x^{16} + x^{15} + x^{14} + x^{9},\\
p_6(x) &= x^{96} + x^{92} + x^{84} + x^{82} + x^{72} + x^{70} + x^{66} + x^{64} + x^{62} + x^{56} + x^{50}\\
&+ x^{48} + x^{46} + x^{40} + x^{36} + x^{34} + x^{30} + x^{28} + x^{24} + x^{22} + x^{18} + x^{16}\\
&+ x^{10} + x^{6},\\
p_9(x) &= x^{88} + x^{86} + x^{82} + x^{81} + x^{80} + x^{76} + x^{75} + x^{74} + x^{70} + x^{69} + x^{68}\\
&+ x^{64} + x^{63} + x^{62} + x^{60} + x^{57} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{47}\\
&+ x^{46} + x^{43} + x^{42} + x^{41} + x^{38} + x^{35} + x^{34} + x^{33} + x^{32} + x^{28} + x^{27}\\
&+ x^{26} + x^{24} + x^{21} + x^{20} + x^{18} + x^{17} + x^{15} + x^{14} + x^{11} + x^{10} + x^{9}\\
&+ x^{8} + x^{3},\\
p_{12}(x) &= x^{80} + x^{64} + x^{60} + x^{56} + x^{52} + x^{48} + x^{44} + x^{40} + x^{36} + x^{12} + x^{8} + x^{4}\\
&+ 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1].

For $\phi(M^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{117} + x^{116} + x^{114} + x^{112} + x^{111} + x^{109} + x^{107} + x^{104} + x^{103} + x^{102}\\
&+ x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{91} + x^{87} + x^{84} + x^{82} + x^{81}\\
&+ x^{80} + x^{79} + x^{78} + x^{76} + x^{74} + x^{70} + x^{69} + x^{67} + x^{66} + x^{65} + x^{64}\\
&+ x^{59} + x^{58} + x^{56} + x^{54} + x^{50} + x^{49} + x^{48} + x^{47} + x^{45} + x^{43} + x^{42}\\
&+ x^{39} + x^{37} + x^{36} + x^{33},\\
p_3(x) &= x^{93} + x^{92} + x^{90} + x^{88} + x^{86} + x^{85} + x^{84} + x^{83} + x^{79} + x^{77} + x^{73}\\
&+ x^{71} + x^{69} + x^{68} + x^{67} + x^{66} + x^{64} + x^{63} + x^{61} + x^{60} + x^{58} + x^{56}\\
&+ x^{54} + x^{52} + x^{48} + x^{45} + x^{44} + x^{43} + x^{39} + x^{38} + x^{36} + x^{32} + x^{28}\\
&+ x^{24} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18} + x^{17} + x^{13} + x^{12},\\
p_6(x) &= x^{85} + x^{84} + x^{82} + x^{80} + x^{78} + x^{77} + x^{76} + x^{75} + x^{73} + x^{72} + x^{68}\\
&+ x^{67} + x^{66} + x^{60} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{50} + x^{49}\\
&+ x^{47} + x^{46} + x^{45} + x^{38} + x^{37} + x^{36} + x^{35} + x^{33} + x^{29} + x^{28} + x^{25}\\
&+ x^{24} + x^{23} + x^{22} + x^{21} + x^{16} + x^{14} + x^{13} + x^{7} + x^{3} + x + 1,\\
p_9(x) &= x^{73} + x^{72} + x^{71} + x^{67} + x^{65} + x^{64} + x^{61} + x^{60} + x^{59} + x^{57} + x^{56}\\
&+ x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{45} + x^{44} + x^{43} + x^{39} + x^{37} + x^{36}\\
&+ x^{25} + x^{24} + x^{23} + x^{19} + x^{17} + x^{16} + x^{13} + x^{12} + x^{11} + x^{7} + x^{5} + x^{4},\\
p_{12}(x) &= x^{65} + x^{64} + x^{63} + x^{59} + x^{57} + x^{56} + x^{53} + x^{52} + x^{51} + x^{47} + x^{43}\\
&+ x^{41} + x^{40} + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{17} + x^{16} + x^{15} + x^{11}\\
&+ x^{9} + x^{8} + x^{5} + x^{4} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 1, 1, 0, 1].

For $\phi(W^o, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{117} + x^{116} + x^{114} + x^{112} + x^{111} + x^{108} + x^{104} + x^{102} + x^{101} + x^{99} \\
&\quad + x^{98} + x^{94} + x^{93} + x^{89} + x^{84} + x^{83} + x^{80} + x^{79} + x^{78} + x^{76} + x^{74} \\
&\quad + x^{71} + x^{66} + x^{63} + x^{62} + x^{58} + x^{55} + x^{51} + x^{50} + x^{48} + x^{47} + x^{46} \\
&\quad + x^{43} + x^{42} + x^{41} + x^{37} + x^{33} + x^{28} + x^{23} + x^{20} + x^{18} + x^{16} + x^{14} \\
&\quad + x^{13} + x^{12}, \\
p_3(x) &= x^{93} + x^{92} + x^{90} + x^{88} + x^{86} + x^{81} + x^{80} + x^{79} + x^{77} + x^{75} + x^{74} \\
&\quad + x^{73} + x^{72} + x^{69} + x^{64} + x^{63} + x^{61} + x^{60} + x^{59} + x^{55} + x^{54} + x^{51} \\
&\quad + x^{50} + x^{48} + x^{43} + x^{42} + x^{41} + x^{38} + x^{31} + x^{23} + x^{22} + x^{20} + x^{18} \\
&\quad + x^{15} + x^{12} + x^{10} + x^{9} + x^{8}, \\
p_6(x) &= x^{85} + x^{84} + x^{82} + x^{80} + x^{78} + x^{68} + x^{64} + x^{63} + x^{61} + x^{60} + x^{57} \\
&\quad + x^{56} + x^{54} + x^{53} + x^{51} + x^{50} + x^{48} + x^{45} + x^{44} + x^{42} + x^{37} + x^{34} \\
&\quad + x^{32} + x^{30} + x^{29} + x^{27} + x^{26} + x^{22} + x^{20} + x^{16} + x^{14} + x^{13} + x^{11} \\
&\quad + x^{10} + x^{8} + x^{7} + x^{6} + x^{5} + x^{4} + x^{3} + x + 1, \\
p_9(x) &= x^{73} + x^{72} + x^{71} + x^{67} + x^{65} + x^{64} + x^{61} + x^{60} + x^{59} + x^{57} + x^{56} \\
&\quad + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{45} + x^{44} + x^{43} + x^{39} + x^{37} + x^{36} \\
&\quad + x^{25} + x^{24} + x^{23} + x^{19} + x^{17} + x^{16} + x^{13} + x^{12} + x^{11} + x^{7} + x^{5} + x^{4}, \\
p_{12}(x) &= x^{65} + x^{64} + x^{63} + x^{59} + x^{57} + x^{56} + x^{53} + x^{52} + x^{51} + x^{47} + x^{43} \\
&\quad + x^{41} + x^{40} + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{17} + x^{16} + x^{15} + x^{11} \\
&\quad + x^{9} + x^{8} + x^{5} + x^{4} + x^{3} + x + 1,
\end{aligned}$$

and the initial terms are [1, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 1, 1].

