PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a, b) = (z, z^6 + z^2 + 1)$

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Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a, b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a, b) in $\mathbb{F}_2[x] \setminus \mathbb{F}_2$. In this paper we prove the conjecture for $(a, b) = (z, z^6 + z^2 + 1)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \ldots)$ of elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{\ddots + \cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a, b)-Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \ldots) = (a, b, b, a, b, a, a, b, \ldots)$$

that we shall denote

$$\mathrm{CF}(a, b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When* $(a,b)=(z,z^6+z^2+1)$*, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

where, for $\mathrm{CF}(a,b)$*,*

$$\begin{aligned}
p_0(z) &= z^{24}+z^{21}+z^{19}+z^{17}+z^{14}+z^{11}+z^9+z^8+z^7+z^6+z^5+z^3+z^2+z\\
&\quad +1,\\
p_1(z) &= z^{27}+z^{25}+z^{22}+z^{20}+z^{18}+z^{15}+z^{12}+z^{11}+z^{10}+z^9+z^8+z^7+z^6\\
&\quad +z^4+z^2+z,\\
p_2(z) &= z^{28}+z^{26}+z^{16}+z^{12}+z^{10}+z^8+z^2,\\
p_3(z) &= z^{27}+z^{25}+z^{22}+z^{20}+z^{18}+z^{15}+z^{12}+z^{11}+z^{10}+z^9+z^8+z^7+z^6\\
&\quad +z^4+z^2+z,\\
p_4(z) &= z^{26}+z^{21}+z^{19}+z^{17}+z^{16}+z^{14}+z^{11}+z^{10}+z^9+z^8+z^7+z^6+z^5\\
&\quad +z^3+z,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$*,*

$$\begin{aligned}
p_0(z) &= z^{21}+z^{19}+z^{17}+z^{14}+z^{11}+z^9+z^7+z^6+z^5+z^4+z^3+z^2+z,\\
p_1(z) &= z^{27}+z^{25}+z^{22}+z^{20}+z^{18}+z^{15}+z^{12}+z^{11}+z^{10}+z^9+z^8+z^7+z^6\\
&\quad +z^4+z^2+z,\\
p_2(z) &= z^{28}+z^{26}+z^{16}+z^{12}+z^{10}+z^8+z^2,\\
p_3(z) &= z^{27}+z^{25}+z^{22}+z^{20}+z^{18}+z^{15}+z^{12}+z^{11}+z^{10}+z^9+z^8+z^7+z^6\\
&\quad +z^4+z^2+z,\\
p_4(z) &= z^{26}+z^{24}+z^{21}+z^{19}+z^{17}+z^{16}+z^{14}+z^{11}+z^{10}+z^9+z^7+z^6+z^5\\
&\quad +z^4+z^3+z+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x) &= x^{\deg(t_{2^n-1})}\begin{pmatrix} t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad \cdots\times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x) &= x^{\deg(\bar t_{2^n-1})}\begin{pmatrix} \bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix} \bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times
\end{aligned}$$

$$\cdots \times x^{\deg(\bar{t}_0)} \begin{pmatrix} \bar{t}_0(1/x) & 1 \\ 1 & 0 \end{pmatrix},$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n \geq 0$

$$M_{n+1}(x) = W_n(x) \cdot M_n(x),$$
$$W_{n+1}(x) = M_n(x) \cdot W_n(x).$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n} \tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n} \tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \leq i,j \leq 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \leq i,j \leq 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \leq i,j \leq 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(175, 175, 175, 175, 175)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \geq 0$. If for some $n \geq 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \geq 0$, $q_1(0) = 1$, and $q_j(0) = 0$

for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0} M^o_{0,0}$ and $W^o_{0,1} M^o_{1,0}$. We know that

$$P(x, y) = \mathrm{Res}_z \left(\phi(W^o, 0, 0)(x, z),\ z^{12} \cdot \phi(M^o, 0, 0)(x, y/z) \right)$$

is an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0} M^o_{0,0}$, that will be called $\phi_0(x, y)$. To prove that $\phi_0(x, y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x, y)$ of multiplicity m and that $Q(x, y) := P(x, y)/\phi_0(x, y)^m$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0} M^o_{0,0}$ to order 630 and substitute it for y in $Q(x, y)$. We get a series of valuation less than 630, which proves that $Q(x, y)$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We find the minimal polynomial $\phi_1(x, y)$ of $W^o_{0,1} M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e, 0, 0)$ is the minimal polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. We know that

$$S(x, y) = \mathrm{Res}_z \left(\phi_0(x, z),\ \phi_1(x, y + z) \right)$$

is an annihilating of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. We verify that $\phi(M^e, 0, 0)$ is an irreducible factor of $S(x, y)$ of multiplicity μ, and that $Q(x, y) := S(x, y)/\phi(M^e, 0, 0)^\mu$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$ to order 770 and substitute it for y in $Q(x, y)$. We get a series of valuation less than 770, and therefore $Q(x, y)$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e, 0, 0)$, we know that the two series are one and the same. $\square$

Define

$$R^e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned} M^e[:14u] = {} & M^e[:7u] + x^{2u} M^e[:5u] + x^{6u} M^e[:u] + x^{2u} M^e[:7u] + x^{4u} M^e[:5u] \\ & + x^{8u} M^e[:u] + x^{4u} M^e[:7u] + x^{6u} M^e[:5u] + x^{10u} M^e[:u] \end{aligned}$$

$$
\begin{aligned}
&+ x^{8u}M^e[:6u] + x^{10u}M^e[:4u] + (x^{7u}+x^{9u}+x^{11u})R^e,\\
W^e[:14u] &= W^e[:7u] + x^{2u}W^e[:5u] + x^{6u}W^e[:u] + x^{2u}W^e[:7u] + x^{4u}W^e[:5u]\\
&+ x^{8u}W^e[:u] + x^{4u}W^e[:7u] + x^{6u}W^e[:5u] + x^{10u}W^e[:u]\\
&+ x^{8u}W^e[:6u] + x^{10u}W^e[:4u] + (x^{7u}+x^{9u}+x^{11u})R^e.
\end{aligned}
$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$
\begin{aligned}
M^o[:14u] &= M^o[:7u] + x^{2u}M^o[:5u] + x^{6u}M^o[:u] + x^{2u}M^o[:7u] + x^{4u}M^o[:5u]\\
&+ x^{8u}M^o[:u] + x^{4u}M^o[:7u] + x^{6u}M^o[:5u] + x^{10u}M^o[:u]\\
&+ x^{8u}M^o[:6u] + x^{10u}M^o[:4u] + (x^{7u}+x^{9u}+x^{11u})R^o,\\
W^o[:14u] &= W^o[:7u] + x^{2u}W^o[:5u] + x^{6u}W^o[:u] + x^{2u}W^o[:7u] + x^{4u}W^o[:5u]\\
&+ x^{8u}W^o[:u] + x^{4u}W^o[:7u] + x^{6u}W^o[:5u] + x^{10u}W^o[:u]\\
&+ x^{8u}W^o[:6u] + x^{10u}W^o[:4u] + (x^{7u}+x^{9u}+x^{11u})R^o.
\end{aligned}
$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$
\begin{aligned}
T[:14u] = {}& T[:7u] + x^{2u}T[:5u] + x^{6u}T[:u] + x^{2u}T[:7u] + x^{4u}T[:5u] + x^{8u}T[:u]\\
&+ x^{4u}T[:7u] + x^{6u}T[:5u] + x^{10u}T[:u] + x^{8u}T[:6u] + x^{10u}T[:4u].
\end{aligned}
$$

The proof of the other 15 cases are similar. We break down the above identity into 7 parts:

$$
\begin{aligned}
0 &= x^{6u}T[u:2u] + x^{2u}T[5u:6u] + T[7u:8u],\\
0 &= x^{6u}T[2u:3u] + x^{2u}T[6u:7u] + T[8u:9u],\\
0 &= x^{8u}T[u:2u] + x^{6u}T[3u:4u] + x^{4u}T[5u:6u] + T[9u:10u],\\
0 &= x^{8u}T[2u:3u] + x^{6u}T[4u:5u] + x^{4u}T[6u:7u] + T[10u:11u],\\
0 &= x^{10u}T[u:2u] + x^{8u}T[3u:4u] + T[11u:12u],\\
0 &= x^{10u}T[2u:3u] + x^{8u}T[4u:5u] + T[12u:13u],\\
0 &= x^{10u}T[3u:4u] + x^{8u}T[5u:6u] + T[13u:14u],
\end{aligned}
$$

which can be rewritten as

$$0 = T[[1w]_2] + T[[101w]_2] + T[[111w]_2], \tag{2.7}$$

$$0 = T[[10w]_2] + T[[110w]_2] + T[[1000w]_2], \tag{2.8}$$

$$0 = T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1001w]_2], \tag{2.9}$$

$$0 = T[[10w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1010w]_2], \tag{2.10}$$

$$0 = T[[1w]_2] + T[[11w]_2] + T[[1011w]_2], \tag{2.11}$$

$$0 = T[[10w]_2] + T[[100w]_2] + T[[1100w]_2], \tag{2.12}$$

$$0 = T[[11w]_2] + T[[101w]_2] + T[[1101w]_2], \tag{2.13}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$0 = T[[1w]_2] + T[[101w]_2] + T[[111w]_2], \tag{2.14}$$

$$0 = T[[10w]_2] + T[[110w]_2] + T[[1000w]_2], \tag{2.15}$$
$$0 = T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1001w]_2], \tag{2.16}$$
$$0 = T[[10w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1010w]_2], \tag{2.17}$$
$$0 = T[[1w]_2] + T[[11w]_2] + T[[1011w]_2], \tag{2.18}$$
$$0 = T[[10w]_2] + T[[100w]_2] + T[[1100w]_2], \tag{2.19}$$
$$0 = T[[11w]_2] + T[[101w]_2] + T[[1101w]_2], \tag{2.20}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 606 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.20) can be written as

$$0 = \tau(A(s, 1)) + \tau(A(s, 101)) + \tau(A(s, 111)), \tag{2.21}$$
$$0 = \tau(A(s, 10)) + \tau(A(s, 110)) + \tau(A(s, 1000)), \tag{2.22}$$
$$0 = \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 101)) + \tau(A(s, 1001)), \tag{2.23}$$
$$0 = \tau(A(s, 10)) + \tau(A(s, 100)) + \tau(A(s, 110)) + \tau(A(s, 1010)), \tag{2.24}$$
$$0 = \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 1011)), \tag{2.25}$$
$$0 = \tau(A(s, 10)) + \tau(A(s, 100)) + \tau(A(s, 1100)), \tag{2.26}$$
$$0 = \tau(A(s, 11)) + \tau(A(s, 101)) + \tau(A(s, 1101)), \tag{2.27}$$

for all $s \in E_{2k}$, and

$$0 = \tau(A(s, 1)) + \tau(A(s, 101)) + \tau(A(s, 111)), \tag{2.28}$$
$$0 = \tau(A(s, 10)) + \tau(A(s, 110)) + \tau(A(s, 1000)), \tag{2.29}$$
$$0 = \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 101)) + \tau(A(s, 1001)), \tag{2.30}$$
$$0 = \tau(A(s, 10)) + \tau(A(s, 100)) + \tau(A(s, 110)) + \tau(A(s, 1010)), \tag{2.31}$$
$$0 = \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 1011)), \tag{2.32}$$
$$0 = \tau(A(s, 10)) + \tau(A(s, 100)) + \tau(A(s, 1100)), \tag{2.33}$$
$$0 = \tau(A(s, 11)) + \tau(A(s, 101)) + \tau(A(s, 1101)), \tag{2.34}$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{26}), E_{26}) = (A(i, 0^{18}), E_{18})$, so that we only have to verify that identities (2.21) through (2.34) hold for $2 \leq k \leq 13$, which turns out to be true. □

