# PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a,b)=(z,z^6+z^4+z^3+z^2+z)$

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Abstract. The continued fraction conjecture claims that the continued fraction defined by the $(a,b)$-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements $(a,b)$ in $\mathbb{F}_2[x]\setminus\mathbb{F}_2$. In this paper we prove the conjecture for $(a,b)=(z,z^6+z^4+z^3+z^2+z)$.

## 1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a}=(a_0(z),a_1(z),a_2(z),\ldots)$ of elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z))=\cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{a_2(z)+\cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z))=\cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{\ddots+\cfrac{1}{a_n(z)}}}}\in\mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1}+c_{-2}z^{-2}+c_{-3}z^{-3}+\cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let $a$ and $b$ be two distinct elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$. We consider the continued fraction (1.1) associated with the $(a,b)$-Thue-Morse sequence

$$\mathbf{t}=(t_0(z),t_1(z),t_2(z),\ldots)=(a,b,b,a,b,a,a,b,\ldots)$$

that we shall denote

$$\mathrm{CF}(a,b):=\mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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**Theorem 1.1.** *When $(a,b)=(z,z^6+z^4+z^3+z^2+z)$, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

*where, for* $\mathrm{CF}(a,b)$*,*

$$\begin{aligned}
p_0(z)&=z^{18}+z^{15}+z^{12}+z^{11}+z^{10}+z^8+z^7+z^4+1,\\
p_1(z)&=z^{21}+z^{19}+z^{16}+z^{12}+z^{11}+z^8,\\
p_2(z)&=z^{22}+z^{20}+z^{14}+z^{12}+z^8,\\
p_3(z)&=z^{21}+z^{19}+z^{16}+z^{12}+z^{11}+z^8,\\
p_4(z)&=z^{20}+z^{15}+z^{11}+z^{10}+z^7+z^4+1,
\end{aligned}$$

*and for* $\mathrm{CF}(b,a)$*,*

$$\begin{aligned}
p_0(z)&=z^{15}+z^{12}+z^{11}+z^8+z^7+z^6+z^4+z^2+1,\\
p_1(z)&=z^{21}+z^{19}+z^{16}+z^{12}+z^{11}+z^8,\\
p_2(z)&=z^{22}+z^{20}+z^{14}+z^{12}+z^8,\\
p_3(z)&=z^{21}+z^{19}+z^{16}+z^{12}+z^{11}+z^8,\\
p_4(z)&=z^{20}+z^{18}+z^{15}+z^{11}+z^7+z^6+z^4+z^2+1.
\end{aligned}$$

## 2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix}P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix}:=\begin{pmatrix}t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix}t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x)&=x^{\deg(t_{2^n-1})}\begin{pmatrix}t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix}t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(t_0)}\begin{pmatrix}t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x)&=x^{\deg(\bar t_{2^n-1})}\begin{pmatrix}\bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix}\bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(\bar t_0)}\begin{pmatrix}\bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the $(b,a)$-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\geq 0$

$$\begin{aligned}
M_{n+1}(x)&=W_n(x)\cdot M_n(x),\\
W_{n+1}(x)&=M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer $d_n$. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four $2\times 2$ matrices $M^e, M^o, W^e, W^o$ as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(175, 175, 175, 175, 175)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices $M^e, M^o, W^e, W^o$. The first one is about relations between them; the second, about the structure of the each matrix.

**Lemma 2.1.** *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

*Proof.* We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z \left(\phi(W^o,0,0)(x,z),\ z^{12}\cdot\phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity $m$ and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 630 and substitute it for $y$ in $Q(x,y)$. We get a series of valuation less than 630, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z\left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity $\mu$, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$ to order 770 and substitute it for $y$ in $Q(x,y)$. We get a series of valuation less than 770, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. $\square$

Define

$$R^e = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

**Lemma 2.2.** *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned}M^e[:14u] = {}& M^e[:7u] + x^{3u}M^e[:4u] + x^{4u}M^e[:3u] + x^{6u}M^e[:u] + x^{3u}M^e[:7u]\\ &+ x^{6u}M^e[:4u] + x^{7u}M^e[:3u] + x^{9u}M^e[:u] + x^{4u}M^e[:7u]\\ &+ x^{7u}M^e[:4u] + x^{8u}M^e[:3u] + x^{10u}M^e[:u] + x^{8u}M^e[:6u]\\ &+ x^{11u}M^e[:3u] + x^{9u}M^e[:5u] + x^{13u}M^e[:u] + x^{12u}M^e[:2u]\\ &+ (x^{7u}+x^{10u}+x^{11u})R^e,\end{aligned}$$

$$W^e[:14u] = W^e[:7u] + x^{3u}W^e[:4u] + x^{4u}W^e[:3u] + x^{6u}W^e[:u] + x^{3u}W^e[:7u]$$

$$
\begin{aligned}
&+ x^{6u}W^e[:4u] + x^{7u}W^e[:3u] + x^{9u}W^e[:u] + x^{4u}W^e[:7u]\\
&+ x^{7u}W^e[:4u] + x^{8u}W^e[:3u] + x^{10u}W^e[:u] + x^{8u}W^e[:6u]\\
&+ x^{11u}W^e[:3u] + x^{9u}W^e[:5u] + x^{13u}W^e[:u] + x^{12u}W^e[:2u]\\
&+ (x^{7u} + x^{10u} + x^{11u})R^e.
\end{aligned}
$$

*For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.*

$$
\begin{aligned}
M^o[:14u] &= M^o[:7u] + x^{3u}M^o[:4u] + x^{4u}M^o[:3u] + x^{6u}M^o[:u] + x^{3u}M^o[:7u]\\
&+ x^{6u}M^o[:4u] + x^{7u}M^o[:3u] + x^{9u}M^o[:u] + x^{4u}M^o[:7u]\\
&+ x^{7u}M^o[:4u] + x^{8u}M^o[:3u] + x^{10u}M^o[:u] + x^{8u}M^o[:6u]\\
&+ x^{11u}M^o[:3u] + x^{9u}M^o[:5u] + x^{13u}M^o[:u] + x^{12u}M^o[:2u]\\
&+ (x^{7u} + x^{10u} + x^{11u})R^o,\\
W^o[:14u] &= W^o[:7u] + x^{3u}W^o[:4u] + x^{4u}W^o[:3u] + x^{6u}W^o[:u] + x^{3u}W^o[:7u]\\
&+ x^{6u}W^o[:4u] + x^{7u}W^o[:3u] + x^{9u}W^o[:u] + x^{4u}W^o[:7u]\\
&+ x^{7u}W^o[:4u] + x^{8u}W^o[:3u] + x^{10u}W^o[:u] + x^{8u}W^o[:6u]\\
&+ x^{11u}W^o[:3u] + x^{9u}W^o[:5u] + x^{13u}W^o[:u] + x^{12u}W^o[:2u]\\
&+ (x^{7u} + x^{10u} + x^{11u})R^o.
\end{aligned}
$$

*Proof.* To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many $k$'s into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$
\begin{aligned}
T[:14u] = T&[:7u] + x^{3u}T[:4u] + x^{4u}T[:3u] + x^{6u}T[:u] + x^{3u}T[:7u] + x^{6u}T[:4u]\\
&+ x^{7u}T[:3u] + x^{9u}T[:u] + x^{4u}T[:7u] + x^{7u}T[:4u] + x^{8u}T[:3u]\\
&+ x^{10u}T[:u] + x^{8u}T[:6u] + x^{11u}T[:3u] + x^{9u}T[:5u] + x^{13u}T[:u]\\
&+ x^{12u}T[:2u].
\end{aligned}
$$