For $\phi(W^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{132} + x^{130} + x^{128} + x^{125} + x^{123} + x^{120} + x^{119} + x^{116} + x^{113} + x^{112} \\
&\quad + x^{111} + x^{110} + x^{108} + x^{105} + x^{104} + x^{102} + x^{101} + x^{99} + x^{93} + x^{90} \\
&\quad + x^{89} + x^{88} + x^{87} + x^{85} + x^{84} + x^{81} + x^{78} + x^{76} + x^{73} + x^{72} + x^{70} \\
&\quad + x^{66} + x^{62} + x^{61} + x^{56} + x^{54} + x^{53} + x^{50} + x^{49} + x^{47} + x^{46} + x^{42} \\
&\quad + x^{41} + x^{40} + x^{39} + x^{37} + x^{35} + x^{33} + x^{32} + x^{31} + x^{29} + x^{28} + x^{27}, \\
p_3(x) &= x^{104} + x^{102} + x^{100} + x^{97} + x^{96} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{88} \\
&\quad + x^{87} + x^{84} + x^{83} + x^{82} + x^{80} + x^{78} + x^{77} + x^{74} + x^{73} + x^{70} + x^{68} \\
&\quad + x^{67} + x^{66} + x^{65} + x^{61} + x^{60} + x^{58} + x^{56} + x^{53} + x^{49} + x^{47} + x^{46} \\
&\quad + x^{45} + x^{44} + x^{43} + x^{41} + x^{38} + x^{37} + x^{35} + x^{33} + x^{32} + x^{28} + x^{26} \\
&\quad + x^{25} + x^{24} + x^{23} + x^{22} + x^{20} + x^{19} + x^{16} + x^{15} + x^{14} + x^{9}, \\
p_6(x) &= x^{96} + x^{92} + x^{84} + x^{82} + x^{72} + x^{70} + x^{66} + x^{64} + x^{62} + x^{56} + x^{50} \\
&\quad + x^{48} + x^{46} + x^{40} + x^{36} + x^{34} + x^{30} + x^{28} + x^{24} + x^{22} + x^{18} + x^{16}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{10} + x^{6},\\
p_9(x) &= x^{88} + x^{86} + x^{82} + x^{81} + x^{80} + x^{76} + x^{75} + x^{74} + x^{70} + x^{69} + x^{68}\\
&+ x^{64} + x^{63} + x^{62} + x^{60} + x^{57} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{47}\\
&+ x^{46} + x^{43} + x^{42} + x^{41} + x^{38} + x^{35} + x^{34} + x^{33} + x^{32} + x^{28} + x^{27}\\
&+ x^{26} + x^{24} + x^{21} + x^{20} + x^{18} + x^{17} + x^{15} + x^{14} + x^{11} + x^{10} + x^{9}\\
&+ x^{8} + x^{3},\\
p_{12}(x) &= x^{80} + x^{64} + x^{60} + x^{56} + x^{52} + x^{48} + x^{44} + x^{40} + x^{36} + x^{12} + x^{8} + x^{4}\\
&+ 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1].

For $\phi(W^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{112} + x^{110} + x^{106} + x^{105} + x^{103} + x^{101} + x^{98} + x^{96} + x^{95} + x^{92}\\
&+ x^{90} + x^{87} + x^{86} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{76} + x^{75} + x^{74}\\
&+ x^{73} + x^{72} + x^{70} + x^{68} + x^{67} + x^{66} + x^{63} + x^{62} + x^{61} + x^{60} + x^{57}\\
&+ x^{56} + x^{54} + x^{50} + x^{49} + x^{48} + x^{45} + x^{44} + x^{40} + x^{36} + x^{32} + x^{31}\\
&+ x^{26} + x^{25} + x^{22} + x^{21} + x^{20} + x^{18},\\
p_3(x) &= x^{84} + x^{82} + x^{78} + x^{77} + x^{74} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{66}\\
&+ x^{65} + x^{62} + x^{61} + x^{58} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{50} + x^{49}\\
&+ x^{46} + x^{45} + x^{44} + x^{40} + x^{39} + x^{36} + x^{33} + x^{32} + x^{31} + x^{28} + x^{25}\\
&+ x^{24} + x^{23} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{9},\\
p_6(x) &= x^{76} + x^{68} + x^{62} + x^{58} + x^{56} + x^{52} + x^{48} + x^{44} + x^{42} + x^{40} + x^{36}\\
&+ x^{34} + x^{26} + x^{20} + x^{16} + x^{6},\\
p_9(x) &= x^{68} + x^{66} + x^{64} + x^{61} + x^{60} + x^{58} + x^{57} + x^{56} + x^{55} + x^{52} + x^{51}\\
&+ x^{50} + x^{48} + x^{45} + x^{44} + x^{42} + x^{41} + x^{40} + x^{39} + x^{35} + x^{20} + x^{18}\\
&+ x^{16} + x^{13} + x^{12} + x^{10} + x^{9} + x^{8} + x^{7} + x^{3},\\
p_{12}(x) &= x^{60} + x^{56} + x^{48} + x^{44} + x^{40} + x^{32} + x^{12} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0].