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \geq 2$, and $u = 2^{2k-1}$,*

$$M_{2k} = M^e[:7u] + x^{2u} M^e[:5u] + x^{6u} M^e[:u] + x^{7u} R^e,$$
$$W_{2k} = W^e[:7u] + x^{2u} W^e[:5u] + x^{6u} W^e[:u] + x^{7u} R^e.$$

For all integer $k \geq 2$, and $u = 2^{2k}$,

$$M_{2k+1} = M^o[:7u] + x^{2u} M^o[:5u] + x^{6u} M^o[:u] + x^{7u} R^o,$$

$$W_{2k+1} = W^o[:7u] + x^{2u}W^o[:5u] + x^{6u}W^o[:u] + x^{7u}R^o.$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \geq 2$, we claim that

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1} \wedge W_{2k+1},$$
$$M_{2k+1} \wedge W_{2k+1} \Rightarrow M_{2k+2} \wedge W_{2k+2}.$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$\begin{aligned} W_{2k}M_{2k} = &\big(W^e[:7v] + x^{2v}W^e[:5v] + x^{6v}W^e[:v] + x^{7u}R^e\big) \times \big(M^e[:7v] \\ &+ x^{2v}M^e[:5v] + x^{6v}M^e[:v] + x^{7u}R^e\big); \end{aligned} \tag{2.35}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{14v}R^o$. Therefore we only need to prove that their difference is $O(x^{14v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{14v}, to

$$W^e[:14v]M^e[:14v] + x^{4v}W^e[:10v]M^e[:10v] + x^{12v}W^e[:2v]M^e[:2v].$$

For all $n \leq 14$, replace the occurrences of $W^e[:n\cdot v]$ and $M^e[:n\cdot v]$ in the above expression by the reduction modulo $x^{n\cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n\cdot v : (n+1)\cdot v]/X^n, \\ b_n &:= M^e[n\cdot v : (n+1)\cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.35) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{14}, b_1, ..., b_{14}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{14})$, which completes the proof. □

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \leq j, k \leq 1$,

$$\begin{aligned} \lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k}, \\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k}, \\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k}, \\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}. \end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a, b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e,0,1)$ and $\phi(M^e,0,0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x,y)=\mathrm{Res}_t\left(\phi(M^e,0,1)(x,t),y^{12}\phi(M^e,0,1)(x,t/y)\right)$$

is an annihilating polynomial of $f(x)=M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x,y)=q_4(x)y^4+q_3(x)y^3+q_2(x)y^2+q_1(x)y+q_0(x),$$

where

$$\begin{aligned}
q_0(x)&=x^{28}+x^{27}+x^{26}+x^{25}+x^{23}+x^{22}+x^{21}+x^{20}+x^{19}+x^{17}+x^{14}+x^{11}\\
&\quad+x^9+x^7+x^4,\\
q_1(x)&=x^{27}+x^{26}+x^{24}+x^{22}+x^{21}+x^{20}+x^{19}+x^{18}+x^{17}+x^{16}+x^{13}+x^{10}\\
&\quad+x^8+x^6+x^3+x,\\
q_2(x)&=x^{26}+x^{20}+x^{18}+x^{16}+x^{12}+x^2+1,\\
q_3(x)&=x^{27}+x^{26}+x^{24}+x^{22}+x^{21}+x^{20}+x^{19}+x^{18}+x^{17}+x^{16}+x^{13}+x^{10}\\
&\quad+x^8+x^6+x^3+x,\\
q_4(x)&=x^{27}+x^{25}+x^{23}+x^{22}+x^{21}+x^{20}+x^{19}+x^{18}+x^{17}+x^{14}+x^{12}+x^{11}\\
&\quad+x^9+x^7+x^2.
\end{aligned}$$

The polynomial $Q(x,y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and $R(x,y):=P(x,y)/Q(x,y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 216, and substitute it for y in $R(z,y)$, we get a series with valuation smaller than 216, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b)=f(1/z)$. ☐

3. Annexe

All 16 polynomials are of the form

$$p_0(x)+p_3(x)y^3+p_6(x)y^6+p_9(x)y^9+p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e,0,0)$:

$$\begin{aligned}
p_0(x)&=x^{162}+x^{158}+x^{152}+x^{148}+x^{146}+x^{140}+x^{136}+x^{134}+x^{132}+x^{126}\\
&\quad+x^{116}+x^{100}+x^{90}+x^{88}+x^{86}+x^{82}+x^{76}+x^{74}+x^{72}+x^{66}+x^{58}\\
&\quad+x^{56}+x^{52}+x^{50}+x^{44}+x^{40}+x^{36}+x^{30}+x^{26}+x^{22}+x^{12},\\
p_3(x)&=x^{154}+x^{146}+x^{140}+x^{136}+x^{128}+x^{100}+x^{96}+x^{88}+x^{84}+x^{80}\\
&\quad+x^{74}+x^{72}+x^{68}+x^{64}+x^{60}+x^{58}+x^{52}+x^{42}+x^{40}+x^{36}+x^{34}\\
&\quad+x^{32}+x^{28}+x^{26}+x^{20}+x^8,\\
p_6(x)&=x^{158}+x^{154}+x^{150}+x^{142}+x^{134}+x^{128}+x^{126}+x^{124}+x^{114}+x^{112}\\
&\quad+x^{102}+x^{100}+x^{98}+x^{94}+x^{88}+x^{86}+x^{84}+x^{80}+x^{78}+x^{70}+x^{68}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{58} + x^{48} + x^{44} + x^{38} + x^{34} + x^{30} + x^{20} + x^{18} + x^{14} + x^{10} + x^{4} \\
&+ x^{2} + 1, \\
p_9(x) = {}& x^{160} + x^{158} + x^{152} + x^{148} + x^{146} + x^{144} + x^{142} + x^{138} + x^{134} + x^{130} \\
&+ x^{128} + x^{126} + x^{122} + x^{100} + x^{92} + x^{84} + x^{80} + x^{78} + x^{70} + x^{66} \\
&+ x^{62} + x^{60} + x^{56} + x^{46} + x^{44} + x^{38} + x^{28} + x^{26} + x^{14} + x^{10} + x^{8} \\
&+ x^{6} + x^{4} + x^{2}, \\
p_{12}(x) = {}& x^{160} + x^{144} + x^{128} + x^{112} + x^{96} + x^{80} + x^{32} + x^{16} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0].

For $\phi(M^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) = {}& x^{147} + x^{146} + x^{143} + x^{142} + x^{141} + x^{140} + x^{138} + x^{137} + x^{135} + x^{133} \\
&+ x^{130} + x^{128} + x^{127} + x^{126} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} \\
&+ x^{117} + x^{110} + x^{109} + x^{107} + x^{103} + x^{102} + x^{101} + x^{99} + x^{96} + x^{95} \\
&+ x^{94} + x^{87} + x^{85} + x^{84} + x^{83} + x^{78} + x^{74} + x^{73} + x^{72} + x^{71} + x^{67} \\
&+ x^{63} + x^{60} + x^{59} + x^{58} + x^{57} + x^{54} + x^{52} + x^{48} + x^{47} + x^{45} + x^{44} \\
&+ x^{42} + x^{41} + x^{37} + x^{35} + x^{34} + x^{28} + x^{27} + x^{26} + x^{25} + x^{22} + x^{21} \\
&+ x^{18}, \\
p_3(x) = {}& x^{145} + x^{144} + x^{143} + x^{138} + x^{137} + x^{135} + x^{134} + x^{128} + x^{126} + x^{125} \\
&+ x^{122} + x^{121} + x^{120} + x^{118} + x^{115} + x^{112} + x^{110} + x^{109} + x^{107} \\
&+ x^{104} + x^{103} + x^{98} + x^{96} + x^{94} + x^{90} + x^{85} + x^{79} + x^{78} + x^{77} + x^{75} \\
&+ x^{74} + x^{73} + x^{72} + x^{70} + x^{66} + x^{61} + x^{57} + x^{56} + x^{54} + x^{53} + x^{51} \\
&+ x^{48} + x^{47} + x^{42} + x^{41} + x^{39} + x^{38} + x^{32} + x^{30} + x^{29} + x^{26} + x^{25} \\
&+ x^{24} + x^{22} + x^{19} + x^{16} + x^{14} + x^{13} + x^{11} + x^{9}, \\
p_6(x) = {}& x^{144} + x^{134} + x^{130} + x^{128} + x^{124} + x^{108} + x^{106} + x^{104} + x^{100} + x^{94} \\
&+ x^{92} + x^{90} + x^{86} + x^{78} + x^{70} + x^{68} + x^{66} + x^{62} + x^{58} + x^{54} + x^{52} \\
&+ x^{50} + x^{42} + x^{40} + x^{36} + x^{32} + x^{22} + x^{20} + x^{12} + x^{6}, \\
p_9(x) = {}& x^{143} + x^{142} + x^{140} + x^{139} + x^{137} + x^{135} + x^{133} + x^{132} + x^{130} + x^{128} \\
&+ x^{127} + x^{126} + x^{122} + x^{117} + x^{115} + x^{114} + x^{112} + x^{110} + x^{107} \\
&+ x^{104} + x^{103} + x^{98} + x^{96} + x^{95} + x^{93} + x^{91} + x^{87} + x^{86} + x^{85} + x^{83} \\
&+ x^{82} + x^{81} + x^{80} + x^{78} + x^{74} + x^{71} + x^{70} + x^{65} + x^{63} + x^{62} + x^{60} \\
&+ x^{58} + x^{57} + x^{56} + x^{53} + x^{51} + x^{47} + x^{46} + x^{44} + x^{42} + x^{38} + x^{32} \\
&+ x^{30} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{20} + x^{17} + x^{16} + x^{15} + x^{14} \\
&+ x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{5} + x^{3}, \\
p_{12}(x) = {}& x^{142} + x^{140} + x^{136} + x^{134} + x^{132} + x^{128} + x^{118} + x^{116} + x^{112} + x^{110} \\
&+ x^{108} + x^{104} + x^{102} + x^{100} + x^{96} + x^{94} + x^{92} + x^{88} + x^{70} + x^{68} \\
&+ x^{64} + x^{46} + x^{44} + x^{40} + x^{38} + x^{36} + x^{32} + x^{30} + x^{28} + x^{24} + x^{6}
\end{aligned}
$$

$$+ x^4 + 1,$$

and the initial terms are [0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 1, 1, 0].