The proof of the other 15 cases are similar. We break down the above identity into 7 parts:

$$
\begin{aligned}
0 &= x^{6u}T[u:2u] + x^{4u}T[3u:4u] + x^{3u}T[4u:5u] + T[7u:8u],\\
0 &= x^{6u}T[2u:3u] + x^{4u}T[4u:5u] + x^{3u}T[5u:6u] + T[8u:9u],\\
0 &= x^{6u}T[3u:4u] + x^{4u}T[5u:6u] + x^{3u}T[6u:7u] + T[9u:10u],\\
0 &= x^{10u}T[:u] + x^{9u}T[u:2u] + x^{7u}T[3u:4u] + x^{4u}T[6u:7u]\\
&\quad + T[10u:11u],\\
0 &= x^{11u}T[:u] + x^{9u}T[2u:3u] + x^{8u}T[3u:4u] + T[11u:12u],\\
0 &= x^{12u}T[:u] + x^{11u}T[u:2u] + x^{9u}T[3u:4u] + x^{8u}T[4u:5u]\\
&\quad + T[12u:13u],\\
0 &= x^{13u}T[:u] + x^{12u}T[u:2u] + x^{11u}T[2u:3u] + x^{9u}T[4u:5u]\\
&\quad + x^{8u}T[5u:6u] + T[13u:14u],
\end{aligned}
$$

which can be rewritten as

$$0 = T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[111w]_2], \tag{2.7}$$

$$0 = T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.8}$$

$$0 = T[[11w]_2] + T[[101w]_2] + T[[110w]_2] + T[[1001w]_2], \tag{2.9}$$

$$0 = T[[w]_2] + T[[1w]_2] + T[[11w]_2] + T[[110w]_2] + T[[1010w]_2], \tag{2.10}$$

$$0 = T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[1011w]_2], \tag{2.11}$$

$$0 = T[[w]_2] + T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[1100w]_2], \tag{2.12}$$

$$\begin{aligned} 0 = T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[100w]_2] + T[[101w]_2] \\ + T[[1101w]_2], \end{aligned} \tag{2.13}$$

for all binary word $w$ of length $2k$ and $w \neq 0^{2k}$, and

$$0 = T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[111w]_2], \tag{2.14}$$

$$0 = T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.15}$$

$$0 = T[[11w]_2] + T[[101w]_2] + T[[110w]_2] + T[[1001w]_2], \tag{2.16}$$

$$0 = T[[w]_2] + T[[1w]_2] + T[[11w]_2] + T[[110w]_2] + T[[1010w]_2], \tag{2.17}$$

$$0 = T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[1011w]_2], \tag{2.18}$$

$$0 = T[[w]_2] + T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[1100w]_2], \tag{2.19}$$

$$\begin{aligned} 0 = T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[100w]_2] + T[[101w]_2] \\ + T[[1101w]_2], \end{aligned} \tag{2.20}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates $T$ from its minimal polynomial and its first terms. This automaton has 286 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading $w$ from right to left starting from the state $s$, and $\tau$ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.20) can be written as

$$0 = \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 111)), \tag{2.21}$$

$$0 = \tau(A(s, 10)) + \tau(A(s, 100)) + \tau(A(s, 101)) + \tau(A(s, 1000)), \tag{2.22}$$

$$0 = \tau(A(s, 11)) + \tau(A(s, 101)) + \tau(A(s, 110)) + \tau(A(s, 1001)), \tag{2.23}$$

$$\begin{aligned} 0 = \tau(A(s, \epsilon)) + \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 110)) \\ + \tau(A(s, 1010)), \end{aligned} \tag{2.24}$$

$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 1011)), \tag{2.25}$$

$$\begin{aligned} 0 = \tau(A(s, \epsilon)) + \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 100)) \\ + \tau(A(s, 1100)), \end{aligned} \tag{2.26}$$

$$\begin{aligned} 0 = \tau(A(s, \epsilon)) + \tau(A(s, 1)) + \tau(A(s, 10)) + \tau(A(s, 100)) \\ + \tau(A(s, 101)) + \tau(A(s, 1101)), \end{aligned} \tag{2.27}$$

for all $s \in E_{2k}$, and

$$0 = \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 111)), \tag{2.28}$$

$$0 = \tau(A(s, 10)) + \tau(A(s, 100)) + \tau(A(s, 101)) + \tau(A(s, 1000)), \tag{2.29}$$

$$0 = \tau(A(s, 11)) + \tau(A(s, 101)) + \tau(A(s, 110)) + \tau(A(s, 1001)), \tag{2.30}$$

$$\begin{aligned} 0 = {}&\tau(A(s, \epsilon)) + \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 110)) \\ &+ \tau(A(s, 1010)), \end{aligned} \tag{2.31}$$

$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 1011)), \tag{2.32}$$

$$\begin{aligned} 0 = {}&\tau(A(s, \epsilon)) + \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 100)) \\ &+ \tau(A(s, 1100)), \end{aligned} \tag{2.33}$$

$$\begin{aligned} 0 = {}&\tau(A(s, \epsilon)) + \tau(A(s, 1)) + \tau(A(s, 10)) + \tau(A(s, 100)) \\ &+ \tau(A(s, 101)) + \tau(A(s, 1101)), \end{aligned} \tag{2.34}$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{18}), E_{18}) = (A(i, 0^{10}), E_{10})$, so that we only have to verify that identities (2.21) through (2.34) hold for $2 \le k \le 9$, which turns out to be true. □

In the following lemma, we express $M_{2k}$, $M_{2k+1}$, $W_{2k}$, and $W_{2k+1}$ in terms of $M^e$, $M^o$, $W^e$, and $W^o$.

**Lemma 2.3.** *For all integer $k \ge 2$, and $u = 2^{2k-1}$,*

$$M_{2k} = M^e[:7u] + x^{3u}M^e[:4u] + x^{4u}M^e[:3u] + x^{6u}M^e[:u] + x^{7u}R^e,$$
$$W_{2k} = W^e[:7u] + x^{3u}W^e[:4u] + x^{4u}W^e[:3u] + x^{6u}W^e[:u] + x^{7u}R^e.$$

*For all integer $k \ge 2$, and $u = 2^{2k}$,*

$$M_{2k+1} = M^o[:7u] + x^{3u}M^o[:4u] + x^{4u}M^o[:3u] + x^{6u}M^o[:u] + x^{7u}R^o,$$
$$W_{2k+1} = W^o[:7u] + x^{3u}W^o[:4u] + x^{4u}W^o[:3u] + x^{6u}W^o[:u] + x^{7u}R^o.$$

*Proof.* We call the four identities in Lemma 2.3 also by the name $M_{2k}$, $W_{2k}$, $M_{2k+1}$, and $W_{2k+1}$. For $n = 2$, the identities can be verified directly. For $n \ge 2$, we claim that

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1} \wedge W_{2k+1},$$
$$M_{2k+1} \wedge W_{2k+1} \Rightarrow M_{2k+2} \wedge W_{2k+2}.$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity $M_{2k+1}$ is equal to

$$\begin{aligned} W_{2k}M_{2k} = {}&\big(W^e[:7v] + x^{3v}W^e[:4v] + x^{4v}W^e[:3v] + x^{6v}W^e[:v] + x^{7u}R^e\big) \times \big( \\ &M^e[:7v] + x^{3v}M^e[:4v] + x^{4v}M^e[:3v] + x^{6v}M^e[:v] + x^{7u}R^e\big); \end{aligned} \tag{2.35}$$

Call this expression $lhs$. Note that both sides of identity $M_{2k+1}$ have the same term of highest degree $x^{14v}R^o$. Therefore we only need to prove that their difference is $O(x^{14v})$. Using Lemma 2.1 it can be seen that the right side of identity $M_{2k+1}$ is congruent, modulo $x^{14v}$, to

$$\begin{aligned} &W^e[:14v]M^e[:14v] + x^{6v}W^e[:8v]M^e[:8v] + x^{8v}W^e[:6v]M^e[:6v] \\ &\qquad + x^{12v}W^e[:2v]M^e[:2v]. \end{aligned}$$

For all $n \leq 14$, replace the occurrences of $W^e[: n \cdot v]$ and $M^e[: n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call $rhs$. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions $lhs$ (2.35) and $rhs$ as polynomials in $\mathbb{F}_2[a_1, ..., a_{14}, b_1, ..., b_{14}, c][X]$. Note that it is not a problem that $a_j$ commutes with $b_k$ while $W^e$ does not commute with $M^e$, because in the expressions concerned, the products of $W^e$-terms and $M^e$-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{14})$, which completes the proof. $\square$

*Proof of Theorem 1.1.* We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \leq j, k \leq 1$,

$$\begin{aligned} \lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k}, \\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k}, \\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k}, \\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}. \end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a, b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x, y) = \mathrm{Res}_t\left(\phi(M^e, 0, 1)(x, t), y^{12}\phi(M^e, 0, 1)(x, t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x, y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned} q_0(x) &= x^{22} + x^{18} + x^{15} + x^{14} + x^{12} + x^{11} + x^{10} + x^7 + x^4, \\ q_1(x) &= x^{14} + x^{11} + x^{10} + x^6 + x^3 + x, \\ q_2(x) &= x^{14} + x^{10} + x^8 + x^2 + 1, \\ q_3(x) &= x^{14} + x^{11} + x^{10} + x^6 + x^3 + x, \\ q_4(x) &= x^{22} + x^{18} + x^{15} + x^{12} + x^{11} + x^7 + x^2. \end{aligned}$$

