# PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a,b)=(z,z^6+1)$

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Abstract. The continued fraction conjecture claims that the continued fraction defined by the $(a,b)$-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements $(a,b)$ in $\mathbb{F}_2[x]\setminus\mathbb{F}_2$. In this paper we prove the conjecture for $(a,b)=(z,z^6+1)$.

## 1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a}=(a_0(z),a_1(z),a_2(z),\ldots)$ of elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z))=\cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{a_2(z)+\cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z))=\cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{\ddots+\cfrac{1}{a_n(z)}}}}\in\mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1}+c_{-2}z^{-2}+c_{-3}z^{-3}+\cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let $a$ and $b$ be two distinct elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$. We consider the continued fraction (1.1) associated with the $(a,b)$-Thue-Morse sequence

$$\mathbf{t}=(t_0(z),t_1(z),t_2(z),\ldots)=(a,b,b,a,b,a,a,b,\ldots)$$

that we shall denote

$$\mathrm{CF}(a,b):=\mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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**Theorem 1.1.** *When $(a, b) = (z, z^6+1)$, the two power series* $\mathrm{CF}(a, b)$ *and* $\mathrm{CF}(b, a)$ *are algebraic over $\mathbb{F}_2(z)$, with minimal polynomials of the form*

$$p_4(z)y^4 + p_3(z)y^3 + p_2(z)y^2 + p_1(z)y + p_0(z) = 0$$

*where, for* $\mathrm{CF}(a, b)$,

$$\begin{aligned}
p_0(z) &= z^{24} + z^{21} + z^{19} + z^{15} + z^{14} + z^{13} + z^7 + z^2 + z + 1,\\
p_1(z) &= z^{27} + z^{25} + z^{22} + z^{20} + z^{16} + z^{15} + z^{14} + z^8 + z^2 + z,\\
p_2(z) &= z^{28} + z^{26} + z^{16} + z^2,\\
p_3(z) &= z^{27} + z^{25} + z^{22} + z^{20} + z^{16} + z^{15} + z^{14} + z^8 + z^2 + z,\\
p_4(z) &= z^{26} + z^{21} + z^{19} + z^{16} + z^{15} + z^{14} + z^{13} + z^7 + z,
\end{aligned}$$

*and for* $\mathrm{CF}(b, a)$,

$$\begin{aligned}
p_0(z) &= z^{21} + z^{19} + z^{15} + z^{14} + z^{13} + z^7 + z^4 + z^2 + z,\\
p_1(z) &= z^{27} + z^{25} + z^{22} + z^{20} + z^{16} + z^{15} + z^{14} + z^8 + z^2 + z,\\
p_2(z) &= z^{28} + z^{26} + z^{16} + z^2,\\
p_3(z) &= z^{27} + z^{25} + z^{22} + z^{20} + z^{16} + z^{15} + z^{14} + z^8 + z^2 + z,\\
p_4(z) &= z^{26} + z^{24} + z^{21} + z^{19} + z^{16} + z^{15} + z^{14} + z^{13} + z^7 + z^4 + z + 1.
\end{aligned}$$

## 2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z) \\ P_{n-1}(z) & Q_{n-1}(z) \end{pmatrix} := \begin{pmatrix} t_n(z) & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t_{n-1}(z) & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} t_0(z) & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a, b) = \frac{P_n(z)}{Q_n(z)} \in \mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a, b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x) &= x^{\deg(t_{2^n-1})} \begin{pmatrix} t_{2^n-1}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times x^{\deg(t_{2^n-2})} \begin{pmatrix} t_{2^n-2}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times\\
&\qquad \cdots \times x^{\deg(t_0)} \begin{pmatrix} t_0(1/x) & 1 \\ 1 & 0 \end{pmatrix},\\
W_n(x) &= x^{\deg(\bar t_{2^n-1})} \begin{pmatrix} \bar t_{2^n-1}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times x^{\deg(\bar t_{2^n-2})} \begin{pmatrix} \bar t_{2^n-2}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times\\
&\qquad \cdots \times x^{\deg(\bar t_0)} \begin{pmatrix} \bar t_0(1/x) & 1 \\ 1 & 0 \end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the $(b, a)$-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n \geq 0$

$$\begin{aligned}
M_{n+1}(x) &= W_n(x) \cdot M_n(x),\\
W_{n+1}(x) &= M_n(x) \cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer $d_n$. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four $2\times 2$ matrices $M^e, M^o, W^e, W^o$ as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(175, 175, 175, 175, 175)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices $M^e, M^o, W^e, W^o$. The first one is about relations between them; the second, about the structure of the each matrix.

**Lemma 2.1.** *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

*Proof.* We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z\left(\phi(W^o,0,0)(x,z),\ z^{12}\cdot\phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity $m$ and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 630 and substitute it for $y$ in $Q(x,y)$. We get a series of valuation less than 630, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z\left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity $\mu$, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ to order 770 and substitute it for $y$ in $Q(x,y)$. We get a series of valuation less than 770, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. ☐

Define

$$R^e = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

**Lemma 2.2.** *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned}
M^e[:14u] = {} & M^e[:7u] + x^{2u}M^e[:5u] + x^{4u}M^e[:3u] + x^{2u}M^e[:7u] + x^{4u}M^e[:5u]\\
& + x^{6u}M^e[:3u] + x^{6u}M^e[:7u] + x^{8u}M^e[:5u] + x^{10u}M^e[:3u]\\
& + x^{10u}M^e[:4u] + x^{12u}M^e[:2u] + (x^{7u}+x^{9u}+x^{13u})R^e,\\
W^e[:14u] = {} & W^e[:7u] + x^{2u}W^e[:5u] + x^{4u}W^e[:3u] + x^{2u}W^e[:7u] + x^{4u}W^e[:5u]\\
& + x^{6u}W^e[:3u] + x^{6u}W^e[:7u] + x^{8u}W^e[:5u] + x^{10u}W^e[:3u]\\
& + x^{10u}W^e[:4u] + x^{12u}W^e[:2u] + (x^{7u}+x^{9u}+x^{13u})R^e.
\end{aligned}$$

*For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.*

$$\begin{aligned}
M^o[:14u] &= M^o[:7u] + x^{2u}M^o[:5u] + x^{4u}M^o[:3u] + x^{2u}M^o[:7u] + x^{4u}M^o[:5u] \\
&\quad + x^{6u}M^o[:3u] + x^{6u}M^o[:7u] + x^{8u}M^o[:5u] + x^{10u}M^o[:3u] \\
&\quad + x^{10u}M^o[:4u] + x^{12u}M^o[:2u] + (x^{7u} + x^{9u} + x^{13u})R^o, \\
W^o[:14u] &= W^o[:7u] + x^{2u}W^o[:5u] + x^{4u}W^o[:3u] + x^{2u}W^o[:7u] + x^{4u}W^o[:5u] \\
&\quad + x^{6u}W^o[:3u] + x^{6u}W^o[:7u] + x^{8u}W^o[:5u] + x^{10u}W^o[:3u] \\
&\quad + x^{10u}W^o[:4u] + x^{12u}W^o[:2u] + (x^{7u} + x^{9u} + x^{13u})R^o.
\end{aligned}$$

*Proof.* To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many $k$'s into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$\begin{aligned}
T[:14u] &= T[:7u] + x^{2u}T[:5u] + x^{4u}T[:3u] + x^{2u}T[:7u] + x^{4u}T[:5u] + x^{6u}T[:3u] \\
&\quad + x^{6u}T[:7u] + x^{8u}T[:5u] + x^{10u}T[:3u] + x^{10u}T[:4u] + x^{12u}T[:2u].
\end{aligned}$$

The proof of the other 15 cases are similar. We break down the above identity into 7 parts:

$$\begin{aligned}
0 &= x^{4u}T[3u:4u] + x^{2u}T[5u:6u] + T[7u:8u], \\
0 &= x^{8u}T[:u] + x^{4u}T[4u:5u] + x^{2u}T[6u:7u] + T[8u:9u], \\
0 &= x^{8u}T[u:2u] + x^{6u}T[3u:4u] + T[9u:10u], \\
0 &= x^{8u}T[2u:3u] + x^{6u}T[4u:5u] + T[10u:11u], \\
0 &= x^{8u}T[3u:4u] + x^{6u}T[5u:6u] + T[11u:12u], \\
0 &= x^{12u}T[:u] + x^{8u}T[4u:5u] + x^{6u}T[6u:7u] + T[12u:13u], \\
0 &= x^{12u}T[u:2u] + x^{10u}T[3u:4u] + T[13u:14u],
\end{aligned}$$

which can be rewritten as

$$0 = T[[11w]_2] + T[[101w]_2] + T[[111w]_2], \tag{2.7}$$
$$0 = T[[w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1000w]_2], \tag{2.8}$$
$$0 = T[[1w]_2] + T[[11w]_2] + T[[1001w]_2], \tag{2.9}$$
$$0 = T[[10w]_2] + T[[100w]_2] + T[[1010w]_2], \tag{2.10}$$
$$0 = T[[11w]_2] + T[[101w]_2] + T[[1011w]_2], \tag{2.11}$$
$$0 = T[[w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1100w]_2], \tag{2.12}$$
$$0 = T[[1w]_2] + T[[11w]_2] + T[[1101w]_2], \tag{2.13}$$

for all binary word $w$ of length $2k$ and $w \neq 0^{2k}$, and

$$0 = T[[11w]_2] + T[[101w]_2] + T[[111w]_2], \tag{2.14}$$
$$0 = T[[w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1000w]_2], \tag{2.15}$$
$$0 = T[[1w]_2] + T[[11w]_2] + T[[1001w]_2], \tag{2.16}$$
$$0 = T[[10w]_2] + T[[100w]_2] + T[[1010w]_2], \tag{2.17}$$
$$0 = T[[11w]_2] + T[[101w]_2] + T[[1011w]_2], \tag{2.18}$$
$$0 = T[[w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1100w]_2], \tag{2.19}$$

$$0 = T[[1w]_2] + T[[11w]_2] + T[[1101w]_2], \tag{2.20}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates $T$ from its minimal polynomial and its first terms. This automaton has 888 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading $w$ from right to left starting from the state $s$, and $\tau$ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.20) can be written as

$$0 = \tau(A(s, 11)) + \tau(A(s, 101)) + \tau(A(s, 111)), \tag{2.21}$$

$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 100)) + \tau(A(s, 110)) + \tau(A(s, 1000)), \tag{2.22}$$

$$0 = \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 1001)), \tag{2.23}$$

$$0 = \tau(A(s, 10)) + \tau(A(s, 100)) + \tau(A(s, 1010)), \tag{2.24}$$

$$0 = \tau(A(s, 11)) + \tau(A(s, 101)) + \tau(A(s, 1011)), \tag{2.25}$$

$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 100)) + \tau(A(s, 110)) + \tau(A(s, 1100)), \tag{2.26}$$

$$0 = \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 1101)), \tag{2.27}$$

for all $s \in E_{2k}$, and

$$0 = \tau(A(s, 11)) + \tau(A(s, 101)) + \tau(A(s, 111)), \tag{2.28}$$

$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 100)) + \tau(A(s, 110)) + \tau(A(s, 1000)), \tag{2.29}$$

$$0 = \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 1001)), \tag{2.30}$$

$$0 = \tau(A(s, 10)) + \tau(A(s, 100)) + \tau(A(s, 1010)), \tag{2.31}$$

$$0 = \tau(A(s, 11)) + \tau(A(s, 101)) + \tau(A(s, 1011)), \tag{2.32}$$

$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 100)) + \tau(A(s, 110)) + \tau(A(s, 1100)), \tag{2.33}$$

$$0 = \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 1101)), \tag{2.34}$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{48}), E_{48}) = (A(i, 0^{20}), E_{20})$, so that we only have to verify that identities (2.21) through (2.34) hold for $2 \leq k \leq 24$, which turns out to be true. □

In the following lemma, we express $M_{2k}$, $M_{2k+1}$, $W_{2k}$, and $W_{2k+1}$ in terms of $M^e$, $M^o$, $W^e$, and $W^o$.