For $\phi(M^e, 1, 0)$:

$$\begin{aligned}
p_0(x) = x^{147} &+ x^{146} + x^{143} + x^{142} + x^{141} + x^{140} + x^{138} + x^{137} + x^{135} + x^{133} \\
&+ x^{130} + x^{128} + x^{127} + x^{126} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} \\
&+ x^{117} + x^{110} + x^{109} + x^{107} + x^{103} + x^{102} + x^{101} + x^{99} + x^{96} + x^{95} \\
&+ x^{94} + x^{87} + x^{85} + x^{84} + x^{83} + x^{78} + x^{74} + x^{73} + x^{72} + x^{71} + x^{67} \\
&+ x^{63} + x^{60} + x^{59} + x^{58} + x^{57} + x^{54} + x^{52} + x^{48} + x^{47} + x^{45} + x^{44} \\
&+ x^{42} + x^{41} + x^{37} + x^{35} + x^{34} + x^{28} + x^{27} + x^{26} + x^{25} + x^{22} + x^{21} \\
&+ x^{18}, \\
p_3(x) = x^{145} &+ x^{144} + x^{143} + x^{138} + x^{137} + x^{135} + x^{134} + x^{128} + x^{126} + x^{125} \\
&+ x^{122} + x^{121} + x^{120} + x^{118} + x^{115} + x^{112} + x^{110} + x^{109} + x^{107} \\
&+ x^{104} + x^{103} + x^{98} + x^{96} + x^{94} + x^{90} + x^{85} + x^{79} + x^{78} + x^{77} + x^{75} \\
&+ x^{74} + x^{73} + x^{72} + x^{70} + x^{66} + x^{61} + x^{57} + x^{56} + x^{54} + x^{53} + x^{51} \\
&+ x^{48} + x^{47} + x^{42} + x^{41} + x^{39} + x^{38} + x^{32} + x^{30} + x^{29} + x^{26} + x^{25} \\
&+ x^{24} + x^{22} + x^{19} + x^{16} + x^{14} + x^{13} + x^{11} + x^{9}, \\
p_6(x) = x^{144} &+ x^{134} + x^{130} + x^{128} + x^{124} + x^{108} + x^{106} + x^{104} + x^{100} + x^{94} \\
&+ x^{92} + x^{90} + x^{86} + x^{78} + x^{70} + x^{68} + x^{66} + x^{62} + x^{58} + x^{54} + x^{52} \\
&+ x^{50} + x^{42} + x^{40} + x^{36} + x^{32} + x^{22} + x^{20} + x^{12} + x^{6}, \\
p_9(x) = x^{143} &+ x^{142} + x^{140} + x^{139} + x^{137} + x^{135} + x^{133} + x^{132} + x^{130} + x^{128} \\
&+ x^{127} + x^{126} + x^{122} + x^{117} + x^{115} + x^{114} + x^{112} + x^{110} + x^{107} \\
&+ x^{104} + x^{103} + x^{98} + x^{96} + x^{95} + x^{93} + x^{91} + x^{87} + x^{86} + x^{85} + x^{83} \\
&+ x^{82} + x^{81} + x^{80} + x^{78} + x^{74} + x^{71} + x^{70} + x^{65} + x^{63} + x^{62} + x^{60} \\
&+ x^{58} + x^{57} + x^{56} + x^{53} + x^{51} + x^{47} + x^{46} + x^{44} + x^{42} + x^{38} + x^{32} \\
&+ x^{30} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{20} + x^{17} + x^{16} + x^{15} + x^{14} \\
&+ x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{5} + x^{3}, \\
p_{12}(x) = x^{142} &+ x^{140} + x^{136} + x^{134} + x^{132} + x^{128} + x^{118} + x^{116} + x^{112} + x^{110} \\
&+ x^{108} + x^{104} + x^{102} + x^{100} + x^{96} + x^{94} + x^{92} + x^{88} + x^{70} + x^{68} \\
&+ x^{64} + x^{46} + x^{44} + x^{40} + x^{38} + x^{36} + x^{32} + x^{30} + x^{28} + x^{24} + x^{6} \\
&+ x^{4} + 1,
\end{aligned}$$

and the initial terms are [0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 1, 1, 0].

For $\phi(M^e, 1, 1)$:

$$\begin{aligned}
p_0(x) = x^{132} &+ x^{130} + x^{128} + x^{126} + x^{122} + x^{116} + x^{114} + x^{112} + x^{110} + x^{106} \\
&+ x^{104} + x^{98} + x^{94} + x^{92} + x^{90} + x^{82} + x^{80} + x^{78} + x^{76} + x^{72} + x^{68} \\
&+ x^{66} + x^{62} + x^{58} + x^{56} + x^{52} + x^{50} + x^{48} + x^{46} + x^{44} + x^{42} + x^{34}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{32} + x^{30} + x^{24},\\
p_3(x) &= x^{126} + x^{124} + x^{120} + x^{110} + x^{108} + x^{106} + x^{104} + x^{100} + x^{98} + x^{94}\\
&+ x^{92} + x^{90} + x^{86} + x^{80} + x^{76} + x^{74} + x^{72} + x^{70} + x^{66} + x^{60} + x^{58}\\
&+ x^{56} + x^{54} + x^{52} + x^{50} + x^{48} + x^{44} + x^{36} + x^{30} + x^{28} + x^{24} + x^{20}\\
&+ x^{18} + x^{10} + x^{8} + x^{6},\\
p_6(x) &= x^{126} + x^{122} + x^{120} + x^{118} + x^{116} + x^{114} + x^{112} + x^{108} + x^{104} + x^{102}\\
&+ x^{90} + x^{84} + x^{82} + x^{78} + x^{76} + x^{74} + x^{70} + x^{66} + x^{60} + x^{48} + x^{46}\\
&+ x^{44} + x^{40} + x^{38} + x^{36} + x^{30} + x^{28} + x^{26} + x^{18} + x^{16} + x^{10} + x^{6}\\
&+ x^{2} + 1,\\
p_9(x) &= x^{124} + x^{122} + x^{120} + x^{118} + x^{116} + x^{112} + x^{108} + x^{106} + x^{104} + x^{102}\\
&+ x^{98} + x^{94} + x^{82} + x^{80} + x^{76} + x^{72} + x^{70} + x^{68} + x^{66} + x^{64} + x^{60}\\
&+ x^{58} + x^{56} + x^{52} + x^{46} + x^{38} + x^{28} + x^{20} + x^{14} + x^{12} + x^{8} + x^{6} + x^{4}\\
&+ x^{2},\\
p_{12}(x) &= x^{124} + x^{120} + x^{112} + x^{92} + x^{88} + x^{80} + x^{76} + x^{72} + x^{64} + x^{28} + x^{24}\\
&+ x^{16} + x^{12} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{132} + x^{130} + x^{126} + x^{120} + x^{116} + x^{106} + x^{104} + x^{100} + x^{92} + x^{88}\\
&+ x^{82} + x^{72} + x^{70} + x^{62} + x^{58} + x^{50} + x^{42} + x^{36} + x^{34} + x^{30} + x^{26}\\
&+ x^{24} + x^{22} + x^{16} + x^{12},\\
p_3(x) &= x^{126} + x^{124} + x^{122} + x^{116} + x^{114} + x^{110} + x^{106} + x^{104} + x^{102} + x^{96}\\
&+ x^{94} + x^{88} + x^{84} + x^{82} + x^{78} + x^{74} + x^{72} + x^{66} + x^{64} + x^{62} + x^{60}\\
&+ x^{58} + x^{56} + x^{54} + x^{48} + x^{46} + x^{38} + x^{30} + x^{28} + x^{22} + x^{20} + x^{16}\\
&+ x^{10} + x^{8},\\
p_6(x) &= x^{126} + x^{120} + x^{118} + x^{104} + x^{98} + x^{92} + x^{86} + x^{84} + x^{76} + x^{74} + x^{68}\\
&+ x^{66} + x^{64} + x^{62} + x^{52} + x^{50} + x^{44} + x^{40} + x^{36} + x^{30} + x^{28} + x^{26}\\
&+ x^{24} + x^{22} + x^{20} + x^{16} + x^{10} + x^{4} + x^{2} + 1,\\
p_9(x) &= x^{124} + x^{122} + x^{120} + x^{118} + x^{116} + x^{112} + x^{108} + x^{106} + x^{104} + x^{102}\\
&+ x^{98} + x^{94} + x^{82} + x^{80} + x^{76} + x^{72} + x^{70} + x^{68} + x^{66} + x^{64} + x^{60}\\
&+ x^{58} + x^{56} + x^{52} + x^{46} + x^{38} + x^{28} + x^{20} + x^{14} + x^{12} + x^{8} + x^{6} + x^{4}\\
&+ x^{2},\\
p_{12}(x) &= x^{124} + x^{120} + x^{112} + x^{92} + x^{88} + x^{80} + x^{76} + x^{72} + x^{64} + x^{28} + x^{24}\\
&+ x^{16} + x^{12} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 1)$:

$$\begin{aligned}
p_0(x) = {} & x^{147} + x^{145} + x^{144} + x^{140} + x^{138} + x^{137} + x^{136} + x^{135} + x^{134} + x^{133} \\
& + x^{132} + x^{131} + x^{126} + x^{125} + x^{123} + x^{121} + x^{120} + x^{117} + x^{114} \\
& + x^{110} + x^{107} + x^{105} + x^{104} + x^{99} + x^{95} + x^{94} + x^{93} + x^{89} + x^{88} \\
& + x^{85} + x^{81} + x^{78} + x^{77} + x^{74} + x^{73} + x^{70} + x^{68} + x^{64} + x^{62} + x^{61} \\
& + x^{60} + x^{57} + x^{56} + x^{51} + x^{50} + x^{49} + x^{41} + x^{39} + x^{36} + x^{34} + x^{32} \\
& + x^{31} + x^{27}, \\
p_3(x) = {} & x^{145} + x^{144} + x^{143} + x^{138} + x^{137} + x^{135} + x^{134} + x^{128} + x^{126} + x^{125} \\
& + x^{122} + x^{121} + x^{120} + x^{118} + x^{115} + x^{112} + x^{110} + x^{109} + x^{107} \\
& + x^{104} + x^{103} + x^{98} + x^{96} + x^{94} + x^{90} + x^{85} + x^{79} + x^{78} + x^{77} + x^{75} \\
& + x^{74} + x^{73} + x^{72} + x^{70} + x^{66} + x^{61} + x^{57} + x^{56} + x^{54} + x^{53} + x^{51} \\
& + x^{48} + x^{47} + x^{42} + x^{41} + x^{39} + x^{38} + x^{32} + x^{30} + x^{29} + x^{26} + x^{25} \\
& + x^{24} + x^{22} + x^{19} + x^{16} + x^{14} + x^{13} + x^{11} + x^{9}, \\
p_6(x) = {} & x^{144} + x^{134} + x^{130} + x^{128} + x^{124} + x^{108} + x^{106} + x^{104} + x^{100} + x^{94} \\
& + x^{92} + x^{90} + x^{86} + x^{78} + x^{70} + x^{68} + x^{66} + x^{62} + x^{58} + x^{54} + x^{52} \\
& + x^{50} + x^{42} + x^{40} + x^{36} + x^{32} + x^{22} + x^{20} + x^{12} + x^{6}, \\
p_9(x) = {} & x^{143} + x^{142} + x^{140} + x^{139} + x^{137} + x^{135} + x^{133} + x^{132} + x^{130} + x^{128} \\
& + x^{127} + x^{126} + x^{122} + x^{117} + x^{115} + x^{114} + x^{112} + x^{110} + x^{107} \\
& + x^{104} + x^{103} + x^{98} + x^{96} + x^{95} + x^{93} + x^{91} + x^{87} + x^{86} + x^{85} + x^{83} \\
& + x^{82} + x^{81} + x^{80} + x^{78} + x^{74} + x^{71} + x^{70} + x^{65} + x^{63} + x^{62} + x^{60} \\
& + x^{58} + x^{57} + x^{56} + x^{53} + x^{51} + x^{47} + x^{46} + x^{44} + x^{42} + x^{38} + x^{32} \\
& + x^{30} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{20} + x^{17} + x^{16} + x^{15} + x^{14} \\
& + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{5} + x^{3}, \\
p_{12}(x) = {} & x^{142} + x^{140} + x^{136} + x^{134} + x^{132} + x^{128} + x^{118} + x^{116} + x^{112} + x^{110} \\
& + x^{108} + x^{104} + x^{102} + x^{100} + x^{96} + x^{94} + x^{92} + x^{88} + x^{70} + x^{68} \\
& + x^{64} + x^{46} + x^{44} + x^{40} + x^{38} + x^{36} + x^{32} + x^{30} + x^{28} + x^{24} + x^{6} \\
& + x^{4} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 1, 1, 0].