The polynomial $Q(x, y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x, y)$ of multiplicity $m$ and $R(x, y) := P(x, y)/Q(x, y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate

$f(x)$ to order 216, and substitute it for $y$ in $R(z,y)$, we get a series with valuation smaller than 216, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b)=f(1/z)$. $\square$

## 3. Annexe

All 16 polynomials are of the form

$$p_0(x)+p_3(x)y^3+p_6(x)y^6+p_9(x)y^9+p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e,0,0)$:

$$\begin{aligned}
p_0(x) &= x^{132}+x^{124}+x^{118}+x^{116}+x^{112}+x^{110}+x^{108}+x^{104}+x^{94}+x^{90}\\
&\quad+x^{88}+x^{86}+x^{82}+x^{80}+x^{78}+x^{76}+x^{70}+x^{68}+x^{66}+x^{62}+x^{52}\\
&\quad+x^{50}+x^{48}+x^{44}+x^{42}+x^{38}+x^{30}+x^{22}+x^{12},\\
p_3(x) &= x^{100}+x^{94}+x^{92}+x^{90}+x^{88}+x^{86}+x^{82}+x^{80}+x^{78}+x^{76}+x^{74}\\
&\quad+x^{68}+x^{62}+x^{60}+x^{54}+x^{52}+x^{48}+x^{46}+x^{44}+x^{42}+x^{40}+x^{38}\\
&\quad+x^{34}+x^{30}+x^{26}+x^{24}+x^{22}+x^{8},\\
p_6(x) &= x^{94}+x^{88}+x^{82}+x^{80}+x^{78}+x^{76}+x^{66}+x^{60}+x^{58}+x^{56}+x^{52}\\
&\quad+x^{48}+x^{44}+x^{42}+x^{36}+x^{34}+x^{32}+x^{28}+x^{26}+x^{24}+x^{20}+x^{14}\\
&\quad+x^{10}+x^{4}+x^{2}+1,\\
p_9(x) &= x^{78}+x^{68}+x^{64}+x^{60}+x^{58}+x^{54}+x^{52}+x^{48}+x^{46}+x^{42}+x^{40}\\
&\quad+x^{38}+x^{36}+x^{34}+x^{30}+x^{22}+x^{20}+x^{16}+x^{14}+x^{10}+x^{8}+x^{6}+x^{4}\\
&\quad+x^{2},\\
p_{12}(x) &= x^{72}+x^{64}+x^{32}+x^{24}+1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0].

For $\phi(M^e,0,1)$:

$$\begin{aligned}
p_0(x) &= x^{117}+x^{116}+x^{115}+x^{114}+x^{113}+x^{111}+x^{110}+x^{108}+x^{103}+x^{98}\\
&\quad+x^{96}+x^{95}+x^{94}+x^{93}+x^{91}+x^{89}+x^{88}+x^{85}+x^{76}+x^{73}+x^{72}\\
&\quad+x^{69}+x^{68}+x^{63}+x^{61}+x^{56}+x^{53}+x^{51}+x^{50}+x^{49}+x^{47}+x^{46}\\
&\quad+x^{42}+x^{41}+x^{40}+x^{39}+x^{38}+x^{37}+x^{34}+x^{33}+x^{26}+x^{25}+x^{24}\\
&\quad+x^{23}+x^{22}+x^{20}+x^{18},\\
p_3(x) &= x^{75}+x^{74}+x^{73}+x^{70}+x^{68}+x^{65}+x^{63}+x^{62}+x^{60}+x^{59}+x^{58}\\
&\quad+x^{54}+x^{52}+x^{49}+x^{47}+x^{46}+x^{44}+x^{41}+x^{35}+x^{34}+x^{33}+x^{30}\\
&\quad+x^{28}+x^{27}+x^{26}+x^{23}+x^{15}+x^{14}+x^{12}+x^{9},\\
p_6(x) &= x^{69}+x^{68}+x^{67}+x^{66}+x^{65}+x^{60}+x^{58}+x^{57}+x^{54}+x^{51}+x^{50}\\
&\quad+x^{47}+x^{45}+x^{41}+x^{39}+x^{37}+x^{36}+x^{35}+x^{30}+x^{28}+x^{27}+x^{24}\\
&\quad+x^{23}+x^{22}+x^{19}+x^{16}+x^{15}+x^{14}+x^{13}+x^{12}+x^{11}+x^{9}+x^{8}+x^{6},\\
p_9(x) &= x^{63}+x^{62}+x^{61}+x^{58}+x^{57}+x^{55}+x^{53}+x^{52}+x^{51}+x^{49}+x^{47}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{46} + x^{45} + x^{42} + x^{40} + x^{37} + x^{33} + x^{32} + x^{31} + x^{29} + x^{21} + x^{20}\\
&+ x^{18} + x^{17} + x^{16} + x^{12} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6} + x^{3},\\
p_{12}(x) &= x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{51} + x^{50} + x^{49} + x^{47} + x^{46} + x^{45}\\
&+ x^{43} + x^{42} + x^{41} + x^{39} + x^{38} + x^{36} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29}\\
&+ x^{27} + x^{26} + x^{25} + x^{23} + x^{22} + x^{20} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13}\\
&+ x^{11} + x^{10} + x^{8} + x^{5} + x^{4} + x^{3} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 1, 1, 1, 0, 1, 0].

For $\phi(M^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{117} + x^{116} + x^{115} + x^{114} + x^{113} + x^{111} + x^{110} + x^{108} + x^{103} + x^{98}\\
&+ x^{96} + x^{95} + x^{94} + x^{93} + x^{91} + x^{89} + x^{88} + x^{85} + x^{76} + x^{73} + x^{72}\\
&+ x^{69} + x^{68} + x^{63} + x^{61} + x^{56} + x^{53} + x^{51} + x^{50} + x^{49} + x^{47} + x^{46}\\
&+ x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{34} + x^{33} + x^{26} + x^{25} + x^{24}\\
&+ x^{23} + x^{22} + x^{20} + x^{18},\\
p_3(x) &= x^{75} + x^{74} + x^{73} + x^{70} + x^{68} + x^{65} + x^{63} + x^{62} + x^{60} + x^{59} + x^{58}\\
&+ x^{54} + x^{52} + x^{49} + x^{47} + x^{46} + x^{44} + x^{41} + x^{35} + x^{34} + x^{33} + x^{30}\\
&+ x^{28} + x^{27} + x^{26} + x^{23} + x^{15} + x^{14} + x^{12} + x^{9},\\
p_6(x) &= x^{69} + x^{68} + x^{67} + x^{66} + x^{65} + x^{60} + x^{58} + x^{57} + x^{54} + x^{51} + x^{50}\\
&+ x^{47} + x^{45} + x^{41} + x^{39} + x^{37} + x^{36} + x^{35} + x^{30} + x^{28} + x^{27} + x^{24}\\
&+ x^{23} + x^{22} + x^{19} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{9} + x^{8} + x^{6},\\
p_9(x) &= x^{63} + x^{62} + x^{61} + x^{58} + x^{57} + x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{47}\\
&+ x^{46} + x^{45} + x^{42} + x^{40} + x^{37} + x^{33} + x^{32} + x^{31} + x^{29} + x^{21} + x^{20}\\
&+ x^{18} + x^{17} + x^{16} + x^{12} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6} + x^{3},\\
p_{12}(x) &= x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{51} + x^{50} + x^{49} + x^{47} + x^{46} + x^{45}\\
&+ x^{43} + x^{42} + x^{41} + x^{39} + x^{38} + x^{36} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29}\\
&+ x^{27} + x^{26} + x^{25} + x^{23} + x^{22} + x^{20} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13}\\
&+ x^{11} + x^{10} + x^{8} + x^{5} + x^{4} + x^{3} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 1, 1, 1, 0, 1, 0].