**Lemma 2.3.** *For all integer $k \geq 2$, and $u = 2^{2k-1}$,*

$$M_{2k} = M^e[:7u] + x^{2u} M^e[:5u] + x^{4u} M^e[:3u] + x^{7u} R^e,$$
$$W_{2k} = W^e[:7u] + x^{2u} W^e[:5u] + x^{4u} W^e[:3u] + x^{7u} R^e.$$

*For all integer $k \geq 2$, and $u = 2^{2k}$,*

$$M_{2k+1} = M^o[:7u] + x^{2u} M^o[:5u] + x^{4u} M^o[:3u] + x^{7u} R^o,$$
$$W_{2k+1} = W^o[:7u] + x^{2u} W^o[:5u] + x^{4u} W^o[:3u] + x^{7u} R^o.$$

*Proof.* We call the four identities in Lemma 2.3 also by the name $M_{2k}$, $W_{2k}$, $M_{2k+1}$, and $W_{2k+1}$. For $n = 2$, the identities can be verified directly. For $n \geq 2$, we claim that

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1} \wedge W_{2k+1},$$

$$M_{2k+1} \wedge W_{2k+1} \Rightarrow M_{2k+2} \wedge W_{2k+2}.$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity $M_{2k+1}$ is equal to

$$\begin{aligned} W_{2k}M_{2k} = &\big(W^e[:7v] + x^{2v}W^e[:5v] + x^{4v}W^e[:3v] + x^{7u}R^e\big) \times \big(M^e[:7v] \\ &+ x^{2v}M^e[:5v] + x^{4v}M^e[:3v] + x^{7u}R^e\big); \end{aligned} \tag{2.35}$$

Call this expression $lhs$. Note that both sides of identity $M_{2k+1}$ have the same term of highest degree $x^{14v}R^o$. Therefore we only need to prove that their difference is $O(x^{14v})$. Using Lemma 2.1 it can be seen that the right side of identity $M_{2k+1}$ is congruent, modulo $x^{14v}$, to

$$W^e[:14v]M^e[:14v] + x^{4v}W^e[:10v]M^e[:10v] + x^{8v}W^e[:6v]M^e[:6v].$$

For all $n \leq 14$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call $rhs$. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions $lhs$ (2.35) and $rhs$ as polynomials in $\mathbb{F}_2[a_1, ..., a_{14}, b_1, ..., b_{14}, c][X]$. Note that it is not a problem that $a_j$ commutes with $b_k$ while $W^e$ does not commute with $M^e$, because in the expressions concerned, the products of $W^e$-terms and $M^e$-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{14})$, which completes the proof. □

*Proof of Theorem 1.1.* We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \leq j, k \leq 1$,

$$\begin{aligned} \lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k}, \\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k}, \\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k}, \\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}. \end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a, b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x, y) = \mathrm{Res}_t\left(\phi(M^e, 0, 1)(x, t), y^{12}\phi(M^e, 0, 1)(x, t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x,y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned}
q_0(x) &= x^{28} + x^{27} + x^{26} + x^{21} + x^{15} + x^{14} + x^{13} + x^9 + x^7 + x^4,\\
q_1(x) &= x^{27} + x^{26} + x^{20} + x^{14} + x^{13} + x^{12} + x^8 + x^6 + x^3 + x,\\
q_2(x) &= x^{26} + x^{12} + x^2 + 1,\\
q_3(x) &= x^{27} + x^{26} + x^{20} + x^{14} + x^{13} + x^{12} + x^8 + x^6 + x^3 + x,\\
q_4(x) &= x^{27} + x^{21} + x^{15} + x^{14} + x^{13} + x^{12} + x^9 + x^7 + x^2.
\end{aligned}$$

The polynomial $Q(x,y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x,y)$ of multiplicity $m$ and $R(x,y) := P(x,y)/Q(x,y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 216, and substitute it for $y$ in $R(z,y)$, we get a series with valuation smaller than 216, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b) = f(1/z)$. $\square$

## 3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e, 0, 0)$:

$$\begin{aligned}
p_0(x) = {} & x^{162} + x^{150} + x^{136} + x^{134} + x^{132} + x^{122} + x^{108} + x^{106} + x^{102} + x^{94}\\
& + x^{90} + x^{82} + x^{74} + x^{72} + x^{70} + x^{66} + x^{62} + x^{58} + x^{46} + x^{42} + x^{40}\\
& + x^{38} + x^{36} + x^{34} + x^{26} + x^{22} + x^{12},\\
p_3(x) = {} & x^{156} + x^{154} + x^{148} + x^{144} + x^{140} + x^{136} + x^{132} + x^{130} + x^{128} + x^{124}\\
& + x^{120} + x^{116} + x^{112} + x^{104} + x^{100} + x^{98} + x^{88} + x^{84} + x^{76} + x^{74}\\
& + x^{72} + x^{64} + x^{58} + x^{50} + x^{44} + x^{36} + x^{34} + x^{32} + x^{26} + x^{24} + x^{20}\\
& + x^8,\\
p_6(x) = {} & x^{158} + x^{156} + x^{148} + x^{146} + x^{144} + x^{136} + x^{134} + x^{128} + x^{124} + x^{122}\\
& + x^{120} + x^{118} + x^{112} + x^{108} + x^{106} + x^{102} + x^{100} + x^{90} + x^{88} + x^{78}\\
& + x^{72} + x^{66} + x^{64} + x^{60} + x^{54} + x^{52} + x^{48} + x^{42} + x^{38} + x^{36} + x^{32}\\
& + x^{30} + x^{28} + x^{26} + x^{24} + x^{22} + x^{18} + x^{14} + x^{10} + x^4 + x^2 + 1,\\
p_9(x) = {} & x^{160} + x^{158} + x^{144} + x^{132} + x^{130} + x^{120} + x^{116} + x^{112} + x^{110} + x^{108}\\
& + x^{106} + x^{104} + x^{102} + x^{88} + x^{76} + x^{74} + x^{62} + x^{60} + x^{52} + x^{50} + x^{48}\\
& + x^{40} + x^{36} + x^{34} + x^{28} + x^{26} + x^{24} + x^{20} + x^{14} + x^{10} + x^8 + x^6 + x^4\\
& + x^2,
\end{aligned}$$

$$p_{12}(x) = x^{160} + x^{112} + x^{64} + x^{48} + x^{16} + 1,$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0].

For $\phi(M^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{147} + x^{146} + x^{144} + x^{142} + x^{141} + x^{140} + x^{138} + x^{136} + x^{135} + x^{134}\\
&\quad + x^{126} + x^{122} + x^{121} + x^{117} + x^{112} + x^{111} + x^{108} + x^{106} + x^{99} + x^{96}\\
&\quad + x^{88} + x^{87} + x^{86} + x^{84} + x^{83} + x^{82} + x^{80} + x^{75} + x^{71} + x^{70} + x^{66}\\
&\quad + x^{63} + x^{60} + x^{57} + x^{52} + x^{50} + x^{46} + x^{45} + x^{44} + x^{42} + x^{36} + x^{34}\\
&\quad + x^{32} + x^{30} + x^{27} + x^{26} + x^{25} + x^{21} + x^{18},\\
p_3(x) &= x^{145} + x^{144} + x^{139} + x^{137} + x^{136} + x^{130} + x^{125} + x^{123} + x^{122} + x^{121}\\
&\quad + x^{120} + x^{119} + x^{118} + x^{117} + x^{115} + x^{112} + x^{111} + x^{110} + x^{107}\\
&\quad + x^{105} + x^{81} + x^{80} + x^{75} + x^{67} + x^{66} + x^{61} + x^{57} + x^{56} + x^{55} + x^{54}\\
&\quad + x^{51} + x^{48} + x^{43} + x^{41} + x^{40} + x^{34} + x^{29} + x^{27} + x^{26} + x^{25} + x^{24}\\
&\quad + x^{23} + x^{22} + x^{21} + x^{19} + x^{16} + x^{15} + x^{14} + x^{11} + x^{9},\\
p_6(x) &= x^{144} + x^{142} + x^{140} + x^{136} + x^{134} + x^{132} + x^{130} + x^{126} + x^{124} + x^{116}\\
&\quad + x^{114} + x^{112} + x^{106} + x^{104} + x^{98} + x^{92} + x^{86} + x^{84} + x^{82} + x^{80}\\
&\quad + x^{78} + x^{76} + x^{74} + x^{70} + x^{68} + x^{64} + x^{62} + x^{60} + x^{46} + x^{42} + x^{40}\\
&\quad + x^{36} + x^{22} + x^{18} + x^{14} + x^{12} + x^{10} + x^{6},\\
p_9(x) &= x^{143} + x^{142} + x^{141} + x^{140} + x^{137} + x^{135} + x^{127} + x^{126} + x^{121} + x^{119}\\
&\quad + x^{118} + x^{117} + x^{116} + x^{115} + x^{114} + x^{112} + x^{111} + x^{109} + x^{107}\\
&\quad + x^{99} + x^{98} + x^{93} + x^{87} + x^{86} + x^{85} + x^{84} + x^{81} + x^{79} + x^{75} + x^{74}\\
&\quad + x^{71} + x^{70} + x^{69} + x^{65} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{56}\\
&\quad + x^{55} + x^{53} + x^{50} + x^{47} + x^{46} + x^{44} + x^{43} + x^{42} + x^{41} + x^{38} + x^{36}\\
&\quad + x^{33} + x^{30} + x^{27} + x^{26} + x^{25} + x^{20} + x^{17} + x^{16} + x^{14} + x^{13} + x^{12}\\
&\quad + x^{10} + x^{8} + x^{7} + x^{5} + x^{3},\\
p_{12}(x) &= x^{142} + x^{136} + x^{118} + x^{112} + x^{46} + x^{40} + x^{30} + x^{24} + x^{22} + x^{16} + x^{6}\\
&\quad + 1,
\end{aligned}$$

and the initial terms are [0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 1, 1, 0].

For $\phi(M^e, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{147} + x^{146} + x^{144} + x^{142} + x^{141} + x^{140} + x^{138} + x^{136} + x^{135} + x^{134}\\
&\quad + x^{126} + x^{122} + x^{121} + x^{117} + x^{112} + x^{111} + x^{108} + x^{106} + x^{99} + x^{96}\\
&\quad + x^{88} + x^{87} + x^{86} + x^{84} + x^{83} + x^{82} + x^{80} + x^{75} + x^{71} + x^{70} + x^{66}\\
&\quad + x^{63} + x^{60} + x^{57} + x^{52} + x^{50} + x^{46} + x^{45} + x^{44} + x^{42} + x^{36} + x^{34}\\
&\quad + x^{32} + x^{30} + x^{27} + x^{26} + x^{25} + x^{21} + x^{18},\\
p_3(x) &= x^{145} + x^{144} + x^{139} + x^{137} + x^{136} + x^{130} + x^{125} + x^{123} + x^{122} + x^{121}\\
&\quad + x^{120} + x^{119} + x^{118} + x^{117} + x^{115} + x^{112} + x^{111} + x^{110} + x^{107}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{105} + x^{81} + x^{80} + x^{75} + x^{67} + x^{66} + x^{61} + x^{57} + x^{56} + x^{55} + x^{54}\\
&+ x^{51} + x^{48} + x^{43} + x^{41} + x^{40} + x^{34} + x^{29} + x^{27} + x^{26} + x^{25} + x^{24}\\
&+ x^{23} + x^{22} + x^{21} + x^{19} + x^{16} + x^{15} + x^{14} + x^{11} + x^{9},\\
p_6(x) &= x^{144} + x^{142} + x^{140} + x^{136} + x^{134} + x^{132} + x^{130} + x^{126} + x^{124} + x^{116}\\
&+ x^{114} + x^{112} + x^{106} + x^{104} + x^{98} + x^{92} + x^{86} + x^{84} + x^{82} + x^{80}\\
&+ x^{78} + x^{76} + x^{74} + x^{70} + x^{68} + x^{64} + x^{62} + x^{60} + x^{46} + x^{42} + x^{40}\\
&+ x^{36} + x^{22} + x^{18} + x^{14} + x^{12} + x^{10} + x^{6},\\
p_9(x) &= x^{143} + x^{142} + x^{141} + x^{140} + x^{137} + x^{135} + x^{127} + x^{126} + x^{121} + x^{119}\\
&+ x^{118} + x^{117} + x^{116} + x^{115} + x^{114} + x^{112} + x^{111} + x^{109} + x^{107}\\
&+ x^{99} + x^{98} + x^{93} + x^{87} + x^{86} + x^{85} + x^{84} + x^{81} + x^{79} + x^{75} + x^{74}\\
&+ x^{71} + x^{70} + x^{69} + x^{65} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{56}\\
&+ x^{55} + x^{53} + x^{50} + x^{47} + x^{46} + x^{44} + x^{43} + x^{42} + x^{41} + x^{38} + x^{36}\\
&+ x^{33} + x^{30} + x^{27} + x^{26} + x^{25} + x^{20} + x^{17} + x^{16} + x^{14} + x^{13} + x^{12}\\
&+ x^{10} + x^{8} + x^{7} + x^{5} + x^{3},\\
p_{12}(x) &= x^{142} + x^{136} + x^{118} + x^{112} + x^{46} + x^{40} + x^{30} + x^{24} + x^{22} + x^{16} + x^{6}\\
&+ 1,
\end{aligned}
$$

and the initial terms are [0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 1, 1, 0].