For $\phi(W^e, 1, 0)$:

$$\begin{aligned}
p_0(x) = {} & x^{147} + x^{145} + x^{144} + x^{140} + x^{138} + x^{137} + x^{136} + x^{135} + x^{134} + x^{133} \\
& + x^{132} + x^{131} + x^{126} + x^{125} + x^{123} + x^{121} + x^{120} + x^{117} + x^{114} \\
& + x^{110} + x^{107} + x^{105} + x^{104} + x^{99} + x^{95} + x^{94} + x^{93} + x^{89} + x^{88} \\
& + x^{85} + x^{81} + x^{78} + x^{77} + x^{74} + x^{73} + x^{70} + x^{68} + x^{64} + x^{62} + x^{61} \\
& + x^{60} + x^{57} + x^{56} + x^{51} + x^{50} + x^{49} + x^{41} + x^{39} + x^{36} + x^{34} + x^{32} \\
& + x^{31} + x^{27},
\end{aligned}$$

$$
\begin{aligned}
p_3(x) = {} & x^{145} + x^{144} + x^{143} + x^{138} + x^{137} + x^{135} + x^{134} + x^{128} + x^{126} + x^{125} \\
& + x^{122} + x^{121} + x^{120} + x^{118} + x^{115} + x^{112} + x^{110} + x^{109} + x^{107} \\
& + x^{104} + x^{103} + x^{98} + x^{96} + x^{94} + x^{90} + x^{85} + x^{79} + x^{78} + x^{77} + x^{75} \\
& + x^{74} + x^{73} + x^{72} + x^{70} + x^{66} + x^{61} + x^{57} + x^{56} + x^{54} + x^{53} + x^{51} \\
& + x^{48} + x^{47} + x^{42} + x^{41} + x^{39} + x^{38} + x^{32} + x^{30} + x^{29} + x^{26} + x^{25} \\
& + x^{24} + x^{22} + x^{19} + x^{16} + x^{14} + x^{13} + x^{11} + x^{9}, \\
p_6(x) = {} & x^{144} + x^{134} + x^{130} + x^{128} + x^{124} + x^{108} + x^{106} + x^{104} + x^{100} + x^{94} \\
& + x^{92} + x^{90} + x^{86} + x^{78} + x^{70} + x^{68} + x^{66} + x^{62} + x^{58} + x^{54} + x^{52} \\
& + x^{50} + x^{42} + x^{40} + x^{36} + x^{32} + x^{22} + x^{20} + x^{12} + x^{6}, \\
p_9(x) = {} & x^{143} + x^{142} + x^{140} + x^{139} + x^{137} + x^{135} + x^{133} + x^{132} + x^{130} + x^{128} \\
& + x^{127} + x^{126} + x^{122} + x^{117} + x^{115} + x^{114} + x^{112} + x^{110} + x^{107} \\
& + x^{104} + x^{103} + x^{98} + x^{96} + x^{95} + x^{93} + x^{91} + x^{87} + x^{86} + x^{85} + x^{83} \\
& + x^{82} + x^{81} + x^{80} + x^{78} + x^{74} + x^{71} + x^{70} + x^{65} + x^{63} + x^{62} + x^{60} \\
& + x^{58} + x^{57} + x^{56} + x^{53} + x^{51} + x^{47} + x^{46} + x^{44} + x^{42} + x^{38} + x^{32} \\
& + x^{30} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{20} + x^{17} + x^{16} + x^{15} + x^{14} \\
& + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{5} + x^{3}, \\
p_{12}(x) = {} & x^{142} + x^{140} + x^{136} + x^{134} + x^{132} + x^{128} + x^{118} + x^{116} + x^{112} + x^{110} \\
& + x^{108} + x^{104} + x^{102} + x^{100} + x^{96} + x^{94} + x^{92} + x^{88} + x^{70} + x^{68} \\
& + x^{64} + x^{46} + x^{44} + x^{40} + x^{38} + x^{36} + x^{32} + x^{30} + x^{28} + x^{24} + x^{6} \\
& + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 1, 1, 0].

For $\phi(W^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) = {} & x^{162} + x^{160} + x^{154} + x^{152} + x^{148} + x^{146} + x^{144} + x^{140} + x^{134} + x^{132} \\
& + x^{128} + x^{122} + x^{120} + x^{114} + x^{108} + x^{106} + x^{100} + x^{94} + x^{92} + x^{90} \\
& + x^{86} + x^{76} + x^{74} + x^{70} + x^{64} + x^{56} + x^{54} + x^{46} + x^{42}, \\
p_3(x) = {} & x^{158} + x^{156} + x^{142} + x^{134} + x^{126} + x^{124} + x^{120} + x^{112} + x^{92} + x^{88} \\
& + x^{80} + x^{78} + x^{76} + x^{70} + x^{68} + x^{62} + x^{46} + x^{44} + x^{38} + x^{36} + x^{32} \\
& + x^{20} + x^{16} + x^{14} + x^{8} + x^{6}, \\
p_6(x) = {} & x^{152} + x^{150} + x^{146} + x^{144} + x^{136} + x^{132} + x^{124} + x^{120} + x^{114} + x^{102} \\
& + x^{98} + x^{96} + x^{94} + x^{92} + x^{88} + x^{86} + x^{80} + x^{74} + x^{64} + x^{62} + x^{56} \\
& + x^{52} + x^{48} + x^{46} + x^{44} + x^{42} + x^{36} + x^{32} + x^{30} + x^{28} + x^{26} + x^{24} \\
& + x^{20} + x^{18} + x^{16} + x^{12} + x^{10} + x^{6} + x^{2} + 1, \\
p_9(x) = {} & x^{160} + x^{158} + x^{152} + x^{148} + x^{146} + x^{144} + x^{142} + x^{138} + x^{134} + x^{130} \\
& + x^{128} + x^{126} + x^{122} + x^{100} + x^{92} + x^{84} + x^{80} + x^{78} + x^{70} + x^{66} \\
& + x^{62} + x^{60} + x^{56} + x^{46} + x^{44} + x^{38} + x^{28} + x^{26} + x^{14} + x^{10} + x^{8}
\end{aligned}
$$

$$
\begin{aligned}
&\quad + x^6 + x^4 + x^2,\\
p_{12}(x) &= x^{160} + x^{144} + x^{128} + x^{112} + x^{96} + x^{80} + x^{32} + x^{16} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0].

For $\phi(M^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{147} + x^{146} + x^{145} + x^{140} + x^{139} + x^{138} + x^{136} + x^{135} + x^{133} + x^{131}\\
&\quad + x^{127} + x^{124} + x^{123} + x^{122} + x^{121} + x^{119} + x^{113} + x^{111} + x^{108}\\
&\quad + x^{107} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{92}\\
&\quad + x^{89} + x^{88} + x^{85} + x^{84} + x^{83} + x^{82} + x^{80} + x^{79} + x^{78} + x^{75} + x^{73}\\
&\quad + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{64} + x^{63} + x^{61} + x^{58} + x^{55}\\
&\quad + x^{54} + x^{53} + x^{52} + x^{51} + x^{49} + x^{46} + x^{44} + x^{39} + x^{36} + x^{35} + x^{33}\\
&\quad + x^{31} + x^{28} + x^{26} + x^{25} + x^{23} + x^{22} + x^{21} + x^{19} + x^{18} + x^{14} + x^{12},\\
p_3(x) &= x^{146} + x^{145} + x^{140} + x^{139} + x^{138} + x^{135} + x^{134} + x^{129} + x^{127} + x^{126}\\
&\quad + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{113} + x^{112} + x^{111}\\
&\quad + x^{110} + x^{108} + x^{106} + x^{105} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95}\\
&\quad + x^{92} + x^{91} + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{80} + x^{79} + x^{78} + x^{75}\\
&\quad + x^{74} + x^{73} + x^{72} + x^{71} + x^{67} + x^{66} + x^{64} + x^{60} + x^{57} + x^{56} + x^{55}\\
&\quad + x^{54} + x^{50} + x^{49} + x^{48} + x^{43} + x^{40} + x^{39} + x^{38} + x^{34} + x^{33} + x^{31}\\
&\quad + x^{30} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{20} + x^{18} + x^{17} + x^{16}\\
&\quad + x^{15} + x^{10} + x^{8},\\
p_6(x) &= x^{146} + x^{145} + x^{141} + x^{139} + x^{137} + x^{133} + x^{129} + x^{127} + x^{123} + x^{121}\\
&\quad + x^{118} + x^{116} + x^{115} + x^{114} + x^{112} + x^{107} + x^{106} + x^{102} + x^{95} + x^{91}\\
&\quad + x^{90} + x^{88} + x^{87} + x^{84} + x^{81} + x^{79} + x^{76} + x^{75} + x^{73} + x^{72} + x^{71}\\
&\quad + x^{70} + x^{68} + x^{66} + x^{63} + x^{61} + x^{59} + x^{57} + x^{56} + x^{52} + x^{50} + x^{49}\\
&\quad + x^{48} + x^{45} + x^{44} + x^{41} + x^{39} + x^{38} + x^{36} + x^{35} + x^{34} + x^{30} + x^{29}\\
&\quad + x^{28} + x^{26} + x^{25} + x^{24} + x^{21} + x^{19} + x^{17} + x^{15} + x^{14} + x^{13} + x^{12}\\
&\quad + x^{11} + x^{10} + x^{8} + 1,\\
p_9(x) &= x^{146} + x^{144} + x^{142} + x^{138} + x^{132} + x^{122} + x^{120} + x^{116} + x^{114} + x^{112}\\
&\quad + x^{110} + x^{106} + x^{100} + x^{98} + x^{96} + x^{94} + x^{88} + x^{74} + x^{72} + x^{68} + x^{50}\\
&\quad + x^{48} + x^{46} + x^{42} + x^{36} + x^{34} + x^{32} + x^{30} + x^{24} + x^{10} + x^{8} + x^{4},\\
p_{12}(x) &= x^{146} + x^{144} + x^{140} + x^{138} + x^{136} + x^{134} + x^{128} + x^{122} + x^{120} + x^{118}\\
&\quad + x^{114} + x^{108} + x^{106} + x^{104} + x^{102} + x^{98} + x^{92} + x^{74} + x^{72} + x^{70}\\
&\quad + x^{64} + x^{50} + x^{48} + x^{44} + x^{42} + x^{40} + x^{38} + x^{34} + x^{28} + x^{10} + x^{8}\\
&\quad + x^{6} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 1, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 1, 1].