For $\phi(M^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{102} + x^{100} + x^{98} + x^{96} + x^{92} + x^{88} + x^{82} + x^{60} + x^{58} + x^{50} + x^{40}\\
&+ x^{36} + x^{32} + x^{28} + x^{24},\\
p_3(x) &= x^{70} + x^{68} + x^{66} + x^{62} + x^{60} + x^{58} + x^{54} + x^{52} + x^{50} + x^{40} + x^{38}\\
&+ x^{36} + x^{28} + x^{24} + x^{22} + x^{20} + x^{8} + x^{6},\\
p_6(x) &= x^{64} + x^{62} + x^{60} + x^{54} + x^{52} + x^{46} + x^{42} + x^{38} + x^{36} + x^{22} + x^{10} + x^{8}\\
&+ x^{6} + x^{4} + x^{2} + 1,\\
p_9(x) &= x^{48} + x^{46} + x^{44} + x^{42} + x^{40} + x^{38} + x^{34} + x^{28} + x^{22} + x^{20} + x^{14} + x^{8}
\end{aligned}
$$

$$\begin{aligned}
&+ x^4 + x^2,\\
p_{12}(x) &= x^{42} + x^{40} + x^{38} + x^{36} + x^{32} + x^{26} + x^{24} + x^{22} + x^{20} + x^{16} + x^{10} + x^8\\
&\quad + x^6 + x^4 + 1,
\end{aligned}$$

and the initial terms are [0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{102} + x^{100} + x^{92} + x^{84} + x^{82} + x^{76} + x^{68} + x^{66} + x^{64} + x^{60} + x^{56}\\
&\quad + x^{52} + x^{50} + x^{44} + x^{42} + x^{32} + x^{30} + x^{26} + x^{24} + x^{22} + x^{20} + x^{18}\\
&\quad + x^{12},\\
p_3(x) &= x^{70} + x^{68} + x^{64} + x^{60} + x^{50} + x^{48} + x^{42} + x^{40} + x^{38} + x^{36} + x^{34}\\
&\quad + x^{32} + x^{30} + x^{28} + x^{26} + x^{24} + x^{18} + x^{12} + x^{10} + x^8,\\
p_6(x) &= x^{64} + x^{62} + x^{58} + x^{56} + x^{54} + x^{52} + x^{48} + x^{40} + x^{38} + x^{36} + x^{28}\\
&\quad + x^{26} + x^{24} + x^{22} + x^{16} + x^{14} + x^{12} + x^{10} + x^2 + 1,\\
p_9(x) &= x^{48} + x^{46} + x^{44} + x^{42} + x^{40} + x^{38} + x^{34} + x^{28} + x^{22} + x^{20} + x^{14} + x^8\\
&\quad + x^4 + x^2,\\
p_{12}(x) &= x^{42} + x^{40} + x^{38} + x^{36} + x^{32} + x^{26} + x^{24} + x^{22} + x^{20} + x^{16} + x^{10} + x^8\\
&\quad + x^6 + x^4 + 1,
\end{aligned}$$

and the initial terms are [1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{117} + x^{116} + x^{115} + x^{114} + x^{112} + x^{107} + x^{106} + x^{103} + x^{102} + x^{101}\\
&\quad + x^{99} + x^{98} + x^{96} + x^{93} + x^{92} + x^{91} + x^{89} + x^{88} + x^{87} + x^{85} + x^{82}\\
&\quad + x^{77} + x^{76} + x^{75} + x^{70} + x^{69} + x^{68} + x^{63} + x^{61} + x^{60} + x^{59} + x^{57}\\
&\quad + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{46} + x^{45} + x^{43} + x^{42} + x^{41} + x^{40}\\
&\quad + x^{38} + x^{33} + x^{32} + x^{27},\\
p_3(x) &= x^{75} + x^{74} + x^{73} + x^{70} + x^{68} + x^{65} + x^{63} + x^{62} + x^{60} + x^{59} + x^{58}\\
&\quad + x^{54} + x^{52} + x^{49} + x^{47} + x^{46} + x^{44} + x^{41} + x^{35} + x^{34} + x^{33} + x^{30}\\
&\quad + x^{28} + x^{27} + x^{26} + x^{23} + x^{15} + x^{14} + x^{12} + x^9,\\
p_6(x) &= x^{69} + x^{68} + x^{67} + x^{66} + x^{65} + x^{60} + x^{58} + x^{57} + x^{54} + x^{51} + x^{50}\\
&\quad + x^{47} + x^{45} + x^{41} + x^{39} + x^{37} + x^{36} + x^{35} + x^{30} + x^{28} + x^{27} + x^{24}\\
&\quad + x^{23} + x^{22} + x^{19} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^9 + x^8 + x^6,\\
p_9(x) &= x^{63} + x^{62} + x^{61} + x^{58} + x^{57} + x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{47}\\
&\quad + x^{46} + x^{45} + x^{42} + x^{40} + x^{37} + x^{33} + x^{32} + x^{31} + x^{29} + x^{21} + x^{20}\\
&\quad + x^{18} + x^{17} + x^{16} + x^{12} + x^{10} + x^9 + x^8 + x^7 + x^6 + x^3,\\
p_{12}(x) &= x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{51} + x^{50} + x^{49} + x^{47} + x^{46} + x^{45}\\
&\quad + x^{43} + x^{42} + x^{41} + x^{39} + x^{38} + x^{36} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29}\\
&\quad + x^{27} + x^{26} + x^{25} + x^{23} + x^{22} + x^{20} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13}
\end{aligned}$$

$$+ x^{11} + x^{10} + x^8 + x^5 + x^4 + x^3 + x^2 + 1,$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 1, 1].

For $\phi(W^e, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{117} + x^{116} + x^{115} + x^{114} + x^{112} + x^{107} + x^{106} + x^{103} + x^{102} + x^{101}\\
&\quad + x^{99} + x^{98} + x^{96} + x^{93} + x^{92} + x^{91} + x^{89} + x^{88} + x^{87} + x^{85} + x^{82}\\
&\quad + x^{77} + x^{76} + x^{75} + x^{70} + x^{69} + x^{68} + x^{63} + x^{61} + x^{60} + x^{59} + x^{57}\\
&\quad + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{46} + x^{45} + x^{43} + x^{42} + x^{41} + x^{40}\\
&\quad + x^{38} + x^{33} + x^{32} + x^{27},\\
p_3(x) &= x^{75} + x^{74} + x^{73} + x^{70} + x^{68} + x^{65} + x^{63} + x^{62} + x^{60} + x^{59} + x^{58}\\
&\quad + x^{54} + x^{52} + x^{49} + x^{47} + x^{46} + x^{44} + x^{41} + x^{35} + x^{34} + x^{33} + x^{30}\\
&\quad + x^{28} + x^{27} + x^{26} + x^{23} + x^{15} + x^{14} + x^{12} + x^9,\\
p_6(x) &= x^{69} + x^{68} + x^{67} + x^{66} + x^{65} + x^{60} + x^{58} + x^{57} + x^{54} + x^{51} + x^{50}\\
&\quad + x^{47} + x^{45} + x^{41} + x^{39} + x^{37} + x^{36} + x^{35} + x^{30} + x^{28} + x^{27} + x^{24}\\
&\quad + x^{23} + x^{22} + x^{19} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^9 + x^8 + x^6,\\
p_9(x) &= x^{63} + x^{62} + x^{61} + x^{58} + x^{57} + x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{47}\\
&\quad + x^{46} + x^{45} + x^{42} + x^{40} + x^{37} + x^{33} + x^{32} + x^{31} + x^{29} + x^{21} + x^{20}\\
&\quad + x^{18} + x^{17} + x^{16} + x^{12} + x^{10} + x^9 + x^8 + x^7 + x^6 + x^3,\\
p_{12}(x) &= x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{51} + x^{50} + x^{49} + x^{47} + x^{46} + x^{45}\\
&\quad + x^{43} + x^{42} + x^{41} + x^{39} + x^{38} + x^{36} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29}\\
&\quad + x^{27} + x^{26} + x^{25} + x^{23} + x^{22} + x^{20} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13}\\
&\quad + x^{11} + x^{10} + x^8 + x^5 + x^4 + x^3 + x^2 + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 1, 1].

For $\phi(W^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{132} + x^{128} + x^{118} + x^{116} + x^{104} + x^{102} + x^{100} + x^{96} + x^{94} + x^{90}\\
&\quad + x^{88} + x^{86} + x^{82} + x^{80} + x^{76} + x^{72} + x^{66} + x^{62} + x^{56} + x^{54} + x^{50}\\
&\quad + x^{48} + x^{42},\\
p_3(x) &= x^{100} + x^{96} + x^{94} + x^{92} + x^{86} + x^{82} + x^{78} + x^{72} + x^{68} + x^{66} + x^{64}\\
&\quad + x^{56} + x^{54} + x^{48} + x^{44} + x^{40} + x^{32} + x^{30} + x^{28} + x^{26} + x^{22} + x^{20}\\
&\quad + x^{16} + x^{14} + x^8 + x^6,\\
p_6(x) &= x^{94} + x^{90} + x^{88} + x^{84} + x^{82} + x^{78} + x^{76} + x^{74} + x^{72} + x^{68} + x^{64}\\
&\quad + x^{62} + x^{60} + x^{58} + x^{56} + x^{52} + x^{46} + x^{38} + x^{36} + x^{26} + x^{24} + x^{20}\\
&\quad + x^{16} + x^{12} + x^{10} + x^6 + x^2 + 1,\\
p_9(x) &= x^{78} + x^{68} + x^{64} + x^{60} + x^{58} + x^{54} + x^{52} + x^{48} + x^{46} + x^{42} + x^{40}\\
&\quad + x^{38} + x^{36} + x^{34} + x^{30} + x^{22} + x^{20} + x^{16} + x^{14} + x^{10} + x^8 + x^6 + x^4\\
&\quad + x^2,
\end{aligned}$$

$$p_{12}(x) = x^{72} + x^{64} + x^{32} + x^{24} + 1,$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0].