For $\phi(W^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{117} + x^{116} + x^{114} + x^{112} + x^{111} + x^{109} + x^{107} + x^{104} + x^{103} + x^{102}\\
&+ x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{91} + x^{87} + x^{84} + x^{82} + x^{81}\\
&+ x^{80} + x^{79} + x^{78} + x^{76} + x^{74} + x^{70} + x^{69} + x^{67} + x^{66} + x^{65} + x^{64}\\
&+ x^{59} + x^{58} + x^{56} + x^{54} + x^{50} + x^{49} + x^{48} + x^{47} + x^{45} + x^{43} + x^{42}\\
&+ x^{39} + x^{37} + x^{36} + x^{33},\\
p_3(x) &= x^{93} + x^{92} + x^{90} + x^{88} + x^{86} + x^{85} + x^{84} + x^{83} + x^{79} + x^{77} + x^{73}\\
&+ x^{71} + x^{69} + x^{68} + x^{67} + x^{66} + x^{64} + x^{63} + x^{61} + x^{60} + x^{58} + x^{56}\\
&+ x^{54} + x^{52} + x^{48} + x^{45} + x^{44} + x^{43} + x^{39} + x^{38} + x^{36} + x^{32} + x^{28}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{24}+x^{22}+x^{21}+x^{20}+x^{19}+x^{18}+x^{17}+x^{13}+x^{12},\\
p_6(x)={}&x^{85}+x^{84}+x^{82}+x^{80}+x^{78}+x^{77}+x^{76}+x^{75}+x^{73}+x^{72}+x^{68}\\
&+x^{67}+x^{66}+x^{60}+x^{58}+x^{57}+x^{56}+x^{55}+x^{54}+x^{53}+x^{50}+x^{49}\\
&+x^{47}+x^{46}+x^{45}+x^{38}+x^{37}+x^{36}+x^{35}+x^{33}+x^{29}+x^{28}+x^{25}\\
&+x^{24}+x^{23}+x^{22}+x^{21}+x^{16}+x^{14}+x^{13}+x^{7}+x^{3}+x+1,\\
p_9(x)={}&x^{73}+x^{72}+x^{71}+x^{67}+x^{65}+x^{64}+x^{61}+x^{60}+x^{59}+x^{57}+x^{56}\\
&+x^{53}+x^{52}+x^{51}+x^{49}+x^{48}+x^{45}+x^{44}+x^{43}+x^{39}+x^{37}+x^{36}\\
&+x^{25}+x^{24}+x^{23}+x^{19}+x^{17}+x^{16}+x^{13}+x^{12}+x^{11}+x^{7}+x^{5}+x^{4},\\
p_{12}(x)={}&x^{65}+x^{64}+x^{63}+x^{59}+x^{57}+x^{56}+x^{53}+x^{52}+x^{51}+x^{47}+x^{43}\\
&+x^{41}+x^{40}+x^{37}+x^{36}+x^{35}+x^{33}+x^{32}+x^{17}+x^{16}+x^{15}+x^{11}\\
&+x^{9}+x^{8}+x^{5}+x^{4}+x^{3}+x+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 1, 1, 0, 1].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	295	[472, 39]	590	[768, 38]	885	[1013, 642]
1	[3, 4]	296	[125, 429]	591	[749, 205]	886	[1014, 637]
2	[5, 6]	297	[473, 410]	592	[769, 498]	887	[1015, 248]
3	[7, 8]	298	[474, 405]	593	[770, 771]	888	[643, 942]
4	[9, 10]	299	[475, 476]	594	[772, 187]	889	[28, 255]
5	[11, 12]	300	[477, 478]	595	[773, 774]	890	[1016, 310]
6	[13, 14]	301	[479, 152]	596	[775, 122]	891	[605, 604]
7	[15, 16]	302	[480, 13]	597	[776, 153]	892	[1017, 70]
8	[17, 18]	303	[121, 382]	598	[777, 488]	893	[656, 661]
9	[19, 20]	304	[436, 481]	599	[778, 50]	894	[747, 80]
10	[21, 22]	305	[482, 483]	600	[779, 400]	895	[1018, 80]
11	[23, 24]	306	[214, 167]	601	[780, 781]	896	[745, 995]
12	[25, 26]	307	[484, 126]	602	[32, 202]	897	[968, 741]
13	[27, 28]	308	[457, 485]	603	[782, 783]	898	[676, 74]
14	[29, 30]	309	[486, 192]	604	[784, 481]	899	[1019, 90]
15	[31, 32]	310	[487, 479]	605	[785, 786]	900	[969, 117]
16	[33, 34]	311	[488, 419]	606	[50, 750]	901	[1020, 590]
17	[35, 36]	312	[489, 490]	607	[787, 788]	902	[1021, 280]
18	[37, 38]	313	[491, 175]	608	[44, 180]	903	[987, 367]
19	[39, 40]	314	[490, 485]	609	[556, 789]	904	[1022, 962]
20	[41, 42]	315	[174, 488]	610	[764, 49]	905	[1000, 962]
21	[43, 44]	316	[492, 493]	611	[790, 791]	906	[1023, 719]
22	[45, 46]	317	[454, 494]	612	[60, 186]	907	[958, 18]
23	[47, 48]	318	[495, 14]	613	[419, 792]	908	[1024, 1025]
24	[49, 50]	319	[496, 13]	614	[793, 794]	909	[1026, 366]

25	[51, 52]	320	[497, 498]	615	[795, 430]	910	[1005, 629]
26	[53, 54]	321	[499, 500]	616	[543, 525]	911	[106, 94]
27	[55, 56]	322	[154, 415]	617	[796, 428]	912	[329, 118]
28	[57, 58]	323	[501, 455]	618	[382, 429]	913	[637, 296]
29	[59, 60]	324	[502, 40]	619	[797, 798]	914	[1027, 1028]
30	[61, 62]	325	[503, 504]	620	[799, 800]	915	[1029, 99]
31	[63, 64]	326	[505, 506]	621	[801, 802]	916	[1024, 711]
32	[65, 66]	327	[507, 508]	622	[803, 521]	917	[998, 580]
33	[67, 68]	328	[509, 510]	623	[804, 148]	918	[1030, 99]
34	[69, 70]	329	[511, 512]	624	[805, 806]	919	[954, 103]
35	[71, 72]	330	[10, 217]	625	[131, 550]	920	[1031, 260]
36	[73, 74]	331	[513, 514]	626	[786, 51]	921	[726, 731]
37	[75, 76]	332	[395, 515]	627	[807, 520]	922	[369, 288]
38	[77, 78]	333	[405, 515]	628	[808, 809]	923	[1032, 243]
39	[19, 79]	334	[516, 517]	629	[810, 205]	924	[585, 979]
40	[80, 81]	335	[518, 519]	630	[811, 812]	925	[327, 285]
41	[82, 83]	336	[520, 521]	631	[774, 176]	926	[295, 625]
42	[25, 84]	337	[522, 523]	632	[813, 814]	927	[1033, 731]
43	[85, 86]	338	[455, 422]	633	[498, 397]	928	[974, 360]
44	[87, 88]	339	[524, 525]	634	[815, 816]	929	[630, 22]
45	[89, 90]	340	[526, 527]	635	[817, 467]	930	[686, 642]
46	[91, 92]	341	[449, 528]	636	[818, 819]	931	[1034, 280]
47	[93, 94]	342	[529, 479]	637	[820, 425]	932	[636, 79]
48	[95, 96]	343	[530, 519]	638	[184, 383]	933	[1035, 296]
49	[97, 98]	344	[531, 532]	639	[821, 765]	934	[1036, 304]
50	[99, 100]	345	[182, 533]	640	[822, 151]	935	[624, 731]
51	[101, 102]	346	[534, 535]	641	[823, 824]	936	[991, 259]
52	[28, 103]	347	[536, 431]	642	[825, 156]	937	[1037, 735]
53	[104, 105]	348	[537, 179]	643	[826, 827]	938	[302, 717]
54	[81, 106]	349	[538, 434]	644	[431, 828]	939	[1038, 322]
55	[107, 108]	350	[539, 487]	645	[829, 830]	940	[1039, 779]
56	[109, 78]	351	[540, 541]	646	[528, 533]	941	[840, 42]
57	[110, 77]	352	[218, 194]	647	[831, 817]	942	[1040, 466]
58	[106, 111]	353	[542, 543]	648	[162, 752]	943	[827, 1041]
59	[112, 113]	354	[544, 545]	649	[832, 833]	944	[1042, 1043]
60	[114, 115]	355	[546, 494]	650	[834, 764]	945	[867, 1044]
61	[116, 117]	356	[40, 208]	651	[222, 835]	946	[1045, 1046]
62	[118, 119]	357	[547, 548]	652	[504, 466]	947	[1047, 217]
63	[120, 121]	358	[549, 550]	653	[836, 837]	948	[1048, 1049]
64	[122, 123]	359	[551, 552]	654	[838, 569]	949	[1050, 531]
65	[124, 125]	360	[553, 506]	655	[839, 840]	950	[414, 882]
66	[126, 127]	361	[389, 222]	656	[841, 842]	951	[1051, 1052]
67	[128, 129]	362	[554, 123]	657	[176, 843]	952	[1053, 388]
68	[130, 131]	363	[555, 556]	658	[844, 845]	953	[451, 188]