For $\phi(M^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) = {} & x^{144} + x^{141} + x^{140} + x^{138} + x^{132} + x^{130} + x^{126} + x^{124} + x^{118} + x^{117} \\
& + x^{115} + x^{110} + x^{109} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{95} + x^{93} \\
& + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{77} + x^{74} + x^{73} \\
& + x^{70} + x^{67} + x^{60} + x^{54} + x^{53} + x^{52} + x^{49} + x^{47} + x^{46} + x^{44} + x^{41} \\
& + x^{38} + x^{37} + x^{34} + x^{33} + x^{30} + x^{28} + x^{23} + x^{22} + x^{21} + x^{20} + x^{18}, \\
p_3(x) = {} & x^{137} + x^{136} + x^{134} + x^{132} + x^{128} + x^{122} + x^{121} + x^{116} + x^{114} + x^{113} \\
& + x^{110} + x^{105} + x^{97} + x^{96} + x^{94} + x^{92} + x^{89} + x^{86} + x^{84} + x^{82} + x^{78} \\
& + x^{72} + x^{70} + x^{68} + x^{65} + x^{62} + x^{60} + x^{58} + x^{54} + x^{49} + x^{41} + x^{40} \\
& + x^{38} + x^{36} + x^{32} + x^{26} + x^{25} + x^{20} + x^{18} + x^{17} + x^{14} + x^{9}, \\
p_6(x) = {} & x^{138} + x^{136} + x^{134} + x^{130} + x^{128} + x^{126} + x^{124} + x^{122} + x^{116} + x^{114} \\
& + x^{112} + x^{108} + x^{100} + x^{88} + x^{84} + x^{80} + x^{74} + x^{68} + x^{66} + x^{56} + x^{54} \\
& + x^{50} + x^{48} + x^{42} + x^{40} + x^{38} + x^{36} + x^{34} + x^{28} + x^{22} + x^{20} + x^{18} \\
& + x^{16} + x^{14} + x^{10} + x^{6}, \\
p_9(x) = {} & x^{139} + x^{138} + x^{137} + x^{133} + x^{131} + x^{130} + x^{128} + x^{126} + x^{125} + x^{124} \\
& + x^{120} + x^{115} + x^{107} + x^{106} + x^{105} + x^{101} + x^{99} + x^{98} + x^{96} + x^{94} \\
& + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{85} + x^{82} + x^{80} + x^{78} + x^{77} \\
& + x^{76} + x^{72} + x^{67} + x^{43} + x^{42} + x^{41} + x^{37} + x^{35} + x^{34} + x^{32} + x^{30} \\
& + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{21} + x^{18} + x^{16} + x^{14} + x^{13} \\
& + x^{12} + x^{8} + x^{3}, \\
p_{12}(x) = {} & x^{140} + x^{132} + x^{128} + x^{116} + x^{112} + x^{108} + x^{100} + x^{96} + x^{92} + x^{68} \\
& + x^{64} + x^{44} + x^{36} + x^{32} + x^{28} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1].

For $\phi(M^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) = {} & x^{162} + x^{161} + x^{160} + x^{159} + x^{158} + x^{155} + x^{154} + x^{148} + x^{144} + x^{143} \\
& + x^{142} + x^{140} + x^{138} + x^{137} + x^{136} + x^{135} + x^{134} + x^{130} + x^{128} \\
& + x^{126} + x^{125} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{115} \\
& + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} + x^{105} + x^{104} + x^{100} + x^{98} + x^{97} \\
& + x^{96} + x^{95} + x^{93} + x^{89} + x^{88} + x^{86} + x^{85} + x^{84} + x^{83} + x^{81} + x^{79} \\
& + x^{78} + x^{77} + x^{76} + x^{72} + x^{70} + x^{69} + x^{68} + x^{62} + x^{61} + x^{59} + x^{56} \\
& + x^{55} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} \\
& + x^{42} + x^{40} + x^{38} + x^{34} + x^{33} + x^{32} + x^{29} + x^{27}, \\
p_3(x) = {} & x^{161} + x^{160} + x^{158} + x^{156} + x^{153} + x^{150} + x^{148} + x^{146} + x^{145} + x^{144} \\
& + x^{140} + x^{137} + x^{136} + x^{130} + x^{128} + x^{126} + x^{122} + x^{121} + x^{120} \\
& + x^{118} + x^{114} + x^{113} + x^{112} + x^{110} + x^{106} + x^{100} + x^{98} + x^{97} + x^{94}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{88} + x^{86} + x^{84} + x^{81} + x^{78} + x^{76} + x^{74} + x^{73} + x^{72} + x^{68} + x^{65}\\
&+ x^{64} + x^{58} + x^{57} + x^{52} + x^{50} + x^{48} + x^{44} + x^{41} + x^{40} + x^{34} + x^{32}\\
&+ x^{30} + x^{26} + x^{20} + x^{18} + x^{17} + x^{14} + x^{9},\\
p_6(x) &= x^{162} + x^{160} + x^{158} + x^{148} + x^{144} + x^{142} + x^{138} + x^{134} + x^{126} + x^{124}\\
&+ x^{122} + x^{114} + x^{98} + x^{96} + x^{92} + x^{88} + x^{82} + x^{78} + x^{70} + x^{68} + x^{66}\\
&+ x^{62} + x^{60} + x^{48} + x^{46} + x^{38} + x^{34} + x^{32} + x^{28} + x^{26} + x^{20} + x^{18}\\
&+ x^{16} + x^{14} + x^{10} + x^{6},\\
p_9(x) &= x^{163} + x^{162} + x^{161} + x^{157} + x^{153} + x^{152} + x^{150} + x^{148} + x^{147} + x^{146}\\
&+ x^{142} + x^{141} + x^{140} + x^{138} + x^{137} + x^{136} + x^{133} + x^{131} + x^{129}\\
&+ x^{128} + x^{126} + x^{124} + x^{121} + x^{118} + x^{116} + x^{115} + x^{113} + x^{110}\\
&+ x^{108} + x^{106} + x^{102} + x^{101} + x^{100} + x^{99} + x^{96} + x^{90} + x^{89} + x^{85}\\
&+ x^{83} + x^{82} + x^{80} + x^{78} + x^{77} + x^{76} + x^{72} + x^{66} + x^{65} + x^{61} + x^{57}\\
&+ x^{56} + x^{54} + x^{52} + x^{49} + x^{46} + x^{44} + x^{42} + x^{38} + x^{37} + x^{36} + x^{35}\\
&+ x^{32} + x^{26} + x^{25} + x^{21} + x^{19} + x^{18} + x^{16} + x^{14} + x^{13} + x^{12} + x^{8}\\
&+ x^{3},\\
p_{12}(x) &= x^{164} + x^{152} + x^{148} + x^{144} + x^{136} + x^{132} + x^{120} + x^{116} + x^{100} + x^{96}\\
&+ x^{88} + x^{84} + x^{80} + x^{64} + x^{56} + x^{48} + x^{36} + x^{32} + x^{24} + x^{20} + x^{16}\\
&+ x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 1].

For $\phi(M^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{147} + x^{142} + x^{141} + x^{140} + x^{136} + x^{135} + x^{134} + x^{133} + x^{128} + x^{125}\\
&+ x^{121} + x^{113} + x^{111} + x^{109} + x^{105} + x^{102} + x^{97} + x^{96} + x^{95} + x^{93}\\
&+ x^{91} + x^{86} + x^{85} + x^{84} + x^{81} + x^{80} + x^{78} + x^{77} + x^{76} + x^{74} + x^{73}\\
&+ x^{70} + x^{69} + x^{66} + x^{62} + x^{56} + x^{55} + x^{53} + x^{50} + x^{48} + x^{45} + x^{44}\\
&+ x^{43} + x^{37} + x^{36} + x^{35} + x^{33},\\
p_3(x) &= x^{146} + x^{145} + x^{144} + x^{142} + x^{140} + x^{139} + x^{138} + x^{136} + x^{135} + x^{129}\\
&+ x^{127} + x^{126} + x^{124} + x^{123} + x^{122} + x^{121} + x^{119} + x^{118} + x^{114}\\
&+ x^{113} + x^{112} + x^{111} + x^{110} + x^{105} + x^{102} + x^{100} + x^{99} + x^{98} + x^{97}\\
&+ x^{96} + x^{95} + x^{92} + x^{91} + x^{90} + x^{88} + x^{86} + x^{82} + x^{80} + x^{79} + x^{76}\\
&+ x^{75} + x^{73} + x^{71} + x^{67} + x^{66} + x^{64} + x^{57} + x^{55} + x^{54} + x^{52} + x^{49}\\
&+ x^{48} + x^{46} + x^{43} + x^{39} + x^{34} + x^{33} + x^{31} + x^{30} + x^{28} + x^{27} + x^{26}\\
&+ x^{25} + x^{23} + x^{22} + x^{20} + x^{17} + x^{16} + x^{15} + x^{12},\\
p_6(x) &= x^{146} + x^{145} + x^{144} + x^{142} + x^{141} + x^{140} + x^{139} + x^{138} + x^{137} + x^{133}\\
&+ x^{129} + x^{128} + x^{127} + x^{126} + x^{124} + x^{123} + x^{121} + x^{120} + x^{115}\\
&+ x^{114} + x^{108} + x^{107} + x^{104} + x^{102} + x^{100} + x^{95} + x^{94} + x^{92} + x^{91}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{88} + x^{87} + x^{86} + x^{84} + x^{81} + x^{79} + x^{75} + x^{73} + x^{71} + x^{70} + x^{68}\\
&+ x^{64} + x^{63} + x^{62} + x^{61} + x^{59} + x^{57} + x^{49} + x^{46} + x^{45} + x^{41} + x^{40}\\
&+ x^{39} + x^{35} + x^{34} + x^{30} + x^{29} + x^{28} + x^{25} + x^{21} + x^{20} + x^{19} + x^{18}\\
&+ x^{17} + x^{15} + x^{13} + x^{11} + x^{8} + x^{6} + x^{4} + 1,\\
p_9(x) &= x^{146} + x^{144} + x^{142} + x^{138} + x^{132} + x^{122} + x^{120} + x^{116} + x^{114} + x^{112}\\
&+ x^{110} + x^{106} + x^{100} + x^{98} + x^{96} + x^{94} + x^{88} + x^{74} + x^{72} + x^{68} + x^{50}\\
&+ x^{48} + x^{46} + x^{42} + x^{36} + x^{34} + x^{32} + x^{30} + x^{24} + x^{10} + x^{8} + x^{4},\\
p_{12}(x) &= x^{146} + x^{144} + x^{140} + x^{138} + x^{136} + x^{134} + x^{128} + x^{122} + x^{120} + x^{118}\\
&+ x^{114} + x^{108} + x^{106} + x^{104} + x^{102} + x^{98} + x^{92} + x^{74} + x^{72} + x^{70}\\
&+ x^{64} + x^{50} + x^{48} + x^{44} + x^{42} + x^{40} + x^{38} + x^{34} + x^{28} + x^{10} + x^{8}\\
&+ x^{6} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 1].