For $\phi(M^o, 0, 0)$:

$$\begin{aligned}
p_0(x) = {} & x^{117} + x^{116} + x^{113} + x^{110} + x^{108} + x^{105} + x^{100} + x^{93} + x^{90} + x^{89} \\
& + x^{87} + x^{85} + x^{80} + x^{78} + x^{77} + x^{74} + x^{71} + x^{66} + x^{65} + x^{64} + x^{63} \\
& + x^{62} + x^{61} + x^{60} + x^{54} + x^{52} + x^{49} + x^{44} + x^{43} + x^{41} + x^{40} + x^{39} \\
& + x^{36} + x^{35} + x^{29} + x^{23} + x^{21} + x^{20} + x^{19} + x^{18} + x^{16} + x^{15} + x^{12}, \\
p_3(x) = {} & x^{85} + x^{84} + x^{81} + x^{79} + x^{77} + x^{75} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} \\
& + x^{67} + x^{66} + x^{63} + x^{58} + x^{56} + x^{55} + x^{53} + x^{51} + x^{48} + x^{47} + x^{46} \\
& + x^{45} + x^{44} + x^{41} + x^{39} + x^{36} + x^{34} + x^{32} + x^{31} + x^{29} + x^{28} + x^{27} \\
& + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{18} + x^{16} + x^{15} + x^{12} + x^{11} + x^{8}, \\
p_6(x) = {} & x^{73} + x^{72} + x^{69} + x^{67} + x^{65} + x^{63} + x^{61} + x^{60} + x^{59} + x^{57} + x^{56} \\
& + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{47} + x^{44} + x^{42} + x^{41} + x^{39} + x^{36} \\
& + x^{33} + x^{32} + x^{31} + x^{28} + x^{26} + x^{24} + x^{23} + x^{21} + x^{20} + x^{19} + x^{16} \\
& + x^{13} + x^{11} + x^{10} + x^{8} + x^{7} + x^{5} + x^{3} + x^{2} + 1, \\
p_9(x) = {} & x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{51} + x^{50} + x^{49} + x^{47} + x^{46} + x^{44} \\
& + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{31} + x^{30} + x^{28} + x^{21} + x^{20} + x^{19} \\
& + x^{18} + x^{17} + x^{15} + x^{14} + x^{12} + x^{9} + x^{8} + x^{7} + x^{6} + x^{4}, \\
p_{12}(x) = {} & x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39} + x^{38} + x^{37} + x^{35} + x^{34} + x^{32} \\
& + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{23} + x^{22} + x^{21} + x^{19} + x^{18} + x^{16} \\
& + x^{13} + x^{12} + x^{11} + x^{10} + x^{9} + x^{7} + x^{6} + x^{5} + x^{3} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 1, 1, 0, 1, 1].

For $\phi(M^o, 0, 1)$:

$$\begin{aligned}
p_0(x) = {} & x^{112} + x^{110} + x^{104} + x^{103} + x^{100} + x^{97} + x^{96} + x^{89} + x^{88} + x^{87} + x^{86} \\
& + x^{85} + x^{81} + x^{80} + x^{79} + x^{76} + x^{75} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} \\
& + x^{67} + x^{58} + x^{54} + x^{53} + x^{51} + x^{48} + x^{47} + x^{46} + x^{42} + x^{41} + x^{39} \\
& + x^{38} + x^{36} + x^{35} + x^{34} + x^{32} + x^{27} + x^{26} + x^{24} + x^{22} + x^{21} + x^{20} \\
& + x^{18}, \\
p_3(x) = {} & x^{76} + x^{73} + x^{72} + x^{70} + x^{67} + x^{64} + x^{62} + x^{60} + x^{56} + x^{54} + x^{51} \\
& + x^{48} + x^{46} + x^{41} + x^{36} + x^{33} + x^{32} + x^{30} + x^{28} + x^{27} + x^{25} + x^{19} \\
& + x^{17} + x^{16} + x^{14} + x^{9}, \\
p_6(x) = {} & x^{64} + x^{60} + x^{58} + x^{56} + x^{54} + x^{46} + x^{44} + x^{40} + x^{38} + x^{34} + x^{30} \\
& + x^{28} + x^{26} + x^{24} + x^{18} + x^{16} + x^{10} + x^{6}, \\
p_9(x) = {} & x^{52} + x^{49} + x^{48} + x^{42} + x^{40} + x^{36} + x^{35} + x^{33} + x^{32} + x^{26} + x^{24} \\
& + x^{20} + x^{19} + x^{17} + x^{16} + x^{10} + x^{8} + x^{3},
\end{aligned}$$

$$p_{12}(x) = x^{40} + x^{36} + x^{32} + x^{24} + x^{20} + x^{16} + x^8 + x^4 + 1,$$

and the initial terms are [0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1].

For $\phi(M^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{132} + x^{130} + x^{128} + x^{126} + x^{123} + x^{120} + x^{119} + x^{117} + x^{116} + x^{115} \\
&\quad + x^{113} + x^{111} + x^{108} + x^{107} + x^{106} + x^{105} + x^{103} + x^{101} + x^{100} \\
&\quad + x^{97} + x^{96} + x^{93} + x^{91} + x^{88} + x^{85} + x^{84} + x^{82} + x^{79} + x^{78} + x^{77} \\
&\quad + x^{74} + x^{69} + x^{68} + x^{65} + x^{60} + x^{59} + x^{55} + x^{54} + x^{53} + x^{51} + x^{50} \\
&\quad + x^{48} + x^{47} + x^{44} + x^{43} + x^{39} + x^{38} + x^{37} + x^{35} + x^{34} + x^{33} + x^{32} \\
&\quad + x^{31} + x^{30} + x^{27}, \\
p_3(x) &= x^{96} + x^{93} + x^{90} + x^{89} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} \\
&\quad + x^{79} + x^{75} + x^{73} + x^{71} + x^{70} + x^{68} + x^{64} + x^{63} + x^{62} + x^{61} + x^{59} \\
&\quad + x^{58} + x^{57} + x^{56} + x^{51} + x^{50} + x^{47} + x^{44} + x^{43} + x^{42} + x^{41} + x^{38} \\
&\quad + x^{36} + x^{34} + x^{33} + x^{31} + x^{26} + x^{25} + x^{24} + x^{22} + x^{21} + x^{19} + x^{16} \\
&\quad + x^{14} + x^9, \\
p_6(x) &= x^{84} + x^{78} + x^{76} + x^{72} + x^{66} + x^{64} + x^{60} + x^{58} + x^{56} + x^{50} + x^{46} \\
&\quad + x^{44} + x^{42} + x^{40} + x^{36} + x^{34} + x^{30} + x^{18} + x^{16} + x^{14} + x^{10} + x^6, \\
p_9(x) &= x^{72} + x^{69} + x^{65} + x^{62} + x^{61} + x^{60} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} \\
&\quad + x^{53} + x^{51} + x^{50} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{41} + x^{39} + x^{38} \\
&\quad + x^{37} + x^{34} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27} + x^{25} + x^{24} + x^{23} + x^{22} \\
&\quad + x^{18} + x^{17} + x^{15} + x^{11} + x^{10} + x^8 + x^3, \\
p_{12}(x) &= x^{60} + x^{52} + x^{48} + x^{44} + x^{28} + x^4 + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 0, 0, 0, 1, 0].

