PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a, b) = (z, z^6 + z^5 + z^3 + z + 1)$

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ABSTRACT. The continued fraction conjecture claims that the continued fraction defined by the (a, b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a, b) in $\mathbb{F}_2[x] \setminus \mathbb{F}_2$. In this paper we prove the conjecture for $(a, b) = (z, z^6 + z^5 + z^3 + z + 1)$.

1. INTRODUCTION

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \ldots)$ of elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{\ddots + \cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a, b)-Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \ldots) = (a, b, b, a, b, a, a, b, \ldots)$$

that we shall denote

$$\mathrm{CF}(a, b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When $(a,b)=(z,z^6+z^5+z^3+z+1)$, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

where, for $\mathrm{CF}(a,b)$*,*

$$\begin{aligned}
p_0(z) &= z^{24}+z^{21}+z^{18}+z^{17}+z^{14}+z^{13}+z^{12}+z^{10}+z^6+z^5+z^3+z+1,\\
p_1(z) &= z^{27}+z^{25}+z^{23}+z^{22}+z^{19}+z^{18}+z^{14}+z^{13}+z^{11}+z^6+z^4+z^3+z^2\\
&\quad +z,\\
p_2(z) &= z^{28}+z^{26}+z^{24}+z^{22}+z^{10}+z^8+z^2,\\
p_3(z) &= z^{27}+z^{25}+z^{23}+z^{22}+z^{19}+z^{18}+z^{14}+z^{13}+z^{11}+z^6+z^4+z^3+z^2\\
&\quad +z,\\
p_4(z) &= z^{26}+z^{22}+z^{21}+z^{20}+z^{18}+z^{17}+z^{16}+z^{14}+z^{13}+z^6+z^5+z^4+z^3\\
&\quad +z^2+z,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$*,*

$$\begin{aligned}
p_0(z) &= z^{21}+z^{20}+z^{18}+z^{17}+z^{14}+z^{13}+z^{10}+z^6+z^5+z^3+z,\\
p_1(z) &= z^{27}+z^{25}+z^{23}+z^{22}+z^{19}+z^{18}+z^{14}+z^{13}+z^{11}+z^6+z^4+z^3+z^2\\
&\quad +z,\\
p_2(z) &= z^{28}+z^{26}+z^{24}+z^{22}+z^{10}+z^8+z^2,\\
p_3(z) &= z^{27}+z^{25}+z^{23}+z^{22}+z^{19}+z^{18}+z^{14}+z^{13}+z^{11}+z^6+z^4+z^3+z^2\\
&\quad +z,\\
p_4(z) &= z^{26}+z^{24}+z^{22}+z^{21}+z^{18}+z^{17}+z^{16}+z^{14}+z^{13}+z^{12}+z^6+z^5+z^4\\
&\quad +z^3+z^2+z+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x) &= x^{\deg(t_{2^n-1})}\begin{pmatrix} t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad \cdots\times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x) &= x^{\deg(\bar t_{2^n-1})}\begin{pmatrix} \bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix} \bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad \cdots\times x^{\deg(\bar t_0)}\begin{pmatrix} \bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n \geq 0$

$$\begin{aligned} M_{n+1}(x) &= W_n(x) \cdot M_n(x), \\ W_{n+1}(x) &= M_n(x) \cdot W_n(x). \end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$\begin{aligned} M_n(x)_{0,1} &= x^{d_n} \tilde{P}_{2^n-1}(x), \\ M_n(x)_{0,0} &= x^{d_n} \tilde{Q}_{2^n-1}(x), \end{aligned}$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \leq i,j \leq 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \leq i,j \leq 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \leq i,j \leq 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(175, 175, 175, 175, 175)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \geq 0$. If for some $n \geq 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \geq 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z\left(\phi(W^o,0,0)(x,z),\ z^{12}\cdot\phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 630 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 630, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z\left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ to order 770 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 770, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. $\square$

Define

$$R^e = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

Lemma 2.2. *For all integers $k \ge 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned}M^e[:14u] = {} & M^e[:7u] + x^u M^e[:6u] + x^{2u}M^e[:5u] + x^{4u}M^e[:3u] + x^{5u}M^e[:2u]\\ & + x^u M^e[:7u] + x^{2u}M^e[:6u] + x^{3u}M^e[:5u] + x^{5u}M^e[:3u]\\ & + x^{6u}M^e[:2u] + x^{3u}M^e[:7u] + x^{4u}M^e[:6u] + x^{5u}M^e[:5u]\\ & + x^{7u}M^e[:3u] + x^{8u}M^e[:2u] + x^{5u}M^e[:7u] + x^{6u}M^e[:6u]\end{aligned}$$

$$
\begin{aligned}
&+ x^{7u}M^e[:5u] + x^{9u}M^e[:3u] + x^{10u}M^e[:2u] + x^{7u}M^e[:7u]\\
&+ x^{8u}M^e[:6u] + x^{10u}M^e[:4u] + x^{11u}M^e[:3u] + x^{13u}M^e[:u]\\
&+ (x^{7u} + x^{8u} + x^{10u} + x^{12u})R^e,\\
W^e[:14u] &= W^e[:7u] + x^uW^e[:6u] + x^{2u}W^e[:5u] + x^{4u}W^e[:3u] + x^{5u}W^e[:2u]\\
&+ x^uW^e[:7u] + x^{2u}W^e[:6u] + x^{3u}W^e[:5u] + x^{5u}W^e[:3u]\\
&+ x^{6u}W^e[:2u] + x^{3u}W^e[:7u] + x^{4u}W^e[:6u] + x^{5u}W^e[:5u]\\
&+ x^{7u}W^e[:3u] + x^{8u}W^e[:2u] + x^{5u}W^e[:7u] + x^{6u}W^e[:6u]\\
&+ x^{7u}W^e[:5u] + x^{9u}W^e[:3u] + x^{10u}W^e[:2u] + x^{7u}W^e[:7u]\\
&+ x^{8u}W^e[:6u] + x^{10u}W^e[:4u] + x^{11u}W^e[:3u] + x^{13u}W^e[:u]\\
&+ (x^{7u} + x^{8u} + x^{10u} + x^{12u})R^e.
\end{aligned}
$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$
\begin{aligned}
M^o[:14u] &= M^o[:7u] + x^uM^o[:6u] + x^{2u}M^o[:5u] + x^{4u}M^o[:3u] + x^{5u}M^o[:2u]\\
&+ x^uM^o[:7u] + x^{2u}M^o[:6u] + x^{3u}M^o[:5u] + x^{5u}M^o[:3u]\\
&+ x^{6u}M^o[:2u] + x^{3u}M^o[:7u] + x^{4u}M^o[:6u] + x^{5u}M^o[:5u]\\
&+ x^{7u}M^o[:3u] + x^{8u}M^o[:2u] + x^{5u}M^o[:7u] + x^{6u}M^o[:6u]\\
&+ x^{7u}M^o[:5u] + x^{9u}M^o[:3u] + x^{10u}M^o[:2u] + x^{7u}M^o[:7u]\\
&+ x^{8u}M^o[:6u] + x^{10u}M^o[:4u] + x^{11u}M^o[:3u] + x^{13u}M^o[:u]\\
&+ (x^{7u} + x^{8u} + x^{10u} + x^{12u})R^o,\\
W^o[:14u] &= W^o[:7u] + x^uW^o[:6u] + x^{2u}W^o[:5u] + x^{4u}W^o[:3u] + x^{5u}W^o[:2u]\\
&+ x^uW^o[:7u] + x^{2u}W^o[:6u] + x^{3u}W^o[:5u] + x^{5u}W^o[:3u]\\
&+ x^{6u}W^o[:2u] + x^{3u}W^o[:7u] + x^{4u}W^o[:6u] + x^{5u}W^o[:5u]\\
&+ x^{7u}W^o[:3u] + x^{8u}W^o[:2u] + x^{5u}W^o[:7u] + x^{6u}W^o[:6u]\\
&+ x^{7u}W^o[:5u] + x^{9u}W^o[:3u] + x^{10u}W^o[:2u] + x^{7u}W^o[:7u]\\
&+ x^{8u}W^o[:6u] + x^{10u}W^o[:4u] + x^{11u}W^o[:3u] + x^{13u}W^o[:u]\\
&+ (x^{7u} + x^{8u} + x^{10u} + x^{12u})R^o.
\end{aligned}
$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$
\begin{aligned}
T[:14u] &= T[:7u] + x^uT[:6u] + x^{2u}T[:5u] + x^{4u}T[:3u] + x^{5u}T[:2u] + x^uT[:7u]\\
&+ x^{2u}T[:6u] + x^{3u}T[:5u] + x^{5u}T[:3u] + x^{6u}T[:2u] + x^{3u}T[:7u]\\
&+ x^{4u}T[:6u] + x^{5u}T[:5u] + x^{7u}T[:3u] + x^{8u}T[:2u] + x^{5u}T[:7u]\\
&+ x^{6u}T[:6u] + x^{7u}T[:5u] + x^{9u}T[:3u] + x^{10u}T[:2u] + x^{7u}T[:7u]\\
&+ x^{8u}T[:6u] + x^{10u}T[:4u] + x^{11u}T[:3u] + x^{13u}T[:u].
\end{aligned}
$$

The proof of the other 15 cases are similar. We break down the above identity into 7 parts:

$$\begin{aligned}
0 &= x^{7u}T[:u] + x^{5u}T[2u:3u] + x^{4u}T[3u:4u] + x^{2u}T[5u:6u] \\
&\quad + x^{u}T[6u:7u] + T[7u:8u], \\
0 &= x^{7u}T[u:2u] + x^{6u}T[2u:3u] + x^{4u}T[4u:5u] + x^{3u}T[5u:6u] \\
&\quad + T[8u:9u], \\
0 &= x^{9u}T[:u] + x^{7u}T[2u:3u] + x^{6u}T[3u:4u] + x^{4u}T[5u:6u] \\
&\quad + x^{3u}T[6u:7u] + T[9u:10u], \\
0 &= x^{9u}T[u:2u] + x^{8u}T[2u:3u] + x^{6u}T[4u:5u] + x^{5u}T[5u:6u] \\
&\quad + T[10u:11u], \\
0 &= x^{11u}T[:u] + x^{9u}T[2u:3u] + x^{8u}T[3u:4u] + x^{6u}T[5u:6u] \\
&\quad + x^{5u}T[6u:7u] + T[11u:12u], \\
0 &= x^{11u}T[u:2u] + x^{10u}T[2u:3u] + x^{8u}T[4u:5u] + x^{7u}T[5u:6u] \\
&\quad + T[12u:13u], \\
0 &= x^{13u}T[:u] + x^{11u}T[2u:3u] + x^{10u}T[3u:4u] + x^{8u}T[5u:6u] \\
&\quad + x^{7u}T[6u:7u] + T[13u:14u],
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
0 &= T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2] \\
&\quad + T[[111w]_2], \qquad (2.7) \\
0 &= T[[1w]_2] + T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1000w]_2], \qquad (2.8) \\
0 &= T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2] \\
&\quad + T[[1001w]_2], \qquad (2.9) \\
0 &= T[[1w]_2] + T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1010w]_2], \qquad (2.10) \\
0 &= T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2] \\
&\quad + T[[1011w]_2], \qquad (2.11) \\
0 &= T[[1w]_2] + T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1100w]_2], \qquad (2.12) \\
0 &= T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2] \\
&\quad + T[[1101w]_2], \qquad (2.13)
\end{aligned}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$\begin{aligned}
0 &= T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2] \\
&\quad + T[[111w]_2], \qquad (2.14) \\
0 &= T[[1w]_2] + T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1000w]_2], \qquad (2.15) \\
0 &= T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2] \\
&\quad + T[[1001w]_2], \qquad (2.16) \\
0 &= T[[1w]_2] + T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1010w]_2], \qquad (2.17) \\
0 &= T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2] \\
&\quad + T[[1011w]_2], \qquad (2.18)
\end{aligned}$$

$$0 = T[[1w]_2] + T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1100w]_2], \tag{2.19}$$

$$\begin{aligned}0 &= T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2]\\ &\quad + T[[1101w]_2],\end{aligned} \tag{2.20}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 688 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.20) can be written as

$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101))\\ &\quad + \tau(A(s,110)) + \tau(A(s,111)),\end{aligned} \tag{2.21}$$

$$\begin{aligned}0 &= \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,101))\\ &\quad + \tau(A(s,1000)),\end{aligned} \tag{2.22}$$

$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101))\\ &\quad + \tau(A(s,110)) + \tau(A(s,1001)),\end{aligned} \tag{2.23}$$

$$\begin{aligned}0 &= \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,101))\\ &\quad + \tau(A(s,1010)),\end{aligned} \tag{2.24}$$

$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101))\\ &\quad + \tau(A(s,110)) + \tau(A(s,1011)),\end{aligned} \tag{2.25}$$

$$\begin{aligned}0 &= \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,101))\\ &\quad + \tau(A(s,1100)),\end{aligned} \tag{2.26}$$

$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101))\\ &\quad + \tau(A(s,110)) + \tau(A(s,1101)),\end{aligned} \tag{2.27}$$

for all $s \in E_{2k}$, and

$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101))\\ &\quad + \tau(A(s,110)) + \tau(A(s,111)),\end{aligned} \tag{2.28}$$

$$\begin{aligned}0 &= \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,101))\\ &\quad + \tau(A(s,1000)),\end{aligned} \tag{2.29}$$

$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101))\\ &\quad + \tau(A(s,110)) + \tau(A(s,1001)),\end{aligned} \tag{2.30}$$

$$\begin{aligned}0 &= \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,101))\\ &\quad + \tau(A(s,1010)),\end{aligned} \tag{2.31}$$

$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101))\\ &\quad + \tau(A(s,110)) + \tau(A(s,1011)),\end{aligned} \tag{2.32}$$

$$\begin{aligned}0 &= \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,101))\\ &\quad + \tau(A(s,1100)),\end{aligned} \tag{2.33}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101))$$

$$+\tau(A(s,110))+\tau(A(s,1101)), \tag{2.34}$$

for $s=A(i,0^{2k})$. We find that $(A(i,0^{20}),E_{20})=(A(i,0^{16}),E_{16})$, so that we only have to verify that identities (2.21) through (2.34) hold for $2\le k\le 10$, which turns out to be true. □