For $\phi(W^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{147} + x^{146} + x^{145} + x^{140} + x^{139} + x^{138} + x^{136} + x^{135} + x^{133} + x^{131}\\
&+ x^{127} + x^{124} + x^{123} + x^{122} + x^{121} + x^{119} + x^{113} + x^{111} + x^{108}\\
&+ x^{107} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{92}\\
&+ x^{89} + x^{88} + x^{85} + x^{84} + x^{83} + x^{82} + x^{80} + x^{79} + x^{78} + x^{75} + x^{73}\\
&+ x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{64} + x^{63} + x^{61} + x^{58} + x^{55}\\
&+ x^{54} + x^{53} + x^{52} + x^{51} + x^{49} + x^{46} + x^{44} + x^{39} + x^{36} + x^{35} + x^{33}\\
&+ x^{31} + x^{28} + x^{26} + x^{25} + x^{23} + x^{22} + x^{21} + x^{19} + x^{18} + x^{14} + x^{12},\\
p_3(x) &= x^{146} + x^{145} + x^{140} + x^{139} + x^{138} + x^{135} + x^{134} + x^{129} + x^{127} + x^{126}\\
&+ x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{113} + x^{112} + x^{111}\\
&+ x^{110} + x^{108} + x^{106} + x^{105} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95}\\
&+ x^{92} + x^{91} + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{80} + x^{79} + x^{78} + x^{75}\\
&+ x^{74} + x^{73} + x^{72} + x^{71} + x^{67} + x^{66} + x^{64} + x^{60} + x^{57} + x^{56} + x^{55}\\
&+ x^{54} + x^{50} + x^{49} + x^{48} + x^{43} + x^{40} + x^{39} + x^{38} + x^{34} + x^{33} + x^{31}\\
&+ x^{30} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{20} + x^{18} + x^{17} + x^{16}\\
&+ x^{15} + x^{10} + x^{8},\\
p_6(x) &= x^{146} + x^{145} + x^{141} + x^{139} + x^{137} + x^{133} + x^{129} + x^{127} + x^{123} + x^{121}\\
&+ x^{118} + x^{116} + x^{115} + x^{114} + x^{112} + x^{107} + x^{106} + x^{102} + x^{95} + x^{91}\\
&+ x^{90} + x^{88} + x^{87} + x^{84} + x^{81} + x^{79} + x^{76} + x^{75} + x^{73} + x^{72} + x^{71}\\
&+ x^{70} + x^{68} + x^{66} + x^{63} + x^{61} + x^{59} + x^{57} + x^{56} + x^{52} + x^{50} + x^{49}\\
&+ x^{48} + x^{45} + x^{44} + x^{41} + x^{39} + x^{38} + x^{36} + x^{35} + x^{34} + x^{30} + x^{29}\\
&+ x^{28} + x^{26} + x^{25} + x^{24} + x^{21} + x^{19} + x^{17} + x^{15} + x^{14} + x^{13} + x^{12}\\
&+ x^{11} + x^{10} + x^{8} + 1,
\end{aligned}
$$

$$\begin{aligned}
p_9(x) &= x^{146} + x^{144} + x^{142} + x^{138} + x^{132} + x^{122} + x^{120} + x^{116} + x^{114} + x^{112}\\
&\quad + x^{110} + x^{106} + x^{100} + x^{98} + x^{96} + x^{94} + x^{88} + x^{74} + x^{72} + x^{68} + x^{50}\\
&\quad + x^{48} + x^{46} + x^{42} + x^{36} + x^{34} + x^{32} + x^{30} + x^{24} + x^{10} + x^{8} + x^{4},\\
p_{12}(x) &= x^{146} + x^{144} + x^{140} + x^{138} + x^{136} + x^{134} + x^{128} + x^{122} + x^{120} + x^{118}\\
&\quad + x^{114} + x^{108} + x^{106} + x^{104} + x^{102} + x^{98} + x^{92} + x^{74} + x^{72} + x^{70}\\
&\quad + x^{64} + x^{50} + x^{48} + x^{44} + x^{42} + x^{40} + x^{38} + x^{34} + x^{28} + x^{10} + x^{8}\\
&\quad + x^{6} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 1, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 1, 1].
For $\phi(W^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{162} + x^{161} + x^{160} + x^{159} + x^{158} + x^{155} + x^{154} + x^{148} + x^{144} + x^{143}\\
&\quad + x^{142} + x^{140} + x^{138} + x^{137} + x^{136} + x^{135} + x^{134} + x^{130} + x^{128}\\
&\quad + x^{126} + x^{125} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{115}\\
&\quad + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} + x^{105} + x^{104} + x^{100} + x^{98} + x^{97}\\
&\quad + x^{96} + x^{95} + x^{93} + x^{89} + x^{88} + x^{86} + x^{85} + x^{84} + x^{83} + x^{81} + x^{79}\\
&\quad + x^{78} + x^{77} + x^{76} + x^{72} + x^{70} + x^{69} + x^{68} + x^{62} + x^{61} + x^{59} + x^{56}\\
&\quad + x^{55} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43}\\
&\quad + x^{42} + x^{40} + x^{38} + x^{34} + x^{33} + x^{32} + x^{29} + x^{27},\\
p_3(x) &= x^{161} + x^{160} + x^{158} + x^{156} + x^{153} + x^{150} + x^{148} + x^{146} + x^{145} + x^{144}\\
&\quad + x^{140} + x^{137} + x^{136} + x^{130} + x^{128} + x^{126} + x^{122} + x^{121} + x^{120}\\
&\quad + x^{118} + x^{114} + x^{113} + x^{112} + x^{110} + x^{106} + x^{100} + x^{98} + x^{97} + x^{94}\\
&\quad + x^{88} + x^{86} + x^{84} + x^{81} + x^{78} + x^{76} + x^{74} + x^{73} + x^{72} + x^{68} + x^{65}\\
&\quad + x^{64} + x^{58} + x^{57} + x^{52} + x^{50} + x^{48} + x^{44} + x^{41} + x^{40} + x^{34} + x^{32}\\
&\quad + x^{30} + x^{26} + x^{20} + x^{18} + x^{17} + x^{14} + x^{9},\\
p_6(x) &= x^{162} + x^{160} + x^{158} + x^{148} + x^{144} + x^{142} + x^{138} + x^{134} + x^{126} + x^{124}\\
&\quad + x^{122} + x^{114} + x^{98} + x^{96} + x^{92} + x^{88} + x^{82} + x^{78} + x^{70} + x^{68} + x^{66}\\
&\quad + x^{62} + x^{60} + x^{48} + x^{46} + x^{38} + x^{34} + x^{32} + x^{28} + x^{26} + x^{20} + x^{18}\\
&\quad + x^{16} + x^{14} + x^{10} + x^{6},\\
p_9(x) &= x^{163} + x^{162} + x^{161} + x^{157} + x^{153} + x^{152} + x^{150} + x^{148} + x^{147} + x^{146}\\
&\quad + x^{142} + x^{141} + x^{140} + x^{138} + x^{137} + x^{136} + x^{133} + x^{131} + x^{129}\\
&\quad + x^{128} + x^{126} + x^{124} + x^{121} + x^{118} + x^{116} + x^{115} + x^{113} + x^{110}\\
&\quad + x^{108} + x^{106} + x^{102} + x^{101} + x^{100} + x^{99} + x^{96} + x^{90} + x^{89} + x^{85}\\
&\quad + x^{83} + x^{82} + x^{80} + x^{78} + x^{77} + x^{76} + x^{72} + x^{66} + x^{65} + x^{61} + x^{57}\\
&\quad + x^{56} + x^{54} + x^{52} + x^{49} + x^{46} + x^{44} + x^{42} + x^{38} + x^{37} + x^{36} + x^{35}\\
&\quad + x^{32} + x^{26} + x^{25} + x^{21} + x^{19} + x^{18} + x^{16} + x^{14} + x^{13} + x^{12} + x^{8}\\
&\quad + x^{3},
\end{aligned}$$

$$\begin{aligned}p_{12}(x) = {} & x^{164} + x^{152} + x^{148} + x^{144} + x^{136} + x^{132} + x^{120} + x^{116} + x^{100} + x^{96} \\ & + x^{88} + x^{84} + x^{80} + x^{64} + x^{56} + x^{48} + x^{36} + x^{32} + x^{24} + x^{20} + x^{16} \\ & + x^4 + 1,\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 1].

For $\phi(W^o, 1, 0)$:

$$\begin{aligned}p_0(x) = {} & x^{144} + x^{141} + x^{140} + x^{138} + x^{132} + x^{130} + x^{126} + x^{124} + x^{118} + x^{117} \\ & + x^{115} + x^{110} + x^{109} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{95} + x^{93} \\ & + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{77} + x^{74} + x^{73} \\ & + x^{70} + x^{67} + x^{60} + x^{54} + x^{53} + x^{52} + x^{49} + x^{47} + x^{46} + x^{44} + x^{41} \\ & + x^{38} + x^{37} + x^{34} + x^{33} + x^{30} + x^{28} + x^{23} + x^{22} + x^{21} + x^{20} + x^{18},\\
p_3(x) = {} & x^{137} + x^{136} + x^{134} + x^{132} + x^{128} + x^{122} + x^{121} + x^{116} + x^{114} + x^{113} \\ & + x^{110} + x^{105} + x^{97} + x^{96} + x^{94} + x^{92} + x^{89} + x^{86} + x^{84} + x^{82} + x^{78} \\ & + x^{72} + x^{70} + x^{68} + x^{65} + x^{62} + x^{60} + x^{58} + x^{54} + x^{49} + x^{41} + x^{40} \\ & + x^{38} + x^{36} + x^{32} + x^{26} + x^{25} + x^{20} + x^{18} + x^{17} + x^{14} + x^{9},\\
p_6(x) = {} & x^{138} + x^{136} + x^{134} + x^{130} + x^{128} + x^{126} + x^{124} + x^{122} + x^{116} + x^{114} \\ & + x^{112} + x^{108} + x^{100} + x^{88} + x^{84} + x^{80} + x^{74} + x^{68} + x^{66} + x^{56} + x^{54} \\ & + x^{50} + x^{48} + x^{42} + x^{40} + x^{38} + x^{36} + x^{34} + x^{28} + x^{22} + x^{20} + x^{18} \\ & + x^{16} + x^{14} + x^{10} + x^{6},\\
p_9(x) = {} & x^{139} + x^{138} + x^{137} + x^{133} + x^{131} + x^{130} + x^{128} + x^{126} + x^{125} + x^{124} \\ & + x^{120} + x^{115} + x^{107} + x^{106} + x^{105} + x^{101} + x^{99} + x^{98} + x^{96} + x^{94} \\ & + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{85} + x^{82} + x^{80} + x^{78} + x^{77} \\ & + x^{76} + x^{72} + x^{67} + x^{43} + x^{42} + x^{41} + x^{37} + x^{35} + x^{34} + x^{32} + x^{30} \\ & + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{21} + x^{18} + x^{16} + x^{14} + x^{13} \\ & + x^{12} + x^{8} + x^{3},\\
p_{12}(x) = {} & x^{140} + x^{132} + x^{128} + x^{116} + x^{112} + x^{108} + x^{100} + x^{96} + x^{92} + x^{68} \\ & + x^{64} + x^{44} + x^{36} + x^{32} + x^{28} + x^4 + 1,\end{aligned}$$

and the initial terms are [0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1].