69	[132, 133]	364	[481, 387]	659	[42, 562]	954	[756, 424]
70	[134, 135]	365	[557, 558]	660	[846, 396]	955	[1054, 516]
71	[136, 137]	366	[559, 412]	661	[847, 58]	956	[1055, 462]
72	[138, 139]	367	[550, 560]	662	[848, 849]	957	[1056, 1057]
73	[140, 141]	368	[519, 411]	663	[170, 62]	958	[1058, 136]
74	[142, 143]	369	[561, 562]	664	[141, 168]	959	[226, 572]
75	[144, 145]	370	[563, 564]	665	[850, 851]	960	[564, 409]
76	[146, 61]	371	[565, 566]	666	[852, 160]	961	[1059, 1060]
77	[147, 148]	372	[567, 568]	667	[853, 772]	962	[1061, 224]
78	[149, 150]	373	[391, 569]	668	[854, 493]	963	[1062, 1063]
79	[151, 152]	374	[570, 571]	669	[855, 856]	964	[1064, 174]
80	[153, 154]	375	[572, 573]	670	[857, 858]	965	[869, 52]
81	[155, 156]	376	[574, 575]	671	[859, 393]	966	[1065, 45]
82	[157, 158]	377	[346, 25]	672	[845, 754]	967	[1052, 1011]
83	[159, 60]	378	[576, 577]	673	[835, 779]	968	[837, 1066]
84	[160, 161]	379	[578, 289]	674	[525, 127]	969	[1067, 474]
85	[143, 162]	380	[579, 96]	675	[860, 768]	970	[1068, 1069]
86	[163, 164]	381	[580, 295]	676	[861, 862]	971	[1070, 407]
87	[165, 166]	382	[228, 581]	677	[863, 505]	972	[1071, 453]
88	[167, 168]	383	[329, 582]	678	[864, 819]	973	[1072, 1073]
89	[169, 170]	384	[583, 584]	679	[865, 392]	974	[1074, 1075]
90	[171, 172]	385	[585, 115]	680	[866, 512]	975	[1076, 1077]
91	[173, 174]	386	[586, 19]	681	[545, 867]	976	[856, 911]
92	[175, 176]	387	[587, 588]	682	[868, 869]	977	[1078, 1079]
93	[177, 178]	388	[589, 343]	683	[833, 870]	978	[1080, 1081]
94	[179, 180]	389	[295, 590]	684	[871, 185]	979	[794, 218]
95	[181, 182]	390	[237, 261]	685	[771, 135]	980	[1082, 384]
96	[183, 184]	391	[591, 364]	686	[872, 530]	981	[1083, 1084]
97	[185, 186]	392	[587, 592]	687	[445, 179]	982	[1085, 1086]
98	[187, 188]	393	[88, 582]	688	[873, 874]	983	[1087, 561]
99	[189, 190]	394	[593, 594]	689	[875, 876]	984	[1088, 223]
100	[191, 192]	395	[595, 596]	690	[877, 46]	985	[1089, 460]
101	[193, 194]	396	[597, 598]	691	[878, 463]	986	[1090, 1091]
102	[195, 51]	397	[599, 600]	692	[879, 880]	987	[1092, 1093]
103	[196, 122]	398	[601, 338]	693	[881, 882]	988	[1094, 1095]
104	[197, 198]	399	[112, 341]	694	[883, 454]	989	[902, 423]
105	[199, 200]	400	[576, 278]	695	[884, 31]	990	[1077, 480]
106	[201, 202]	401	[16, 602]	696	[885, 843]	991	[548, 1096]
107	[203, 204]	402	[603, 604]	697	[886, 383]	992	[1097, 1098]
108	[205, 206]	403	[605, 606]	698	[887, 888]	993	[1099, 865]
109	[207, 208]	404	[607, 608]	699	[889, 445]	994	[1100, 443]
110	[209, 210]	405	[609, 277]	700	[890, 55]	995	[158, 560]
111	[211, 212]	406	[610, 611]	701	[891, 792]	996	[1101, 1102]
112	[213, 214]	407	[69, 612]	702	[892, 893]	997	[417, 555]