For $\phi(M^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{132} + x^{130} + x^{126} + x^{122} + x^{108} + x^{98} + x^{96} + x^{94} + x^{84} + x^{74} + x^{72}\\
&+ x^{68} + x^{60} + x^{56} + x^{50} + x^{48} + x^{46} + x^{42} + x^{34} + x^{30} + x^{24},\\
p_3(x) &= x^{126} + x^{124} + x^{122} + x^{118} + x^{114} + x^{112} + x^{110} + x^{106} + x^{104} + x^{92}\\
&+ x^{68} + x^{66} + x^{62} + x^{54} + x^{50} + x^{48} + x^{44} + x^{36} + x^{30} + x^{28} + x^{26}\\
&+ x^{22} + x^{20} + x^{16} + x^{14} + x^{10} + x^{8} + x^{6},\\
p_6(x) &= x^{126} + x^{120} + x^{114} + x^{110} + x^{108} + x^{96} + x^{94} + x^{92} + x^{84} + x^{72} + x^{70}\\
&+ x^{64} + x^{60} + x^{56} + x^{54} + x^{52} + x^{48} + x^{46} + x^{44} + x^{40} + x^{38} + x^{30}\\
&+ x^{28} + x^{22} + x^{18} + x^{14} + x^{8} + x^{6} + x^{2} + 1,\\
p_9(x) &= x^{124} + x^{122} + x^{112} + x^{110} + x^{108} + x^{94} + x^{82} + x^{80} + x^{70} + x^{66} + x^{58}\\
&+ x^{56} + x^{54} + x^{52} + x^{46} + x^{38} + x^{34} + x^{32} + x^{28} + x^{24} + x^{22} + x^{18}\\
&+ x^{16} + x^{10} + x^{8} + x^{6} + x^{4} + x^{2},\\
p_{12}(x) &= x^{124} + x^{112} + x^{28} + x^{16} + x^{12} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{132} + x^{130} + x^{128} + x^{126} + x^{124} + x^{114} + x^{112} + x^{104} + x^{98} + x^{96}\\
&+ x^{88} + x^{86} + x^{84} + x^{76} + x^{74} + x^{72} + x^{70} + x^{66} + x^{62} + x^{50} + x^{42}\\
&+ x^{36} + x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{22} + x^{20} + x^{16} + x^{12},
\end{aligned}
$$

$$\begin{aligned}
p_3(x) &= x^{126} + x^{124} + x^{120} + x^{116} + x^{114} + x^{112} + x^{104} + x^{94} + x^{82} + x^{70} \\
&\quad + x^{64} + x^{60} + x^{58} + x^{48} + x^{46} + x^{38} + x^{34} + x^{30} + x^{28} + x^{26} + x^{24} \\
&\quad + x^{22} + x^{20} + x^{18} + x^{10} + x^{8}, \\
p_6(x) &= x^{126} + x^{122} + x^{120} + x^{118} + x^{114} + x^{106} + x^{104} + x^{96} + x^{92} + x^{84} \\
&\quad + x^{82} + x^{80} + x^{72} + x^{66} + x^{64} + x^{62} + x^{60} + x^{58} + x^{50} + x^{44} + x^{40} \\
&\quad + x^{34} + x^{32} + x^{30} + x^{28} + x^{24} + x^{22} + x^{12} + x^{8} + x^{4} + x^{2} + 1, \\
p_9(x) &= x^{124} + x^{122} + x^{112} + x^{110} + x^{108} + x^{94} + x^{82} + x^{80} + x^{70} + x^{66} + x^{58} \\
&\quad + x^{56} + x^{54} + x^{52} + x^{46} + x^{38} + x^{34} + x^{32} + x^{28} + x^{24} + x^{22} + x^{18} \\
&\quad + x^{16} + x^{10} + x^{8} + x^{6} + x^{4} + x^{2}, \\
p_{12}(x) &= x^{124} + x^{112} + x^{28} + x^{16} + x^{12} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{147} + x^{145} + x^{144} + x^{143} + x^{142} + x^{139} + x^{138} + x^{137} + x^{136} + x^{133} \\
&\quad + x^{132} + x^{128} + x^{127} + x^{126} + x^{124} + x^{123} + x^{122} + x^{120} + x^{119} \\
&\quad + x^{118} + x^{116} + x^{112} + x^{109} + x^{105} + x^{103} + x^{91} + x^{90} + x^{85} + x^{84} \\
&\quad + x^{82} + x^{81} + x^{76} + x^{74} + x^{66} + x^{62} + x^{61} + x^{60} + x^{57} + x^{56} + x^{51} \\
&\quad + x^{50} + x^{49} + x^{48} + x^{47} + x^{44} + x^{43} + x^{41} + x^{38} + x^{35} + x^{34} + x^{32} \\
&\quad + x^{31} + x^{27}, \\
p_3(x) &= x^{145} + x^{144} + x^{139} + x^{137} + x^{136} + x^{130} + x^{125} + x^{123} + x^{122} + x^{121} \\
&\quad + x^{120} + x^{119} + x^{118} + x^{117} + x^{115} + x^{112} + x^{111} + x^{110} + x^{107} \\
&\quad + x^{105} + x^{81} + x^{80} + x^{75} + x^{67} + x^{66} + x^{61} + x^{57} + x^{56} + x^{55} + x^{54} \\
&\quad + x^{51} + x^{48} + x^{43} + x^{41} + x^{40} + x^{34} + x^{29} + x^{27} + x^{26} + x^{25} + x^{24} \\
&\quad + x^{23} + x^{22} + x^{21} + x^{19} + x^{16} + x^{15} + x^{14} + x^{11} + x^{9}, \\
p_6(x) &= x^{144} + x^{142} + x^{140} + x^{136} + x^{134} + x^{132} + x^{130} + x^{126} + x^{124} + x^{116} \\
&\quad + x^{114} + x^{112} + x^{106} + x^{104} + x^{98} + x^{92} + x^{86} + x^{84} + x^{82} + x^{80} \\
&\quad + x^{78} + x^{76} + x^{74} + x^{70} + x^{68} + x^{64} + x^{62} + x^{60} + x^{46} + x^{42} + x^{40} \\
&\quad + x^{36} + x^{22} + x^{18} + x^{14} + x^{12} + x^{10} + x^{6}, \\
p_9(x) &= x^{143} + x^{142} + x^{141} + x^{140} + x^{137} + x^{135} + x^{127} + x^{126} + x^{121} + x^{119} \\
&\quad + x^{118} + x^{117} + x^{116} + x^{115} + x^{114} + x^{112} + x^{111} + x^{109} + x^{107} \\
&\quad + x^{99} + x^{98} + x^{93} + x^{87} + x^{86} + x^{85} + x^{84} + x^{81} + x^{79} + x^{75} + x^{74} \\
&\quad + x^{71} + x^{70} + x^{69} + x^{65} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{56} \\
&\quad + x^{55} + x^{53} + x^{50} + x^{47} + x^{46} + x^{44} + x^{43} + x^{42} + x^{41} + x^{38} + x^{36} \\
&\quad + x^{33} + x^{30} + x^{27} + x^{26} + x^{25} + x^{20} + x^{17} + x^{16} + x^{14} + x^{13} + x^{12} \\
&\quad + x^{10} + x^{8} + x^{7} + x^{5} + x^{3}, \\
p_{12}(x) &= x^{142} + x^{136} + x^{118} + x^{112} + x^{46} + x^{40} + x^{30} + x^{24} + x^{22} + x^{16} + x^{6}
\end{aligned}$$

$$+ 1,$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 1, 0].
For $\phi(W^e, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{147} + x^{145} + x^{144} + x^{143} + x^{142} + x^{139} + x^{138} + x^{137} + x^{136} + x^{133} \\
&\quad + x^{132} + x^{128} + x^{127} + x^{126} + x^{124} + x^{123} + x^{122} + x^{120} + x^{119} \\
&\quad + x^{118} + x^{116} + x^{112} + x^{109} + x^{105} + x^{103} + x^{91} + x^{90} + x^{85} + x^{84} \\
&\quad + x^{82} + x^{81} + x^{76} + x^{74} + x^{66} + x^{62} + x^{61} + x^{60} + x^{57} + x^{56} + x^{51} \\
&\quad + x^{50} + x^{49} + x^{48} + x^{47} + x^{44} + x^{43} + x^{41} + x^{38} + x^{35} + x^{34} + x^{32} \\
&\quad + x^{31} + x^{27}, \\
p_3(x) &= x^{145} + x^{144} + x^{139} + x^{137} + x^{136} + x^{130} + x^{125} + x^{123} + x^{122} + x^{121} \\
&\quad + x^{120} + x^{119} + x^{118} + x^{117} + x^{115} + x^{112} + x^{111} + x^{110} + x^{107} \\
&\quad + x^{105} + x^{81} + x^{80} + x^{75} + x^{67} + x^{66} + x^{61} + x^{57} + x^{56} + x^{55} + x^{54} \\
&\quad + x^{51} + x^{48} + x^{43} + x^{41} + x^{40} + x^{34} + x^{29} + x^{27} + x^{26} + x^{25} + x^{24} \\
&\quad + x^{23} + x^{22} + x^{21} + x^{19} + x^{16} + x^{15} + x^{14} + x^{11} + x^{9}, \\
p_6(x) &= x^{144} + x^{142} + x^{140} + x^{136} + x^{134} + x^{132} + x^{130} + x^{126} + x^{124} + x^{116} \\
&\quad + x^{114} + x^{112} + x^{106} + x^{104} + x^{98} + x^{92} + x^{86} + x^{84} + x^{82} + x^{80} \\
&\quad + x^{78} + x^{76} + x^{74} + x^{70} + x^{68} + x^{64} + x^{62} + x^{60} + x^{46} + x^{42} + x^{40} \\
&\quad + x^{36} + x^{22} + x^{18} + x^{14} + x^{12} + x^{10} + x^{6}, \\
p_9(x) &= x^{143} + x^{142} + x^{141} + x^{140} + x^{137} + x^{135} + x^{127} + x^{126} + x^{121} + x^{119} \\
&\quad + x^{118} + x^{117} + x^{116} + x^{115} + x^{114} + x^{112} + x^{111} + x^{109} + x^{107} \\
&\quad + x^{99} + x^{98} + x^{93} + x^{87} + x^{86} + x^{85} + x^{84} + x^{81} + x^{79} + x^{75} + x^{74} \\
&\quad + x^{71} + x^{70} + x^{69} + x^{65} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{56} \\
&\quad + x^{55} + x^{53} + x^{50} + x^{47} + x^{46} + x^{44} + x^{43} + x^{42} + x^{41} + x^{38} + x^{36} \\
&\quad + x^{33} + x^{30} + x^{27} + x^{26} + x^{25} + x^{20} + x^{17} + x^{16} + x^{14} + x^{13} + x^{12} \\
&\quad + x^{10} + x^{8} + x^{7} + x^{5} + x^{3}, \\
p_{12}(x) &= x^{142} + x^{136} + x^{118} + x^{112} + x^{46} + x^{40} + x^{30} + x^{24} + x^{22} + x^{16} + x^{6} \\
&\quad + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 1, 0].
For $\phi(W^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{162} + x^{160} + x^{158} + x^{152} + x^{148} + x^{146} + x^{144} + x^{138} + x^{130} + x^{126} \\
&\quad + x^{124} + x^{122} + x^{118} + x^{114} + x^{110} + x^{106} + x^{100} + x^{94} + x^{82} + x^{78} \\
&\quad + x^{76} + x^{70} + x^{64} + x^{62} + x^{58} + x^{56} + x^{50} + x^{46} + x^{42}, \\
p_3(x) &= x^{158} + x^{152} + x^{148} + x^{136} + x^{128} + x^{124} + x^{116} + x^{112} + x^{110} + x^{104} \\
&\quad + x^{102} + x^{96} + x^{72} + x^{62} + x^{60} + x^{56} + x^{52} + x^{40} + x^{36} + x^{32} + x^{28} \\
&\quad + x^{20} + x^{16} + x^{14} + x^{8} + x^{6},
\end{aligned}$$

$$
\begin{aligned}
p_6(x) &= x^{156} + x^{154} + x^{148} + x^{146} + x^{140} + x^{136} + x^{134} + x^{132} + x^{130} + x^{124} \\
&\quad + x^{122} + x^{118} + x^{112} + x^{110} + x^{106} + x^{100} + x^{98} + x^{90} + x^{84} + x^{78} \\
&\quad + x^{76} + x^{74} + x^{66} + x^{62} + x^{60} + x^{58} + x^{54} + x^{50} + x^{44} + x^{42} + x^{40} \\
&\quad + x^{38} + x^{36} + x^{34} + x^{30} + x^{24} + x^{22} + x^{18} + x^{16} + x^{12} + x^{10} + x^{6} \\
&\quad + x^{2} + 1, \\
p_9(x) &= x^{160} + x^{158} + x^{144} + x^{132} + x^{130} + x^{120} + x^{116} + x^{112} + x^{110} + x^{108} \\
&\quad + x^{106} + x^{104} + x^{102} + x^{88} + x^{76} + x^{74} + x^{62} + x^{60} + x^{52} + x^{50} + x^{48} \\
&\quad + x^{40} + x^{36} + x^{34} + x^{28} + x^{26} + x^{24} + x^{20} + x^{14} + x^{10} + x^{8} + x^{6} + x^{4} \\
&\quad + x^{2}, \\
p_{12}(x) &= x^{160} + x^{112} + x^{64} + x^{48} + x^{16} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0].