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k\ge 2$, and $u=2^{2k-1}$,*

$$\begin{aligned}M_{2k}&=M^e[:7u]+x^uM^e[:6u]+x^{2u}M^e[:5u]+x^{4u}M^e[:3u]+x^{5u}M^e[:2u]\\&\quad+x^{7u}R^e,\\W_{2k}&=W^e[:7u]+x^uW^e[:6u]+x^{2u}W^e[:5u]+x^{4u}W^e[:3u]+x^{5u}W^e[:2u]\\&\quad+x^{7u}R^e.\end{aligned}$$

For all integer $k\ge 2$, and $u=2^{2k}$,

$$\begin{aligned}M_{2k+1}&=M^o[:7u]+x^uM^o[:6u]+x^{2u}M^o[:5u]+x^{4u}M^o[:3u]+x^{5u}M^o[:2u]\\&\quad+x^{7u}R^o,\\W_{2k+1}&=W^o[:7u]+x^uW^o[:6u]+x^{2u}W^o[:5u]+x^{4u}W^o[:3u]+x^{5u}W^o[:2u]\\&\quad+x^{7u}R^o.\end{aligned}$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n=2$, the identities can be verified directly. For $n\ge 2$, we claim that

$$\begin{aligned}M_{2k}\wedge W_{2k}&\Rightarrow M_{2k+1}\wedge W_{2k+1},\\M_{2k+1}\wedge W_{2k+1}&\Rightarrow M_{2k+2}\wedge W_{2k+2}.\end{aligned}$$

We give the proof of

$$M_{2k}\wedge W_{2k}\Rightarrow M_{2k+1},$$

the other ones being similar. Set $u=2^{2k}$ and $v=2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$\begin{aligned}W_{2k}M_{2k}&=\big(W^e[:7v]+x^vW^e[:6v]+x^{2v}W^e[:5v]+x^{4v}W^e[:3v]\\&\quad+x^{5v}W^e[:2v]+x^{7u}R^e\big)\times\big(M^e[:7v]+x^vM^e[:6v]+x^{2v}M^e[:5v]\\&\quad+x^{4v}M^e[:3v]+x^{5v}M^e[:2v]+x^{7u}R^e\big);\end{aligned}\tag{2.35}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{14v}R^o$. Therefore we only need to prove that their difference is $O(x^{14v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{14v}, to

$$\begin{aligned}&W^e[:14v]M^e[:14v]+x^{2v}W^e[:12v]M^e[:12v]+x^{4v}W^e[:10v]M^e[:10v]\\&\qquad+x^{8v}W^e[:6v]M^e[:6v]+x^{10v}W^e[:4v]M^e[:4v].\end{aligned}$$

For all $n\le 14$, replace the occurrences of $W^e[:n\cdot v]$ and $M^e[:n\cdot v]$ in the above expression by the reduction modulo $x^{n\cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$X:=x^v,$$

$$\begin{aligned} a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.35) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{14}, b_1, ..., b_{14}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{14})$, which completes the proof. $\square$

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \leq j, k \leq 1$,

$$\begin{aligned} \lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k}, \\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k}, \\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k}, \\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}. \end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a, b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x, y) = \mathrm{Res}_t\left(\phi(M^e, 0, 1)(x, t), y^{12}\phi(M^e, 0, 1)(x, t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x, y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned} q_0(x) &= x^{28} + x^{27} + x^{25} + x^{23} + x^{22} + x^{18} + x^{16} + x^{15} + x^{14} + x^{11} + x^{10} + x^7 \\ &\quad + x^4, \\ q_1(x) &= x^{27} + x^{26} + x^{25} + x^{24} + x^{22} + x^{17} + x^{15} + x^{14} + x^{10} + x^9 + x^6 + x^5 \\ &\quad + x^3 + x, \\ q_2(x) &= x^{26} + x^{20} + x^{18} + x^6 + x^4 + x^2 + 1, \\ q_3(x) &= x^{27} + x^{26} + x^{25} + x^{24} + x^{22} + x^{17} + x^{15} + x^{14} + x^{10} + x^9 + x^6 + x^5 \\ &\quad + x^3 + x, \\ q_4(x) &= x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{15} + x^{14} + x^{12} + x^{11} + x^{10} + x^8 \\ &\quad + x^7 + x^6 + x^2. \end{aligned}$$

The polynomial $Q(x, y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x, y)$ of multiplicity m and $R(x, y) := P(x, y)/Q(x, y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 216, and substitute it for y in $R(z,y)$, we get a series with valuation smaller than 216, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b)=f(1/z)$. $\square$

3. Annexe

All 16 polynomials are of the form

$$p_0(x)+p_3(x)y^3+p_6(x)y^6+p_9(x)y^9+p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e,0,0)$:

$$\begin{aligned}
p_0(x) &= x^{162}+x^{160}+x^{154}+x^{152}+x^{150}+x^{148}+x^{146}+x^{144}+x^{142}+x^{140}\\
&\quad+x^{138}+x^{136}+x^{132}+x^{128}+x^{122}+x^{120}+x^{118}+x^{116}+x^{114}\\
&\quad+x^{106}+x^{104}+x^{102}+x^{96}+x^{94}+x^{90}+x^{88}+x^{86}+x^{82}+x^{78}+x^{68}\\
&\quad+x^{66}+x^{58}+x^{56}+x^{50}+x^{42}+x^{36}+x^{30}+x^{24}+x^{22}+x^{20}+x^{12},\\
p_3(x) &= x^{158}+x^{148}+x^{142}+x^{140}+x^{138}+x^{130}+x^{126}+x^{122}+x^{120}+x^{116}\\
&\quad+x^{114}+x^{112}+x^{108}+x^{104}+x^{92}+x^{90}+x^{86}+x^{84}+x^{80}+x^{78}\\
&\quad+x^{72}+x^{64}+x^{62}+x^{60}+x^{58}+x^{54}+x^{48}+x^{46}+x^{42}+x^{34}+x^{22}\\
&\quad+x^{18}+x^{16}+x^{8},\\
p_6(x) &= x^{156}+x^{152}+x^{148}+x^{146}+x^{144}+x^{142}+x^{140}+x^{138}+x^{136}+x^{134}\\
&\quad+x^{132}+x^{128}+x^{126}+x^{120}+x^{116}+x^{114}+x^{104}+x^{102}+x^{100}\\
&\quad+x^{98}+x^{94}+x^{88}+x^{86}+x^{80}+x^{74}+x^{72}+x^{70}+x^{68}+x^{66}+x^{64}\\
&\quad+x^{56}+x^{52}+x^{44}+x^{42}+x^{38}+x^{36}+x^{34}+x^{30}+x^{26}+x^{18}+x^{16}\\
&\quad+x^{12}+x^{8}+x^{4}+x^{2}+1,\\
p_9(x) &= x^{160}+x^{158}+x^{154}+x^{148}+x^{140}+x^{138}+x^{132}+x^{130}+x^{126}+x^{120}\\
&\quad+x^{108}+x^{102}+x^{98}+x^{92}+x^{86}+x^{84}+x^{76}+x^{72}+x^{70}+x^{68}+x^{66}\\
&\quad+x^{64}+x^{62}+x^{52}+x^{46}+x^{40}+x^{38}+x^{36}+x^{34}+x^{28}+x^{26}+x^{24}\\
&\quad+x^{22}+x^{18}+x^{14}+x^{12}+x^{8}+x^{6}+x^{4}+x^{2},\\
p_{12}(x) &= x^{160}+x^{152}+x^{144}+x^{128}+x^{120}+x^{104}+x^{88}+x^{72}+x^{64}+x^{32}\\
&\quad+x^{24}+x^{8}+1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0].

For $\phi(M^e,0,1)$:

$$\begin{aligned}
p_0(x) &= x^{147}+x^{146}+x^{143}+x^{142}+x^{141}+x^{140}+x^{139}+x^{138}+x^{136}+x^{134}\\
&\quad+x^{133}+x^{130}+x^{126}+x^{123}+x^{121}+x^{120}+x^{117}+x^{116}+x^{115}\\
&\quad+x^{113}+x^{111}+x^{109}+x^{106}+x^{104}+x^{103}+x^{102}+x^{99}+x^{96}+x^{93}\\
&\quad+x^{92}+x^{91}+x^{90}+x^{89}+x^{88}+x^{86}+x^{81}+x^{80}+x^{78}+x^{77}+x^{76}\\
&\quad+x^{72}+x^{68}+x^{67}+x^{65}+x^{64}+x^{63}+x^{62}+x^{58}+x^{57}+x^{55}+x^{54}
\end{aligned}$$

$$+ x^{52} + x^{50} + x^{47} + x^{46} + x^{45} + x^{43} + x^{42} + x^{41} + x^{39} + x^{35} + x^{34} + x^{32} + x^{31} + x^{30} + x^{27} + x^{22} + x^{19} + x^{18},$$

$$p_3(x) = x^{145} + x^{142} + x^{140} + x^{136} + x^{134} + x^{133} + x^{131} + x^{128} + x^{126} + x^{122} + x^{120} + x^{117} + x^{116} + x^{115} + x^{105} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{94} + x^{93} + x^{91} + x^{89} + x^{88} + x^{84} + x^{82} + x^{78} + x^{76} + x^{73} + x^{72} + x^{68} + x^{66} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{54} + x^{50} + x^{49} + x^{48} + x^{46} + x^{42} + x^{41} + x^{40} + x^{38} + x^{37} + x^{35} + x^{33} + x^{32} + x^{29} + x^{28} + x^{27} + x^{13} + x^{11} + x^{10} + x^{9},$$

$$p_6(x) = x^{144} + x^{143} + x^{142} + x^{140} + x^{139} + x^{137} + x^{135} + x^{134} + x^{132} + x^{131} + x^{130} + x^{126} + x^{125} + x^{123} + x^{121} + x^{119} + x^{117} + x^{116} + x^{114} + x^{113} + x^{111} + x^{110} + x^{108} + x^{107} + x^{106} + x^{104} + x^{103} + x^{101} + x^{100} + x^{94} + x^{93} + x^{92} + x^{90} + x^{89} + x^{88} + x^{82} + x^{81} + x^{79} + x^{77} + x^{76} + x^{74} + x^{73} + x^{71} + x^{70} + x^{56} + x^{55} + x^{54} + x^{52} + x^{51} + x^{49} + x^{47} + x^{46} + x^{36} + x^{35} + x^{34} + x^{30} + x^{29} + x^{28} + x^{26} + x^{25} + x^{24} + x^{20} + x^{19} + x^{17} + x^{16} + x^{12} + x^{11} + x^{9} + x^{7} + x^{6},$$

$$p_9(x) = x^{143} + x^{141} + x^{140} + x^{139} + x^{137} + x^{134} + x^{133} + x^{132} + x^{131} + x^{130} + x^{128} + x^{124} + x^{122} + x^{121} + x^{119} + x^{116} + x^{114} + x^{113} + x^{111} + x^{109} + x^{108} + x^{105} + x^{104} + x^{101} + x^{100} + x^{96} + x^{95} + x^{94} + x^{92} + x^{91} + x^{89} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{80} + x^{79} + x^{73} + x^{71} + x^{70} + x^{66} + x^{64} + x^{61} + x^{60} + x^{57} + x^{56} + x^{52} + x^{50} + x^{47} + x^{46} + x^{45} + x^{41} + x^{39} + x^{38} + x^{37} + x^{35} + x^{33} + x^{32} + x^{31} + x^{27} + x^{24} + x^{23} + x^{22} + x^{20} + x^{16} + x^{14} + x^{13} + x^{11} + x^{9} + x^{8} + x^{5} + x^{4} + x^{3},$$

$$p_{12}(x) = x^{142} + x^{141} + x^{139} + x^{138} + x^{136} + x^{135} + x^{133} + x^{132} + x^{130} + x^{129} + x^{127} + x^{126} + x^{124} + x^{123} + x^{121} + x^{120} + x^{118} + x^{117} + x^{115} + x^{114} + x^{112} + x^{111} + x^{110} + x^{108} + x^{107} + x^{105} + x^{104} + x^{102} + x^{101} + x^{99} + x^{98} + x^{96} + x^{95} + x^{94} + x^{92} + x^{91} + x^{89} + x^{88} + x^{86} + x^{85} + x^{83} + x^{82} + x^{80} + x^{79} + x^{78} + x^{76} + x^{75} + x^{73} + x^{72} + x^{70} + x^{69} + x^{67} + x^{66} + x^{64} + x^{63} + x^{62} + x^{60} + x^{59} + x^{57} + x^{56} + x^{54} + x^{53} + x^{51} + x^{50} + x^{48} + x^{47} + x^{45} + x^{44} + x^{42} + x^{41} + x^{39} + x^{38} + x^{36} + x^{35} + x^{33} + x^{32} + x^{30} + x^{29} + x^{27} + x^{26} + x^{24} + x^{23} + x^{21} + x^{20} + x^{18} + x^{17} + x^{15} + x^{13} + x^{12} + x^{10} + x^{9} + x^{7} + x^{6} + x^{4} + x^{3} + x + 1,$$

and the initial terms are [0, 1, 0, 0, 0, 1, 0, 0, 1, 1, 1, 0, 0, 1, 0, 1].