113	[215, 216]	408	[317, 613]	703	[894, 139]	998	[814, 472]
114	[217, 218]	409	[304, 591]	704	[895, 502]	999	[14, 889]
115	[219, 220]	410	[614, 585]	705	[562, 56]	1000	[1103, 501]
116	[221, 222]	411	[97, 236]	706	[896, 789]	1001	[898, 1104]
117	[223, 224]	412	[258, 286]	707	[897, 898]	1002	[1105, 1106]
118	[225, 216]	413	[615, 616]	708	[899, 546]	1003	[1107, 390]
119	[168, 226]	414	[617, 618]	709	[900, 163]	1004	[1108, 777]
120	[227, 228]	415	[284, 101]	710	[901, 902]	1005	[1109, 418]
121	[229, 230]	416	[619, 254]	711	[190, 543]	1006	[541, 932]
122	[231, 232]	417	[620, 108]	712	[188, 766]	1007	[812, 196]
123	[233, 234]	418	[621, 622]	713	[903, 860]	1008	[1110, 53]
124	[235, 236]	419	[314, 92]	714	[904, 905]	1009	[1111, 408]
125	[237, 238]	420	[79, 623]	715	[906, 907]	1010	[1112, 1113]
126	[239, 231]	421	[624, 272]	716	[54, 171]	1011	[1011, 1011]
127	[240, 241]	422	[234, 326]	717	[908, 573]	1012	[208, 1114]
128	[242, 243]	423	[108, 277]	718	[909, 910]	1013	[1115, 495]
129	[244, 245]	424	[367, 625]	719	[911, 184]	1014	[1116, 1092]
130	[246, 247]	425	[626, 588]	720	[8, 401]	1015	[1117, 9]
131	[72, 248]	426	[627, 66]	721	[412, 912]	1016	[1118, 159]
132	[249, 250]	427	[628, 629]	722	[913, 143]	1017	[1119, 1120]
133	[251, 252]	428	[630, 631]	723	[914, 549]	1018	[1121, 155]
134	[253, 254]	429	[236, 373]	724	[483, 490]	1019	[1122, 142]
135	[255, 256]	430	[632, 245]	725	[915, 916]	1020	[1113, 818]
136	[257, 258]	431	[321, 633]	726	[917, 918]	1021	[1123, 524]
137	[259, 260]	432	[634, 635]	727	[459, 497]	1022	[1124, 1125]
138	[261, 262]	433	[636, 20]	728	[135, 154]	1023	[127, 1126]
139	[263, 101]	434	[637, 336]	729	[919, 757]	1024	[1127, 1128]
140	[264, 265]	435	[325, 638]	730	[920, 886]	1025	[876, 500]
141	[266, 267]	436	[639, 587]	731	[463, 791]	1026	[1129, 1130]
142	[268, 269]	437	[640, 636]	732	[921, 200]	1027	[1131, 215]
143	[270, 102]	438	[603, 606]	733	[754, 922]	1028	[918, 381]
144	[271, 272]	439	[641, 642]	734	[923, 207]	1029	[1132, 1133]
145	[273, 274]	440	[643, 116]	735	[924, 389]	1030	[1134, 1135]
146	[88, 118]	441	[644, 247]	736	[925, 926]	1031	[180, 511]
147	[275, 276]	442	[75, 333]	737	[927, 542]	1032	[1136, 1064]
148	[277, 278]	443	[645, 646]	738	[928, 929]	1033	[1137, 921]
149	[279, 243]	444	[647, 648]	739	[930, 885]	1034	[1138, 57]
150	[280, 28]	445	[105, 282]	740	[931, 149]	1035	[1139, 138]
151	[23, 274]	446	[649, 28]	741	[212, 932]	1036	[442, 1117]
152	[281, 282]	447	[104, 281]	742	[933, 790]	1037	[1140, 147]
153	[283, 284]	448	[270, 650]	743	[934, 935]	1038	[788, 211]
154	[285, 286]	449	[575, 591]	744	[936, 503]	1039	[714, 989]
155	[287, 261]	450	[651, 652]	745	[937, 891]	1040	[980, 673]
156	[288, 289]	451	[653, 654]	746	[569, 938]	1041	[236, 728]

157	[290, 111]	452	[655, 26]	747	[939, 537]	1042	[1141, 608]
158	[291, 265]	453	[656, 616]	748	[737, 320]	1043	[634, 243]
159	[117, 288]	454	[657, 311]	749	[77, 940]	1044	[231, 599]
160	[292, 293]	455	[353, 633]	750	[367, 590]	1045	[1001, 255]
161	[294, 295]	456	[658, 659]	751	[300, 941]	1046	[287, 238]
162	[296, 259]	457	[660, 661]	752	[253, 86]	1047	[608, 575]
163	[297, 246]	458	[662, 263]	753	[942, 369]	1048	[997, 296]
164	[117, 298]	459	[663, 664]	754	[232, 600]	1049	[970, 293]
165	[299, 105]	460	[665, 114]	755	[943, 98]	1050	[1142, 581]
166	[300, 301]	461	[666, 342]	756	[639, 626]	1051	[615, 661]
167	[302, 303]	462	[667, 668]	757	[944, 626]	1052	[1143, 68]
168	[265, 304]	463	[334, 30]	758	[945, 361]	1053	[675, 295]
169	[305, 306]	464	[669, 654]	759	[946, 280]	1054	[1144, 246]
170	[307, 308]	465	[670, 671]	760	[232, 947]	1055	[1145, 629]
171	[309, 310]	466	[672, 324]	761	[948, 253]	1056	[1146, 703]
172	[103, 256]	467	[673, 232]	762	[647, 716]	1057	[275, 696]
173	[311, 312]	468	[103, 674]	763	[949, 950]	1058	[1147, 288]
174	[313, 314]	469	[675, 367]	764	[582, 119]	1059	[1148, 953]
175	[92, 315]	470	[676, 663]	765	[951, 360]	1060	[632, 967]
176	[316, 317]	471	[677, 234]	766	[942, 117]	1061	[272, 1028]
177	[318, 319]	472	[323, 678]	767	[952, 233]	1062	[738, 642]
178	[320, 321]	473	[679, 97]	768	[98, 728]	1063	[1143, 90]
179	[322, 111]	474	[663, 680]	769	[670, 953]	1064	[641, 687]
180	[323, 324]	475	[681, 369]	770	[954, 255]	1065	[1149, 606]
181	[325, 326]	476	[682, 368]	771	[257, 285]	1066	[298, 705]
182	[327, 258]	477	[683, 373]	772	[955, 333]	1067	[107, 609]
183	[328, 329]	478	[684, 366]	773	[956, 618]	1068	[1036, 575]
184	[330, 241]	479	[685, 236]	774	[957, 70]	1069	[370, 105]
185	[331, 318]	480	[76, 29]	775	[958, 358]	1070	[1150, 70]
186	[259, 248]	481	[638, 652]	776	[959, 960]	1071	[1019, 68]
187	[238, 332]	482	[686, 687]	777	[961, 962]	1072	[975, 97]
188	[333, 334]	483	[688, 606]	778	[273, 24]	1073	[1030, 246]
189	[303, 335]	484	[689, 100]	779	[96, 234]	1074	[67, 90]
190	[336, 72]	485	[690, 315]	780	[344, 234]	1075	[961, 659]
191	[337, 338]	486	[582, 691]	781	[742, 338]	1076	[586, 636]
192	[339, 103]	487	[692, 368]	782	[963, 250]	1077	[330, 349]
193	[340, 341]	488	[312, 613]	783	[307, 964]	1078	[1146, 306]
194	[342, 84]	489	[91, 690]	784	[368, 965]	1079	[251, 668]
195	[18, 294]	490	[693, 657]	785	[966, 596]	1080	[101, 650]
196	[343, 319]	491	[694, 316]	786	[244, 967]	1081	[685, 98]
197	[344, 345]	492	[613, 91]	787	[968, 591]	1082	[1151, 585]
198	[346, 342]	493	[311, 317]	788	[589, 318]	1083	[118, 691]
199	[321, 347]	494	[694, 311]	789	[358, 294]	1084	[336, 259]
200	[348, 106]	495	[695, 696]	790	[682, 309]	1085	[986, 995]