For $\phi(M^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{147} + x^{146} + x^{145} + x^{144} + x^{143} + x^{141} + x^{140} + x^{139} + x^{134} + x^{133} \\
&\quad + x^{129} + x^{128} + x^{126} + x^{125} + x^{124} + x^{123} + x^{122} + x^{119} + x^{117} \\
&\quad + x^{114} + x^{110} + x^{107} + x^{106} + x^{105} + x^{103} + x^{101} + x^{95} + x^{94} + x^{93} \\
&\quad + x^{87} + x^{84} + x^{81} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{67} + x^{66} \\
&\quad + x^{64} + x^{62} + x^{60} + x^{58} + x^{55} + x^{53} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} \\
&\quad + x^{45} + x^{44} + x^{42} + x^{40} + x^{39} + x^{38} + x^{37} + x^{34} + x^{33} + x^{32} + x^{31} \\
&\quad + x^{28} + x^{25} + x^{23} + x^{21} + x^{20} + x^{19} + x^{16} + x^{14} + x^{12}, \\
p_3(x) &= x^{146} + x^{145} + x^{144} + x^{142} + x^{137} + x^{136} + x^{132} + x^{131} + x^{130} + x^{123} \\
&\quad + x^{122} + x^{121} + x^{120} + x^{119} + x^{113} + x^{112} + x^{111} + x^{110} + x^{108} \\
&\quad + x^{104} + x^{100} + x^{94} + x^{88} + x^{84} + x^{81} + x^{70} + x^{68} + x^{67} + x^{64} + x^{62} \\
&\quad + x^{60} + x^{57} + x^{55} + x^{54} + x^{50} + x^{49} + x^{44} + x^{41} + x^{38} + x^{35} + x^{30} \\
&\quad + x^{28} + x^{27} + x^{26} + x^{25} + x^{23} + x^{22} + x^{18} + x^{17} + x^{15} + x^{14} + x^{12} \\
&\quad + x^{10} + x^{8}, \\
p_6(x) &= x^{146} + x^{145} + x^{144} + x^{142} + x^{141} + x^{140} + x^{136} + x^{132} + x^{131} + x^{130} \\
&\quad + x^{128} + x^{127} + x^{126} + x^{124} + x^{121} + x^{119} + x^{118} + x^{117} + x^{115} \\
&\quad + x^{114} + x^{113} + x^{112} + x^{104} + x^{100} + x^{99} + x^{98} + x^{94} + x^{92} + x^{88} \\
&\quad + x^{87} + x^{86} + x^{85} + x^{80} + x^{75} + x^{74} + x^{72} + x^{71} + x^{68} + x^{66} + x^{63} \\
&\quad + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{52} + x^{51} + x^{49} + x^{45} \\
&\quad + x^{44} + x^{43} + x^{42} + x^{39} + x^{37} + x^{35} + x^{33} + x^{30} + x^{27} + x^{26} + x^{25} \\
&\quad + x^{24} + x^{22} + x^{21} + x^{19} + x^{17} + x^{16} + x^{15} + x^{13} + x^{12} + x^{11} + x^{4} + 1, \\
p_9(x) &= x^{146} + x^{142} + x^{140} + x^{136} + x^{122} + x^{116} + x^{50} + x^{46} + x^{44} + x^{40} \\
&\quad + x^{34} + x^{30} + x^{28} + x^{26} + x^{24} + x^{20} + x^{10} + x^{4}, \\
p_{12}(x) &= x^{146} + x^{140} + x^{122} + x^{118} + x^{116} + x^{112} + x^{50} + x^{44} + x^{34} + x^{28}
\end{aligned}
$$

$$+ x^{26} + x^{22} + x^{20} + x^{16} + x^{10} + x^{6} + x^{4} + 1,$$

and the initial terms are [1, 0, 1, 0, 0, 0, 0, 1, 1, 1, 1, 0, 1, 1, 1, 0].

For $\phi(M^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{144} + x^{141} + x^{140} + x^{139} + x^{136} + x^{133} + x^{132} + x^{130} + x^{125} + x^{124} \\
&\quad + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{115} + x^{114} + x^{113} \\
&\quad + x^{111} + x^{110} + x^{108} + x^{107} + x^{103} + x^{100} + x^{98} + x^{96} + x^{95} + x^{94} \\
&\quad + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{82} + x^{81} + x^{74} + x^{73} + x^{72} + x^{71} \\
&\quad + x^{69} + x^{68} + x^{65} + x^{64} + x^{62} + x^{61} + x^{60} + x^{59} + x^{57} + x^{56} + x^{52} \\
&\quad + x^{50} + x^{49} + x^{47} + x^{46} + x^{45} + x^{44} + x^{42} + x^{41} + x^{40} + x^{38} + x^{37} \\
&\quad + x^{35} + x^{32} + x^{31} + x^{30} + x^{28} + x^{27} + x^{25} + x^{23} + x^{22} + x^{21} + x^{20} \\
&\quad + x^{18}, \\
p_3(x) &= x^{137} + x^{136} + x^{130} + x^{129} + x^{128} + x^{124} + x^{122} + x^{118} + x^{116} + x^{113} \\
&\quad + x^{110} + x^{105} + x^{73} + x^{72} + x^{66} + x^{60} + x^{54} + x^{49} + x^{41} + x^{40} + x^{34} \\
&\quad + x^{33} + x^{32} + x^{28} + x^{26} + x^{22} + x^{20} + x^{17} + x^{14} + x^{9}, \\
p_6(x) &= x^{138} + x^{136} + x^{134} + x^{132} + x^{124} + x^{120} + x^{114} + x^{110} + x^{106} + x^{104} \\
&\quad + x^{92} + x^{80} + x^{78} + x^{76} + x^{68} + x^{64} + x^{56} + x^{54} + x^{50} + x^{48} + x^{44} \\
&\quad + x^{42} + x^{40} + x^{38} + x^{32} + x^{30} + x^{26} + x^{24} + x^{20} + x^{18} + x^{16} + x^{14} \\
&\quad + x^{10} + x^{6}, \\
p_9(x) &= x^{139} + x^{138} + x^{137} + x^{136} + x^{132} + x^{130} + x^{126} + x^{125} + x^{120} + x^{115} \\
&\quad + x^{43} + x^{42} + x^{41} + x^{40} + x^{36} + x^{34} + x^{30} + x^{29} + x^{27} + x^{26} + x^{25} \\
&\quad + x^{20} + x^{19} + x^{18} + x^{14} + x^{13} + x^{8} + x^{3}, \\
p_{12}(x) &= x^{140} + x^{116} + x^{112} + x^{44} + x^{28} + x^{20} + x^{16} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1].

For $\phi(M^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{162} + x^{161} + x^{160} + x^{157} + x^{156} + x^{153} + x^{151} + x^{150} + x^{148} + x^{146} \\
&\quad + x^{145} + x^{144} + x^{141} + x^{138} + x^{137} + x^{136} + x^{135} + x^{134} + x^{121} \\
&\quad + x^{119} + x^{118} + x^{116} + x^{114} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} \\
&\quad + x^{106} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{97} + x^{95} + x^{94} \\
&\quad + x^{92} + x^{89} + x^{85} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{77} + x^{74} + x^{73} \\
&\quad + x^{71} + x^{70} + x^{69} + x^{68} + x^{65} + x^{61} + x^{60} + x^{58} + x^{56} + x^{55} + x^{53} \\
&\quad + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{44} + x^{43} + x^{42} + x^{40} + x^{38} + x^{37} \\
&\quad + x^{35} + x^{34} + x^{32} + x^{31} + x^{29} + x^{27}, \\
p_3(x) &= x^{161} + x^{160} + x^{154} + x^{153} + x^{152} + x^{148} + x^{146} + x^{142} + x^{140} + x^{136} \\
&\quad + x^{134} + x^{130} + x^{128} + x^{124} + x^{122} + x^{118} + x^{116} + x^{113} + x^{110} \\
&\quad + x^{105} + x^{97} + x^{96} + x^{90} + x^{84} + x^{78} + x^{72} + x^{66} + x^{65} + x^{64} + x^{60}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{58} + x^{57} + x^{56} + x^{54} + x^{52} + x^{50} + x^{49} + x^{46} + x^{44} + x^{40} + x^{38}\\
&+ x^{34} + x^{32} + x^{28} + x^{26} + x^{22} + x^{20} + x^{17} + x^{14} + x^{9},\\
p_6(x) = {} & x^{162} + x^{160} + x^{158} + x^{156} + x^{148} + x^{144} + x^{136} + x^{132} + x^{130} + x^{128}\\
&+ x^{124} + x^{120} + x^{116} + x^{114} + x^{110} + x^{106} + x^{102} + x^{100} + x^{88} + x^{76}\\
&+ x^{74} + x^{72} + x^{66} + x^{62} + x^{34} + x^{32} + x^{26} + x^{24} + x^{20} + x^{18} + x^{16}\\
&+ x^{14} + x^{10} + x^{6},\\
p_9(x) = {} & x^{163} + x^{162} + x^{161} + x^{160} + x^{156} + x^{154} + x^{150} + x^{149} + x^{144} + x^{138}\\
&+ x^{137} + x^{136} + x^{132} + x^{130} + x^{126} + x^{125} + x^{120} + x^{115} + x^{67} + x^{66}\\
&+ x^{65} + x^{64} + x^{60} + x^{58} + x^{54} + x^{53} + x^{51} + x^{50} + x^{49} + x^{44} + x^{41}\\
&+ x^{40} + x^{38} + x^{37} + x^{36} + x^{34} + x^{32} + x^{30} + x^{29} + x^{26} + x^{25} + x^{20}\\
&+ x^{19} + x^{18} + x^{14} + x^{13} + x^{8} + x^{3},\\
p_{12}(x) = {} & x^{164} + x^{136} + x^{116} + x^{112} + x^{68} + x^{52} + x^{40} + x^{24} + x^{20} + x^{16} + x^{4}\\
&+ 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 1].

For $\phi(M^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) = {} & x^{147} + x^{143} + x^{142} + x^{139} + x^{137} + x^{136} + x^{134} + x^{132} + x^{131} + x^{129}\\
&+ x^{128} + x^{127} + x^{126} + x^{125} + x^{121} + x^{120} + x^{119} + x^{115} + x^{111}\\
&+ x^{109} + x^{106} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{96} + x^{95} + x^{93}\\
&+ x^{91} + x^{86} + x^{85} + x^{82} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{73} + x^{72}\\
&+ x^{71} + x^{67} + x^{63} + x^{61} + x^{59} + x^{55} + x^{54} + x^{53} + x^{48} + x^{47} + x^{46}\\
&+ x^{41} + x^{40} + x^{37} + x^{36} + x^{35} + x^{33},\\
p_3(x) = {} & x^{146} + x^{145} + x^{142} + x^{138} + x^{137} + x^{132} + x^{131} + x^{130} + x^{124} + x^{123}\\
&+ x^{121} + x^{119} + x^{118} + x^{116} + x^{114} + x^{113} + x^{112} + x^{111} + x^{108}\\
&+ x^{106} + x^{100} + x^{94} + x^{88} + x^{84} + x^{81} + x^{80} + x^{74} + x^{70} + x^{68} + x^{67}\\
&+ x^{66} + x^{64} + x^{62} + x^{57} + x^{56} + x^{55} + x^{49} + x^{44} + x^{42} + x^{41} + x^{40}\\
&+ x^{38} + x^{35} + x^{30} + x^{27} + x^{25} + x^{24} + x^{23} + x^{20} + x^{17} + x^{15} + x^{12},\\
p_6(x) = {} & x^{146} + x^{145} + x^{142} + x^{141} + x^{138} + x^{136} + x^{134} + x^{132} + x^{131} + x^{128}\\
&+ x^{127} + x^{121} + x^{119} + x^{117} + x^{116} + x^{115} + x^{114} + x^{113} + x^{112}\\
&+ x^{110} + x^{108} + x^{106} + x^{104} + x^{100} + x^{99} + x^{94} + x^{88} + x^{87} + x^{85}\\
&+ x^{84} + x^{78} + x^{75} + x^{72} + x^{71} + x^{70} + x^{66} + x^{64} + x^{63} + x^{61} + x^{59}\\
&+ x^{57} + x^{56} + x^{54} + x^{51} + x^{49} + x^{48} + x^{45} + x^{44} + x^{43} + x^{42} + x^{39}\\
&+ x^{37} + x^{35} + x^{34} + x^{33} + x^{28} + x^{27} + x^{26} + x^{25} + x^{22} + x^{21} + x^{19}\\
&+ x^{17} + x^{15} + x^{13} + x^{12} + x^{11} + x^{8} + x^{6} + 1,\\
p_9(x) = {} & x^{146} + x^{142} + x^{140} + x^{136} + x^{122} + x^{116} + x^{50} + x^{46} + x^{44} + x^{40}\\
&+ x^{34} + x^{30} + x^{28} + x^{26} + x^{24} + x^{20} + x^{10} + x^{4},
\end{aligned}
$$

$$p_{12}(x) = x^{146} + x^{140} + x^{122} + x^{118} + x^{116} + x^{112} + x^{50} + x^{44} + x^{34} + x^{28} + x^{26} + x^{22} + x^{20} + x^{16} + x^{10} + x^{6} + x^{4} + 1,$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 1, 0].
For $\phi(W^o, 0, 0)$:

$$\begin{aligned}
p_0(x) = {} & x^{147} + x^{146} + x^{145} + x^{144} + x^{143} + x^{141} + x^{140} + x^{139} + x^{134} + x^{133} \\
& + x^{129} + x^{128} + x^{126} + x^{125} + x^{124} + x^{123} + x^{122} + x^{119} + x^{117} \\
& + x^{114} + x^{110} + x^{107} + x^{106} + x^{105} + x^{103} + x^{101} + x^{95} + x^{94} + x^{93} \\
& + x^{87} + x^{84} + x^{81} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{67} + x^{66} \\
& + x^{64} + x^{62} + x^{60} + x^{58} + x^{55} + x^{53} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} \\
& + x^{45} + x^{44} + x^{42} + x^{40} + x^{39} + x^{38} + x^{37} + x^{34} + x^{33} + x^{32} + x^{31} \\
& + x^{28} + x^{25} + x^{23} + x^{21} + x^{20} + x^{19} + x^{16} + x^{14} + x^{12}, \\
p_3(x) = {} & x^{146} + x^{145} + x^{144} + x^{142} + x^{137} + x^{136} + x^{132} + x^{131} + x^{130} + x^{123} \\
& + x^{122} + x^{121} + x^{120} + x^{119} + x^{113} + x^{112} + x^{111} + x^{110} + x^{108} \\
& + x^{104} + x^{100} + x^{94} + x^{88} + x^{84} + x^{81} + x^{70} + x^{68} + x^{67} + x^{64} + x^{62} \\
& + x^{60} + x^{57} + x^{55} + x^{54} + x^{50} + x^{49} + x^{44} + x^{41} + x^{38} + x^{35} + x^{30} \\
& + x^{28} + x^{27} + x^{26} + x^{25} + x^{23} + x^{22} + x^{18} + x^{17} + x^{15} + x^{14} + x^{12} \\
& + x^{10} + x^{8}, \\
p_6(x) = {} & x^{146} + x^{145} + x^{144} + x^{142} + x^{141} + x^{140} + x^{136} + x^{132} + x^{131} + x^{130} \\
& + x^{128} + x^{127} + x^{126} + x^{124} + x^{121} + x^{119} + x^{118} + x^{117} + x^{115} \\
& + x^{114} + x^{113} + x^{112} + x^{104} + x^{100} + x^{99} + x^{98} + x^{94} + x^{92} + x^{88} \\
& + x^{87} + x^{86} + x^{85} + x^{80} + x^{75} + x^{74} + x^{72} + x^{71} + x^{68} + x^{66} + x^{63} \\
& + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{52} + x^{51} + x^{49} + x^{45} \\
& + x^{44} + x^{43} + x^{42} + x^{39} + x^{37} + x^{35} + x^{33} + x^{30} + x^{27} + x^{26} + x^{25} \\
& + x^{24} + x^{22} + x^{21} + x^{19} + x^{17} + x^{16} + x^{15} + x^{13} + x^{12} + x^{11} + x^{4} + 1, \\
p_9(x) = {} & x^{146} + x^{142} + x^{140} + x^{136} + x^{122} + x^{116} + x^{50} + x^{46} + x^{44} + x^{40} \\
& + x^{34} + x^{30} + x^{28} + x^{26} + x^{24} + x^{20} + x^{10} + x^{4}, \\
p_{12}(x) = {} & x^{146} + x^{140} + x^{122} + x^{118} + x^{116} + x^{112} + x^{50} + x^{44} + x^{34} + x^{28} \\
& + x^{26} + x^{22} + x^{20} + x^{16} + x^{10} + x^{6} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 1, 0, 0, 0, 0, 1, 1, 1, 1, 0, 1, 1, 1, 0].
For $\phi(W^o, 0, 1)$:

$$\begin{aligned}
p_0(x) = {} & x^{162} + x^{161} + x^{160} + x^{157} + x^{156} + x^{153} + x^{151} + x^{150} + x^{148} + x^{146} \\
& + x^{145} + x^{144} + x^{141} + x^{138} + x^{137} + x^{136} + x^{135} + x^{134} + x^{121} \\
& + x^{119} + x^{118} + x^{116} + x^{114} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} \\
& + x^{106} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{97} + x^{95} + x^{94} \\
& + x^{92} + x^{89} + x^{85} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{77} + x^{74} + x^{73}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{71} + x^{70} + x^{69} + x^{68} + x^{65} + x^{61} + x^{60} + x^{58} + x^{56} + x^{55} + x^{53}\\
&+ x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{44} + x^{43} + x^{42} + x^{40} + x^{38} + x^{37}\\
&+ x^{35} + x^{34} + x^{32} + x^{31} + x^{29} + x^{27},\\
p_3(x) = {}& x^{161} + x^{160} + x^{154} + x^{153} + x^{152} + x^{148} + x^{146} + x^{142} + x^{140} + x^{136}\\
&+ x^{134} + x^{130} + x^{128} + x^{124} + x^{122} + x^{118} + x^{116} + x^{113} + x^{110}\\
&+ x^{105} + x^{97} + x^{96} + x^{90} + x^{84} + x^{78} + x^{72} + x^{66} + x^{65} + x^{64} + x^{60}\\
&+ x^{58} + x^{57} + x^{56} + x^{54} + x^{52} + x^{50} + x^{49} + x^{46} + x^{44} + x^{40} + x^{38}\\
&+ x^{34} + x^{32} + x^{28} + x^{26} + x^{22} + x^{20} + x^{17} + x^{14} + x^{9},\\
p_6(x) = {}& x^{162} + x^{160} + x^{158} + x^{156} + x^{148} + x^{144} + x^{136} + x^{132} + x^{130} + x^{128}\\
&+ x^{124} + x^{120} + x^{116} + x^{114} + x^{110} + x^{106} + x^{102} + x^{100} + x^{88} + x^{76}\\
&+ x^{74} + x^{72} + x^{66} + x^{62} + x^{34} + x^{32} + x^{26} + x^{24} + x^{20} + x^{18} + x^{16}\\
&+ x^{14} + x^{10} + x^{6},\\
p_9(x) = {}& x^{163} + x^{162} + x^{161} + x^{160} + x^{156} + x^{154} + x^{150} + x^{149} + x^{144} + x^{138}\\
&+ x^{137} + x^{136} + x^{132} + x^{130} + x^{126} + x^{125} + x^{120} + x^{115} + x^{67} + x^{66}\\
&+ x^{65} + x^{64} + x^{60} + x^{58} + x^{54} + x^{53} + x^{51} + x^{50} + x^{49} + x^{44} + x^{41}\\
&+ x^{40} + x^{38} + x^{37} + x^{36} + x^{34} + x^{32} + x^{30} + x^{29} + x^{26} + x^{25} + x^{20}\\
&+ x^{19} + x^{18} + x^{14} + x^{13} + x^{8} + x^{3},\\
p_{12}(x) = {}& x^{164} + x^{136} + x^{116} + x^{112} + x^{68} + x^{52} + x^{40} + x^{24} + x^{20} + x^{16} + x^{4}\\
&+ 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 1].

For $\phi(W^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) = {}& x^{144} + x^{141} + x^{140} + x^{139} + x^{136} + x^{133} + x^{132} + x^{130} + x^{125} + x^{124}\\
&+ x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{115} + x^{114} + x^{113}\\
&+ x^{111} + x^{110} + x^{108} + x^{107} + x^{103} + x^{100} + x^{98} + x^{96} + x^{95} + x^{94}\\
&+ x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{82} + x^{81} + x^{74} + x^{73} + x^{72} + x^{71}\\
&+ x^{69} + x^{68} + x^{65} + x^{64} + x^{62} + x^{61} + x^{60} + x^{59} + x^{57} + x^{56} + x^{52}\\
&+ x^{50} + x^{49} + x^{47} + x^{46} + x^{45} + x^{44} + x^{42} + x^{41} + x^{40} + x^{38} + x^{37}\\
&+ x^{35} + x^{32} + x^{31} + x^{30} + x^{28} + x^{27} + x^{25} + x^{23} + x^{22} + x^{21} + x^{20}\\
&+ x^{18},\\
p_3(x) = {}& x^{137} + x^{136} + x^{130} + x^{129} + x^{128} + x^{124} + x^{122} + x^{118} + x^{116} + x^{113}\\
&+ x^{110} + x^{105} + x^{73} + x^{72} + x^{66} + x^{60} + x^{54} + x^{49} + x^{41} + x^{40} + x^{34}\\
&+ x^{33} + x^{32} + x^{28} + x^{26} + x^{22} + x^{20} + x^{17} + x^{14} + x^{9},\\
p_6(x) = {}& x^{138} + x^{136} + x^{134} + x^{132} + x^{124} + x^{120} + x^{114} + x^{110} + x^{106} + x^{104}\\
&+ x^{92} + x^{80} + x^{78} + x^{76} + x^{68} + x^{64} + x^{56} + x^{54} + x^{50} + x^{48} + x^{44}\\
&+ x^{42} + x^{40} + x^{38} + x^{32} + x^{30} + x^{26} + x^{24} + x^{20} + x^{18} + x^{16} + x^{14}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{10} + x^{6},\\
p_9(x) &= x^{139} + x^{138} + x^{137} + x^{136} + x^{132} + x^{130} + x^{126} + x^{125} + x^{120} + x^{115}\\
&+ x^{43} + x^{42} + x^{41} + x^{40} + x^{36} + x^{34} + x^{30} + x^{29} + x^{27} + x^{26} + x^{25}\\
&+ x^{20} + x^{19} + x^{18} + x^{14} + x^{13} + x^{8} + x^{3},\\
p_{12}(x) &= x^{140} + x^{116} + x^{112} + x^{44} + x^{28} + x^{20} + x^{16} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1].

For $\phi(W^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{147} + x^{143} + x^{142} + x^{139} + x^{137} + x^{136} + x^{134} + x^{132} + x^{131} + x^{129}\\
&+ x^{128} + x^{127} + x^{126} + x^{125} + x^{121} + x^{120} + x^{119} + x^{115} + x^{111}\\
&+ x^{109} + x^{106} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{96} + x^{95} + x^{93}\\
&+ x^{91} + x^{86} + x^{85} + x^{82} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{73} + x^{72}\\
&+ x^{71} + x^{67} + x^{63} + x^{61} + x^{59} + x^{55} + x^{54} + x^{53} + x^{48} + x^{47} + x^{46}\\
&+ x^{41} + x^{40} + x^{37} + x^{36} + x^{35} + x^{33},\\
p_3(x) &= x^{146} + x^{145} + x^{142} + x^{138} + x^{137} + x^{132} + x^{131} + x^{130} + x^{124} + x^{123}\\
&+ x^{121} + x^{119} + x^{118} + x^{116} + x^{114} + x^{113} + x^{112} + x^{111} + x^{108}\\
&+ x^{106} + x^{100} + x^{94} + x^{88} + x^{84} + x^{81} + x^{80} + x^{74} + x^{70} + x^{68} + x^{67}\\
&+ x^{66} + x^{64} + x^{62} + x^{57} + x^{56} + x^{55} + x^{49} + x^{44} + x^{42} + x^{41} + x^{40}\\
&+ x^{38} + x^{35} + x^{30} + x^{27} + x^{25} + x^{24} + x^{23} + x^{20} + x^{17} + x^{15} + x^{12},\\
p_6(x) &= x^{146} + x^{145} + x^{142} + x^{141} + x^{138} + x^{136} + x^{134} + x^{132} + x^{131} + x^{128}\\
&+ x^{127} + x^{121} + x^{119} + x^{117} + x^{116} + x^{115} + x^{114} + x^{113} + x^{112}\\
&+ x^{110} + x^{108} + x^{106} + x^{104} + x^{100} + x^{99} + x^{94} + x^{88} + x^{87} + x^{85}\\
&+ x^{84} + x^{78} + x^{75} + x^{72} + x^{71} + x^{70} + x^{66} + x^{64} + x^{63} + x^{61} + x^{59}\\
&+ x^{57} + x^{56} + x^{54} + x^{51} + x^{49} + x^{48} + x^{45} + x^{44} + x^{43} + x^{42} + x^{39}\\
&+ x^{37} + x^{35} + x^{34} + x^{33} + x^{28} + x^{27} + x^{26} + x^{25} + x^{22} + x^{21} + x^{19}\\
&+ x^{17} + x^{15} + x^{13} + x^{12} + x^{11} + x^{8} + x^{6} + 1,\\
p_9(x) &= x^{146} + x^{142} + x^{140} + x^{136} + x^{122} + x^{116} + x^{50} + x^{46} + x^{44} + x^{40}\\
&+ x^{34} + x^{30} + x^{28} + x^{26} + x^{24} + x^{20} + x^{10} + x^{4},\\
p_{12}(x) &= x^{146} + x^{140} + x^{122} + x^{118} + x^{116} + x^{112} + x^{50} + x^{44} + x^{34} + x^{28}\\
&+ x^{26} + x^{22} + x^{20} + x^{16} + x^{10} + x^{6} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 1, 0].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