For $\phi(M^e, 1, 0)$:

$$p_0(x) = x^{147} + x^{146} + x^{143} + x^{142} + x^{141} + x^{140} + x^{139} + x^{138} + x^{136} + x^{134}$$

$$
\begin{aligned}
&+ x^{133} + x^{130} + x^{126} + x^{123} + x^{121} + x^{120} + x^{117} + x^{116} + x^{115}\\
&+ x^{113} + x^{111} + x^{109} + x^{106} + x^{104} + x^{103} + x^{102} + x^{99} + x^{96} + x^{93}\\
&+ x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{86} + x^{81} + x^{80} + x^{78} + x^{77} + x^{76}\\
&+ x^{72} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{62} + x^{58} + x^{57} + x^{55} + x^{54}\\
&+ x^{52} + x^{50} + x^{47} + x^{46} + x^{45} + x^{43} + x^{42} + x^{41} + x^{39} + x^{35} + x^{34}\\
&+ x^{32} + x^{31} + x^{30} + x^{27} + x^{22} + x^{19} + x^{18},\\
p_3(x) = {}& x^{145} + x^{142} + x^{140} + x^{136} + x^{134} + x^{133} + x^{131} + x^{128} + x^{126} + x^{122}\\
&+ x^{120} + x^{117} + x^{116} + x^{115} + x^{105} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98}\\
&+ x^{97} + x^{96} + x^{94} + x^{93} + x^{91} + x^{89} + x^{88} + x^{84} + x^{82} + x^{78} + x^{76}\\
&+ x^{73} + x^{72} + x^{68} + x^{66} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56}\\
&+ x^{54} + x^{50} + x^{49} + x^{48} + x^{46} + x^{42} + x^{41} + x^{40} + x^{38} + x^{37} + x^{35}\\
&+ x^{33} + x^{32} + x^{29} + x^{28} + x^{27} + x^{13} + x^{11} + x^{10} + x^{9},\\
p_6(x) = {}& x^{144} + x^{143} + x^{142} + x^{140} + x^{139} + x^{137} + x^{135} + x^{134} + x^{132} + x^{131}\\
&+ x^{130} + x^{126} + x^{125} + x^{123} + x^{121} + x^{119} + x^{117} + x^{116} + x^{114}\\
&+ x^{113} + x^{111} + x^{110} + x^{108} + x^{107} + x^{106} + x^{104} + x^{103} + x^{101}\\
&+ x^{100} + x^{94} + x^{93} + x^{92} + x^{90} + x^{89} + x^{88} + x^{82} + x^{81} + x^{79} + x^{77}\\
&+ x^{76} + x^{74} + x^{73} + x^{71} + x^{70} + x^{56} + x^{55} + x^{54} + x^{52} + x^{51} + x^{49}\\
&+ x^{47} + x^{46} + x^{36} + x^{35} + x^{34} + x^{30} + x^{29} + x^{28} + x^{26} + x^{25} + x^{24}\\
&+ x^{20} + x^{19} + x^{17} + x^{16} + x^{12} + x^{11} + x^{9} + x^{7} + x^{6},\\
p_9(x) = {}& x^{143} + x^{141} + x^{140} + x^{139} + x^{137} + x^{134} + x^{133} + x^{132} + x^{131} + x^{130}\\
&+ x^{128} + x^{124} + x^{122} + x^{121} + x^{119} + x^{116} + x^{114} + x^{113} + x^{111}\\
&+ x^{109} + x^{108} + x^{105} + x^{104} + x^{101} + x^{100} + x^{96} + x^{95} + x^{94} + x^{92}\\
&+ x^{91} + x^{89} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{80} + x^{79} + x^{73} + x^{71}\\
&+ x^{70} + x^{66} + x^{64} + x^{61} + x^{60} + x^{57} + x^{56} + x^{52} + x^{50} + x^{47} + x^{46}\\
&+ x^{45} + x^{41} + x^{39} + x^{38} + x^{37} + x^{35} + x^{33} + x^{32} + x^{31} + x^{27} + x^{24}\\
&+ x^{23} + x^{22} + x^{20} + x^{16} + x^{14} + x^{13} + x^{11} + x^{9} + x^{8} + x^{5} + x^{4} + x^{3},\\
p_{12}(x) = {}& x^{142} + x^{141} + x^{139} + x^{138} + x^{136} + x^{135} + x^{133} + x^{132} + x^{130} + x^{129}\\
&+ x^{127} + x^{126} + x^{124} + x^{123} + x^{121} + x^{120} + x^{118} + x^{117} + x^{115}\\
&+ x^{114} + x^{112} + x^{111} + x^{110} + x^{108} + x^{107} + x^{105} + x^{104} + x^{102}\\
&+ x^{101} + x^{99} + x^{98} + x^{96} + x^{95} + x^{94} + x^{92} + x^{91} + x^{89} + x^{88} + x^{86}\\
&+ x^{85} + x^{83} + x^{82} + x^{80} + x^{79} + x^{78} + x^{76} + x^{75} + x^{73} + x^{72} + x^{70}\\
&+ x^{69} + x^{67} + x^{66} + x^{64} + x^{63} + x^{62} + x^{60} + x^{59} + x^{57} + x^{56} + x^{54}\\
&+ x^{53} + x^{51} + x^{50} + x^{48} + x^{47} + x^{45} + x^{44} + x^{42} + x^{41} + x^{39} + x^{38}\\
&+ x^{36} + x^{35} + x^{33} + x^{32} + x^{30} + x^{29} + x^{27} + x^{26} + x^{24} + x^{23} + x^{21}\\
&+ x^{20} + x^{18} + x^{17} + x^{15} + x^{13} + x^{12} + x^{10} + x^{9} + x^{7} + x^{6} + x^{4} + x^{3}
\end{aligned}
$$

$$+ x + 1,$$

and the initial terms are [0, 1, 0, 0, 0, 1, 0, 0, 1, 1, 1, 0, 0, 1, 0, 1].

For $\phi(M^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{132} + x^{112} + x^{110} + x^{102} + x^{100} + x^{98} + x^{96} + x^{92} + x^{88} + x^{70} + x^{66} \\
&\quad + x^{64} + x^{62} + x^{54} + x^{50} + x^{48} + x^{46} + x^{44} + x^{38} + x^{26} + x^{24}, \\
p_3(x) &= x^{126} + x^{116} + x^{114} + x^{110} + x^{106} + x^{102} + x^{98} + x^{96} + x^{92} + x^{90} \\
&\quad + x^{88} + x^{84} + x^{82} + x^{80} + x^{78} + x^{76} + x^{74} + x^{72} + x^{70} + x^{64} + x^{62} \\
&\quad + x^{58} + x^{56} + x^{54} + x^{46} + x^{40} + x^{36} + x^{30} + x^{22} + x^{6}, \\
p_6(x) &= x^{126} + x^{124} + x^{120} + x^{118} + x^{116} + x^{112} + x^{106} + x^{104} + x^{100} + x^{96} \\
&\quad + x^{94} + x^{92} + x^{86} + x^{80} + x^{78} + x^{76} + x^{60} + x^{58} + x^{54} + x^{52} + x^{50} \\
&\quad + x^{46} + x^{44} + x^{42} + x^{40} + x^{36} + x^{32} + x^{24} + x^{22} + x^{20} + x^{16} + x^{8} \\
&\quad + x^{4} + 1, \\
p_9(x) &= x^{124} + x^{120} + x^{116} + x^{114} + x^{112} + x^{106} + x^{102} + x^{100} + x^{96} + x^{92} \\
&\quad + x^{86} + x^{82} + x^{76} + x^{74} + x^{72} + x^{70} + x^{68} + x^{66} + x^{64} + x^{62} + x^{60} \\
&\quad + x^{58} + x^{56} + x^{52} + x^{50} + x^{48} + x^{44} + x^{38} + x^{36} + x^{26} + x^{24} + x^{22} \\
&\quad + x^{18} + x^{14} + x^{12} + x^{10} + x^{8} + x^{2}, \\
p_{12}(x) &= x^{124} + x^{122} + x^{118} + x^{114} + x^{112} + x^{108} + x^{106} + x^{102} + x^{98} + x^{96} \\
&\quad + x^{92} + x^{90} + x^{86} + x^{82} + x^{80} + x^{76} + x^{74} + x^{70} + x^{66} + x^{64} + x^{60} \\
&\quad + x^{58} + x^{54} + x^{50} + x^{48} + x^{44} + x^{42} + x^{38} + x^{34} + x^{32} + x^{12} + x^{10} \\
&\quad + x^{6} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{132} + x^{128} + x^{126} + x^{124} + x^{122} + x^{116} + x^{114} + x^{106} + x^{102} + x^{98} \\
&\quad + x^{92} + x^{88} + x^{86} + x^{84} + x^{76} + x^{72} + x^{68} + x^{64} + x^{60} + x^{58} + x^{54} \\
&\quad + x^{52} + x^{48} + x^{44} + x^{42} + x^{20} + x^{16} + x^{14} + x^{12}, \\
p_3(x) &= x^{126} + x^{122} + x^{118} + x^{110} + x^{102} + x^{100} + x^{88} + x^{84} + x^{80} + x^{78} \\
&\quad + x^{70} + x^{68} + x^{64} + x^{60} + x^{56} + x^{50} + x^{48} + x^{46} + x^{42} + x^{36} + x^{34} \\
&\quad + x^{32} + x^{30} + x^{24} + x^{20} + x^{18} + x^{16} + x^{14} + x^{12} + x^{8}, \\
p_6(x) &= x^{126} + x^{124} + x^{122} + x^{118} + x^{116} + x^{114} + x^{110} + x^{106} + x^{98} + x^{96} \\
&\quad + x^{88} + x^{86} + x^{82} + x^{68} + x^{62} + x^{60} + x^{58} + x^{56} + x^{50} + x^{46} + x^{44} \\
&\quad + x^{40} + x^{38} + x^{36} + x^{32} + x^{20} + x^{16} + x^{14} + x^{10} + 1, \\
p_9(x) &= x^{124} + x^{120} + x^{116} + x^{114} + x^{112} + x^{106} + x^{102} + x^{100} + x^{96} + x^{92} \\
&\quad + x^{86} + x^{82} + x^{76} + x^{74} + x^{72} + x^{70} + x^{68} + x^{66} + x^{64} + x^{62} + x^{60} \\
&\quad + x^{58} + x^{56} + x^{52} + x^{50} + x^{48} + x^{44} + x^{38} + x^{36} + x^{26} + x^{24} + x^{22} \\
&\quad + x^{18} + x^{14} + x^{12} + x^{10} + x^{8} + x^{2},
\end{aligned}$$

$$\begin{aligned}
p_{12}(x) = {} & x^{124} + x^{122} + x^{118} + x^{114} + x^{112} + x^{108} + x^{106} + x^{102} + x^{98} + x^{96} \\
& + x^{92} + x^{90} + x^{86} + x^{82} + x^{80} + x^{76} + x^{74} + x^{70} + x^{66} + x^{64} + x^{60} \\
& + x^{58} + x^{54} + x^{50} + x^{48} + x^{44} + x^{42} + x^{38} + x^{34} + x^{32} + x^{12} + x^{10} \\
& + x^{6} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 1)$:

$$\begin{aligned}
p_0(x) = {} & x^{147} + x^{144} + x^{142} + x^{139} + x^{138} + x^{137} + x^{135} + x^{134} + x^{131} + x^{130} \\
& + x^{129} + x^{128} + x^{126} + x^{125} + x^{117} + x^{115} + x^{113} + x^{111} + x^{109} \\
& + x^{108} + x^{104} + x^{103} + x^{100} + x^{99} + x^{96} + x^{95} + x^{94} + x^{93} + x^{89} \\
& + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{82} + x^{80} + x^{78} + x^{77} + x^{75} + x^{70} \\
& + x^{65} + x^{64} + x^{63} + x^{59} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} \\
& + x^{49} + x^{47} + x^{43} + x^{41} + x^{39} + x^{38} + x^{37} + x^{35} + x^{34} + x^{33} + x^{32} \\
& + x^{31} + x^{29} + x^{27}, \\
p_3(x) = {} & x^{145} + x^{142} + x^{140} + x^{136} + x^{134} + x^{133} + x^{131} + x^{128} + x^{126} + x^{122} \\
& + x^{120} + x^{117} + x^{116} + x^{115} + x^{105} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} \\
& + x^{97} + x^{96} + x^{94} + x^{93} + x^{91} + x^{89} + x^{88} + x^{84} + x^{82} + x^{78} + x^{76} \\
& + x^{73} + x^{72} + x^{68} + x^{66} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} \\
& + x^{54} + x^{50} + x^{49} + x^{48} + x^{46} + x^{42} + x^{41} + x^{40} + x^{38} + x^{37} + x^{35} \\
& + x^{33} + x^{32} + x^{29} + x^{28} + x^{27} + x^{13} + x^{11} + x^{10} + x^{9}, \\
p_6(x) = {} & x^{144} + x^{143} + x^{142} + x^{140} + x^{139} + x^{137} + x^{135} + x^{134} + x^{132} + x^{131} \\
& + x^{130} + x^{126} + x^{125} + x^{123} + x^{121} + x^{119} + x^{117} + x^{116} + x^{114} \\
& + x^{113} + x^{111} + x^{110} + x^{108} + x^{107} + x^{106} + x^{104} + x^{103} + x^{101} \\
& + x^{100} + x^{94} + x^{93} + x^{92} + x^{90} + x^{89} + x^{88} + x^{82} + x^{81} + x^{79} + x^{77} \\
& + x^{76} + x^{74} + x^{73} + x^{71} + x^{70} + x^{56} + x^{55} + x^{54} + x^{52} + x^{51} + x^{49} \\
& + x^{47} + x^{46} + x^{36} + x^{35} + x^{34} + x^{30} + x^{29} + x^{28} + x^{26} + x^{25} + x^{24} \\
& + x^{20} + x^{19} + x^{17} + x^{16} + x^{12} + x^{11} + x^{9} + x^{7} + x^{6}, \\
p_9(x) = {} & x^{143} + x^{141} + x^{140} + x^{139} + x^{137} + x^{134} + x^{133} + x^{132} + x^{131} + x^{130} \\
& + x^{128} + x^{124} + x^{122} + x^{121} + x^{119} + x^{116} + x^{114} + x^{113} + x^{111} \\
& + x^{109} + x^{108} + x^{105} + x^{104} + x^{101} + x^{100} + x^{96} + x^{95} + x^{94} + x^{92} \\
& + x^{91} + x^{89} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{80} + x^{79} + x^{73} + x^{71} \\
& + x^{70} + x^{66} + x^{64} + x^{61} + x^{60} + x^{57} + x^{56} + x^{52} + x^{50} + x^{47} + x^{46} \\
& + x^{45} + x^{41} + x^{39} + x^{38} + x^{37} + x^{35} + x^{33} + x^{32} + x^{31} + x^{27} + x^{24} \\
& + x^{23} + x^{22} + x^{20} + x^{16} + x^{14} + x^{13} + x^{11} + x^{9} + x^{8} + x^{5} + x^{4} + x^{3}, \\
p_{12}(x) = {} & x^{142} + x^{141} + x^{139} + x^{138} + x^{136} + x^{135} + x^{133} + x^{132} + x^{130} + x^{129} \\
& + x^{127} + x^{126} + x^{124} + x^{123} + x^{121} + x^{120} + x^{118} + x^{117} + x^{115}
\end{aligned}$$