201	[240, 349]	496	[359, 697]	791	[296, 72]	1086	[695, 276]
202	[64, 303]	497	[673, 599]	792	[613, 314]	1087	[976, 741]
203	[350, 309]	498	[366, 685]	793	[969, 369]	1088	[235, 98]
204	[351, 352]	499	[698, 699]	794	[970, 253]	1089	[1035, 336]
205	[320, 353]	500	[338, 581]	795	[971, 697]	1090	[982, 618]
206	[66, 336]	501	[700, 701]	796	[972, 671]	1091	[660, 616]
207	[354, 355]	502	[702, 703]	797	[334, 722]	1092	[372, 240]
208	[79, 356]	503	[704, 672]	798	[575, 741]	1093	[741, 364]
209	[357, 358]	504	[288, 705]	799	[973, 25]	1094	[977, 735]
210	[359, 360]	505	[258, 706]	800	[374, 230]	1095	[621, 618]
211	[326, 361]	506	[76, 334]	801	[974, 697]	1096	[255, 674]
212	[318, 362]	507	[707, 96]	802	[723, 355]	1097	[1022, 659]
213	[363, 364]	508	[708, 703]	803	[281, 960]	1098	[957, 612]
214	[365, 366]	509	[709, 293]	804	[280, 339]	1099	[1152, 965]
215	[294, 367]	510	[710, 711]	805	[975, 685]	1100	[658, 962]
216	[368, 310]	511	[712, 321]	806	[292, 253]	1101	[1153, 580]
217	[20, 356]	512	[654, 81]	807	[976, 591]	1102	[614, 114]
218	[369, 298]	513	[713, 637]	808	[977, 646]	1103	[971, 360]
219	[370, 281]	514	[714, 596]	809	[89, 68]	1104	[638, 361]
220	[345, 326]	515	[594, 685]	810	[946, 27]	1105	[1148, 671]
221	[371, 64]	516	[715, 716]	811	[978, 979]	1106	[700, 269]
222	[343, 362]	517	[246, 100]	812	[980, 231]	1107	[1154, 326]
223	[372, 330]	518	[654, 348]	813	[981, 282]	1108	[1150, 612]
224	[98, 373]	519	[717, 6]	814	[640, 19]	1109	[1149, 604]
225	[374, 375]	520	[718, 672]	815	[982, 622]	1110	[1155, 81]
226	[303, 6]	521	[591, 719]	816	[645, 735]	1111	[690, 693]
227	[376, 377]	522	[681, 117]	817	[983, 942]	1112	[239, 673]
228	[378, 164]	523	[665, 585]	818	[263, 270]	1113	[626, 592]
229	[379, 380]	524	[720, 240]	819	[608, 304]	1114	[81, 322]
230	[381, 382]	525	[298, 289]	820	[63, 302]	1115	[1145, 1006]
231	[383, 150]	526	[721, 256]	821	[984, 942]	1116	[1154, 638]
232	[384, 385]	527	[65, 637]	822	[985, 281]	1117	[74, 680]
233	[386, 387]	528	[293, 86]	823	[986, 267]	1118	[353, 347]
234	[388, 389]	529	[29, 722]	824	[667, 252]	1119	[988, 596]
235	[390, 391]	530	[723, 724]	825	[338, 611]	1120	[354, 724]
236	[392, 393]	531	[725, 712]	826	[987, 295]	1121	[1156, 288]
237	[394, 395]	532	[238, 262]	827	[718, 323]	1122	[966, 989]
238	[396, 397]	533	[326, 652]	828	[580, 367]	1123	[1156, 298]
239	[398, 399]	534	[649, 339]	829	[988, 989]	1124	[992, 1006]
240	[133, 400]	535	[726, 272]	830	[21, 631]	1125	[597, 950]
241	[401, 195]	536	[358, 580]	831	[990, 674]	1126	[284, 270]
242	[402, 403]	537	[704, 323]	832	[371, 302]	1127	[1157, 705]
243	[404, 405]	538	[708, 306]	833	[672, 678]	1128	[607, 265]
244	[406, 221]	539	[679, 685]	834	[594, 97]	1129	[1147, 298]

245	[407, 408]	540	[727, 728]	835	[108, 576]	1130	[944, 587]
246	[62, 409]	541	[729, 18]	836	[991, 72]	1131	[599, 947]
247	[410, 411]	542	[730, 301]	837	[110, 109]	1132	[1010, 81]
248	[137, 412]	543	[731, 732]	838	[620, 609]	1133	[1008, 25]
249	[413, 414]	544	[733, 84]	839	[20, 623]	1134	[1153, 294]
250	[380, 415]	545	[662, 284]	840	[345, 638]	1135	[692, 309]
251	[416, 417]	546	[734, 735]	841	[992, 629]	1136	[1037, 646]
252	[418, 419]	547	[736, 322]	842	[993, 308]	1137	[1155, 348]
253	[420, 161]	548	[331, 343]	843	[315, 657]	1138	[1038, 106]
254	[421, 422]	549	[737, 712]	844	[994, 267]	1139	[283, 263]
255	[56, 423]	550	[74, 664]	845	[993, 964]	1140	[249, 602]
256	[424, 425]	551	[738, 687]	846	[994, 995]	1141	[1158, 925]
257	[426, 427]	552	[617, 622]	847	[342, 26]	1142	[1159, 1160]
258	[428, 226]	553	[95, 233]	848	[996, 368]	1143	[1161, 1162]
259	[425, 167]	554	[739, 671]	849	[82, 584]	1144	[1163, 191]
260	[429, 187]	555	[285, 706]	850	[997, 336]	1145	[1164, 854]
261	[430, 431]	556	[64, 717]	851	[297, 99]	1146	[1165, 1166]
262	[432, 61]	557	[736, 106]	852	[998, 294]	1147	[1130, 1167]
263	[433, 434]	558	[73, 663]	853	[999, 115]	1148	[1168, 1169]
264	[435, 436]	559	[725, 320]	854	[1000, 659]	1149	[1170, 763]
265	[437, 212]	560	[712, 353]	855	[1001, 103]	1150	[1171, 10]
266	[438, 439]	561	[740, 109]	856	[720, 330]	1151	[1172, 219]
267	[440, 39]	562	[741, 719]	857	[1002, 616]	1152	[400, 1173]
268	[441, 442]	563	[574, 304]	858	[688, 604]	1153	[1174, 177]
269	[443, 175]	564	[742, 228]	859	[684, 594]	1154	[1046, 1175]
270	[444, 445]	565	[743, 114]	860	[740, 77]	1155	[31, 201]
271	[446, 447]	566	[698, 352]	861	[1003, 116]	1156	[558, 468]
272	[448, 449]	567	[744, 27]	862	[1004, 602]	1157	[387, 1176]
273	[450, 451]	568	[745, 267]	863	[955, 76]	1158	[943, 236]
274	[422, 385]	569	[261, 332]	864	[1004, 250]	1159	[717, 335]
275	[452, 386]	570	[746, 625]	865	[328, 88]	1160	[339, 255]
276	[453, 454]	571	[747, 654]	866	[1005, 1006]	1161	[963, 602]
277	[204, 171]	572	[304, 741]	867	[109, 940]	1162	[949, 598]
278	[206, 455]	573	[272, 732]	868	[1007, 28]	1163	[1007, 339]
279	[456, 457]	574	[748, 749]	869	[1008, 342]	1164	[956, 622]
280	[458, 459]	575	[470, 750]	870	[609, 576]	1165	[1002, 661]
281	[148, 381]	576	[751, 752]	871	[71, 259]	1166	[1013, 687]
282	[460, 190]	577	[753, 754]	872	[972, 953]	1167	[72, 260]
283	[461, 193]	578	[532, 146]	873	[242, 635]	1168	[1017, 612]
284	[415, 195]	579	[755, 756]	874	[268, 701]	1169	[734, 646]
285	[462, 463]	580	[757, 172]	875	[277, 577]	1170	[305, 703]
286	[427, 206]	581	[377, 449]	876	[348, 322]	1171	[1032, 635]
287	[464, 465]	582	[510, 528]	877	[1009, 1009]	1172	[677, 345]
288	[466, 467]	583	[758, 759]	878	[233, 345]	1173	[234, 638]