| $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | [1, 2] | 178 | [284, 105] | 356 | [497, 95] | 534 | [652, 653] | 712 | [201, 773] |
| 1 | [3, 4] | 179 | [285, 9] | 357 | [323, 96] | 535 | [654, 655] | 713 | [774, 775] |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 2 | [5, 6] | 180 | [258, 128] | 358 | [498, 252] | 536 | [479, 656] | 714 | [776, 480] |
| 3 | [7, 8] | 181 | [286, 263] | 359 | [499, 106] | 537 | [657, 18] | 715 | [777, 778] |
| 4 | [9, 10] | 182 | [284, 123] | 360 | [500, 501] | 538 | [658, 659] | 716 | [716, 716] |
| 5 | [11, 2] | 183 | [287, 288] | 361 | [29, 38] | 539 | [660, 661] | 717 | [779, 780] |
| 6 | [12, 13] | 184 | [283, 36] | 362 | [502, 503] | 540 | [662, 663] | 718 | [781, 422] |
| 7 | [14, 15] | 185 | [289, 128] | 363 | [504, 38] | 541 | [664, 665] | 719 | [782, 366] |
| 8 | [16, 17] | 186 | [290, 291] | 364 | [505, 109] | 542 | [666, 481] | 720 | [783, 197] |
| 9 | [18, 19] | 187 | [292, 293] | 365 | [137, 43] | 543 | [667, 668] | 721 | [784, 785] |
| 10 | [20, 21] | 188 | [93, 294] | 366 | [103, 121] | 544 | [669, 670] | 722 | [786, 702] |
| 11 | [22, 23] | 189 | [252, 277] | 367 | [276, 104] | 545 | [671, 214] | 723 | [787, 788] |
| 12 | [24, 25] | 190 | [105, 295] | 368 | [506, 279] | 546 | [672, 176] | 724 | [789, 790] |
| 13 | [26, 27] | 191 | [296, 297] | 369 | [298, 273] | 547 | [673, 674] | 725 | [791, 792] |
| 14 | [28, 29] | 192 | [298, 125] | 370 | [262, 90] | 548 | [675, 148] | 726 | [793, 794] |
| 15 | [30, 31] | 193 | [299, 253] | 371 | [507, 97] | 549 | [676, 677] | 727 | [795, 796] |
| 16 | [7, 32] | 194 | [251, 10] | 372 | [508, 137] | 550 | [678, 679] | 728 | [797, 798] |
| 17 | [33, 34] | 195 | [300, 269] | 373 | [509, 510] | 551 | [680, 681] | 729 | [799, 800] |
| 18 | [35, 36] | 196 | [279, 273] | 374 | [311, 309] | 552 | [682, 683] | 730 | [801, 802] |
| 19 | [37, 38] | 197 | [258, 275] | 375 | [245, 251] | 553 | [684, 685] | 731 | [803, 804] |
| 20 | [39, 40] | 198 | [295, 291] | 376 | [320, 89] | 554 | [686, 687] | 732 | [805, 806] |
| 21 | [41, 42] | 199 | [262, 40] | 377 | [511, 510] | 555 | [688, 689] | 733 | [807, 808] |
| 22 | [43, 38] | 200 | [301, 9] | 378 | [127, 31] | 556 | [690, 691] | 734 | [809, 810] |
| 23 | [9, 44] | 201 | [31, 34] | 379 | [32, 251] | 557 | [692, 375] | 735 | [811, 202] |
| 24 | [13, 45] | 202 | [250, 104] | 380 | [93, 282] | 558 | [693, 694] | 736 | [812, 813] |
| 25 | [12, 46] | 203 | [120, 96] | 381 | [128, 101] | 559 | [695, 696] | 737 | [814, 815] |
| 26 | [45, 47] | 204 | [302, 281] | 382 | [512, 250] | 560 | [697, 698] | 738 | [816, 55] |
| 27 | [46, 48] | 205 | [303, 294] | 383 | [513, 42] | 561 | [699, 240] | 739 | [817, 818] |
| 28 | [49, 50] | 206 | [124, 294] | 384 | [514, 515] | 562 | [700, 701] | 740 | [819, 820] |
| 29 | [51, 52] | 207 | [42, 29] | 385 | [258, 119] | 563 | [702, 703] | 741 | [821, 822] |
| 30 | [53, 54] | 208 | [304, 297] | 386 | [32, 133] | 564 | [532, 95] | 742 | [823, 824] |
| 31 | [55, 56] | 209 | [252, 273] | 387 | [35, 262] | 565 | [251, 29] | 743 | [825, 826] |
| 32 | [57, 58] | 210 | [305, 119] | 388 | [516, 260] | 566 | [261, 309] | 744 | [827, 828] |
| 33 | [59, 60] | 211 | [43, 88] | 389 | [30, 128] | 567 | [704, 535] | 745 | [829, 830] |
| 34 | [61, 62] | 212 | [306, 307] | 390 | [517, 98] | 568 | [305, 275] | 746 | [831, 832] |
| 35 | [63, 64] | 213 | [308, 309] | 391 | [517, 322] | 569 | [274, 282] | 747 | [833, 834] |
| 36 | [65, 66] | 214 | [310, 42] | 392 | [518, 293] | 570 | [33, 90] | 748 | [835, 65] |
| 37 | [67, 68] | 215 | [311, 36] | 393 | [299, 282] | 571 | [252, 267] | 749 | [836, 837] |
| 38 | [69, 70] | 216 | [312, 10] | 394 | [263, 275] | 572 | [100, 135] | 750 | [838, 348] |
| 39 | [71, 72] | 217 | [313, 314] | 395 | [124, 273] | 573 | [705, 517] | 751 | [839, 840] |
| 40 | [73, 74] | 218 | [9, 269] | 396 | [134, 101] | 574 | [285, 133] | 752 | [841, 842] |
| 41 | [75, 20] | 219 | [315, 114] | 397 | [519, 515] | 575 | [706, 493] | 753 | [843, 844] |
| 42 | [76, 77] | 220 | [95, 112] | 398 | [520, 89] | 576 | [35, 89] | 754 | [845, 163] |
| 43 | [78, 79] | 221 | [316, 129] | 399 | [279, 125] | 577 | [707, 708] | 755 | [846, 146] |
| 44 | [80, 81] | 222 | [275, 90] | 400 | [326, 40] | 578 | [43, 327] | 756 | [847, 848] |
| 45 | [82, 83] | 223 | [317, 114] | 401 | [521, 120] | 579 | [709, 710] | 757 | [746, 714] |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 46 | [83, 84] | 224 | [318, 249] | 402 | [310, 123] | 580 | [316, 118] | 758 | [44, 129] |
| 47 | [27, 85] | 225 | [312, 37] | 403 | [91, 116] | 581 | [711, 124] | 759 | [513, 251] |
| 48 | [25, 26] | 226 | [314, 294] | 404 | [30, 119] | 582 | [495, 295] | 760 | [93, 267] |
| 49 | [86, 31] | 227 | [319, 136] | 405 | [522, 487] | 583 | [311, 89] | 761 | [849, 708] |
| 50 | [87, 88] | 228 | [320, 309] | 406 | [523, 132] | 584 | [712, 321] | 762 | [117, 38] |
| 51 | [89, 90] | 229 | [106, 321] | 407 | [524, 136] | 585 | [42, 44] | 763 | [712, 129] |
| 52 | [91, 9] | 230 | [113, 322] | 408 | [262, 34] | 586 | [289, 119] | 764 | [42, 106] |
| 53 | [92, 93] | 231 | [323, 112] | 409 | [525, 517] | 587 | [91, 105] | 765 | [721, 728] |
| 54 | [94, 44] | 232 | [324, 43] | 410 | [526, 527] | 588 | [298, 107] | 766 | [329, 140] |
| 55 | [95, 96] | 233 | [268, 267] | 411 | [290, 321] | 589 | [713, 494] | 767 | [729, 724] |
| 56 | [97, 98] | 234 | [325, 8] | 412 | [528, 529] | 590 | [301, 116] | 768 | [13, 48] |
| 57 | [99, 91] | 235 | [326, 247] | 413 | [290, 88] | 591 | [520, 309] | 769 | [850, 494] |
| 58 | [100, 101] | 236 | [308, 262] | 414 | [530, 314] | 592 | [504, 246] | 770 | [103, 96] |
| 59 | [102, 96] | 237 | [269, 327] | 415 | [508, 123] | 593 | [44, 88] | 771 | [737, 311] |
| 60 | [103, 104] | 238 | [328, 48] | 416 | [266, 253] | 594 | [94, 269] | 772 | [8, 9] |
| 61 | [105, 106] | 239 | [329, 45] | 417 | [309, 115] | 595 | [500, 714] | 773 | [513, 105] |
| 62 | [93, 107] | 240 | [330, 140] | 418 | [531, 95] | 596 | [138, 46] | 774 | [507, 315] |
| 63 | [108, 109] | 241 | [140, 13] | 419 | [245, 116] | 597 | [502, 715] | 775 | [276, 121] |
| 64 | [110, 29] | 242 | [48, 45] | 420 | [508, 105] | 598 | [716, 46] | 776 | [727, 722] |
| 65 | [111, 112] | 243 | [331, 179] | 421 | [289, 31] | 599 | [257, 263] | 777 | [735, 289] |
| 66 | [113, 114] | 244 | [332, 333] | 422 | [119, 101] | 600 | [111, 104] | 778 | [318, 322] |
| 67 | [36, 115] | 245 | [334, 335] | 423 | [255, 123] | 601 | [100, 115] | 779 | [851, 517] |
| 68 | [8, 116] | 246 | [186, 336] | 424 | [299, 107] | 602 | [310, 137] | 780 | [102, 104] |
| 69 | [117, 118] | 247 | [337, 338] | 425 | [256, 34] | 603 | [717, 263] | 781 | [852, 255] |
| 70 | [119, 115] | 248 | [339, 340] | 426 | [532, 276] | 604 | [718, 133] | 782 | [853, 276] |
| 71 | [120, 121] | 249 | [341, 342] | 427 | [301, 251] | 605 | [719, 35] | 783 | [137, 295] |
| 72 | [122, 98] | 250 | [343, 344] | 428 | [533, 307] | 606 | [136, 116] | 784 | [117, 321] |
| 73 | [123, 10] | 251 | [15, 345] | 429 | [263, 31] | 607 | [109, 253] | 785 | [499, 37] |
| 74 | [124, 125] | 252 | [346, 347] | 430 | [534, 535] | 608 | [720, 8] | 786 | [742, 527] |
| 75 | [126, 41] | 253 | [348, 349] | 431 | [494, 36] | 609 | [721, 722] | 787 | [854, 544] |
| 76 | [127, 128] | 254 | [350, 351] | 432 | [494, 262] | 610 | [295, 321] | 788 | [329, 139] |
| 77 | [10, 129] | 255 | [352, 353] | 433 | [536, 38] | 611 | [723, 724] | 789 | [549, 30] |
| 78 | [130, 131] | 256 | [354, 355] | 434 | [537, 91] | 612 | [712, 327] | 790 | [122, 114] |
| 79 | [132, 133] | 257 | [356, 357] | 435 | [272, 125] | 613 | [725, 722] | 791 | [10, 321] |
| 80 | [134, 135] | 258 | [358, 359] | 436 | [36, 131] | 614 | [316, 246] | 792 | [251, 269] |
| 81 | [136, 137] | 259 | [360, 361] | 437 | [538, 539] | 615 | [8, 137] | 793 | [116, 10] |
| 82 | [138, 13] | 260 | [362, 363] | 438 | [320, 262] | 616 | [726, 41] | 794 | [311, 262] |
| 83 | [139, 139] | 261 | [364, 365] | 439 | [487, 107] | 617 | [309, 131] | 795 | [504, 291] |
| 84 | [13, 140] | 262 | [366, 367] | 440 | [119, 135] | 618 | [727, 728] | 796 | [123, 43] |
| 85 | [47, 46] | 263 | [368, 195] | 441 | [540, 279] | 619 | [269, 88] | 797 | [707, 542] |
| 86 | [141, 61] | 264 | [369, 370] | 442 | [42, 37] | 620 | [729, 730] | 798 | [330, 45] |
| 87 | [142, 143] | 265 | [371, 372] | 443 | [541, 542] | 621 | [87, 291] | 799 | [741, 503] |
| 88 | [144, 145] | 266 | [373, 374] | 444 | [106, 291] | 622 | [731, 501] | 800 | [140, 47] |
| 89 | [146, 147] | 267 | [375, 376] | 445 | [543, 544] | 623 | [87, 327] | 801 | [286, 258] |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 90 | [148, 149] | 268 | [377, 378] | 446 | [536, 246] | 624 | [732, 556] | 802 | [102, 112] |
| 91 | [150, 151] | 269 | [17, 379] | 447 | [545, 124] | 625 | [558, 277] | 803 | [536, 88] |
| 92 | [152, 153] | 270 | [380, 381] | 448 | [122, 322] | 626 | [268, 253] | 804 | [105, 37] |
| 93 | [154, 155] | 271 | [382, 383] | 449 | [315, 249] | 627 | [39, 247] | 805 | [109, 294] |
| 94 | [156, 69] | 272 | [384, 385] | 450 | [546, 284] | 628 | [733, 728] | 806 | [275, 40] |
| 95 | [157, 158] | 273 | [386, 387] | 451 | [547, 487] | 629 | [536, 118] | 807 | [37, 291] |
| 96 | [159, 160] | 274 | [388, 389] | 452 | [28, 10] | 630 | [734, 268] | 808 | [495, 29] |
| 97 | [161, 162] | 275 | [390, 391] | 453 | [548, 275] | 631 | [116, 29] | 809 | [850, 320] |
| 98 | [163, 164] | 276 | [392, 393] | 454 | [549, 289] | 632 | [32, 42] | 810 | [255, 105] |
| 99 | [165, 166] | 277 | [182, 394] | 455 | [132, 251] | 633 | [491, 282] | 811 | [512, 120] |
| 100 | [167, 168] | 278 | [395, 396] | 456 | [550, 539] | 634 | [735, 30] | 812 | [855, 730] |
| 101 | [64, 169] | 279 | [397, 398] | 457 | [548, 128] | 635 | [9, 106] | 813 | [562, 46] |
| 102 | [170, 171] | 280 | [399, 400] | 458 | [551, 552] | 636 | [499, 269] | 814 | [705, 113] |
| 103 | [172, 173] | 281 | [401, 402] | 459 | [274, 107] | 637 | [263, 128] | 815 | [113, 249] |
| 104 | [174, 175] | 282 | [403, 404] | 460 | [326, 90] | 638 | [310, 133] | 816 | [754, 97] |
| 105 | [176, 144] | 283 | [405, 337] | 461 | [553, 42] | 639 | [290, 38] | 817 | [295, 246] |
| 106 | [177, 178] | 284 | [406, 78] | 462 | [554, 552] | 640 | [312, 106] | 818 | [116, 295] |
| 107 | [179, 180] | 285 | [407, 408] | 463 | [487, 277] | 641 | [541, 708] | 819 | [316, 327] |
| 108 | [181, 182] | 286 | [409, 390] | 464 | [555, 556] | 642 | [736, 529] | 820 | [28, 43] |
| 109 | [183, 184] | 287 | [410, 411] | 465 | [314, 107] | 643 | [563, 13] | 821 | [856, 710] |
| 110 | [185, 186] | 288 | [412, 413] | 466 | [133, 269] | 644 | [737, 35] | 822 | [561, 140] |
| 111 | [187, 188] | 289 | [414, 225] | 467 | [557, 268] | 645 | [248, 98] | 823 | [106, 118] |
| 112 | [189, 190] | 290 | [336, 415] | 468 | [36, 101] | 646 | [738, 289] | 824 | [110, 295] |
| 113 | [191, 192] | 291 | [237, 199] | 469 | [558, 273] | 647 | [285, 123] | 825 | [723, 730] |
| 114 | [193, 194] | 292 | [416, 417] | 470 | [133, 106] | 648 | [266, 294] | 826 | [139, 45] |
| 115 | [195, 196] | 293 | [418, 419] | 471 | [559, 490] | 649 | [718, 105] | 827 | [857, 320] |
| 116 | [197, 198] | 294 | [420, 421] | 472 | [124, 267] | 650 | [739, 542] | 828 | [323, 121] |
| 117 | [199, 200] | 295 | [422, 423] | 473 | [261, 36] | 651 | [504, 129] | 829 | [531, 276] |
| 118 | [50, 201] | 296 | [424, 425] | 474 | [560, 327] | 652 | [740, 714] | 830 | [97, 322] |
| 119 | [202, 203] | 297 | [426, 427] | 475 | [300, 295] | 653 | [712, 291] | 831 | [29, 327] |
| 120 | [204, 205] | 298 | [428, 429] | 476 | [520, 36] | 654 | [741, 715] | 832 | [133, 29] |
| 121 | [206, 207] | 299 | [430, 431] | 477 | [97, 249] | 655 | [117, 129] | 833 | [743, 529] |
| 122 | [208, 209] | 300 | [432, 433] | 478 | [317, 322] | 656 | [285, 116] | 834 | [46, 140] |
| 123 | [210, 211] | 301 | [434, 51] | 479 | [309, 135] | 657 | [94, 37] | 835 | [858, 113] |
| 124 | [212, 213] | 302 | [435, 436] | 480 | [561, 139] | 658 | [742, 2] | 836 | [859, 120] |
| 125 | [214, 215] | 303 | [437, 438] | 481 | [562, 47] | 659 | [37, 118] | 837 | [317, 98] |
| 126 | [216, 76] | 304 | [439, 440] | 482 | [563, 47] | 660 | [743, 744] | 838 | [857, 494] |
| 127 | [217, 218] | 305 | [441, 442] | 483 | [140, 46] | 661 | [560, 321] | 839 | [553, 9] |
| 128 | [219, 220] | 306 | [443, 444] | 484 | [561, 48] | 662 | [745, 35] | 840 | [487, 273] |
| 129 | [221, 222] | 307 | [445, 446] | 485 | [564, 219] | 663 | [513, 123] | 841 | [560, 118] |
| 130 | [223, 224] | 308 | [447, 73] | 486 | [565, 566] | 664 | [87, 246] | 842 | [137, 10] |
| 131 | [225, 226] | 309 | [448, 449] | 487 | [567, 568] | 665 | [324, 10] | 843 | [324, 106] |
| 132 | [227, 80] | 310 | [450, 177] | 488 | [569, 570] | 666 | [746, 501] | 844 | [548, 31] |
| 133 | [228, 229] | 311 | [451, 452] | 489 | [571, 572] | 667 | [747, 744] | 845 | [284, 9] |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 134 | [230, 231] | 312 | [453, 221] | 490 | [573, 574] | 668 | [12, 716] | 846 | [860, 250] |
| 135 | [232, 233] | 313 | [454, 455] | 491 | [575, 576] | 669 | [719, 311] | 847 | [11, 527] |
| 136 | [234, 235] | 314 | [456, 457] | 492 | [577, 578] | 670 | [317, 249] | 848 | [10, 246] |
| 137 | [236, 237] | 315 | [458, 459] | 493 | [579, 580] | 671 | [748, 311] | 849 | [861, 862] |
| 138 | [238, 239] | 316 | [460, 461] | 494 | [581, 582] | 672 | [110, 44] | 850 | [863, 864] |
| 139 | [240, 241] | 317 | [462, 463] | 495 | [583, 584] | 673 | [749, 30] | 851 | [865, 866] |
| 140 | [239, 242] | 318 | [464, 465] | 496 | [585, 586] | 674 | [553, 137] | 852 | [867, 210] |
| 141 | [243, 93] | 319 | [466, 236] | 497 | [587, 588] | 675 | [750, 252] | 853 | [868, 174] |
| 142 | [244, 101] | 320 | [467, 232] | 498 | [589, 590] | 676 | [751, 97] | 854 | [869, 870] |
| 143 | [245, 133] | 321 | [19, 468] | 499 | [591, 592] | 677 | [318, 114] | 855 | [871, 872] |
| 144 | [44, 246] | 322 | [469, 470] | 500 | [593, 594] | 678 | [752, 2] | 856 | [873, 874] |
| 145 | [89, 247] | 323 | [471, 472] | 501 | [595, 596] | 679 | [560, 88] | 857 | [875, 448] |
| 146 | [248, 249] | 324 | [473, 474] | 502 | [597, 598] | 680 | [303, 277] | 858 | [876, 193] |
| 147 | [250, 121] | 325 | [475, 476] | 503 | [599, 600] | 681 | [130, 115] | 859 | [877, 878] |
| 148 | [251, 44] | 326 | [477, 478] | 504 | [601, 602] | 682 | [525, 113] | 860 | [879, 206] |
| 149 | [252, 253] | 327 | [229, 479] | 505 | [603, 604] | 683 | [718, 9] | 861 | [269, 129] |
| 150 | [254, 255] | 328 | [480, 82] | 506 | [605, 386] | 684 | [753, 132] | 862 | [300, 44] |
| 151 | [256, 90] | 329 | [481, 482] | 507 | [606, 607] | 685 | [31, 40] | 863 | [521, 250] |
| 152 | [257, 258] | 330 | [483, 484] | 508 | [608, 67] | 686 | [314, 277] | 864 | [120, 112] |
| 153 | [136, 42] | 331 | [485, 258] | 509 | [609, 610] | 687 | [31, 247] | 865 | [255, 133] |
| 154 | [259, 260] | 332 | [323, 104] | 510 | [611, 612] | 688 | [558, 294] | 866 | [314, 125] |
| 155 | [261, 262] | 333 | [248, 114] | 511 | [613, 614] | 689 | [244, 131] | 867 | [105, 43] |
| 156 | [263, 119] | 334 | [486, 32] | 512 | [615, 159] | 690 | [754, 315] | 868 | [132, 123] |
| 157 | [264, 265] | 335 | [128, 135] | 513 | [616, 617] | 691 | [553, 105] | 869 | [880, 724] |
| 158 | [266, 267] | 336 | [128, 131] | 514 | [618, 619] | 692 | [755, 320] | 870 | [138, 47] |
| 159 | [268, 107] | 337 | [133, 295] | 515 | [620, 621] | 693 | [756, 288] | 871 | [709, 544] |
| 160 | [137, 269] | 338 | [487, 282] | 516 | [622, 623] | 694 | [289, 275] | 872 | [47, 716] |
| 161 | [270, 271] | 339 | [488, 271] | 517 | [624, 625] | 695 | [491, 267] | 873 | [528, 744] |
| 162 | [272, 273] | 340 | [268, 125] | 518 | [626, 627] | 696 | [89, 34] | 874 | [48, 139] |
| 163 | [274, 125] | 341 | [303, 267] | 519 | [628, 629] | 697 | [256, 247] | 875 | [881, 315] |
| 164 | [9, 43] | 342 | [116, 37] | 520 | [630, 631] | 698 | [718, 251] | 876 | [41, 251] |
| 165 | [123, 37] | 343 | [489, 490] | 521 | [632, 633] | 699 | [48, 140] | 877 | [245, 137] |
| 166 | [86, 275] | 344 | [491, 253] | 522 | [634, 403] | 700 | [47, 13] | 878 | [272, 277] |
| 167 | [276, 112] | 345 | [29, 118] | 523 | [635, 228] | 701 | [562, 716] | 879 | [301, 42] |
| 168 | [111, 96] | 346 | [492, 493] | 524 | [636, 637] | 702 | [563, 716] | 880 | [882, 883] |
| 169 | [109, 277] | 347 | [86, 119] | 525 | [638, 469] | 703 | [328, 139] | 881 | [884, 341] |
| 170 | [278, 265] | 348 | [41, 137] | 526 | [639, 640] | 704 | [757, 758] | 882 | [885, 503] |
| 171 | [279, 253] | 349 | [494, 309] | 527 | [641, 238] | 705 | [759, 760] | 883 | [330, 48] |
| 172 | [280, 281] | 350 | [495, 43] | 528 | [642, 643] | 706 | [761, 762] | 884 | [508, 116] |
| 173 | [109, 282] | 351 | [494, 89] | 529 | [644, 645] | 707 | [763, 764] | 885 | [886, 887] |
| 174 | [279, 282] | 352 | [496, 284] | 530 | [646, 420] | 708 | [765, 766] | 886 | [543, 710] |
| 175 | [123, 44] | 353 | [244, 115] | 531 | [647, 648] | 709 | [767, 768] | 887 | [45, 13] |
| 176 | [283, 89] | 354 | [318, 98] | 532 | [649, 189] | 710 | [769, 770] | | |
| 177 | [275, 247] | 355 | [102, 121] | 533 | [650, 651] | 711 | [771, 772] | | |