$$+ x^{114} + x^{112} + x^{111} + x^{110} + x^{108} + x^{107} + x^{105} + x^{104} + x^{102}$$
$$+ x^{101} + x^{99} + x^{98} + x^{96} + x^{95} + x^{94} + x^{92} + x^{91} + x^{89} + x^{88} + x^{86}$$
$$+ x^{85} + x^{83} + x^{82} + x^{80} + x^{79} + x^{78} + x^{76} + x^{75} + x^{73} + x^{72} + x^{70}$$
$$+ x^{69} + x^{67} + x^{66} + x^{64} + x^{63} + x^{62} + x^{60} + x^{59} + x^{57} + x^{56} + x^{54}$$
$$+ x^{53} + x^{51} + x^{50} + x^{48} + x^{47} + x^{45} + x^{44} + x^{42} + x^{41} + x^{39} + x^{38}$$
$$+ x^{36} + x^{35} + x^{33} + x^{32} + x^{30} + x^{29} + x^{27} + x^{26} + x^{24} + x^{23} + x^{21}$$
$$+ x^{20} + x^{18} + x^{17} + x^{15} + x^{13} + x^{12} + x^{10} + x^{9} + x^{7} + x^{6} + x^{4} + x^{3}$$
$$+ x + 1,$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 0, 0, 1, 1, 0].
For $\phi(W^e, 1, 0)$:

$$\begin{aligned}
p_0(x) = {} & x^{147} + x^{144} + x^{142} + x^{139} + x^{138} + x^{137} + x^{135} + x^{134} + x^{131} + x^{130} \\
& + x^{129} + x^{128} + x^{126} + x^{125} + x^{117} + x^{115} + x^{113} + x^{111} + x^{109} \\
& + x^{108} + x^{104} + x^{103} + x^{100} + x^{99} + x^{96} + x^{95} + x^{94} + x^{93} + x^{89} \\
& + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{82} + x^{80} + x^{78} + x^{77} + x^{75} + x^{70} \\
& + x^{65} + x^{64} + x^{63} + x^{59} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} \\
& + x^{49} + x^{47} + x^{43} + x^{41} + x^{39} + x^{38} + x^{37} + x^{35} + x^{34} + x^{33} + x^{32} \\
& + x^{31} + x^{29} + x^{27}, \\
p_3(x) = {} & x^{145} + x^{142} + x^{140} + x^{136} + x^{134} + x^{133} + x^{131} + x^{128} + x^{126} + x^{122} \\
& + x^{120} + x^{117} + x^{116} + x^{115} + x^{105} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} \\
& + x^{97} + x^{96} + x^{94} + x^{93} + x^{91} + x^{89} + x^{88} + x^{84} + x^{82} + x^{78} + x^{76} \\
& + x^{73} + x^{72} + x^{68} + x^{66} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} \\
& + x^{54} + x^{50} + x^{49} + x^{48} + x^{46} + x^{42} + x^{41} + x^{40} + x^{38} + x^{37} + x^{35} \\
& + x^{33} + x^{32} + x^{29} + x^{28} + x^{27} + x^{13} + x^{11} + x^{10} + x^{9}, \\
p_6(x) = {} & x^{144} + x^{143} + x^{142} + x^{140} + x^{139} + x^{137} + x^{135} + x^{134} + x^{132} + x^{131} \\
& + x^{130} + x^{126} + x^{125} + x^{123} + x^{121} + x^{119} + x^{117} + x^{116} + x^{114} \\
& + x^{113} + x^{111} + x^{110} + x^{108} + x^{107} + x^{106} + x^{104} + x^{103} + x^{101} \\
& + x^{100} + x^{94} + x^{93} + x^{92} + x^{90} + x^{89} + x^{88} + x^{82} + x^{81} + x^{79} + x^{77} \\
& + x^{76} + x^{74} + x^{73} + x^{71} + x^{70} + x^{56} + x^{55} + x^{54} + x^{52} + x^{51} + x^{49} \\
& + x^{47} + x^{46} + x^{36} + x^{35} + x^{34} + x^{30} + x^{29} + x^{28} + x^{26} + x^{25} + x^{24} \\
& + x^{20} + x^{19} + x^{17} + x^{16} + x^{12} + x^{11} + x^{9} + x^{7} + x^{6}, \\
p_9(x) = {} & x^{143} + x^{141} + x^{140} + x^{139} + x^{137} + x^{134} + x^{133} + x^{132} + x^{131} + x^{130} \\
& + x^{128} + x^{124} + x^{122} + x^{121} + x^{119} + x^{116} + x^{114} + x^{113} + x^{111} \\
& + x^{109} + x^{108} + x^{105} + x^{104} + x^{101} + x^{100} + x^{96} + x^{95} + x^{94} + x^{92} \\
& + x^{91} + x^{89} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{80} + x^{79} + x^{73} + x^{71} \\
& + x^{70} + x^{66} + x^{64} + x^{61} + x^{60} + x^{57} + x^{56} + x^{52} + x^{50} + x^{47} + x^{46}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{45} + x^{41} + x^{39} + x^{38} + x^{37} + x^{35} + x^{33} + x^{32} + x^{31} + x^{27} + x^{24} \\
&+ x^{23} + x^{22} + x^{20} + x^{16} + x^{14} + x^{13} + x^{11} + x^{9} + x^{8} + x^{5} + x^{4} + x^{3}, \\
p_{12}(x) = {}& x^{142} + x^{141} + x^{139} + x^{138} + x^{136} + x^{135} + x^{133} + x^{132} + x^{130} + x^{129} \\
&+ x^{127} + x^{126} + x^{124} + x^{123} + x^{121} + x^{120} + x^{118} + x^{117} + x^{115} \\
&+ x^{114} + x^{112} + x^{111} + x^{110} + x^{108} + x^{107} + x^{105} + x^{104} + x^{102} \\
&+ x^{101} + x^{99} + x^{98} + x^{96} + x^{95} + x^{94} + x^{92} + x^{91} + x^{89} + x^{88} + x^{86} \\
&+ x^{85} + x^{83} + x^{82} + x^{80} + x^{79} + x^{78} + x^{76} + x^{75} + x^{73} + x^{72} + x^{70} \\
&+ x^{69} + x^{67} + x^{66} + x^{64} + x^{63} + x^{62} + x^{60} + x^{59} + x^{57} + x^{56} + x^{54} \\
&+ x^{53} + x^{51} + x^{50} + x^{48} + x^{47} + x^{45} + x^{44} + x^{42} + x^{41} + x^{39} + x^{38} \\
&+ x^{36} + x^{35} + x^{33} + x^{32} + x^{30} + x^{29} + x^{27} + x^{26} + x^{24} + x^{23} + x^{21} \\
&+ x^{20} + x^{18} + x^{17} + x^{15} + x^{13} + x^{12} + x^{10} + x^{9} + x^{7} + x^{6} + x^{4} + x^{3} \\
&+ x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 0, 0, 1, 1, 0].

For $\phi(W^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) = {}& x^{162} + x^{158} + x^{152} + x^{150} + x^{138} + x^{136} + x^{134} + x^{132} + x^{128} + x^{126} \\
&+ x^{116} + x^{114} + x^{112} + x^{110} + x^{108} + x^{106} + x^{102} + x^{98} + x^{96} + x^{94} \\
&+ x^{90} + x^{88} + x^{82} + x^{78} + x^{74} + x^{72} + x^{66} + x^{60} + x^{52} + x^{50} + x^{46} \\
&+ x^{44} + x^{42}, \\
p_3(x) = {}& x^{156} + x^{150} + x^{148} + x^{146} + x^{142} + x^{140} + x^{136} + x^{134} + x^{132} + x^{126} \\
&+ x^{122} + x^{116} + x^{114} + x^{110} + x^{108} + x^{100} + x^{96} + x^{94} + x^{88} + x^{86} \\
&+ x^{84} + x^{80} + x^{78} + x^{76} + x^{72} + x^{68} + x^{66} + x^{64} + x^{62} + x^{58} + x^{56} \\
&+ x^{54} + x^{52} + x^{50} + x^{44} + x^{42} + x^{38} + x^{36} + x^{28} + x^{24} + x^{22} + x^{20} \\
&+ x^{8} + x^{6}, \\
p_6(x) = {}& x^{158} + x^{156} + x^{150} + x^{148} + x^{144} + x^{142} + x^{132} + x^{130} + x^{128} + x^{126} \\
&+ x^{124} + x^{120} + x^{116} + x^{114} + x^{112} + x^{110} + x^{104} + x^{102} + x^{98} + x^{90} \\
&+ x^{86} + x^{80} + x^{76} + x^{74} + x^{70} + x^{68} + x^{60} + x^{56} + x^{50} + x^{48} + x^{46} \\
&+ x^{44} + x^{42} + x^{40} + x^{32} + x^{30} + x^{28} + x^{26} + x^{20} + x^{12} + x^{8} + x^{6} + x^{2} \\
&+ 1, \\
p_9(x) = {}& x^{160} + x^{158} + x^{154} + x^{148} + x^{140} + x^{138} + x^{132} + x^{130} + x^{126} + x^{120} \\
&+ x^{108} + x^{102} + x^{98} + x^{92} + x^{86} + x^{84} + x^{76} + x^{72} + x^{70} + x^{68} + x^{66} \\
&+ x^{64} + x^{62} + x^{52} + x^{46} + x^{40} + x^{38} + x^{36} + x^{34} + x^{28} + x^{26} + x^{24} \\
&+ x^{22} + x^{18} + x^{14} + x^{12} + x^{8} + x^{6} + x^{4} + x^{2}, \\
p_{12}(x) = {}& x^{160} + x^{152} + x^{144} + x^{128} + x^{120} + x^{104} + x^{88} + x^{72} + x^{64} + x^{32} \\
&+ x^{24} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0].

For $\phi(M^o, 0, 0)$:

$$\begin{aligned}
p_0(x) = x^{147} &+ x^{146} + x^{145} + x^{143} + x^{142} + x^{141} + x^{140} + x^{139} + x^{138} + x^{136} \\
&+ x^{133} + x^{124} + x^{123} + x^{116} + x^{115} + x^{113} + x^{112} + x^{108} + x^{106} \\
&+ x^{105} + x^{103} + x^{102} + x^{101} + x^{100} + x^{98} + x^{97} + x^{96} + x^{94} + x^{92} \\
&+ x^{90} + x^{85} + x^{82} + x^{80} + x^{79} + x^{78} + x^{74} + x^{73} + x^{71} + x^{70} + x^{69} \\
&+ x^{68} + x^{66} + x^{65} + x^{57} + x^{52} + x^{47} + x^{45} + x^{43} + x^{42} + x^{41} + x^{35} \\
&+ x^{34} + x^{31} + x^{30} + x^{29} + x^{28} + x^{26} + x^{24} + x^{23} + x^{18} + x^{14} + x^{13} \\
&+ x^{12}, \\
p_3(x) = x^{146} &+ x^{144} + x^{139} + x^{136} + x^{135} + x^{133} + x^{129} + x^{128} + x^{125} + x^{124} \\
&+ x^{123} + x^{122} + x^{120} + x^{118} + x^{117} + x^{116} + x^{114} + x^{113} + x^{111} \\
&+ x^{110} + x^{108} + x^{107} + x^{104} + x^{102} + x^{99} + x^{98} + x^{97} + x^{96} + x^{94} \\
&+ x^{93} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{83} + x^{80} + x^{76} + x^{71} \\
&+ x^{70} + x^{69} + x^{67} + x^{63} + x^{60} + x^{57} + x^{55} + x^{54} + x^{53} + x^{50} + x^{48} \\
&+ x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{38} + x^{37} + x^{34} + x^{33} + x^{32} + x^{31} \\
&+ x^{29} + x^{27} + x^{25} + x^{23} + x^{22} + x^{16} + x^{15} + x^{10} + x^{9} + x^{8}, \\
p_6(x) = x^{146} &+ x^{144} + x^{141} + x^{140} + x^{136} + x^{135} + x^{130} + x^{129} + x^{125} + x^{122} \\
&+ x^{121} + x^{117} + x^{116} + x^{115} + x^{113} + x^{112} + x^{111} + x^{109} + x^{108} \\
&+ x^{103} + x^{102} + x^{100} + x^{99} + x^{98} + x^{97} + x^{94} + x^{93} + x^{92} + x^{91} + x^{88} \\
&+ x^{85} + x^{84} + x^{82} + x^{81} + x^{80} + x^{78} + x^{77} + x^{74} + x^{70} + x^{66} + x^{65} \\
&+ x^{64} + x^{63} + x^{62} + x^{60} + x^{59} + x^{57} + x^{53} + x^{50} + x^{49} + x^{48} + x^{46} \\
&+ x^{44} + x^{40} + x^{38} + x^{37} + x^{34} + x^{30} + x^{27} + x^{25} + x^{23} + x^{22} + x^{20} \\
&+ x^{19} + x^{17} + x^{16} + x^{13} + x^{11} + x^{8} + x^{7} + x^{5} + x^{3} + x + 1, \\
p_9(x) = x^{146} &+ x^{145} + x^{143} + x^{141} + x^{140} + x^{134} + x^{133} + x^{131} + x^{130} + x^{128} \\
&+ x^{127} + x^{126} + x^{124} + x^{123} + x^{122} + x^{120} + x^{119} + x^{118} + x^{116} \\
&+ x^{115} + x^{114} + x^{112} + x^{111} + x^{110} + x^{108} + x^{107} + x^{106} + x^{104} \\
&+ x^{103} + x^{102} + x^{100} + x^{99} + x^{98} + x^{96} + x^{95} + x^{94} + x^{92} + x^{91} + x^{90} \\
&+ x^{88} + x^{87} + x^{86} + x^{84} + x^{83} + x^{82} + x^{80} + x^{79} + x^{78} + x^{76} + x^{75} \\
&+ x^{74} + x^{72} + x^{71} + x^{70} + x^{68} + x^{67} + x^{66} + x^{64} + x^{63} + x^{62} + x^{60} \\
&+ x^{59} + x^{58} + x^{56} + x^{55} + x^{54} + x^{52} + x^{51} + x^{49} + x^{48} + x^{46} + x^{45} \\
&+ x^{43} + x^{42} + x^{40} + x^{39} + x^{37} + x^{36} + x^{34} + x^{33} + x^{31} + x^{29} + x^{28} \\
&+ x^{22} + x^{21} + x^{19} + x^{17} + x^{16} + x^{14} + x^{13} + x^{11} + x^{10} + x^{8} + x^{7} + x^{5} \\
&+ x^{4}, \\
p_{12}(x) = x^{146} &+ x^{145} + x^{143} + x^{142} + x^{140} + x^{139} + x^{138} + x^{136} + x^{135} + x^{134} \\
&+ x^{132} + x^{131} + x^{129} + x^{128} + x^{122} + x^{121} + x^{119} + x^{117} + x^{116} \\
&+ x^{106} + x^{105} + x^{103} + x^{101} + x^{100} + x^{90} + x^{89} + x^{87} + x^{85} + x^{84}
\end{aligned}$$

$$
\begin{aligned}
&+x^{74}+x^{73}+x^{71}+x^{69}+x^{68}+x^{58}+x^{57}+x^{55}+x^{53}+x^{52}+x^{50}\\
&+x^{49}+x^{47}+x^{46}+x^{44}+x^{43}+x^{41}+x^{40}+x^{38}+x^{37}+x^{35}+x^{34}\\
&+x^{32}+x^{31}+x^{30}+x^{28}+x^{27}+x^{26}+x^{24}+x^{23}+x^{22}+x^{20}+x^{19}\\
&+x^{18}+x^{16}+x^{15}+x^{14}+x^{12}+x^{11}+x^{9}+x^{8}+x^{6}+x^{5}+x^{3}+x+1,
\end{aligned}
$$

and the initial terms are [1, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0].