289	[468, 459]	584	[192, 517]	879	[1010, 348]	1174	[87, 329]
290	[38, 433]	585	[760, 220]	880	[337, 228]	1175	[322, 94]
291	[469, 470]	586	[761, 762]	881	[317, 313]	1176	[333, 29]
292	[471, 421]	587	[763, 764]	882	[1009, 1011]		
293	[397, 54]	588	[765, 766]	883	[315, 694]		
294	[36, 131]	589	[767, 554]	884	[1012, 732]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	197	1	394	0	591	1	788	0	985	0
1	0	198	1	395	1	592	0	789	0	986	0
2	0	199	0	396	1	593	0	790	0	987	1
3	0	200	0	397	1	594	0	791	0	988	0
4	0	201	0	398	1	595	1	792	1	989	0
5	0	202	1	399	1	596	1	793	0	990	1
6	0	203	0	400	0	597	1	794	1	991	1
7	0	204	1	401	0	598	0	795	1	992	0
8	0	205	0	402	0	599	1	796	1	993	0
9	0	206	1	403	1	600	0	797	0	994	1
10	1	207	0	404	1	601	1	798	1	995	0
11	0	208	0	405	1	602	0	799	1	996	1
12	0	209	1	406	1	603	0	800	0	997	1
13	0	210	0	407	1	604	0	801	0	998	0
14	1	211	0	408	1	605	1	802	1	999	1
15	0	212	0	409	0	606	1	803	1	1000	1
16	0	213	1	410	0	607	1	804	1	1001	1
17	0	214	1	411	1	608	1	805	1	1002	0
18	1	215	0	412	0	609	1	806	0	1003	1
19	0	216	0	413	1	610	1	807	0	1004	1
20	1	217	1	414	1	611	0	808	0	1005	1
21	1	218	0	415	0	612	1	809	1	1006	0
22	1	219	0	416	0	613	1	810	1	1007	0
23	0	220	1	417	1	614	0	811	0	1008	0
24	1	221	1	418	0	615	1	812	0	1009	1
25	0	222	1	419	1	616	0	813	0	1010	1
26	1	223	1	420	0	617	1	814	0	1011	0
27	0	224	1	421	1	618	0	815	0	1012	0
28	1	225	0	422	0	619	0	816	0	1013	1
29	1	226	1	423	0	620	1	817	0	1014	1
30	1	227	0	424	0	621	0	818	1	1015	0
31	0	228	0	425	1	622	1	819	1	1016	1
32	0	229	1	426	0	623	1	820	0	1017	0
33	0	230	1	427	0	624	1	821	0	1018	0
34	1	231	1	428	0	625	0	822	0	1019	1

35	0	232	0	429	0	626	1	823	0	1020	1
36	0	233	1	430	0	627	0	824	1	1021	0
37	1	234	0	431	0	628	0	825	1	1022	0
38	1	235	0	432	0	629	1	826	1	1023	0
39	0	236	0	433	1	630	0	827	1	1024	0
40	0	237	0	434	0	631	0	828	1	1025	0
41	1	238	1	435	0	632	0	829	0	1026	1
42	0	239	1	436	0	633	1	830	1	1027	1
43	1	240	0	437	0	634	0	831	1	1028	1
44	1	241	0	438	0	635	0	832	1	1029	1
45	1	242	0	439	0	636	1	833	0	1030	1
46	0	243	1	440	1	637	0	834	0	1031	1
47	0	244	1	441	0	638	1	835	0	1032	1
48	0	245	1	442	1	639	0	836	1	1033	0
49	1	246	0	443	0	640	0	837	1	1034	0
50	1	247	0	444	1	641	0	838	1	1035	0
51	0	248	1	445	0	642	1	839	1	1036	1
52	1	249	1	446	1	643	1	840	1	1037	1
53	1	250	1	447	1	644	0	841	0	1038	0
54	1	251	0	448	1	645	0	842	0	1039	0
55	0	252	0	449	1	646	1	843	0	1040	0
56	0	253	0	450	1	647	1	844	1	1041	0
57	1	254	1	451	1	648	0	845	0	1042	1
58	0	255	0	452	1	649	1	846	1	1043	0
59	1	256	0	453	0	650	0	847	1	1044	1
60	1	257	0	454	1	651	1	848	1	1045	1
61	1	258	0	455	1	652	0	849	1	1046	1
62	0	259	1	456	1	653	1	850	1	1047	1
63	0	260	0	457	1	654	1	851	0	1048	1
64	1	261	0	458	1	655	1	852	0	1049	1
65	0	262	0	459	1	656	0	853	1	1050	0
66	1	263	1	460	1	657	1	854	1	1051	1
67	0	264	0	461	0	658	1	855	1	1052	0
68	0	265	0	462	1	659	0	856	0	1053	0
69	1	266	0	463	0	660	1	857	0	1054	0
70	0	267	1	464	1	661	1	858	0	1055	1
71	0	268	0	465	0	662	1	859	0	1056	0
72	0	269	0	466	0	663	1	860	0	1057	1
73	0	270	1	467	0	664	0	861	1	1058	1
74	0	271	1	468	1	665	1	862	1	1059	0
75	1	272	1	469	0	666	0	863	0	1060	0
76	0	273	1	470	1	667	1	864	1	1061	1
77	1	274	0	471	0	668	1	865	0	1062	0
78	1	275	1	472	1	669	1	866	1	1063	0