Output function $n \mapsto \tau(n)$:

| $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 127 | 1 | 254 | 1 | 381 | 1 | 508 | 0 | 635 | 0 | 762 | 0 |
| 1 | 0 | 128 | 1 | 255 | 1 | 382 | 0 | 509 | 0 | 636 | 1 | 763 | 0 |
| 2 | 0 | 129 | 0 | 256 | 0 | 383 | 1 | 510 | 1 | 637 | 0 | 764 | 1 |
| 3 | 0 | 130 | 0 | 257 | 1 | 384 | 1 | 511 | 1 | 638 | 0 | 765 | 0 |
| 4 | 0 | 131 | 0 | 258 | 1 | 385 | 1 | 512 | 0 | 639 | 1 | 766 | 1 |
| 5 | 0 | 132 | 0 | 259 | 0 | 386 | 1 | 513 | 1 | 640 | 0 | 767 | 0 |
| 6 | 0 | 133 | 0 | 260 | 1 | 387 | 0 | 514 | 1 | 641 | 1 | 768 | 0 |
| 7 | 0 | 134 | 0 | 261 | 1 | 388 | 1 | 515 | 0 | 642 | 0 | 769 | 1 |
| 8 | 0 | 135 | 1 | 262 | 0 | 389 | 1 | 516 | 1 | 643 | 1 | 770 | 0 |
| 9 | 0 | 136 | 1 | 263 | 0 | 390 | 1 | 517 | 1 | 644 | 1 | 771 | 1 |
| 10 | 1 | 137 | 0 | 264 | 0 | 391 | 1 | 518 | 1 | 645 | 1 | 772 | 0 |
| 11 | 0 | 138 | 0 | 265 | 1 | 392 | 1 | 519 | 0 | 646 | 0 | 773 | 1 |
| 12 | 0 | 139 | 1 | 266 | 0 | 393 | 0 | 520 | 1 | 647 | 1 | 774 | 1 |
| 13 | 0 | 140 | 1 | 267 | 1 | 394 | 0 | 521 | 1 | 648 | 0 | 775 | 1 |
| 14 | 0 | 141 | 0 | 268 | 1 | 395 | 1 | 522 | 0 | 649 | 0 | 776 | 1 |
| 15 | 1 | 142 | 1 | 269 | 1 | 396 | 0 | 523 | 0 | 650 | 0 | 777 | 0 |
| 16 | 0 | 143 | 1 | 270 | 0 | 397 | 0 | 524 | 1 | 651 | 1 | 778 | 0 |
| 17 | 1 | 144 | 0 | 271 | 0 | 398 | 1 | 525 | 0 | 652 | 0 | 779 | 1 |
| 18 | 0 | 145 | 1 | 272 | 1 | 399 | 0 | 526 | 1 | 653 | 0 | 780 | 1 |
| 19 | 0 | 146 | 1 | 273 | 1 | 400 | 0 | 527 | 1 | 654 | 0 | 781 | 0 |
| 20 | 1 | 147 | 0 | 274 | 1 | 401 | 1 | 528 | 0 | 655 | 0 | 782 | 0 |
| 21 | 0 | 148 | 1 | 275 | 1 | 402 | 0 | 529 | 1 | 656 | 1 | 783 | 0 |
| 22 | 0 | 149 | 0 | 276 | 1 | 403 | 1 | 530 | 0 | 657 | 0 | 784 | 0 |
| 23 | 0 | 150 | 1 | 277 | 0 | 404 | 1 | 531 | 1 | 658 | 1 | 785 | 1 |
| 24 | 0 | 151 | 0 | 278 | 1 | 405 | 0 | 532 | 0 | 659 | 0 | 786 | 1 |
| 25 | 0 | 152 | 1 | 279 | 0 | 406 | 0 | 533 | 0 | 660 | 1 | 787 | 1 |
| 26 | 0 | 153 | 1 | 280 | 0 | 407 | 1 | 534 | 0 | 661 | 0 | 788 | 1 |
| 27 | 1 | 154 | 0 | 281 | 1 | 408 | 0 | 535 | 0 | 662 | 1 | 789 | 1 |
| 28 | 0 | 155 | 1 | 282 | 1 | 409 | 0 | 536 | 1 | 663 | 1 | 790 | 1 |
| 29 | 1 | 156 | 0 | 283 | 0 | 410 | 1 | 537 | 0 | 664 | 1 | 791 | 1 |
| 30 | 1 | 157 | 0 | 284 | 0 | 411 | 1 | 538 | 1 | 665 | 1 | 792 | 1 |
| 31 | 0 | 158 | 0 | 285 | 1 | 412 | 0 | 539 | 1 | 666 | 1 | 793 | 1 |
| 32 | 1 | 159 | 1 | 286 | 0 | 413 | 1 | 540 | 1 | 667 | 1 | 794 | 1 |
| 33 | 1 | 160 | 0 | 287 | 1 | 414 | 0 | 541 | 1 | 668 | 0 | 795 | 1 |
| 34 | 0 | 161 | 0 | 288 | 0 | 415 | 0 | 542 | 1 | 669 | 0 | 796 | 1 |
| 35 | 0 | 162 | 1 | 289 | 0 | 416 | 0 | 543 | 1 | 670 | 0 | 797 | 0 |
| 36 | 0 | 163 | 1 | 290 | 1 | 417 | 1 | 544 | 0 | 671 | 0 | 798 | 1 |
| 37 | 0 | 164 | 0 | 291 | 1 | 418 | 1 | 545 | 0 | 672 | 0 | 799 | 0 |
| 38 | 0 | 165 | 1 | 292 | 0 | 419 | 1 | 546 | 0 | 673 | 1 | 800 | 1 |
| 39 | 1 | 166 | 0 | 293 | 1 | 420 | 0 | 547 | 1 | 674 | 1 | 801 | 0 |
| 40 | 1 | 167 | 1 | 294 | 0 | 421 | 0 | 548 | 0 | 675 | 0 | 802 | 1 |