For $\phi(M^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{144}+x^{141}+x^{137}+x^{133}+x^{131}+x^{127}+x^{124}+x^{121}+x^{118}+x^{116}\\
&+x^{112}+x^{111}+x^{110}+x^{109}+x^{107}+x^{103}+x^{102}+x^{100}+x^{99}+x^{96}\\
&+x^{95}+x^{94}+x^{93}+x^{92}+x^{91}+x^{89}+x^{88}+x^{86}+x^{84}+x^{82}+x^{81}\\
&+x^{80}+x^{74}+x^{73}+x^{72}+x^{70}+x^{69}+x^{68}+x^{67}+x^{65}+x^{64}+x^{63}\\
&+x^{61}+x^{58}+x^{57}+x^{56}+x^{53}+x^{52}+x^{50}+x^{47}+x^{46}+x^{44}+x^{42}\\
&+x^{41}+x^{40}+x^{38}+x^{37}+x^{36}+x^{34}+x^{30}+x^{29}+x^{27}+x^{26}+x^{22}\\
&+x^{21}+x^{20}+x^{18},\\
p_3(x) &= x^{137}+x^{136}+x^{135}+x^{134}+x^{133}+x^{129}+x^{121}+x^{120}+x^{119}+x^{118}\\
&+x^{117}+x^{113}+x^{105}+x^{104}+x^{103}+x^{102}+x^{101}+x^{96}+x^{95}+x^{94}\\
&+x^{93}+x^{89}+x^{65}+x^{64}+x^{63}+x^{62}+x^{61}+x^{57}+x^{41}+x^{40}+x^{39}\\
&+x^{38}+x^{37}+x^{32}+x^{31}+x^{30}+x^{29}+x^{25}+x^{17}+x^{16}+x^{15}+x^{14}\\
&+x^{13}+x^{9},\\
p_6(x) &= x^{138}+x^{136}+x^{134}+x^{132}+x^{130}+x^{120}+x^{114}+x^{110}+x^{102}+x^{100}\\
&+x^{98}+x^{96}+x^{92}+x^{90}+x^{88}+x^{76}+x^{68}+x^{64}+x^{62}+x^{58}+x^{56}\\
&+x^{52}+x^{48}+x^{42}+x^{40}+x^{36}+x^{34}+x^{24}+x^{22}+x^{20}+x^{16}+x^{6},\\
p_9(x) &= x^{139}+x^{138}+x^{135}+x^{133}+x^{129}+x^{128}+x^{125}+x^{123}+x^{121}+x^{120}\\
&+x^{117}+x^{115}+x^{113}+x^{112}+x^{109}+x^{107}+x^{105}+x^{104}+x^{101}\\
&+x^{99}+x^{97}+x^{96}+x^{93}+x^{91}+x^{89}+x^{88}+x^{85}+x^{83}+x^{81}+x^{80}\\
&+x^{77}+x^{75}+x^{73}+x^{72}+x^{69}+x^{67}+x^{65}+x^{64}+x^{61}+x^{59}+x^{57}\\
&+x^{56}+x^{53}+x^{51}+x^{49}+x^{48}+x^{45}+x^{42}+x^{41}+x^{40}+x^{39}+x^{35}\\
&+x^{27}+x^{26}+x^{23}+x^{21}+x^{17}+x^{16}+x^{13}+x^{10}+x^{9}+x^{8}+x^{7}+x^{3},\\
p_{12}(x) &= x^{140}+x^{136}+x^{132}+x^{128}+x^{116}+x^{100}+x^{84}+x^{68}+x^{52}+x^{44}\\
&+x^{40}+x^{32}+x^{28}+x^{24}+x^{20}+x^{16}+x^{12}+x^{8}+1,
\end{aligned}
$$

and the initial terms are [0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0].

For $\phi(M^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{162}+x^{161}+x^{160}+x^{159}+x^{158}+x^{155}+x^{153}+x^{150}+x^{148}+x^{145}\\
&+x^{144}+x^{143}+x^{141}+x^{140}+x^{134}+x^{133}+x^{131}+x^{130}+x^{129}\\
&+x^{125}+x^{124}+x^{121}+x^{120}+x^{119}+x^{117}+x^{113}+x^{112}+x^{106}\\
&+x^{104}+x^{99}+x^{95}+x^{94}+x^{93}+x^{92}+x^{91}+x^{89}+x^{84}+x^{83}+x^{81}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{80} + x^{77} + x^{76} + x^{75} + x^{73} + x^{68} + x^{67} + x^{66} + x^{63} + x^{62} + x^{60}\\
&+ x^{59} + x^{56} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45}\\
&+ x^{44} + x^{42} + x^{41} + x^{40} + x^{37} + x^{33} + x^{32} + x^{29} + x^{28} + x^{27},\\
p_3(x) =& x^{161} + x^{160} + x^{159} + x^{158} + x^{156} + x^{155} + x^{154} + x^{148} + x^{147} + x^{146}\\
&+ x^{144} + x^{143} + x^{142} + x^{136} + x^{135} + x^{134} + x^{132} + x^{131} + x^{130}\\
&+ x^{129} + x^{128} + x^{127} + x^{126} + x^{121} + x^{105} + x^{104} + x^{103} + x^{102}\\
&+ x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{88} + x^{87} + x^{86} + x^{84}\\
&+ x^{83} + x^{82} + x^{76} + x^{75} + x^{74} + x^{73} + x^{68} + x^{67} + x^{66} + x^{65} + x^{60}\\
&+ x^{59} + x^{58} + x^{56} + x^{55} + x^{54} + x^{49} + x^{33} + x^{32} + x^{31} + x^{30} + x^{28}\\
&+ x^{27} + x^{26} + x^{20} + x^{19} + x^{18} + x^{16} + x^{15} + x^{14} + x^{9},\\
p_6(x) =& x^{162} + x^{160} + x^{152} + x^{148} + x^{146} + x^{138} + x^{122} + x^{120} + x^{112} + x^{110}\\
&+ x^{106} + x^{102} + x^{98} + x^{96} + x^{94} + x^{92} + x^{90} + x^{86} + x^{84} + x^{80} + x^{78}\\
&+ x^{74} + x^{72} + x^{70} + x^{64} + x^{62} + x^{56} + x^{52} + x^{48} + x^{46} + x^{44} + x^{38}\\
&+ x^{36} + x^{32} + x^{30} + x^{22} + x^{18} + x^{16} + x^{10} + x^{6},\\
p_9(x) =& x^{163} + x^{162} + x^{158} + x^{157} + x^{155} + x^{152} + x^{151} + x^{150} + x^{148} + x^{145}\\
&+ x^{144} + x^{142} + x^{141} + x^{139} + x^{138} + x^{136} + x^{135} + x^{133} + x^{132}\\
&+ x^{131} + x^{129} + x^{127} + x^{125} + x^{124} + x^{121} + x^{119} + x^{117} + x^{116}\\
&+ x^{113} + x^{111} + x^{109} + x^{108} + x^{105} + x^{103} + x^{101} + x^{100} + x^{97} + x^{95}\\
&+ x^{93} + x^{92} + x^{89} + x^{87} + x^{85} + x^{84} + x^{81} + x^{79} + x^{77} + x^{76} + x^{73}\\
&+ x^{71} + x^{69} + x^{68} + x^{67} + x^{66} + x^{65} + x^{63} + x^{62} + x^{60} + x^{59} + x^{57}\\
&+ x^{56} + x^{54} + x^{53} + x^{51} + x^{50} + x^{48} + x^{47} + x^{45} + x^{44} + x^{42} + x^{41}\\
&+ x^{39} + x^{38} + x^{36} + x^{34} + x^{33} + x^{32} + x^{26} + x^{22} + x^{21} + x^{19} + x^{16}\\
&+ x^{15} + x^{14} + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{3},\\
p_{12}(x) =& x^{164} + x^{152} + x^{136} + x^{124} + x^{116} + x^{112} + x^{108} + x^{100} + x^{96} + x^{92}\\
&+ x^{84} + x^{80} + x^{76} + x^{64} + x^{60} + x^{56} + x^{48} + x^{44} + x^{32} + x^{12} + x^{8}\\
&+ x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 0, 1, 0, 0].

For $\phi(M^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) =& x^{147} + x^{145} + x^{142} + x^{141} + x^{137} + x^{136} + x^{133} + x^{132} + x^{131} + x^{129}\\
&+ x^{128} + x^{127} + x^{125} + x^{124} + x^{123} + x^{121} + x^{117} + x^{115} + x^{111}\\
&+ x^{110} + x^{109} + x^{108} + x^{105} + x^{103} + x^{102} + x^{99} + x^{97} + x^{94} + x^{92}\\
&+ x^{91} + x^{89} + x^{88} + x^{82} + x^{81} + x^{78} + x^{77} + x^{71} + x^{70} + x^{65} + x^{61}\\
&+ x^{60} + x^{58} + x^{57} + x^{55} + x^{54} + x^{53} + x^{52} + x^{48} + x^{47} + x^{45} + x^{44}\\
&+ x^{41} + x^{40} + x^{39} + x^{37} + x^{36} + x^{33},\\
p_3(x) =& x^{146} + x^{143} + x^{141} + x^{137} + x^{136} + x^{132} + x^{130} + x^{128} + x^{127} + x^{124}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{110} + x^{109} + x^{108} + x^{105}\\
&+ x^{102} + x^{100} + x^{97} + x^{95} + x^{94} + x^{92} + x^{87} + x^{86} + x^{85} + x^{83} + x^{80}\\
&+ x^{76} + x^{72} + x^{70} + x^{65} + x^{61} + x^{58} + x^{56} + x^{55} + x^{54} + x^{53} + x^{50}\\
&+ x^{46} + x^{44} + x^{41} + x^{40} + x^{38} + x^{37} + x^{36} + x^{35} + x^{33} + x^{28} + x^{27}\\
&+ x^{26} + x^{22} + x^{21} + x^{19} + x^{17} + x^{16} + x^{13} + x^{12},\\
p_6(x) = &\, x^{146} + x^{143} + x^{133} + x^{132} + x^{128} + x^{126} + x^{123} + x^{121} + x^{117} + x^{116}\\
&+ x^{115} + x^{114} + x^{112} + x^{106} + x^{105} + x^{97} + x^{96} + x^{95} + x^{94} + x^{92}\\
&+ x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{83} + x^{82} + x^{78} + x^{77} + x^{73} + x^{71}\\
&+ x^{70} + x^{69} + x^{67} + x^{65} + x^{58} + x^{57} + x^{54} + x^{51} + x^{49} + x^{48} + x^{45}\\
&+ x^{40} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{31} + x^{30} + x^{29} + x^{20}\\
&+ x^{19} + x^{17} + x^{16} + x^{13} + x^{12} + x^{10} + x^{8} + x^{7} + x^{6} + x^{4} + x^{3} + x + 1,\\
p_9(x) = &\, x^{146} + x^{145} + x^{143} + x^{141} + x^{140} + x^{134} + x^{133} + x^{131} + x^{130} + x^{128}\\
&+ x^{127} + x^{126} + x^{124} + x^{123} + x^{122} + x^{120} + x^{119} + x^{118} + x^{116}\\
&+ x^{115} + x^{114} + x^{112} + x^{111} + x^{110} + x^{108} + x^{107} + x^{106} + x^{104}\\
&+ x^{103} + x^{102} + x^{100} + x^{99} + x^{98} + x^{96} + x^{95} + x^{94} + x^{92} + x^{91} + x^{90}\\
&+ x^{88} + x^{87} + x^{86} + x^{84} + x^{83} + x^{82} + x^{80} + x^{79} + x^{78} + x^{76} + x^{75}\\
&+ x^{74} + x^{72} + x^{71} + x^{70} + x^{68} + x^{67} + x^{66} + x^{64} + x^{63} + x^{62} + x^{60}\\
&+ x^{59} + x^{58} + x^{56} + x^{55} + x^{54} + x^{52} + x^{51} + x^{49} + x^{48} + x^{46} + x^{45}\\
&+ x^{43} + x^{42} + x^{40} + x^{39} + x^{37} + x^{36} + x^{34} + x^{33} + x^{31} + x^{29} + x^{28}\\
&+ x^{22} + x^{21} + x^{19} + x^{17} + x^{16} + x^{14} + x^{13} + x^{11} + x^{10} + x^{8} + x^{7} + x^{5}\\
&+ x^{4},\\
p_{12}(x) = &\, x^{146} + x^{145} + x^{143} + x^{142} + x^{140} + x^{139} + x^{138} + x^{136} + x^{135} + x^{134}\\
&+ x^{132} + x^{131} + x^{129} + x^{128} + x^{122} + x^{121} + x^{119} + x^{117} + x^{116}\\
&+ x^{106} + x^{105} + x^{103} + x^{101} + x^{100} + x^{90} + x^{89} + x^{87} + x^{85} + x^{84}\\
&+ x^{74} + x^{73} + x^{71} + x^{69} + x^{68} + x^{58} + x^{57} + x^{55} + x^{53} + x^{52} + x^{50}\\
&+ x^{49} + x^{47} + x^{46} + x^{44} + x^{43} + x^{41} + x^{40} + x^{38} + x^{37} + x^{35} + x^{34}\\
&+ x^{32} + x^{31} + x^{30} + x^{28} + x^{27} + x^{26} + x^{24} + x^{23} + x^{22} + x^{20} + x^{19}\\
&+ x^{18} + x^{16} + x^{15} + x^{14} + x^{12} + x^{11} + x^{9} + x^{8} + x^{6} + x^{5} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 0, 0, 0].