For $\phi(W^o, 1, 1)$:

$$\begin{aligned}p_0(x) = {} & x^{147} + x^{142} + x^{141} + x^{140} + x^{136} + x^{135} + x^{134} + x^{133} + x^{128} + x^{125} \\ & + x^{121} + x^{113} + x^{111} + x^{109} + x^{105} + x^{102} + x^{97} + x^{96} + x^{95} + x^{93} \\ & + x^{91} + x^{86} + x^{85} + x^{84} + x^{81} + x^{80} + x^{78} + x^{77} + x^{76} + x^{74} + x^{73} \\ & + x^{70} + x^{69} + x^{66} + x^{62} + x^{56} + x^{55} + x^{53} + x^{50} + x^{48} + x^{45} + x^{44} \\ & + x^{43} + x^{37} + x^{36} + x^{35} + x^{33},\\
p_3(x) = {} & x^{146} + x^{145} + x^{144} + x^{142} + x^{140} + x^{139} + x^{138} + x^{136} + x^{135} + x^{129} \\ & + x^{127} + x^{126} + x^{124} + x^{123} + x^{122} + x^{121} + x^{119} + x^{118} + x^{114}\end{aligned}$$

$$
\begin{aligned}
&+ x^{113} + x^{112} + x^{111} + x^{110} + x^{105} + x^{102} + x^{100} + x^{99} + x^{98} + x^{97}\\
&+ x^{96} + x^{95} + x^{92} + x^{91} + x^{90} + x^{88} + x^{86} + x^{82} + x^{80} + x^{79} + x^{76}\\
&+ x^{75} + x^{73} + x^{71} + x^{67} + x^{66} + x^{64} + x^{57} + x^{55} + x^{54} + x^{52} + x^{49}\\
&+ x^{48} + x^{46} + x^{43} + x^{39} + x^{34} + x^{33} + x^{31} + x^{30} + x^{28} + x^{27} + x^{26}\\
&+ x^{25} + x^{23} + x^{22} + x^{20} + x^{17} + x^{16} + x^{15} + x^{12},\\
p_6(x) = &\, x^{146} + x^{145} + x^{144} + x^{142} + x^{141} + x^{140} + x^{139} + x^{138} + x^{137} + x^{133}\\
&+ x^{129} + x^{128} + x^{127} + x^{126} + x^{124} + x^{123} + x^{121} + x^{120} + x^{115}\\
&+ x^{114} + x^{108} + x^{107} + x^{104} + x^{102} + x^{100} + x^{95} + x^{94} + x^{92} + x^{91}\\
&+ x^{88} + x^{87} + x^{86} + x^{84} + x^{81} + x^{79} + x^{75} + x^{73} + x^{71} + x^{70} + x^{68}\\
&+ x^{64} + x^{63} + x^{62} + x^{61} + x^{59} + x^{57} + x^{49} + x^{46} + x^{45} + x^{41} + x^{40}\\
&+ x^{39} + x^{35} + x^{34} + x^{30} + x^{29} + x^{28} + x^{25} + x^{21} + x^{20} + x^{19} + x^{18}\\
&+ x^{17} + x^{15} + x^{13} + x^{11} + x^{8} + x^{6} + x^{4} + 1,\\
p_9(x) = &\, x^{146} + x^{144} + x^{142} + x^{138} + x^{132} + x^{122} + x^{120} + x^{116} + x^{114} + x^{112}\\
&+ x^{110} + x^{106} + x^{100} + x^{98} + x^{96} + x^{94} + x^{88} + x^{74} + x^{72} + x^{68} + x^{50}\\
&+ x^{48} + x^{46} + x^{42} + x^{36} + x^{34} + x^{32} + x^{30} + x^{24} + x^{10} + x^{8} + x^{4},\\
p_{12}(x) = &\, x^{146} + x^{144} + x^{140} + x^{138} + x^{136} + x^{134} + x^{128} + x^{122} + x^{120} + x^{118}\\
&+ x^{114} + x^{108} + x^{106} + x^{104} + x^{102} + x^{98} + x^{92} + x^{74} + x^{72} + x^{70}\\
&+ x^{64} + x^{50} + x^{48} + x^{44} + x^{42} + x^{40} + x^{38} + x^{34} + x^{28} + x^{10} + x^{8}\\
&+ x^{6} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 1].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	122	[198, 199]	244	[37, 296]	366	[133, 471]	488	[550, 205]
1	[3, 4]	123	[71, 200]	245	[322, 134]	367	[271, 472]	489	[551, 552]
2	[5, 6]	124	[201, 202]	246	[140, 48]	368	[108, 130]	490	[553, 554]
3	[7, 8]	125	[203, 204]	247	[323, 53]	369	[473, 269]	491	[555, 556]
4	[9, 10]	126	[205, 206]	248	[324, 325]	370	[474, 97]	492	[557, 558]
5	[11, 2]	127	[207, 208]	249	[55, 326]	371	[475, 45]	493	[200, 559]
6	[12, 13]	128	[209, 210]	250	[327, 328]	372	[50, 52]	494	[560, 561]
7	[14, 15]	129	[211, 212]	251	[329, 330]	373	[265, 312]	495	[562, 563]
8	[16, 17]	130	[213, 214]	252	[331, 332]	374	[138, 134]	496	[564, 565]
9	[18, 19]	131	[215, 216]	253	[333, 334]	375	[294, 312]	497	[297, 142]
10	[20, 21]	132	[217, 218]	254	[335, 336]	376	[473, 278]	498	[312, 130]
11	[22, 23]	133	[219, 220]	255	[337, 338]	377	[476, 35]	499	[248, 135]
12	[24, 25]	134	[221, 222]	256	[339, 340]	378	[477, 317]	500	[53, 51]
13	[26, 27]	135	[63, 223]	257	[341, 342]	379	[39, 256]	501	[484, 318]
14	[28, 29]	136	[224, 225]	258	[343, 344]	380	[252, 301]	502	[145, 103]

15	[30, 31]	137	[226, 227]	259	[345, 346]	381	[478, 294]	503	[274, 105]
16	[32, 33]	138	[228, 229]	260	[347, 348]	382	[322, 479]	504	[124, 106]
17	[34, 35]	139	[170, 230]	261	[349, 350]	383	[480, 136]	505	[40, 301]
18	[36, 37]	140	[231, 232]	262	[351, 352]	384	[273, 130]	506	[459, 253]
19	[38, 39]	141	[233, 234]	263	[353, 354]	385	[255, 301]	507	[566, 258]
20	[40, 41]	142	[235, 236]	264	[355, 356]	386	[48, 49]	508	[138, 471]
21	[42, 43]	143	[237, 238]	265	[357, 358]	387	[481, 272]	509	[49, 52]
22	[44, 35]	144	[239, 226]	266	[359, 360]	388	[482, 252]	510	[567, 256]
23	[45, 46]	145	[240, 241]	267	[361, 362]	389	[483, 306]	511	[119, 313]
24	[47, 48]	146	[242, 243]	268	[363, 364]	390	[476, 116]	512	[30, 132]
25	[47, 49]	147	[244, 245]	269	[365, 366]	391	[484, 115]	513	[568, 280]
26	[50, 51]	148	[246, 247]	270	[367, 368]	392	[469, 276]	514	[481, 472]
27	[52, 53]	149	[32, 248]	271	[369, 370]	393	[109, 249]	515	[97, 143]
28	[54, 55]	150	[37, 116]	272	[371, 372]	394	[97, 284]	516	[487, 109]
29	[56, 57]	151	[29, 249]	273	[373, 374]	395	[274, 248]	517	[569, 311]
30	[58, 59]	152	[250, 37]	274	[375, 221]	396	[28, 306]	518	[132, 459]
31	[60, 61]	153	[251, 252]	275	[376, 377]	397	[277, 465]	519	[570, 490]
32	[62, 63]	154	[139, 253]	276	[378, 244]	398	[281, 276]	520	[571, 42]
33	[64, 65]	155	[51, 43]	277	[379, 380]	399	[45, 280]	521	[252, 253]
34	[66, 67]	156	[147, 254]	278	[381, 382]	400	[31, 284]	522	[130, 256]
35	[68, 69]	157	[255, 256]	279	[383, 384]	401	[261, 129]	523	[468, 248]
36	[70, 71]	158	[257, 258]	280	[385, 386]	402	[251, 143]	524	[145, 276]
37	[72, 73]	159	[259, 260]	281	[387, 388]	403	[485, 264]	525	[52, 43]
38	[74, 75]	160	[261, 262]	282	[389, 390]	404	[106, 123]	526	[143, 41]
39	[17, 76]	161	[112, 252]	283	[391, 392]	405	[148, 48]	527	[128, 262]
40	[77, 78]	162	[263, 264]	284	[178, 393]	406	[486, 253]	528	[266, 39]
41	[61, 79]	163	[265, 266]	285	[218, 394]	407	[320, 136]	529	[252, 572]
42	[27, 80]	164	[267, 99]	286	[395, 193]	408	[482, 143]	530	[571, 51]
43	[81, 82]	165	[268, 269]	287	[396, 213]	409	[487, 29]	531	[307, 306]
44	[83, 84]	166	[270, 116]	288	[397, 398]	410	[270, 35]	532	[573, 119]
45	[85, 86]	167	[271, 272]	289	[399, 400]	411	[488, 98]	533	[574, 46]
46	[87, 88]	168	[273, 39]	290	[401, 402]	412	[323, 42]	534	[266, 139]
47	[89, 81]	169	[274, 275]	291	[403, 404]	413	[281, 137]	535	[575, 37]
48	[25, 90]	170	[102, 276]	292	[405, 406]	414	[308, 318]	536	[576, 121]
49	[91, 92]	171	[131, 97]	293	[407, 408]	415	[287, 108]	537	[268, 278]
50	[93, 91]	172	[275, 135]	294	[409, 410]	416	[489, 490]	538	[111, 108]
51	[90, 94]	173	[277, 278]	295	[411, 412]	417	[140, 13]	539	[282, 266]
52	[94, 95]	174	[44, 103]	296	[202, 413]	418	[294, 266]	540	[494, 110]
53	[53, 53]	175	[279, 280]	297	[414, 415]	419	[322, 135]	541	[131, 31]
54	[96, 97]	176	[119, 139]	298	[416, 417]	420	[270, 123]	542	[293, 311]
55	[98, 99]	177	[28, 109]	299	[418, 419]	421	[288, 249]	543	[139, 572]
56	[100, 101]	178	[281, 103]	300	[420, 421]	422	[266, 313]	544	[577, 41]
57	[102, 103]	179	[282, 119]	301	[419, 422]	423	[107, 312]	545	[255, 254]
58	[104, 105]	180	[283, 249]	302	[423, 335]	424	[491, 287]	546	[47, 43]