For $\phi(M^o, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{117} + x^{116} + x^{113} + x^{110} + x^{108} + x^{107} + x^{106} + x^{104} + x^{102} + x^{101} \\
&\quad + x^{97} + x^{95} + x^{92} + x^{91} + x^{90} + x^{87} + x^{86} + x^{85} + x^{81} + x^{77} + x^{76} \\
&\quad + x^{75} + x^{74} + x^{73} + x^{68} + x^{65} + x^{62} + x^{58} + x^{57} + x^{55} + x^{54} + x^{51} \\
&\quad + x^{49} + x^{47} + x^{43} + x^{41} + x^{39} + x^{38} + x^{36} + x^{35} + x^{33}, \\
p_3(x) &= x^{85} + x^{84} + x^{81} + x^{79} + x^{77} + x^{74} + x^{69} + x^{68} + x^{66} + x^{64} + x^{62} \\
&\quad + x^{57} + x^{56} + x^{53} + x^{45} + x^{44} + x^{43} + x^{41} + x^{40} + x^{39} + x^{36} + x^{35} \\
&\quad + x^{33} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{18} \\
&\quad + x^{16} + x^{14} + x^{12}, \\
p_6(x) &= x^{73} + x^{72} + x^{69} + x^{67} + x^{65} + x^{62} + x^{57} + x^{56} + x^{55} + x^{53} + x^{49} \\
&\quad + x^{43} + x^{40} + x^{36} + x^{34} + x^{32} + x^{29} + x^{25} + x^{23} + x^{20} + x^{19} + x^{18} \\
&\quad + x^{16} + x^{15} + x^{14} + x^{13} + x^{11} + x^5 + x^4 + x^3 + x^2 + 1, \\
p_9(x) &= x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{51} + x^{50} + x^{49} + x^{47} + x^{46} + x^{44}
\end{aligned}$$

$$+ x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{31} + x^{30} + x^{28} + x^{21} + x^{20} + x^{19} + x^{18} + x^{17} + x^{15} + x^{14} + x^{12} + x^{9} + x^{8} + x^{7} + x^{6} + x^{4},$$

$$p_{12}(x) = x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39} + x^{38} + x^{37} + x^{35} + x^{34} + x^{32} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{23} + x^{22} + x^{21} + x^{19} + x^{18} + x^{16} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9} + x^{7} + x^{6} + x^{5} + x^{3} + x^{2} + 1,$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 0, 0, 1, 1].

For $\phi(W^o, 0, 0)$:

$$p_0(x) = x^{117} + x^{116} + x^{113} + x^{110} + x^{108} + x^{105} + x^{100} + x^{93} + x^{90} + x^{89} + x^{87} + x^{85} + x^{80} + x^{78} + x^{77} + x^{74} + x^{71} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{54} + x^{52} + x^{49} + x^{44} + x^{43} + x^{41} + x^{40} + x^{39} + x^{36} + x^{35} + x^{29} + x^{23} + x^{21} + x^{20} + x^{19} + x^{18} + x^{16} + x^{15} + x^{12},$$

$$p_3(x) = x^{85} + x^{84} + x^{81} + x^{79} + x^{77} + x^{75} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{67} + x^{66} + x^{63} + x^{58} + x^{56} + x^{55} + x^{53} + x^{51} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{41} + x^{39} + x^{36} + x^{34} + x^{32} + x^{31} + x^{29} + x^{28} + x^{27} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{18} + x^{16} + x^{15} + x^{12} + x^{11} + x^{8},$$

$$p_6(x) = x^{73} + x^{72} + x^{69} + x^{67} + x^{65} + x^{63} + x^{61} + x^{60} + x^{59} + x^{57} + x^{56} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{47} + x^{44} + x^{42} + x^{41} + x^{39} + x^{36} + x^{33} + x^{32} + x^{31} + x^{28} + x^{26} + x^{24} + x^{23} + x^{21} + x^{20} + x^{19} + x^{16} + x^{13} + x^{11} + x^{10} + x^{8} + x^{7} + x^{5} + x^{3} + x^{2} + 1,$$

$$p_9(x) = x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{51} + x^{50} + x^{49} + x^{47} + x^{46} + x^{44} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{31} + x^{30} + x^{28} + x^{21} + x^{20} + x^{19} + x^{18} + x^{17} + x^{15} + x^{14} + x^{12} + x^{9} + x^{8} + x^{7} + x^{6} + x^{4},$$

$$p_{12}(x) = x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39} + x^{38} + x^{37} + x^{35} + x^{34} + x^{32} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{23} + x^{22} + x^{21} + x^{19} + x^{18} + x^{16} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9} + x^{7} + x^{6} + x^{5} + x^{3} + x^{2} + 1,$$

and the initial terms are [1, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 1, 1, 0, 1, 1].

For $\phi(W^o, 0, 1)$:

$$p_0(x) = x^{132} + x^{130} + x^{128} + x^{126} + x^{123} + x^{120} + x^{119} + x^{117} + x^{116} + x^{115} + x^{113} + x^{111} + x^{108} + x^{107} + x^{106} + x^{105} + x^{103} + x^{101} + x^{100} + x^{97} + x^{96} + x^{93} + x^{91} + x^{88} + x^{85} + x^{84} + x^{82} + x^{79} + x^{78} + x^{77} + x^{74} + x^{69} + x^{68} + x^{65} + x^{60} + x^{59} + x^{55} + x^{54} + x^{53} + x^{51} + x^{50} + x^{48} + x^{47} + x^{44} + x^{43} + x^{39} + x^{38} + x^{37} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{27},$$

$$p_3(x) = x^{96} + x^{93} + x^{90} + x^{89} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{79} + x^{75} + x^{73} + x^{71} + x^{70} + x^{68} + x^{64} + x^{63} + x^{62} + x^{61} + x^{59}$$

$$
\begin{aligned}
&+x^{58}+x^{57}+x^{56}+x^{51}+x^{50}+x^{47}+x^{44}+x^{43}+x^{42}+x^{41}+x^{38}\\
&+x^{36}+x^{34}+x^{33}+x^{31}+x^{26}+x^{25}+x^{24}+x^{22}+x^{21}+x^{19}+x^{16}\\
&+x^{14}+x^{9},\\
p_6(x)&=x^{84}+x^{78}+x^{76}+x^{72}+x^{66}+x^{64}+x^{60}+x^{58}+x^{56}+x^{50}+x^{46}\\
&+x^{44}+x^{42}+x^{40}+x^{36}+x^{34}+x^{30}+x^{18}+x^{16}+x^{14}+x^{10}+x^{6},\\
p_9(x)&=x^{72}+x^{69}+x^{65}+x^{62}+x^{61}+x^{60}+x^{58}+x^{57}+x^{56}+x^{55}+x^{54}\\
&+x^{53}+x^{51}+x^{50}+x^{47}+x^{46}+x^{45}+x^{44}+x^{43}+x^{41}+x^{39}+x^{38}\\
&+x^{37}+x^{34}+x^{31}+x^{30}+x^{29}+x^{28}+x^{27}+x^{25}+x^{24}+x^{23}+x^{22}\\
&+x^{18}+x^{17}+x^{15}+x^{11}+x^{10}+x^{8}+x^{3},\\
p_{12}(x)&=x^{60}+x^{52}+x^{48}+x^{44}+x^{28}+x^{4}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 0, 0, 0, 1, 0].

For $\phi(W^o,1,0)$:

$$
\begin{aligned}
p_0(x)&=x^{112}+x^{110}+x^{104}+x^{103}+x^{100}+x^{97}+x^{96}+x^{89}+x^{88}+x^{87}+x^{86}\\
&+x^{85}+x^{81}+x^{80}+x^{79}+x^{76}+x^{75}+x^{73}+x^{72}+x^{71}+x^{69}+x^{68}\\
&+x^{67}+x^{58}+x^{54}+x^{53}+x^{51}+x^{48}+x^{47}+x^{46}+x^{42}+x^{41}+x^{39}\\
&+x^{38}+x^{36}+x^{35}+x^{34}+x^{32}+x^{27}+x^{26}+x^{24}+x^{22}+x^{21}+x^{20}\\
&+x^{18},\\
p_3(x)&=x^{76}+x^{73}+x^{72}+x^{70}+x^{67}+x^{64}+x^{62}+x^{60}+x^{56}+x^{54}+x^{51}\\
&+x^{48}+x^{46}+x^{41}+x^{36}+x^{33}+x^{32}+x^{30}+x^{28}+x^{27}+x^{25}+x^{19}\\
&+x^{17}+x^{16}+x^{14}+x^{9},\\
p_6(x)&=x^{64}+x^{60}+x^{58}+x^{56}+x^{54}+x^{46}+x^{44}+x^{40}+x^{38}+x^{34}+x^{30}\\
&+x^{28}+x^{26}+x^{24}+x^{18}+x^{16}+x^{10}+x^{6},\\
p_9(x)&=x^{52}+x^{49}+x^{48}+x^{42}+x^{40}+x^{36}+x^{35}+x^{33}+x^{32}+x^{26}+x^{24}\\
&+x^{20}+x^{19}+x^{17}+x^{16}+x^{10}+x^{8}+x^{3},\\
p_{12}(x)&=x^{40}+x^{36}+x^{32}+x^{24}+x^{20}+x^{16}+x^{8}+x^{4}+1,
\end{aligned}
$$

and the initial terms are [0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1].