79	0	276	0	473	0	670	0	867	0	1064	0
80	0	277	1	474	1	671	0	868	0	1065	1
81	1	278	1	475	1	672	0	869	0	1066	1
82	1	279	1	476	0	673	0	870	1	1067	0
83	1	280	1	477	0	674	1	871	0	1068	1
84	0	281	1	478	0	675	0	872	1	1069	0
85	1	282	1	479	0	676	1	873	0	1070	1
86	0	283	0	480	0	677	0	874	0	1071	1
87	1	284	0	481	1	678	1	875	1	1072	1
88	0	285	1	482	1	679	0	876	0	1073	1
89	1	286	0	483	0	680	1	877	1	1074	0
90	1	287	1	484	1	681	1	878	1	1075	0
91	0	288	0	485	1	682	0	879	1	1076	1
92	0	289	1	486	1	683	0	880	1	1077	1
93	0	290	1	487	1	684	0	881	1	1078	0
94	1	291	0	488	0	685	0	882	1	1079	0
95	0	292	0	489	0	686	1	883	0	1080	0
96	0	293	1	490	1	687	0	884	0	1081	0
97	1	294	0	491	0	688	0	885	1	1082	0
98	1	295	1	492	1	689	1	886	1	1083	0
99	1	296	0	493	0	690	1	887	0	1084	1
100	1	297	0	494	0	691	1	888	1	1085	0
101	0	298	1	495	0	692	1	889	1	1086	0
102	1	299	1	496	0	693	1	890	1	1087	0
103	1	300	0	497	0	694	0	891	1	1088	0
104	1	301	0	498	1	695	0	892	0	1089	0
105	0	302	0	499	0	696	1	893	0	1090	0
106	0	303	1	500	1	697	1	894	0	1091	1
107	0	304	0	501	1	698	0	895	0	1092	1
108	0	305	1	502	0	699	1	896	1	1093	0
109	0	306	1	503	0	700	1	897	1	1094	0
110	1	307	1	504	0	701	1	898	1	1095	0
111	0	308	1	505	0	702	0	899	1	1096	0
112	1	309	1	506	0	703	0	900	0	1097	0
113	0	310	1	507	1	704	0	901	1	1098	0
114	1	311	0	508	1	705	0	902	0	1099	0
115	0	312	0	509	0	706	1	903	1	1100	1
116	1	313	0	510	1	707	1	904	0	1101	1
117	1	314	1	511	1	708	1	905	1	1102	0
118	0	315	0	512	1	709	0	906	0	1103	1
119	0	316	1	513	1	710	1	907	1	1104	1
120	0	317	1	514	0	711	1	908	0	1105	0
121	1	318	0	515	0	712	1	909	1	1106	1
122	1	319	0	516	0	713	1	910	1	1107	1

123	1	320	0	517	0	714	0	911	0	1108	1
124	0	321	0	518	1	715	0	912	1	1109	1
125	0	322	1	519	0	716	1	913	0	1110	0
126	1	323	1	520	1	717	0	914	1	1111	1
127	0	324	0	521	1	718	1	915	1	1112	1
128	0	325	0	522	1	719	0	916	0	1113	1
129	1	326	0	523	1	720	0	917	0	1114	1
130	0	327	1	524	0	721	0	918	1	1115	1
131	0	328	0	525	1	722	0	919	0	1116	1
132	1	329	1	526	0	723	1	920	1	1117	0
133	0	330	1	527	0	724	0	921	0	1118	1
134	0	331	1	528	1	725	1	922	0	1119	0
135	0	332	1	529	1	726	0	923	1	1120	0
136	0	333	1	530	1	727	1	924	0	1121	0
137	1	334	0	531	1	728	0	925	1	1122	1
138	0	335	1	532	1	729	0	926	1	1123	0
139	1	336	1	533	0	730	1	927	0	1124	0
140	0	337	1	534	1	731	0	928	0	1125	1
141	0	338	1	535	0	732	0	929	0	1126	0
142	0	339	0	536	0	733	0	930	1	1127	0
143	1	340	0	537	0	734	1	931	0	1128	1
144	1	341	1	538	1	735	0	932	1	1129	1
145	1	342	1	539	0	736	1	933	0	1130	1
146	0	343	1	540	1	737	0	934	1	1131	1
147	1	344	1	541	0	738	0	935	1	1132	1
148	1	345	1	542	1	739	1	936	1	1133	0
149	1	346	1	543	0	740	0	937	1	1134	1
150	1	347	0	544	0	741	0	938	0	1135	1
151	0	348	0	545	1	742	0	939	0	1136	1
152	1	349	1	546	1	743	1	940	0	1137	0
153	0	350	0	547	1	744	1	941	1	1138	0
154	1	351	1	548	1	745	1	942	0	1139	0
155	1	352	0	549	0	746	0	943	1	1140	1
156	0	353	1	550	0	747	0	944	1	1141	1
157	1	354	0	551	0	748	0	945	0	1142	0
158	0	355	1	552	1	749	1	946	1	1143	0
159	1	356	0	553	0	750	0	947	1	1144	0
160	0	357	1	554	1	751	0	948	1	1145	1
161	0	358	0	555	1	752	0	949	0	1146	0
162	0	359	0	556	1	753	0	950	1	1147	1
163	0	360	0	557	1	754	0	951	1	1148	0
164	1	361	1	558	0	755	1	952	0	1149	1
165	1	362	1	559	1	756	0	953	1	1150	1
166	0	363	1	560	1	757	1	954	0	1151	0

167	0	364	1	561	0	758	0	955	0	1152	0
168	0	365	1	562	0	759	1	956	1	1153	1
169	1	366	1	563	0	760	0	957	0	1154	1
170	1	367	0	564	0	761	1	958	1	1155	0
171	1	368	0	565	1	762	1	959	1	1156	0
172	1	369	0	566	0	763	0	960	0	1157	0
173	0	370	0	567	1	764	1	961	0	1158	1
174	0	371	1	568	1	765	1	962	1	1159	0
175	0	372	1	569	0	766	0	963	0	1160	0
176	1	373	1	570	0	767	0	964	0	1161	0
177	0	374	0	571	0	768	1	965	0	1162	0
178	0	375	0	572	0	769	0	966	1	1163	0
179	1	376	0	573	1	770	0	967	0	1164	1
180	1	377	1	574	0	771	0	968	1	1165	0
181	0	378	0	575	1	772	0	969	0	1166	1
182	1	379	1	576	0	773	1	970	1	1167	0
183	0	380	1	577	0	774	0	971	1	1168	0
184	1	381	1	578	1	775	1	972	1	1169	1
185	1	382	0	579	1	776	1	973	1	1170	1
186	1	383	1	580	1	777	0	974	0	1171	1
187	1	384	0	581	1	778	1	975	1	1172	0
188	1	385	0	582	1	779	0	976	0	1173	0
189	1	386	1	583	0	780	1	977	0	1174	1
190	1	387	0	584	0	781	0	978	0	1175	1
191	1	388	0	585	0	782	0	979	1	1176	1
192	0	389	1	586	1	783	1	980	0		
193	0	390	0	587	0	784	0	981	0		
194	1	391	1	588	1	785	1	982	0		
195	1	392	0	589	0	786	1	983	0		
196	1	393	0	590	1	787	1	984	0		

Université de Strasbourg, CNRS, IRMA UMR 7501, F-67000 Strasbourg, France
Email address: guoniu.han@unistra.fr

School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan, PR China
Email address: huyining@hust.edu.cn