59	[106, 35]	181	[36, 125]	303	[424, 425]	425	[321, 479]	547	[34, 123]
60	[107, 108]	182	[266, 130]	304	[426, 20]	426	[47, 42]	548	[321, 134]
61	[109, 110]	183	[284, 285]	305	[427, 428]	427	[148, 42]	549	[107, 119]
62	[111, 112]	184	[147, 253]	306	[429, 430]	428	[305, 43]	550	[53, 13]
63	[113, 46]	185	[286, 287]	307	[431, 432]	429	[464, 269]	551	[140, 51]
64	[114, 115]	186	[288, 289]	308	[433, 434]	430	[492, 276]	552	[313, 301]
65	[34, 116]	187	[46, 127]	309	[435, 436]	431	[282, 108]	553	[575, 106]
66	[117, 118]	188	[290, 291]	310	[437, 355]	432	[9, 46]	554	[299, 130]
67	[119, 39]	189	[100, 115]	311	[438, 439]	433	[300, 467]	555	[468, 105]
68	[120, 121]	190	[96, 119]	312	[440, 333]	434	[493, 301]	556	[258, 31]
69	[122, 123]	191	[292, 293]	313	[410, 441]	435	[7, 294]	557	[314, 118]
70	[124, 125]	192	[148, 52]	314	[442, 443]	436	[494, 289]	558	[31, 252]
71	[126, 127]	193	[294, 119]	315	[444, 339]	437	[320, 281]	559	[494, 249]
72	[128, 129]	194	[248, 134]	316	[445, 446]	438	[252, 256]	560	[464, 303]
73	[38, 130]	195	[284, 256]	317	[86, 447]	439	[43, 53]	561	[290, 103]
74	[131, 132]	196	[49, 49]	318	[448, 449]	440	[96, 31]	562	[483, 109]
75	[133, 134]	197	[125, 123]	319	[450, 451]	441	[29, 110]	563	[122, 35]
76	[33, 135]	198	[117, 295]	320	[452, 403]	442	[484, 101]	564	[483, 283]
77	[136, 137]	199	[146, 130]	321	[453, 454]	443	[495, 31]	565	[578, 97]
78	[138, 135]	200	[125, 296]	322	[455, 456]	444	[111, 132]	566	[579, 171]
79	[108, 139]	201	[144, 281]	323	[457, 26]	445	[39, 285]	567	[413, 580]
80	[51, 49]	202	[99, 121]	324	[308, 101]	446	[50, 53]	568	[581, 582]
81	[140, 49]	203	[297, 298]	325	[458, 35]	447	[145, 137]	569	[583, 584]
82	[13, 51]	204	[299, 39]	326	[97, 459]	448	[496, 265]	570	[585, 195]
83	[141, 142]	205	[300, 301]	327	[250, 122]	449	[461, 260]	571	[586, 587]
84	[132, 143]	206	[13, 53]	328	[460, 317]	450	[48, 53]	572	[243, 588]
85	[144, 145]	207	[302, 143]	329	[30, 112]	451	[39, 253]	573	[589, 331]
86	[146, 39]	208	[106, 296]	330	[288, 110]	452	[36, 106]	574	[590, 591]
87	[147, 41]	209	[277, 303]	331	[125, 116]	453	[268, 303]	575	[592, 593]
88	[52, 51]	210	[282, 31]	332	[283, 110]	454	[125, 35]	576	[594, 595]
89	[48, 13]	211	[304, 293]	333	[33, 134]	455	[473, 465]	577	[596, 597]
90	[48, 43]	212	[305, 48]	334	[31, 459]	456	[122, 116]	578	[598, 599]
91	[148, 51]	213	[44, 276]	335	[259, 249]	457	[53, 52]	579	[32, 275]
92	[42, 49]	214	[306, 249]	336	[312, 139]	458	[497, 498]	580	[259, 110]
93	[50, 42]	215	[307, 283]	337	[44, 291]	459	[150, 499]	581	[480, 281]
94	[49, 43]	216	[290, 276]	338	[461, 110]	460	[451, 500]	582	[108, 39]
95	[43, 51]	217	[111, 97]	339	[461, 249]	461	[501, 502]	583	[600, 290]
96	[149, 150]	218	[306, 110]	340	[132, 284]	462	[503, 237]	584	[31, 143]
97	[15, 151]	219	[308, 309]	341	[104, 33]	463	[504, 207]	585	[50, 49]
98	[152, 153]	220	[106, 116]	342	[462, 108]	464	[505, 506]	586	[49, 13]
99	[154, 155]	221	[310, 311]	343	[104, 275]	465	[507, 508]	587	[571, 53]
100	[156, 157]	222	[312, 313]	344	[458, 116]	466	[509, 510]	588	[112, 459]
101	[158, 159]	223	[112, 284]	345	[100, 309]	467	[194, 511]	589	[487, 283]
102	[160, 161]	224	[314, 295]	346	[136, 276]	468	[512, 513]	590	[575, 122]

103	[162, 77]	225	[315, 143]	347	[463, 99]	469	[514, 515]	591	[312, 39]
104	[163, 164]	226	[316, 317]	348	[108, 313]	470	[516, 60]	592	[480, 290]
105	[165, 166]	227	[145, 291]	349	[464, 465]	471	[517, 518]	593	[280, 264]
106	[167, 168]	228	[114, 318]	350	[8, 119]	472	[519, 520]	594	[139, 285]
107	[169, 170]	229	[37, 35]	351	[466, 98]	473	[521, 522]	595	[12, 49]
108	[171, 172]	230	[275, 134]	352	[47, 13]	474	[523, 524]	596	[102, 291]
109	[173, 174]	231	[52, 52]	353	[40, 467]	475	[525, 526]	597	[133, 135]
110	[175, 176]	232	[12, 43]	354	[305, 49]	476	[527, 528]	598	[307, 29]
111	[177, 178]	233	[114, 309]	355	[97, 252]	477	[529, 530]	599	[492, 103]
112	[179, 180]	234	[265, 97]	356	[281, 291]	478	[531, 532]	600	[601, 602]
113	[181, 182]	235	[319, 45]	357	[468, 33]	479	[533, 534]	601	[600, 136]
114	[183, 184]	236	[12, 48]	358	[469, 103]	480	[535, 536]	602	[603, 127]
115	[185, 186]	237	[136, 103]	359	[265, 108]	481	[537, 538]	603	[604, 605]
116	[187, 188]	238	[105, 134]	360	[105, 135]	482	[539, 540]	604	[300, 254]
117	[189, 190]	239	[320, 145]	361	[250, 125]	483	[541, 542]	605	[323, 51]
118	[191, 192]	240	[141, 298]	362	[112, 143]	484	[543, 544]		
119	[193, 194]	241	[302, 252]	363	[255, 41]	485	[545, 546]		
120	[195, 196]	242	[96, 112]	364	[139, 41]	486	[547, 548]		
121	[182, 197]	243	[321, 135]	365	[470, 265]	487	[549, 175]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	87	0	174	0	261	1	348	1	435	0	522	0
1	0	88	1	175	1	262	1	349	1	436	1	523	1
2	0	89	0	176	0	263	1	350	0	437	0	524	0
3	0	90	0	177	0	264	1	351	1	438	1	525	1
4	0	91	1	178	0	265	1	352	0	439	1	526	1
5	0	92	1	179	1	266	1	353	1	440	0	527	0
6	0	93	0	180	1	267	1	354	1	441	0	528	1
7	0	94	1	181	0	268	0	355	1	442	1	529	1
8	0	95	1	182	1	269	0	356	0	443	1	530	1
9	0	96	0	183	0	270	1	357	1	444	0	531	1
10	1	97	1	184	0	271	1	358	0	445	0	532	0
11	0	98	1	185	0	272	1	359	1	446	0	533	1
12	0	99	1	186	0	273	1	360	0	447	0	534	1
13	0	100	0	187	0	274	0	361	1	448	1	535	1
14	0	101	1	188	1	275	1	362	1	449	1	536	1
15	1	102	1	189	0	276	1	363	0	450	0	537	0
16	0	103	1	190	0	277	0	364	1	451	0	538	0
17	0	104	1	191	1	278	1	365	0	452	0	539	1
18	0	105	0	192	1	279	1	366	1	453	0	540	1
19	1	106	1	193	0	280	0	367	1	454	1	541	1
20	1	107	0	194	1	281	0	368	1	455	1	542	0
21	1	108	1	195	0	282	1	369	1	456	0	543	1

22	0	109	0	196	1	283	1	370	1	457	0	544	1
23	0	110	1	197	1	284	0	371	1	458	1	545	0
24	0	111	0	198	0	285	1	372	0	459	0	546	0
25	0	112	1	199	0	286	0	373	1	460	0	547	0
26	0	113	0	200	1	287	0	374	0	461	1	548	0
27	1	114	0	201	0	288	0	375	0	462	0	549	0
28	0	115	0	202	1	289	0	376	1	463	0	550	0
29	0	116	0	203	1	290	1	377	0	464	1	551	1
30	1	117	0	204	0	291	0	378	1	465	0	552	1
31	0	118	1	205	1	292	1	379	0	466	1	553	1
32	0	119	0	206	0	293	0	380	1	467	1	554	0
33	0	120	0	207	0	294	0	381	1	468	1	555	1
34	0	121	1	208	1	295	0	382	1	469	0	556	1
35	0	122	0	209	0	296	1	383	1	470	0	557	1
36	0	123	1	210	1	297	1	384	1	471	1	558	0
37	0	124	0	211	0	298	1	385	0	472	0	559	1
38	1	125	1	212	1	299	0	386	0	473	1	560	1
39	0	126	1	213	0	300	1	387	0	474	1	561	1
40	1	127	0	214	1	301	1	388	1	475	1	562	1
41	0	128	0	215	1	302	0	389	1	476	0	563	0
42	1	129	0	216	1	303	1	390	0	477	1	564	1
43	1	130	0	217	0	304	0	391	1	478	1	565	1
44	0	131	1	218	1	305	1	392	0	479	1	566	0
45	0	132	0	219	1	306	1	393	0	480	1	567	0
46	0	133	1	220	1	307	1	394	1	481	0	568	1
47	0	134	0	221	0	308	1	395	0	482	1	569	1
48	0	135	0	222	0	309	0	396	0	483	1	570	0
49	1	136	1	223	1	310	0	397	0	484	1	571	1
50	0	137	0	224	1	311	1	398	0	485	0	572	0
51	0	138	0	225	0	312	0	399	0	486	0	573	0
52	1	139	1	226	0	313	1	400	0	487	0	574	1
53	0	140	1	227	0	314	1	401	1	488	0	575	1
54	0	141	0	228	0	315	0	402	1	489	1	576	1
55	1	142	0	229	0	316	0	403	0	490	1	577	1
56	0	143	1	230	1	317	0	404	1	491	1	578	1
57	1	144	0	231	1	318	1	405	1	492	1	579	0
58	1	145	0	232	0	319	0	406	0	493	1	580	0
59	1	146	0	233	0	320	0	407	0	494	1	581	1
60	0	147	0	234	1	321	0	408	1	495	1	582	1
61	0	148	1	235	0	322	1	409	0	496	1	583	1
62	0	149	0	236	0	323	0	410	1	497	1	584	0
63	0	150	0	237	1	324	1	411	0	498	0	585	0
64	0	151	0	238	0	325	1	412	0	499	1	586	1
65	0	152	1	239	0	326	1	413	0	500	0	587	1

66	0	153	1	240	0	327	1	414	1	501	1	588	1
67	0	154	1	241	0	328	0	415	0	502	0	589	0
68	0	155	0	242	0	329	1	416	1	503	0	590	1
69	0	156	0	243	0	330	0	417	1	504	0	591	0
70	0	157	0	244	0	331	1	418	0	505	1	592	1
71	1	158	1	245	1	332	1	419	1	506	0	593	0
72	0	159	0	246	1	333	0	420	1	507	0	594	1
73	1	160	1	247	0	334	0	421	0	508	0	595	0
74	1	161	1	248	1	335	0	422	1	509	1	596	1
75	1	162	1	249	1	336	0	423	0	510	0	597	1
76	0	163	1	250	1	337	0	424	1	511	0	598	1
77	1	164	1	251	1	338	1	425	0	512	1	599	1
78	0	165	0	252	1	339	1	426	0	513	1	600	1
79	1	166	1	253	0	340	0	427	1	514	0	601	1
80	0	167	1	254	0	341	1	428	1	515	1	602	1
81	1	168	1	255	0	342	0	429	1	516	0	603	1
82	0	169	0	256	1	343	1	430	1	517	1	604	1
83	0	170	1	257	1	344	1	431	1	518	0	605	0
84	0	171	1	258	1	345	0	432	0	519	0		
85	0	172	1	259	0	346	1	433	1	520	1		
86	0	173	0	260	0	347	0	434	1	521	1		

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