For $\phi(W^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) = &\, x^{147} + x^{146} + x^{145} + x^{143} + x^{142} + x^{141} + x^{140} + x^{139} + x^{138} + x^{136}\\
&+ x^{133} + x^{124} + x^{123} + x^{116} + x^{115} + x^{113} + x^{112} + x^{108} + x^{106}\\
&+ x^{105} + x^{103} + x^{102} + x^{101} + x^{100} + x^{98} + x^{97} + x^{96} + x^{94} + x^{92}\\
&+ x^{90} + x^{85} + x^{82} + x^{80} + x^{79} + x^{78} + x^{74} + x^{73} + x^{71} + x^{70} + x^{69}\\
&+ x^{68} + x^{66} + x^{65} + x^{57} + x^{52} + x^{47} + x^{45} + x^{43} + x^{42} + x^{41} + x^{35}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{34} + x^{31} + x^{30} + x^{29} + x^{28} + x^{26} + x^{24} + x^{23} + x^{18} + x^{14} + x^{13}\\
&+ x^{12},\\
p_3(x) = {}& x^{146} + x^{144} + x^{139} + x^{136} + x^{135} + x^{133} + x^{129} + x^{128} + x^{125} + x^{124}\\
&+ x^{123} + x^{122} + x^{120} + x^{118} + x^{117} + x^{116} + x^{114} + x^{113} + x^{111}\\
&+ x^{110} + x^{108} + x^{107} + x^{104} + x^{102} + x^{99} + x^{98} + x^{97} + x^{96} + x^{94}\\
&+ x^{93} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{83} + x^{80} + x^{76} + x^{71}\\
&+ x^{70} + x^{69} + x^{67} + x^{63} + x^{60} + x^{57} + x^{55} + x^{54} + x^{53} + x^{50} + x^{48}\\
&+ x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{38} + x^{37} + x^{34} + x^{33} + x^{32} + x^{31}\\
&+ x^{29} + x^{27} + x^{25} + x^{23} + x^{22} + x^{16} + x^{15} + x^{10} + x^{9} + x^{8},\\
p_6(x) = {}& x^{146} + x^{144} + x^{141} + x^{140} + x^{136} + x^{135} + x^{130} + x^{129} + x^{125} + x^{122}\\
&+ x^{121} + x^{117} + x^{116} + x^{115} + x^{113} + x^{112} + x^{111} + x^{109} + x^{108}\\
&+ x^{103} + x^{102} + x^{100} + x^{99} + x^{98} + x^{97} + x^{94} + x^{93} + x^{92} + x^{91} + x^{88}\\
&+ x^{85} + x^{84} + x^{82} + x^{81} + x^{80} + x^{78} + x^{77} + x^{74} + x^{70} + x^{66} + x^{65}\\
&+ x^{64} + x^{63} + x^{62} + x^{60} + x^{59} + x^{57} + x^{53} + x^{50} + x^{49} + x^{48} + x^{46}\\
&+ x^{44} + x^{40} + x^{38} + x^{37} + x^{34} + x^{30} + x^{27} + x^{25} + x^{23} + x^{22} + x^{20}\\
&+ x^{19} + x^{17} + x^{16} + x^{13} + x^{11} + x^{8} + x^{7} + x^{5} + x^{3} + x + 1,\\
p_9(x) = {}& x^{146} + x^{145} + x^{143} + x^{141} + x^{140} + x^{134} + x^{133} + x^{131} + x^{130} + x^{128}\\
&+ x^{127} + x^{126} + x^{124} + x^{123} + x^{122} + x^{120} + x^{119} + x^{118} + x^{116}\\
&+ x^{115} + x^{114} + x^{112} + x^{111} + x^{110} + x^{108} + x^{107} + x^{106} + x^{104}\\
&+ x^{103} + x^{102} + x^{100} + x^{99} + x^{98} + x^{96} + x^{95} + x^{94} + x^{92} + x^{91} + x^{90}\\
&+ x^{88} + x^{87} + x^{86} + x^{84} + x^{83} + x^{82} + x^{80} + x^{79} + x^{78} + x^{76} + x^{75}\\
&+ x^{74} + x^{72} + x^{71} + x^{70} + x^{68} + x^{67} + x^{66} + x^{64} + x^{63} + x^{62} + x^{60}\\
&+ x^{59} + x^{58} + x^{56} + x^{55} + x^{54} + x^{52} + x^{51} + x^{49} + x^{48} + x^{46} + x^{45}\\
&+ x^{43} + x^{42} + x^{40} + x^{39} + x^{37} + x^{36} + x^{34} + x^{33} + x^{31} + x^{29} + x^{28}\\
&+ x^{22} + x^{21} + x^{19} + x^{17} + x^{16} + x^{14} + x^{13} + x^{11} + x^{10} + x^{8} + x^{7} + x^{5}\\
&+ x^{4},\\
p_{12}(x) = {}& x^{146} + x^{145} + x^{143} + x^{142} + x^{140} + x^{139} + x^{138} + x^{136} + x^{135} + x^{134}\\
&+ x^{132} + x^{131} + x^{129} + x^{128} + x^{122} + x^{121} + x^{119} + x^{117} + x^{116}\\
&+ x^{106} + x^{105} + x^{103} + x^{101} + x^{100} + x^{90} + x^{89} + x^{87} + x^{85} + x^{84}\\
&+ x^{74} + x^{73} + x^{71} + x^{69} + x^{68} + x^{58} + x^{57} + x^{55} + x^{53} + x^{52} + x^{50}\\
&+ x^{49} + x^{47} + x^{46} + x^{44} + x^{43} + x^{41} + x^{40} + x^{38} + x^{37} + x^{35} + x^{34}\\
&+ x^{32} + x^{31} + x^{30} + x^{28} + x^{27} + x^{26} + x^{24} + x^{23} + x^{22} + x^{20} + x^{19}\\
&+ x^{18} + x^{16} + x^{15} + x^{14} + x^{12} + x^{11} + x^{9} + x^{8} + x^{6} + x^{5} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0].

For $\phi(W^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{162}+x^{161}+x^{160}+x^{159}+x^{158}+x^{155}+x^{153}+x^{150}+x^{148}+x^{145}\\
&\quad+x^{144}+x^{143}+x^{141}+x^{140}+x^{134}+x^{133}+x^{131}+x^{130}+x^{129}\\
&\quad+x^{125}+x^{124}+x^{121}+x^{120}+x^{119}+x^{117}+x^{113}+x^{112}+x^{106}\\
&\quad+x^{104}+x^{99}+x^{95}+x^{94}+x^{93}+x^{92}+x^{91}+x^{89}+x^{84}+x^{83}+x^{81}\\
&\quad+x^{80}+x^{77}+x^{76}+x^{75}+x^{73}+x^{68}+x^{67}+x^{66}+x^{63}+x^{62}+x^{60}\\
&\quad+x^{59}+x^{56}+x^{53}+x^{52}+x^{51}+x^{50}+x^{49}+x^{48}+x^{47}+x^{46}+x^{45}\\
&\quad+x^{44}+x^{42}+x^{41}+x^{40}+x^{37}+x^{33}+x^{32}+x^{29}+x^{28}+x^{27},\\
p_3(x) &= x^{161}+x^{160}+x^{159}+x^{158}+x^{156}+x^{155}+x^{154}+x^{148}+x^{147}+x^{146}\\
&\quad+x^{144}+x^{143}+x^{142}+x^{136}+x^{135}+x^{134}+x^{132}+x^{131}+x^{130}\\
&\quad+x^{129}+x^{128}+x^{127}+x^{126}+x^{121}+x^{105}+x^{104}+x^{103}+x^{102}\\
&\quad+x^{100}+x^{99}+x^{98}+x^{97}+x^{96}+x^{95}+x^{94}+x^{88}+x^{87}+x^{86}+x^{84}\\
&\quad+x^{83}+x^{82}+x^{76}+x^{75}+x^{74}+x^{73}+x^{68}+x^{67}+x^{66}+x^{65}+x^{60}\\
&\quad+x^{59}+x^{58}+x^{56}+x^{55}+x^{54}+x^{49}+x^{33}+x^{32}+x^{31}+x^{30}+x^{28}\\
&\quad+x^{27}+x^{26}+x^{20}+x^{19}+x^{18}+x^{16}+x^{15}+x^{14}+x^{9},\\
p_6(x) &= x^{162}+x^{160}+x^{152}+x^{148}+x^{146}+x^{138}+x^{122}+x^{120}+x^{112}+x^{110}\\
&\quad+x^{106}+x^{102}+x^{98}+x^{96}+x^{94}+x^{92}+x^{90}+x^{86}+x^{84}+x^{80}+x^{78}\\
&\quad+x^{74}+x^{72}+x^{70}+x^{64}+x^{62}+x^{56}+x^{52}+x^{48}+x^{46}+x^{44}+x^{38}\\
&\quad+x^{36}+x^{32}+x^{30}+x^{22}+x^{18}+x^{16}+x^{10}+x^{6},\\
p_9(x) &= x^{163}+x^{162}+x^{158}+x^{157}+x^{155}+x^{152}+x^{151}+x^{150}+x^{148}+x^{145}\\
&\quad+x^{144}+x^{142}+x^{141}+x^{139}+x^{138}+x^{136}+x^{135}+x^{133}+x^{132}\\
&\quad+x^{131}+x^{129}+x^{127}+x^{125}+x^{124}+x^{121}+x^{119}+x^{117}+x^{116}\\
&\quad+x^{113}+x^{111}+x^{109}+x^{108}+x^{105}+x^{103}+x^{101}+x^{100}+x^{97}+x^{95}\\
&\quad+x^{93}+x^{92}+x^{89}+x^{87}+x^{85}+x^{84}+x^{81}+x^{79}+x^{77}+x^{76}+x^{73}\\
&\quad+x^{71}+x^{69}+x^{68}+x^{67}+x^{66}+x^{65}+x^{63}+x^{62}+x^{60}+x^{59}+x^{57}\\
&\quad+x^{56}+x^{54}+x^{53}+x^{51}+x^{50}+x^{48}+x^{47}+x^{45}+x^{44}+x^{42}+x^{41}\\
&\quad+x^{39}+x^{38}+x^{36}+x^{34}+x^{33}+x^{32}+x^{26}+x^{22}+x^{21}+x^{19}+x^{16}\\
&\quad+x^{15}+x^{14}+x^{12}+x^{11}+x^{10}+x^{9}+x^{8}+x^{3},\\
p_{12}(x) &= x^{164}+x^{152}+x^{136}+x^{124}+x^{116}+x^{112}+x^{108}+x^{100}+x^{96}+x^{92}\\
&\quad+x^{84}+x^{80}+x^{76}+x^{64}+x^{60}+x^{56}+x^{48}+x^{44}+x^{32}+x^{12}+x^{8}\\
&\quad+x^{4}+1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 0, 1, 0, 0].

For $\phi(W^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{144}+x^{141}+x^{137}+x^{133}+x^{131}+x^{127}+x^{124}+x^{121}+x^{118}+x^{116}\\
&\quad+x^{112}+x^{111}+x^{110}+x^{109}+x^{107}+x^{103}+x^{102}+x^{100}+x^{99}+x^{96}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{89} + x^{88} + x^{86} + x^{84} + x^{82} + x^{81} \\
&+ x^{80} + x^{74} + x^{73} + x^{72} + x^{70} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} \\
&+ x^{61} + x^{58} + x^{57} + x^{56} + x^{53} + x^{52} + x^{50} + x^{47} + x^{46} + x^{44} + x^{42} \\
&+ x^{41} + x^{40} + x^{38} + x^{37} + x^{36} + x^{34} + x^{30} + x^{29} + x^{27} + x^{26} + x^{22} \\
&+ x^{21} + x^{20} + x^{18}, \\
p_3(x) = {}& x^{137} + x^{136} + x^{135} + x^{134} + x^{133} + x^{129} + x^{121} + x^{120} + x^{119} + x^{118} \\
&+ x^{117} + x^{113} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{96} + x^{95} + x^{94} \\
&+ x^{93} + x^{89} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{57} + x^{41} + x^{40} + x^{39} \\
&+ x^{38} + x^{37} + x^{32} + x^{31} + x^{30} + x^{29} + x^{25} + x^{17} + x^{16} + x^{15} + x^{14} \\
&+ x^{13} + x^{9}, \\
p_6(x) = {}& x^{138} + x^{136} + x^{134} + x^{132} + x^{130} + x^{120} + x^{114} + x^{110} + x^{102} + x^{100} \\
&+ x^{98} + x^{96} + x^{92} + x^{90} + x^{88} + x^{76} + x^{68} + x^{64} + x^{62} + x^{58} + x^{56} \\
&+ x^{52} + x^{48} + x^{42} + x^{40} + x^{36} + x^{34} + x^{24} + x^{22} + x^{20} + x^{16} + x^{6}, \\
p_9(x) = {}& x^{139} + x^{138} + x^{135} + x^{133} + x^{129} + x^{128} + x^{125} + x^{123} + x^{121} + x^{120} \\
&+ x^{117} + x^{115} + x^{113} + x^{112} + x^{109} + x^{107} + x^{105} + x^{104} + x^{101} \\
&+ x^{99} + x^{97} + x^{96} + x^{93} + x^{91} + x^{89} + x^{88} + x^{85} + x^{83} + x^{81} + x^{80} \\
&+ x^{77} + x^{75} + x^{73} + x^{72} + x^{69} + x^{67} + x^{65} + x^{64} + x^{61} + x^{59} + x^{57} \\
&+ x^{56} + x^{53} + x^{51} + x^{49} + x^{48} + x^{45} + x^{42} + x^{41} + x^{40} + x^{39} + x^{35} \\
&+ x^{27} + x^{26} + x^{23} + x^{21} + x^{17} + x^{16} + x^{13} + x^{10} + x^{9} + x^{8} + x^{7} + x^{3}, \\
p_{12}(x) = {}& x^{140} + x^{136} + x^{132} + x^{128} + x^{116} + x^{100} + x^{84} + x^{68} + x^{52} + x^{44} \\
&+ x^{40} + x^{32} + x^{28} + x^{24} + x^{20} + x^{16} + x^{12} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0].