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 41 | 0 | 168 | 0 | 295 | 0 | 422 | 0 | 549 | 1 | 676 | 1 | 803 | 1 |
| 42 | 1 | 169 | 1 | 296 | 0 | 423 | 1 | 550 | 0 | 677 | 0 | 804 | 0 |
| 43 | 0 | 170 | 1 | 297 | 0 | 424 | 0 | 551 | 1 | 678 | 0 | 805 | 1 |
| 44 | 0 | 171 | 0 | 298 | 0 | 425 | 0 | 552 | 0 | 679 | 0 | 806 | 1 |
| 45 | 0 | 172 | 0 | 299 | 0 | 426 | 0 | 553 | 1 | 680 | 1 | 807 | 0 |
| 46 | 1 | 173 | 1 | 300 | 1 | 427 | 0 | 554 | 0 | 681 | 0 | 808 | 1 |
| 47 | 1 | 174 | 0 | 301 | 0 | 428 | 0 | 555 | 0 | 682 | 0 | 809 | 1 |
| 48 | 0 | 175 | 1 | 302 | 1 | 429 | 0 | 556 | 0 | 683 | 0 | 810 | 1 |
| 49 | 0 | 176 | 0 | 303 | 1 | 430 | 0 | 557 | 0 | 684 | 1 | 811 | 0 |
| 50 | 1 | 177 | 1 | 304 | 1 | 431 | 1 | 558 | 0 | 685 | 0 | 812 | 0 |
| 51 | 1 | 178 | 0 | 305 | 1 | 432 | 1 | 559 | 1 | 686 | 0 | 813 | 1 |
| 52 | 1 | 179 | 1 | 306 | 1 | 433 | 1 | 560 | 0 | 687 | 0 | 814 | 1 |
| 53 | 1 | 180 | 1 | 307 | 1 | 434 | 0 | 561 | 0 | 688 | 0 | 815 | 0 |
| 54 | 0 | 181 | 0 | 308 | 0 | 435 | 1 | 562 | 1 | 689 | 1 | 816 | 0 |
| 55 | 0 | 182 | 0 | 309 | 1 | 436 | 0 | 563 | 1 | 690 | 0 | 817 | 0 |
| 56 | 0 | 183 | 1 | 310 | 0 | 437 | 1 | 564 | 0 | 691 | 1 | 818 | 1 |
| 57 | 1 | 184 | 0 | 311 | 1 | 438 | 0 | 565 | 1 | 692 | 0 | 819 | 0 |
| 58 | 1 | 185 | 0 | 312 | 0 | 439 | 1 | 566 | 1 | 693 | 0 | 820 | 0 |
| 59 | 1 | 186 | 1 | 313 | 1 | 440 | 0 | 567 | 1 | 694 | 0 | 821 | 0 |
| 60 | 0 | 187 | 0 | 314 | 0 | 441 | 1 | 568 | 1 | 695 | 1 | 822 | 0 |
| 61 | 0 | 188 | 0 | 315 | 1 | 442 | 1 | 569 | 1 | 696 | 1 | 823 | 1 |
| 62 | 0 | 189 | 0 | 316 | 0 | 443 | 1 | 570 | 1 | 697 | 0 | 824 | 0 |
| 63 | 0 | 190 | 0 | 317 | 0 | 444 | 1 | 571 | 0 | 698 | 0 | 825 | 1 |
| 64 | 0 | 191 | 0 | 318 | 0 | 445 | 1 | 572 | 1 | 699 | 0 | 826 | 1 |
| 65 | 0 | 192 | 0 | 319 | 0 | 446 | 1 | 573 | 1 | 700 | 1 | 827 | 0 |
| 66 | 0 | 193 | 0 | 320 | 0 | 447 | 0 | 574 | 1 | 701 | 1 | 828 | 1 |
| 67 | 0 | 194 | 1 | 321 | 0 | 448 | 1 | 575 | 1 | 702 | 1 | 829 | 1 |
| 68 | 0 | 195 | 1 | 322 | 0 | 449 | 1 | 576 | 0 | 703 | 0 | 830 | 0 |
| 69 | 0 | 196 | 0 | 323 | 1 | 450 | 0 | 577 | 0 | 704 | 1 | 831 | 1 |
| 70 | 0 | 197 | 1 | 324 | 1 | 451 | 1 | 578 | 0 | 705 | 1 | 832 | 0 |
| 71 | 1 | 198 | 0 | 325 | 1 | 452 | 0 | 579 | 0 | 706 | 1 | 833 | 1 |
| 72 | 1 | 199 | 0 | 326 | 0 | 453 | 0 | 580 | 0 | 707 | 0 | 834 | 1 |
| 73 | 1 | 200 | 0 | 327 | 1 | 454 | 1 | 581 | 1 | 708 | 0 | 835 | 0 |
| 74 | 1 | 201 | 0 | 328 | 0 | 455 | 0 | 582 | 1 | 709 | 0 | 836 | 1 |
| 75 | 0 | 202 | 0 | 329 | 1 | 456 | 0 | 583 | 1 | 710 | 1 | 837 | 0 |
| 76 | 1 | 203 | 1 | 330 | 1 | 457 | 0 | 584 | 0 | 711 | 1 | 838 | 0 |
| 77 | 1 | 204 | 1 | 331 | 0 | 458 | 1 | 585 | 1 | 712 | 0 | 839 | 1 |
| 78 | 0 | 205 | 1 | 332 | 1 | 459 | 1 | 586 | 0 | 713 | 1 | 840 | 1 |
| 79 | 0 | 206 | 1 | 333 | 1 | 460 | 0 | 587 | 1 | 714 | 1 | 841 | 0 |
| 80 | 0 | 207 | 1 | 334 | 1 | 461 | 1 | 588 | 0 | 715 | 0 | 842 | 0 |
| 81 | 1 | 208 | 1 | 335 | 1 | 462 | 0 | 589 | 1 | 716 | 0 | 843 | 1 |
| 82 | 0 | 209 | 0 | 336 | 1 | 463 | 1 | 590 | 0 | 717 | 1 | 844 | 0 |
| 83 | 1 | 210 | 1 | 337 | 0 | 464 | 0 | 591 | 1 | 718 | 0 | 845 | 0 |
| 84 | 0 | 211 | 0 | 338 | 1 | 465 | 0 | 592 | 1 | 719 | 0 | 846 | 0 |

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 85 | 1 | 212 | 1 | 339 | 1 | 466 | 0 | 593 | 0 | 720 | 0 | 847 | 0 |
| 86 | 0 | 213 | 0 | 340 | 1 | 467 | 0 | 594 | 0 | 721 | 0 | 848 | 1 |
| 87 | 1 | 214 | 0 | 341 | 1 | 468 | 0 | 595 | 0 | 722 | 1 | 849 | 1 |
| 88 | 0 | 215 | 1 | 342 | 1 | 469 | 0 | 596 | 0 | 723 | 1 | 850 | 1 |
| 89 | 1 | 216 | 0 | 343 | 0 | 470 | 0 | 597 | 1 | 724 | 1 | 851 | 1 |
| 90 | 1 | 217 | 1 | 344 | 1 | 471 | 1 | 598 | 0 | 725 | 1 | 852 | 0 |
| 91 | 1 | 218 | 0 | 345 | 1 | 472 | 1 | 599 | 1 | 726 | 1 | 853 | 0 |
| 92 | 1 | 219 | 1 | 346 | 0 | 473 | 1 | 600 | 0 | 727 | 1 | 854 | 1 |
| 93 | 0 | 220 | 0 | 347 | 0 | 474 | 0 | 601 | 1 | 728 | 0 | 855 | 0 |
| 94 | 0 | 221 | 0 | 348 | 0 | 475 | 1 | 602 | 0 | 729 | 0 | 856 | 0 |
| 95 | 0 | 222 | 1 | 349 | 1 | 476 | 1 | 603 | 1 | 730 | 0 | 857 | 0 |
| 96 | 1 | 223 | 0 | 350 | 1 | 477 | 0 | 604 | 0 | 731 | 1 | 858 | 0 |
| 97 | 0 | 224 | 0 | 351 | 1 | 478 | 0 | 605 | 0 | 732 | 1 | 859 | 1 |
| 98 | 1 | 225 | 0 | 352 | 1 | 479 | 1 | 606 | 1 | 733 | 0 | 860 | 0 |
| 99 | 1 | 226 | 0 | 353 | 1 | 480 | 0 | 607 | 1 | 734 | 1 | 861 | 1 |
| 100 | 1 | 227 | 0 | 354 | 0 | 481 | 1 | 608 | 0 | 735 | 0 | 862 | 1 |
| 101 | 0 | 228 | 0 | 355 | 1 | 482 | 1 | 609 | 0 | 736 | 0 | 863 | 1 |
| 102 | 1 | 229 | 1 | 356 | 1 | 483 | 1 | 610 | 0 | 737 | 1 | 864 | 1 |
| 103 | 0 | 230 | 0 | 357 | 1 | 484 | 0 | 611 | 1 | 738 | 0 | 865 | 1 |
| 104 | 0 | 231 | 1 | 358 | 1 | 485 | 0 | 612 | 0 | 739 | 0 | 866 | 0 |
| 105 | 0 | 232 | 1 | 359 | 1 | 486 | 1 | 613 | 1 | 740 | 0 | 867 | 0 |
| 106 | 1 | 233 | 1 | 360 | 0 | 487 | 1 | 614 | 0 | 741 | 0 | 868 | 0 |
| 107 | 1 | 234 | 1 | 361 | 1 | 488 | 1 | 615 | 0 | 742 | 1 | 869 | 1 |
| 108 | 0 | 235 | 0 | 362 | 1 | 489 | 0 | 616 | 1 | 743 | 1 | 870 | 0 |
| 109 | 1 | 236 | 0 | 363 | 1 | 490 | 1 | 617 | 1 | 744 | 0 | 871 | 0 |
| 110 | 0 | 237 | 1 | 364 | 1 | 491 | 1 | 618 | 1 | 745 | 1 | 872 | 1 |
| 111 | 0 | 238 | 0 | 365 | 0 | 492 | 0 | 619 | 1 | 746 | 1 | 873 | 0 |
| 112 | 0 | 239 | 1 | 366 | 0 | 493 | 0 | 620 | 0 | 747 | 1 | 874 | 0 |
| 113 | 0 | 240 | 1 | 367 | 1 | 494 | 1 | 621 | 1 | 748 | 0 | 875 | 0 |
| 114 | 0 | 241 | 1 | 368 | 0 | 495 | 1 | 622 | 1 | 749 | 1 | 876 | 0 |
| 115 | 1 | 242 | 0 | 369 | 0 | 496 | 1 | 623 | 1 | 750 | 0 | 877 | 1 |
| 116 | 1 | 243 | 0 | 370 | 0 | 497 | 1 | 624 | 1 | 751 | 1 | 878 | 1 |
| 117 | 0 | 244 | 1 | 371 | 1 | 498 | 1 | 625 | 0 | 752 | 0 | 879 | 0 |
| 118 | 1 | 245 | 1 | 372 | 0 | 499 | 1 | 626 | 1 | 753 | 1 | 880 | 1 |
| 119 | 0 | 246 | 1 | 373 | 0 | 500 | 0 | 627 | 1 | 754 | 0 | 881 | 0 |
| 120 | 1 | 247 | 0 | 374 | 1 | 501 | 0 | 628 | 0 | 755 | 0 | 882 | 1 |
| 121 | 1 | 248 | 1 | 375 | 1 | 502 | 1 | 629 | 1 | 756 | 0 | 883 | 1 |
| 122 | 1 | 249 | 1 | 376 | 0 | 503 | 1 | 630 | 1 | 757 | 1 | 884 | 0 |
| 123 | 1 | 250 | 0 | 377 | 1 | 504 | 1 | 631 | 1 | 758 | 0 | 885 | 1 |
| 124 | 1 | 251 | 1 | 378 | 1 | 505 | 1 | 632 | 1 | 759 | 1 | 886 | 1 |
| 125 | 0 | 252 | 0 | 379 | 1 | 506 | 0 | 633 | 1 | 760 | 0 | 887 | 0 |
| 126 | 0 | 253 | 0 | 380 | 0 | 507 | 1 | 634 | 0 | 761 | 1 | | |

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