For $\phi(W^o,1,1)$:

$$
\begin{aligned}
p_0(x)&=x^{117}+x^{116}+x^{113}+x^{110}+x^{108}+x^{107}+x^{106}+x^{104}+x^{102}+x^{101}\\
&+x^{97}+x^{95}+x^{92}+x^{91}+x^{90}+x^{87}+x^{86}+x^{85}+x^{81}+x^{77}+x^{76}\\
&+x^{75}+x^{74}+x^{73}+x^{68}+x^{65}+x^{62}+x^{58}+x^{57}+x^{55}+x^{54}+x^{51}\\
&+x^{49}+x^{47}+x^{43}+x^{41}+x^{39}+x^{38}+x^{36}+x^{35}+x^{33},\\
p_3(x)&=x^{85}+x^{84}+x^{81}+x^{79}+x^{77}+x^{74}+x^{69}+x^{68}+x^{66}+x^{64}+x^{62}\\
&+x^{57}+x^{56}+x^{53}+x^{45}+x^{44}+x^{43}+x^{41}+x^{40}+x^{39}+x^{36}+x^{35}\\
&+x^{33}+x^{29}+x^{28}+x^{27}+x^{26}+x^{25}+x^{24}+x^{23}+x^{21}+x^{20}+x^{18}\\
&+x^{16}+x^{14}+x^{12},
\end{aligned}
$$

$$\begin{aligned}p_6(x) &= x^{73} + x^{72} + x^{69} + x^{67} + x^{65} + x^{62} + x^{57} + x^{56} + x^{55} + x^{53} + x^{49}\\ &\quad + x^{43} + x^{40} + x^{36} + x^{34} + x^{32} + x^{29} + x^{25} + x^{23} + x^{20} + x^{19} + x^{18}\\ &\quad + x^{16} + x^{15} + x^{14} + x^{13} + x^{11} + x^5 + x^4 + x^3 + x^2 + 1,\\ p_9(x) &= x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{51} + x^{50} + x^{49} + x^{47} + x^{46} + x^{44}\\ &\quad + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{31} + x^{30} + x^{28} + x^{21} + x^{20} + x^{19}\\ &\quad + x^{18} + x^{17} + x^{15} + x^{14} + x^{12} + x^9 + x^8 + x^7 + x^6 + x^4,\\ p_{12}(x) &= x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39} + x^{38} + x^{37} + x^{35} + x^{34} + x^{32}\\ &\quad + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{23} + x^{22} + x^{21} + x^{19} + x^{18} + x^{16}\\ &\quad + x^{13} + x^{12} + x^{11} + x^{10} + x^9 + x^7 + x^6 + x^5 + x^3 + x^2 + 1,\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 0, 0, 1, 1].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ ($\Lambda(n) := [\delta(n, 0), \delta(n, 1)]$):

| $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | [1, 2] | 58 | [102, 103] | 116 | [169, 66] | 174 | [222, 144] | 232 | [59, 173] |
| 1 | [3, 4] | 59 | [38, 104] | 117 | [94, 170] | 175 | [149, 112] | 233 | [68, 177] |
| 2 | [5, 6] | 60 | [105, 106] | 118 | [171, 172] | 176 | [223, 224] | 234 | [256, 96] |
| 3 | [7, 8] | 61 | [107, 108] | 119 | [92, 173] | 177 | [225, 128] | 235 | [173, 255] |
| 4 | [9, 10] | 62 | [109, 110] | 120 | [174, 175] | 178 | [226, 105] | 236 | [259, 93] |
| 5 | [11, 12] | 63 | [12, 111] | 121 | [176, 177] | 179 | [227, 228] | 237 | [252, 257] |
| 6 | [13, 14] | 64 | [112, 45] | 122 | [178, 179] | 180 | [32, 212] | 238 | [262, 15] |
| 7 | [15, 16] | 65 | [113, 114] | 123 | [180, 79] | 181 | [123, 56] | 239 | [250, 16] |
| 8 | [17, 18] | 66 | [115, 116] | 124 | [181, 182] | 182 | [219, 229] | 240 | [263, 264] |
| 9 | [19, 20] | 67 | [8, 117] | 125 | [183, 184] | 183 | [230, 231] | 241 | [265, 91] |
| 10 | [21, 22] | 68 | [118, 105] | 126 | [181, 168] | 184 | [35, 53] | 242 | [266, 16] |
| 11 | [23, 24] | 69 | [119, 120] | 127 | [166, 185] | 185 | [155, 47] | 243 | [194, 267] |
| 12 | [25, 26] | 70 | [121, 122] | 128 | [176, 97] | 186 | [229, 152] | 244 | [262, 250] |
| 13 | [19, 27] | 71 | [123, 124] | 129 | [164, 186] | 187 | [232, 54] | 245 | [204, 100] |
| 14 | [28, 29] | 72 | [47, 125] | 130 | [66, 161] | 188 | [233, 216] | 246 | [254, 77] |
| 15 | [30, 31] | 73 | [53, 126] | 131 | [78, 184] | 189 | [30, 216] | 247 | [268, 161] |
| 16 | [32, 33] | 74 | [127, 128] | 132 | [178, 187] | 190 | [153, 115] | 248 | [247, 269] |
| 17 | [34, 35] | 75 | [129, 130] | 133 | [188, 172] | 191 | [234, 104] | 249 | [141, 149] |
| 18 | [36, 37] | 76 | [131, 132] | 134 | [189, 85] | 192 | [126, 235] | 250 | [140, 244] |
| 19 | [38, 39] | 77 | [111, 47] | 135 | [190, 65] | 193 | [236, 206] | 251 | [39, 220] |
| 20 | [40, 41] | 78 | [129, 133] | 136 | [191, 192] | 194 | [237, 238] | 252 | [212, 270] |
| 21 | [42, 43] | 79 | [134, 125] | 137 | [88, 185] | 195 | [239, 238] | 253 | [271, 224] |
| 22 | [44, 45] | 80 | [135, 32] | 138 | [193, 96] | 196 | [240, 127] | 254 | [218, 245] |
| 23 | [46, 47] | 81 | [131, 136] | 139 | [194, 175] | 197 | [241, 242] | 255 | [104, 220] |
| 24 | [48, 49] | 82 | [137, 138] | 140 | [195, 88] | 198 | [239, 243] | 256 | [37, 57] |
| 25 | [50, 32] | 83 | [45, 139] | 141 | [25, 22] | 199 | [45, 244] | 257 | [272, 138] |
| 26 | [51, 52] | 84 | [140, 139] | 142 | [196, 197] | 200 | [108, 114] | 258 | [31, 217] |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 27 | [53, 54] | 85 | [141, 110] | 143 | [198, 199] | 201 | [33, 213] | 259 | [214, 273] |
| 28 | [55, 56] | 86 | [142, 121] | 144 | [61, 159] | 202 | [245, 229] | 260 | [128, 207] |
| 29 | [57, 58] | 87 | [143, 49] | 145 | [200, 91] | 203 | [246, 145] | 261 | [128, 120] |
| 30 | [59, 60] | 88 | [37, 116] | 146 | [201, 163] | 204 | [247, 231] | 262 | [274, 43] |
| 31 | [61, 62] | 89 | [144, 145] | 147 | [159, 29] | 205 | [248, 65] | 263 | [275, 53] |
| 32 | [63, 64] | 90 | [42, 146] | 148 | [75, 202] | 206 | [249, 72] | 264 | [276, 109] |
| 33 | [65, 66] | 91 | [110, 112] | 149 | [76, 203] | 207 | [80, 250] | 265 | [50, 209] |
| 34 | [67, 68] | 92 | [147, 41] | 150 | [171, 82] | 208 | [191, 251] | 266 | [277, 132] |
| 35 | [69, 70] | 93 | [148, 53] | 151 | [69, 170] | 209 | [169, 201] | 267 | [124, 278] |
| 36 | [71, 72] | 94 | [109, 149] | 152 | [18, 184] | 210 | [20, 251] | 268 | [205, 279] |
| 37 | [73, 74] | 95 | [150, 108] | 153 | [190, 169] | 211 | [62, 200] | 269 | [161, 169] |
| 38 | [15, 75] | 96 | [151, 12] | 154 | [165, 88] | 212 | [160, 101] | 270 | [83, 172] |
| 39 | [76, 77] | 97 | [117, 152] | 155 | [204, 99] | 213 | [201, 161] | 271 | [248, 169] |
| 40 | [78, 79] | 98 | [153, 37] | 156 | [16, 202] | 214 | [252, 96] | 272 | [189, 260] |
| 41 | [80, 15] | 99 | [154, 155] | 157 | [65, 201] | 215 | [253, 187] | 273 | [20, 192] |
| 42 | [81, 82] | 100 | [49, 122] | 158 | [205, 206] | 216 | [254, 203] | 274 | [268, 163] |
| 43 | [83, 82] | 101 | [156, 52] | 159 | [119, 207] | 217 | [60, 255] | 275 | [23, 280] |
| 44 | [84, 85] | 102 | [157, 106] | 160 | [121, 208] | 218 | [256, 257] | 276 | [281, 27] |
| 45 | [21, 26] | 103 | [103, 103] | 161 | [58, 130] | 219 | [167, 182] | 277 | [159, 200] |
| 46 | [86, 87] | 104 | [158, 159] | 162 | [107, 113] | 220 | [75, 258] | 278 | [29, 91] |
| 47 | [88, 89] | 105 | [160, 64] | 163 | [209, 33] | 221 | [73, 186] | 279 | [163, 169] |
| 48 | [90, 91] | 106 | [161, 65] | 164 | [10, 210] | 222 | [180, 184] | 280 | [282, 14] |
| 49 | [92, 60] | 107 | [162, 163] | 165 | [211, 144] | 223 | [259, 72] | 281 | [147, 283] |
| 50 | [71, 93] | 108 | [164, 74] | 166 | [212, 213] | 224 | [188, 82] | 282 | [284, 285] |
| 51 | [94, 70] | 109 | [165, 166] | 167 | [214, 215] | 225 | [84, 260] | 283 | [174, 267] |
| 52 | [95, 96] | 110 | [158, 62] | 168 | [216, 217] | 226 | [162, 161] | 284 | [118, 157] |
| 53 | [68, 97] | 111 | [167, 168] | 169 | [116, 58] | 227 | [195, 166] | 285 | [54, 235] |
| 54 | [98, 99] | 112 | [166, 89] | 170 | [12, 155] | 228 | [253, 179] | | |
| 55 | [62, 29] | 113 | [102, 102] | 171 | [218, 219] | 229 | [16, 258] | | |
| 56 | [98, 100] | 114 | [163, 65] | 172 | [220, 12] | 230 | [81, 172] | | |
| 57 | [63, 101] | 115 | [103, 102] | 173 | [221, 103] | 231 | [261, 72] | | |