For $\phi(W^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) = {}& x^{147} + x^{145} + x^{142} + x^{141} + x^{137} + x^{136} + x^{133} + x^{132} + x^{131} + x^{129} \\
&+ x^{128} + x^{127} + x^{125} + x^{124} + x^{123} + x^{121} + x^{117} + x^{115} + x^{111} \\
&+ x^{110} + x^{109} + x^{108} + x^{105} + x^{103} + x^{102} + x^{99} + x^{97} + x^{94} + x^{92} \\
&+ x^{91} + x^{89} + x^{88} + x^{82} + x^{81} + x^{78} + x^{77} + x^{71} + x^{70} + x^{65} + x^{61} \\
&+ x^{60} + x^{58} + x^{57} + x^{55} + x^{54} + x^{53} + x^{52} + x^{48} + x^{47} + x^{45} + x^{44} \\
&+ x^{41} + x^{40} + x^{39} + x^{37} + x^{36} + x^{33}, \\
p_3(x) = {}& x^{146} + x^{143} + x^{141} + x^{137} + x^{136} + x^{132} + x^{130} + x^{128} + x^{127} + x^{124} \\
&+ x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{110} + x^{109} + x^{108} + x^{105} \\
&+ x^{102} + x^{100} + x^{97} + x^{95} + x^{94} + x^{92} + x^{87} + x^{86} + x^{85} + x^{83} + x^{80} \\
&+ x^{76} + x^{72} + x^{70} + x^{65} + x^{61} + x^{58} + x^{56} + x^{55} + x^{54} + x^{53} + x^{50} \\
&+ x^{46} + x^{44} + x^{41} + x^{40} + x^{38} + x^{37} + x^{36} + x^{35} + x^{33} + x^{28} + x^{27} \\
&+ x^{26} + x^{22} + x^{21} + x^{19} + x^{17} + x^{16} + x^{13} + x^{12},
\end{aligned}
$$

$$\begin{aligned}
p_6(x) &= x^{146} + x^{143} + x^{133} + x^{132} + x^{128} + x^{126} + x^{123} + x^{121} + x^{117} + x^{116} \\
&+ x^{115} + x^{114} + x^{112} + x^{106} + x^{105} + x^{97} + x^{96} + x^{95} + x^{94} + x^{92} \\
&+ x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{83} + x^{82} + x^{78} + x^{77} + x^{73} + x^{71} \\
&+ x^{70} + x^{69} + x^{67} + x^{65} + x^{58} + x^{57} + x^{54} + x^{51} + x^{49} + x^{48} + x^{45} \\
&+ x^{40} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{31} + x^{30} + x^{29} + x^{20} \\
&+ x^{19} + x^{17} + x^{16} + x^{13} + x^{12} + x^{10} + x^{8} + x^{7} + x^{6} + x^{4} + x^{3} + x + 1, \\
p_9(x) &= x^{146} + x^{145} + x^{143} + x^{141} + x^{140} + x^{134} + x^{133} + x^{131} + x^{130} + x^{128} \\
&+ x^{127} + x^{126} + x^{124} + x^{123} + x^{122} + x^{120} + x^{119} + x^{118} + x^{116} \\
&+ x^{115} + x^{114} + x^{112} + x^{111} + x^{110} + x^{108} + x^{107} + x^{106} + x^{104} \\
&+ x^{103} + x^{102} + x^{100} + x^{99} + x^{98} + x^{96} + x^{95} + x^{94} + x^{92} + x^{91} + x^{90} \\
&+ x^{88} + x^{87} + x^{86} + x^{84} + x^{83} + x^{82} + x^{80} + x^{79} + x^{78} + x^{76} + x^{75} \\
&+ x^{74} + x^{72} + x^{71} + x^{70} + x^{68} + x^{67} + x^{66} + x^{64} + x^{63} + x^{62} + x^{60} \\
&+ x^{59} + x^{58} + x^{56} + x^{55} + x^{54} + x^{52} + x^{51} + x^{49} + x^{48} + x^{46} + x^{45} \\
&+ x^{43} + x^{42} + x^{40} + x^{39} + x^{37} + x^{36} + x^{34} + x^{33} + x^{31} + x^{29} + x^{28} \\
&+ x^{22} + x^{21} + x^{19} + x^{17} + x^{16} + x^{14} + x^{13} + x^{11} + x^{10} + x^{8} + x^{7} + x^{5} \\
&+ x^{4}, \\
p_{12}(x) &= x^{146} + x^{145} + x^{143} + x^{142} + x^{140} + x^{139} + x^{138} + x^{136} + x^{135} + x^{134} \\
&+ x^{132} + x^{131} + x^{129} + x^{128} + x^{122} + x^{121} + x^{119} + x^{117} + x^{116} \\
&+ x^{106} + x^{105} + x^{103} + x^{101} + x^{100} + x^{90} + x^{89} + x^{87} + x^{85} + x^{84} \\
&+ x^{74} + x^{73} + x^{71} + x^{69} + x^{68} + x^{58} + x^{57} + x^{55} + x^{53} + x^{52} + x^{50} \\
&+ x^{49} + x^{47} + x^{46} + x^{44} + x^{43} + x^{41} + x^{40} + x^{38} + x^{37} + x^{35} + x^{34} \\
&+ x^{32} + x^{31} + x^{30} + x^{28} + x^{27} + x^{26} + x^{24} + x^{23} + x^{22} + x^{20} + x^{19} \\
&+ x^{18} + x^{16} + x^{15} + x^{14} + x^{12} + x^{11} + x^{9} + x^{8} + x^{6} + x^{5} + x^{3} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 0, 0, 0].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	138	[18, 206]	276	[161, 321]	414	[540, 43]	552	[607, 418]
1	[3, 4]	139	[245, 246]	277	[323, 381]	415	[544, 359]	553	[608, 609]
2	[5, 6]	140	[247, 248]	278	[346, 156]	416	[177, 144]	554	[610, 611]
3	[7, 8]	141	[249, 250]	279	[382, 383]	417	[545, 132]	555	[612, 613]
4	[9, 10]	142	[251, 252]	280	[183, 127]	418	[367, 131]	556	[614, 615]
5	[11, 12]	143	[253, 254]	281	[325, 126]	419	[347, 172]	557	[616, 617]
6	[13, 14]	144	[58, 255]	282	[378, 351]	420	[546, 361]	558	[618, 456]
7	[15, 16]	145	[256, 257]	283	[384, 134]	421	[363, 347]	559	[619, 620]
8	[17, 18]	146	[258, 259]	284	[385, 166]	422	[547, 548]	560	[621, 587]
9	[19, 20]	147	[260, 261]	285	[345, 123]	423	[114, 159]	561	[622, 190]

10	[21, 22]	148	[225, 262]	286	[337, 138]	424	[184, 125]	562	[623, 214]
11	[23, 24]	149	[263, 264]	287	[157, 386]	425	[549, 125]	563	[624, 625]
12	[25, 26]	150	[265, 266]	288	[183, 371]	426	[550, 163]	564	[626, 627]
13	[27, 28]	151	[267, 268]	289	[387, 170]	427	[551, 341]	565	[628, 29]
14	[29, 30]	152	[269, 270]	290	[388, 120]	428	[398, 545]	566	[629, 630]
15	[31, 32]	153	[74, 271]	291	[389, 390]	429	[50, 120]	567	[243, 631]
16	[33, 34]	154	[272, 273]	292	[160, 138]	430	[327, 358]	568	[632, 633]
17	[35, 36]	155	[274, 275]	293	[136, 347]	431	[545, 56]	569	[634, 635]
18	[13, 37]	156	[187, 276]	294	[391, 163]	432	[125, 185]	570	[636, 637]
19	[38, 32]	157	[277, 278]	295	[392, 172]	433	[174, 361]	571	[638, 639]
20	[39, 40]	158	[279, 280]	296	[51, 355]	434	[37, 121]	572	[471, 83]
21	[41, 42]	159	[281, 282]	297	[369, 381]	435	[172, 161]	573	[640, 641]
22	[43, 44]	160	[283, 284]	298	[393, 119]	436	[391, 177]	574	[642, 643]
23	[45, 46]	161	[276, 57]	299	[394, 180]	437	[394, 129]	575	[644, 503]
24	[47, 48]	162	[285, 286]	300	[148, 131]	438	[154, 182]	576	[645, 301]
25	[49, 2]	163	[287, 288]	301	[114, 371]	439	[344, 136]	577	[442, 448]
26	[50, 51]	164	[289, 290]	302	[185, 125]	440	[347, 108]	578	[646, 647]
27	[52, 53]	165	[291, 292]	303	[343, 132]	441	[346, 321]	579	[215, 251]
28	[54, 10]	166	[72, 293]	304	[141, 341]	442	[109, 321]	580	[648, 21]
29	[55, 56]	167	[294, 232]	305	[332, 113]	443	[125, 184]	581	[649, 650]
30	[53, 34]	168	[295, 296]	306	[348, 184]	444	[552, 41]	582	[651, 521]
31	[57, 58]	169	[297, 298]	307	[395, 8]	445	[548, 359]	583	[625, 63]
32	[59, 60]	170	[299, 300]	308	[147, 111]	446	[385, 358]	584	[537, 345]
33	[61, 62]	171	[301, 302]	309	[53, 341]	447	[170, 132]	585	[546, 175]
34	[63, 64]	172	[189, 265]	310	[396, 333]	448	[48, 182]	586	[123, 321]
35	[65, 66]	173	[303, 304]	311	[397, 115]	449	[321, 172]	587	[560, 185]
36	[67, 68]	174	[305, 306]	312	[398, 170]	450	[553, 554]	588	[330, 159]
37	[69, 70]	175	[307, 308]	313	[32, 144]	451	[108, 136]	589	[10, 182]
38	[71, 72]	176	[28, 309]	314	[399, 322]	452	[555, 53]	590	[577, 378]
39	[73, 74]	177	[310, 311]	315	[118, 121]	453	[545, 171]	591	[365, 554]
40	[75, 76]	178	[312, 192]	316	[400, 331]	454	[55, 171]	592	[48, 155]
41	[77, 78]	179	[16, 313]	317	[41, 44]	455	[551, 142]	593	[352, 390]
42	[79, 80]	180	[314, 315]	318	[32, 166]	456	[47, 154]	594	[652, 172]
43	[81, 82]	181	[316, 317]	319	[343, 56]	457	[133, 142]	595	[653, 378]
44	[83, 84]	182	[20, 318]	320	[401, 402]	458	[556, 152]	596	[654, 350]
45	[85, 86]	183	[318, 319]	321	[402, 188]	459	[40, 111]	597	[543, 554]
46	[87, 88]	184	[184, 184]	322	[403, 208]	460	[557, 36]	598	[374, 149]
47	[89, 90]	185	[219, 220]	323	[404, 405]	461	[558, 51]	599	[538, 158]
48	[91, 92]	186	[320, 161]	324	[406, 93]	462	[380, 338]	600	[549, 185]
49	[93, 94]	187	[321, 108]	325	[407, 408]	463	[123, 347]	601	[576, 139]
50	[95, 4]	188	[123, 150]	326	[264, 409]	464	[32, 46]	602	[552, 43]
51	[96, 97]	189	[172, 136]	327	[410, 411]	465	[36, 171]	603	[549, 184]
52	[98, 99]	190	[169, 134]	328	[412, 413]	466	[559, 113]	604	[179, 111]
53	[100, 101]	191	[163, 46]	329	[414, 415]	467	[560, 125]	605	[360, 175]

54	[102, 103]	192	[322, 14]	330	[416, 417]	468	[561, 379]	606	[51, 121]
55	[104, 105]	193	[170, 56]	331	[418, 419]	469	[137, 338]	607	[109, 347]
56	[106, 107]	194	[323, 117]	332	[420, 421]	470	[150, 172]	608	[569, 573]
57	[108, 109]	195	[161, 150]	333	[422, 423]	471	[562, 156]	609	[156, 108]
58	[110, 111]	196	[324, 325]	334	[424, 425]	472	[161, 347]	610	[655, 656]
59	[112, 113]	197	[326, 127]	335	[426, 427]	473	[563, 161]	611	[393, 121]
60	[114, 115]	198	[327, 46]	336	[428, 429]	474	[370, 386]	612	[347, 122]
61	[116, 117]	199	[328, 56]	337	[213, 430]	475	[350, 159]	613	[393, 176]
62	[118, 119]	200	[329, 330]	338	[429, 431]	476	[564, 390]	614	[372, 129]
63	[120, 121]	201	[43, 331]	339	[66, 432]	477	[321, 345]	615	[345, 109]
64	[122, 123]	202	[332, 333]	340	[433, 434]	478	[565, 558]	616	[125, 125]
65	[124, 125]	203	[334, 125]	341	[99, 435]	479	[173, 355]	617	[326, 115]
66	[126, 127]	204	[335, 336]	342	[436, 263]	480	[375, 44]	618	[579, 133]
67	[128, 129]	205	[337, 338]	343	[437, 438]	481	[566, 331]	619	[657, 361]
68	[130, 131]	206	[36, 339]	344	[439, 295]	482	[567, 32]	620	[136, 321]
69	[36, 132]	207	[340, 341]	345	[217, 440]	483	[336, 380]	621	[348, 185]
70	[133, 134]	208	[342, 48]	346	[441, 186]	484	[559, 333]	622	[658, 163]
71	[135, 136]	209	[169, 341]	347	[241, 442]	485	[185, 184]	623	[123, 156]
72	[137, 138]	210	[343, 339]	348	[425, 443]	486	[568, 381]	624	[122, 161]
73	[35, 139]	211	[33, 134]	349	[444, 445]	487	[109, 150]	625	[652, 122]
74	[140, 51]	212	[167, 181]	350	[446, 447]	488	[569, 152]	626	[659, 554]
75	[141, 142]	213	[33, 142]	351	[448, 449]	489	[392, 122]	627	[320, 123]
76	[143, 144]	214	[344, 123]	352	[450, 451]	490	[570, 41]	628	[583, 53]
77	[145, 146]	215	[156, 345]	353	[452, 453]	491	[43, 42]	629	[151, 573]
78	[147, 138]	216	[346, 347]	354	[454, 455]	492	[400, 42]	630	[380, 111]
79	[148, 149]	217	[122, 109]	355	[456, 457]	493	[400, 351]	631	[40, 338]
80	[150, 122]	218	[334, 348]	356	[458, 459]	494	[145, 390]	632	[660, 386]
81	[151, 152]	219	[334, 185]	357	[460, 461]	495	[179, 338]	633	[124, 185]
82	[137, 153]	220	[348, 348]	358	[462, 463]	496	[571, 152]	634	[553, 366]
83	[154, 155]	221	[185, 348]	359	[464, 465]	497	[150, 108]	635	[563, 123]
84	[156, 122]	222	[327, 166]	360	[466, 467]	498	[572, 41]	636	[136, 156]
85	[157, 158]	223	[139, 171]	361	[468, 469]	499	[556, 573]	637	[130, 149]
86	[126, 159]	224	[349, 350]	362	[470, 471]	500	[337, 153]	638	[659, 366]
87	[160, 153]	225	[165, 351]	363	[472, 473]	501	[557, 139]	639	[122, 136]
88	[161, 156]	226	[352, 146]	364	[474, 475]	502	[574, 37]	640	[661, 167]
89	[162, 163]	227	[347, 345]	365	[476, 477]	503	[384, 341]	641	[130, 155]
90	[164, 110]	228	[353, 140]	366	[478, 479]	504	[177, 46]	642	[662, 133]
91	[165, 44]	229	[354, 355]	367	[480, 481]	505	[575, 164]	643	[581, 181]
92	[165, 42]	230	[41, 331]	368	[482, 483]	506	[160, 111]	644	[658, 177]
93	[45, 166]	231	[356, 351]	369	[484, 485]	507	[576, 36]	645	[185, 185]
94	[167, 10]	232	[357, 179]	370	[486, 487]	508	[577, 165]	646	[539, 139]
95	[168, 169]	233	[163, 358]	371	[445, 230]	509	[116, 381]	647	[143, 46]
96	[170, 171]	234	[359, 159]	372	[488, 489]	510	[120, 355]	648	[572, 43]
97	[169, 34]	235	[348, 125]	373	[490, 491]	511	[112, 333]	649	[567, 143]

98	[156, 172]	236	[360, 361]	374	[492, 493]	512	[359, 127]	650	[663, 40]
99	[173, 119]	237	[354, 176]	375	[494, 495]	513	[578, 544]	651	[363, 150]
100	[174, 175]	238	[118, 176]	376	[496, 497]	514	[397, 159]	652	[664, 216]
101	[173, 176]	239	[108, 123]	377	[498, 316]	515	[350, 127]	653	[665, 666]
102	[162, 177]	240	[320, 109]	378	[499, 500]	516	[579, 33]	654	[667, 668]
103	[178, 179]	241	[362, 122]	379	[501, 502]	517	[376, 180]	655	[669, 670]
104	[128, 180]	242	[363, 156]	380	[503, 504]	518	[172, 109]	656	[671, 672]
105	[181, 182]	243	[345, 136]	381	[505, 506]	519	[580, 581]	657	[673, 674]
106	[183, 115]	244	[364, 166]	382	[507, 464]	520	[148, 155]	658	[675, 87]
107	[184, 185]	245	[365, 366]	383	[508, 224]	521	[135, 123]	659	[676, 677]
108	[186, 187]	246	[367, 149]	384	[509, 510]	522	[362, 108]	660	[678, 679]
109	[188, 189]	247	[168, 53]	385	[511, 512]	523	[582, 347]	661	[680, 681]
110	[190, 191]	248	[368, 48]	386	[513, 514]	524	[384, 142]	662	[682, 683]
111	[192, 193]	249	[369, 117]	387	[443, 515]	525	[564, 146]	663	[684, 685]
112	[194, 195]	250	[14, 355]	388	[516, 212]	526	[362, 172]	664	[321, 122]
113	[196, 197]	251	[37, 355]	389	[517, 518]	527	[550, 177]	665	[582, 321]
114	[198, 199]	252	[345, 161]	390	[519, 520]	528	[53, 134]	666	[374, 131]
115	[200, 201]	253	[370, 158]	391	[521, 462]	529	[568, 117]	667	[653, 165]
116	[202, 203]	254	[326, 371]	392	[522, 523]	530	[582, 156]	668	[383, 126]
117	[204, 205]	255	[109, 156]	393	[193, 524]	531	[385, 144]	669	[555, 169]
118	[206, 207]	256	[372, 180]	394	[525, 526]	532	[55, 339]	670	[542, 56]
119	[208, 209]	257	[135, 161]	395	[527, 528]	533	[124, 184]	671	[662, 33]
120	[210, 211]	258	[373, 9]	396	[529, 530]	534	[583, 169]	672	[686, 154]
121	[212, 213]	259	[374, 182]	397	[531, 532]	535	[389, 146]	673	[660, 158]
122	[214, 215]	260	[141, 34]	398	[533, 106]	536	[110, 338]	674	[125, 348]
123	[216, 217]	261	[45, 358]	399	[534, 96]	537	[584, 242]	675	[108, 161]
124	[218, 219]	262	[375, 331]	400	[535, 536]	538	[585, 586]	676	[571, 573]
125	[220, 221]	263	[8, 110]	401	[172, 123]	539	[587, 588]	677	[537, 122]
126	[222, 223]	264	[177, 358]	402	[537, 172]	540	[523, 589]	678	[657, 175]
127	[224, 225]	265	[150, 345]	403	[52, 169]	541	[590, 281]	679	[562, 347]
128	[226, 227]	266	[136, 150]	404	[538, 386]	542	[591, 592]	680	[570, 43]
129	[228, 229]	267	[376, 129]	405	[184, 348]	543	[593, 594]	681	[356, 331]
130	[230, 231]	268	[344, 161]	406	[539, 36]	544	[595, 596]	682	[392, 345]
131	[232, 233]	269	[377, 54]	407	[540, 41]	545	[597, 598]	683	[354, 119]
132	[234, 235]	270	[367, 182]	408	[541, 330]	546	[599, 600]	684	[387, 545]
133	[236, 237]	271	[139, 339]	409	[542, 339]	547	[601, 222]	685	[656, 14]
134	[238, 239]	272	[375, 351]	410	[396, 113]	548	[602, 200]	686	[687, 274]
135	[240, 241]	273	[378, 42]	411	[330, 127]	549	[603, 65]	687	[38, 143]
136	[242, 243]	274	[379, 380]	412	[543, 366]	550	[473, 604]		
137	[209, 244]	275	[143, 166]	413	[10, 155]	551	[605, 606]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	99	0	198	1	297	1	396	1	495	0	594	0

1	0	100	0	199	1	298	0	397	0	496	0	595	1
2	0	101	0	200	1	299	0	398	0	497	1	596	1
3	0	102	1	201	0	300	0	399	0	498	0	597	1
4	0	103	0	202	0	301	1	400	0	499	1	598	0
5	0	104	1	203	0	302	0	401	0	500	0	599	0
6	0	105	0	204	1	303	0	402	0	501	1	600	1
7	0	106	0	205	0	304	1	403	0	502	1	601	0
8	0	107	0	206	1	305	0	404	0	503	0	602	0
9	0	108	0	207	0	306	1	405	0	504	1	603	1
10	1	109	0	208	0	307	1	406	1	505	0	604	0
11	0	110	1	209	1	308	1	407	1	506	0	605	1
12	0	111	0	210	0	309	0	408	0	507	0	606	0
13	0	112	0	211	0	310	1	409	0	508	0	607	0
14	1	113	1	212	0	311	0	410	1	509	0	608	1
15	0	114	1	213	0	312	0	411	1	510	0	609	0
16	0	115	1	214	1	313	0	412	1	511	0	610	1
17	0	116	0	215	0	314	0	413	1	512	0	611	0
18	0	117	1	216	0	315	1	414	1	513	1	612	1
19	0	118	1	217	1	316	0	415	1	514	0	613	0
20	0	119	0	218	0	317	1	416	1	515	0	614	1
21	1	120	0	219	0	318	0	417	1	516	0	615	1
22	0	121	0	220	1	319	0	418	1	517	0	616	1
23	0	122	1	221	0	320	0	419	1	518	0	617	1
24	1	123	0	222	1	321	0	420	0	519	0	618	0
25	0	124	0	223	0	322	0	421	1	520	0	619	1
26	1	125	1	224	0	323	0	422	0	521	0	620	1
27	0	126	1	225	0	324	1	423	1	522	1	621	1
28	1	127	0	226	1	325	1	424	0	523	1	622	0
29	1	128	1	227	1	326	1	425	1	524	0	623	0
30	0	129	1	228	1	327	1	426	1	525	0	624	1
31	0	130	1	229	1	328	1	427	1	526	1	625	0
32	0	131	1	230	1	329	1	428	0	527	1	626	0
33	0	132	0	231	1	330	1	429	1	528	0	627	0
34	0	133	1	232	1	331	1	430	1	529	1	628	0
35	0	134	1	233	0	332	0	431	1	530	1	629	0
36	1	135	0	234	0	333	0	432	1	531	0	630	0
37	1	136	1	235	1	334	0	433	0	532	1	631	1
38	0	137	1	236	1	335	1	434	1	533	0	632	1
39	0	138	0	237	1	336	0	435	0	534	0	633	0
40	1	139	0	238	1	337	0	436	0	535	0	634	1
41	1	140	1	239	0	338	1	437	0	536	1	635	1
42	0	141	1	240	0	339	1	438	1	537	0	636	1
43	0	142	1	241	1	340	0	439	1	538	0	637	1
44	1	143	1	242	1	341	0	440	1	539	1	638	0

45	0	144	1	243	1	342	0	441	0	540	1	639	1
46	0	145	1	244	1	343	0	442	0	541	0	640	1
47	1	146	1	245	0	344	1	443	1	542	0	641	1
48	0	147	1	246	1	345	1	444	0	543	1	642	1
49	0	148	0	247	1	346	0	445	0	544	1	643	1
50	1	149	0	248	1	347	1	446	0	545	1	644	0
51	0	150	1	249	1	348	1	447	0	546	0	645	0
52	0	151	0	250	1	349	0	448	0	547	0	646	1
53	0	152	0	251	1	350	0	449	0	548	0	647	1
54	1	153	1	252	1	351	0	450	1	549	1	648	0
55	1	154	1	253	1	352	1	451	0	550	1	649	1
56	0	155	1	254	1	353	1	452	1	551	1	650	1
57	0	156	0	255	0	354	1	453	1	552	0	651	1
58	1	157	0	256	1	355	1	454	1	553	1	652	0
59	0	158	0	257	0	356	1	455	1	554	1	653	1
60	1	159	1	258	1	357	1	456	1	555	1	654	1
61	0	160	0	259	0	358	0	457	1	556	1	655	1
62	1	161	1	260	1	359	0	458	1	557	1	656	1
63	0	162	1	261	0	360	1	459	1	558	0	657	1
64	1	163	0	262	1	361	0	460	1	559	1	658	0
65	0	164	1	263	0	362	1	461	0	560	1	659	0
66	1	165	0	264	1	363	1	462	0	561	0	660	1
67	1	166	1	265	1	364	1	463	0	562	0	661	1
68	1	167	0	266	1	365	0	464	0	563	1	662	1
69	1	168	1	267	0	366	0	465	1	564	0	663	1
70	1	169	1	268	1	367	1	466	1	565	0	664	0
71	0	170	0	269	0	368	1	467	1	566	0	665	1
72	1	171	1	270	1	369	1	468	0	567	1	666	0
73	0	172	0	271	0	370	1	469	1	568	1	667	1
74	1	173	0	272	1	371	0	470	1	569	1	668	0
75	1	174	0	273	1	372	1	471	0	570	1	669	1
76	1	175	1	274	1	373	1	472	1	571	0	670	0
77	1	176	1	275	1	374	0	473	1	572	0	671	1
78	1	177	1	276	1	375	1	474	1	573	1	672	0
79	0	178	0	277	0	376	0	475	0	574	1	673	1
80	1	179	0	278	0	377	0	476	0	575	0	674	1
81	0	180	0	279	0	378	1	477	0	576	0	675	0
82	1	181	0	280	0	379	1	478	0	577	0	676	0
83	1	182	0	281	1	380	0	479	0	578	1	677	0
84	0	183	0	282	1	381	0	480	1	579	0	678	1
85	0	184	0	283	0	382	0	481	0	580	0	679	0
86	1	185	0	284	0	383	0	482	1	581	1	680	1
87	0	186	0	285	1	384	0	483	0	582	1	681	1
88	1	187	0	286	0	385	0	484	1	583	0	682	1

89	1	188	0	287	0	386	1	485	0	584	0	683	1
90	1	189	0	288	0	387	1	486	1	585	0	684	1
91	0	190	1	289	1	388	0	487	0	586	0	685	1
92	0	191	0	290	0	389	0	488	1	587	1	686	0
93	0	192	0	291	0	390	0	489	1	588	1	687	0
94	0	193	0	292	0	391	0	490	1	589	1		
95	1	194	0	293	1	392	1	491	0	590	0		
96	0	195	1	294	0	393	0	492	0	591	0		
97	1	196	1	295	1	394	0	493	0	592	0		
98	0	197	1	296	0	395	1	494	1	593	1		

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