Output function $n \mapsto \tau(n)$:

| $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 41 | 0 | 82 | 0 | 123 | 0 | 164 | 1 | 205 | 1 | 246 | 0 |
| 1 | 0 | 42 | 1 | 83 | 1 | 124 | 0 | 165 | 0 | 206 | 0 | 247 | 1 |
| 2 | 0 | 43 | 1 | 84 | 1 | 125 | 1 | 166 | 1 | 207 | 0 | 248 | 1 |
| 3 | 0 | 44 | 1 | 85 | 0 | 126 | 0 | 167 | 1 | 208 | 0 | 249 | 0 |
| 4 | 0 | 45 | 1 | 86 | 0 | 127 | 1 | 168 | 0 | 209 | 0 | 250 | 1 |
| 5 | 0 | 46 | 0 | 87 | 1 | 128 | 1 | 169 | 0 | 210 | 1 | 251 | 1 |
| 6 | 0 | 47 | 0 | 88 | 0 | 129 | 1 | 170 | 0 | 211 | 0 | 252 | 1 |
| 7 | 0 | 48 | 1 | 89 | 0 | 130 | 0 | 171 | 0 | 212 | 1 | 253 | 1 |
| 8 | 0 | 49 | 1 | 90 | 1 | 131 | 1 | 172 | 1 | 213 | 1 | 254 | 0 |
| 9 | 0 | 50 | 0 | 91 | 1 | 132 | 0 | 173 | 0 | 214 | 1 | 255 | 1 |

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 1 | 51 | 0 | 92 | 1 | 133 | 0 | 174 | 0 | 215 | 1 | 256 | 0 |
| 11 | 0 | 52 | 0 | 93 | 1 | 134 | 0 | 175 | 1 | 216 | 0 | 257 | 0 |
| 12 | 0 | 53 | 0 | 94 | 0 | 135 | 0 | 176 | 1 | 217 | 1 | 258 | 0 |
| 13 | 0 | 54 | 0 | 95 | 0 | 136 | 0 | 177 | 1 | 218 | 0 | 259 | 1 |
| 14 | 0 | 55 | 0 | 96 | 1 | 137 | 0 | 178 | 0 | 219 | 1 | 260 | 1 |
| 15 | 0 | 56 | 0 | 97 | 0 | 138 | 1 | 179 | 1 | 220 | 1 | 261 | 1 |
| 16 | 0 | 57 | 0 | 98 | 0 | 139 | 1 | 180 | 0 | 221 | 0 | 262 | 1 |
| 17 | 0 | 58 | 1 | 99 | 0 | 140 | 1 | 181 | 0 | 222 | 0 | 263 | 0 |
| 18 | 0 | 59 | 0 | 100 | 1 | 141 | 0 | 182 | 1 | 223 | 1 | 264 | 1 |
| 19 | 0 | 60 | 1 | 101 | 0 | 142 | 0 | 183 | 1 | 224 | 0 | 265 | 0 |
| 20 | 1 | 61 | 0 | 102 | 1 | 143 | 1 | 184 | 1 | 225 | 1 | 266 | 1 |
| 21 | 1 | 62 | 0 | 103 | 0 | 144 | 0 | 185 | 1 | 226 | 0 | 267 | 0 |
| 22 | 1 | 63 | 0 | 104 | 1 | 145 | 1 | 186 | 0 | 227 | 1 | 268 | 1 |
| 23 | 0 | 64 | 1 | 105 | 1 | 146 | 1 | 187 | 0 | 228 | 1 | 269 | 1 |
| 24 | 1 | 65 | 1 | 106 | 1 | 147 | 1 | 188 | 0 | 229 | 0 | 270 | 1 |
| 25 | 0 | 66 | 0 | 107 | 0 | 148 | 1 | 189 | 0 | 230 | 1 | 271 | 1 |
| 26 | 0 | 67 | 0 | 108 | 1 | 149 | 1 | 190 | 0 | 231 | 1 | 272 | 0 |
| 27 | 0 | 68 | 0 | 109 | 0 | 150 | 0 | 191 | 0 | 232 | 0 | 273 | 1 |
| 28 | 0 | 69 | 1 | 110 | 1 | 151 | 1 | 192 | 0 | 233 | 0 | 274 | 1 |
| 29 | 0 | 70 | 1 | 111 | 1 | 152 | 0 | 193 | 1 | 234 | 0 | 275 | 0 |
| 30 | 0 | 71 | 0 | 112 | 1 | 153 | 0 | 194 | 1 | 235 | 0 | 276 | 1 |
| 31 | 0 | 72 | 0 | 113 | 1 | 154 | 0 | 195 | 1 | 236 | 1 | 277 | 1 |
| 32 | 0 | 73 | 0 | 114 | 0 | 155 | 1 | 196 | 0 | 237 | 1 | 278 | 0 |
| 33 | 1 | 74 | 1 | 115 | 0 | 156 | 0 | 197 | 0 | 238 | 1 | 279 | 0 |
| 34 | 0 | 75 | 1 | 116 | 0 | 157 | 1 | 198 | 1 | 239 | 1 | 280 | 0 |
| 35 | 1 | 76 | 1 | 117 | 0 | 158 | 1 | 199 | 1 | 240 | 0 | 281 | 1 |
| 36 | 0 | 77 | 1 | 118 | 0 | 159 | 1 | 200 | 1 | 241 | 0 | 282 | 0 |
| 37 | 0 | 78 | 1 | 119 | 1 | 160 | 1 | 201 | 1 | 242 | 1 | 283 | 0 |
| 38 | 0 | 79 | 0 | 120 | 0 | 161 | 1 | 202 | 1 | 243 | 1 | 284 | 0 |
| 39 | 1 | 80 | 0 | 121 | 1 | 162 | 0 | 203 | 0 | 244 | 1 | 285 | 0 |
| 40 | 1 | 81 | 1 | 122 | 0 | 163 | 0 | 204 | 1 | 245 | 1 | | |

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