PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a,b)=(z^3+z, z^4+z^3+z)$

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Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a,b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a,b) in $\mathbb{F}_2[x]\setminus\mathbb{F}_2$. In this paper we prove the conjecture for $(a,b)=(z^3+z, z^4+z^3+z)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a}=(a_0(z), a_1(z), a_2(z), \ldots)$ of elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{a_2(z)+\cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{\ddots+\cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1}+c_{-2}z^{-2}+c_{-3}z^{-3}+\cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a,b)-Thue-Morse sequence

$$\mathbf{t}=(t_0(z), t_1(z), t_2(z), \ldots)=(a,b,b,a,b,a,a,b,\ldots)$$

that we shall denote

$$\mathrm{CF}(a,b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When* $(a,b)=(z^3+z, z^4+z^3+z)$, *the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$, *with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

where, for $\mathrm{CF}(a,b)$,

$$\begin{aligned}
p_0(z)&=z^{15}+z^{14}+z^{12}+z^{10}+z^8+z^7+z^6+z^5+z^4+z^3+1,\\
p_1(z)&=z^{19}+z^{18}+z^{16}+z^{11}+z^{10}+z^9+z^7+z^4,\\
p_2(z)&=z^{22}+z^{18}+z^{16}+z^{12}+z^6+z^2,\\
p_3(z)&=z^{19}+z^{18}+z^{16}+z^{11}+z^{10}+z^9+z^7+z^4,\\
p_4(z)&=z^{16}+z^{15}+z^8+z^7+z^5+z^4+z^3+1,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$,

$$\begin{aligned}
p_0(z)&=z^{15}+z^{14}+z^{12}+z^8+z^7+z^6+z^5+z^4+z^3+1,\\
p_1(z)&=z^{19}+z^{18}+z^{16}+z^{11}+z^{10}+z^9+z^7+z^4,\\
p_2(z)&=z^{22}+z^{18}+z^{16}+z^{12}+z^6+z^2,\\
p_3(z)&=z^{19}+z^{18}+z^{16}+z^{11}+z^{10}+z^9+z^7+z^4,\\
p_4(z)&=z^{16}+z^{15}+z^{10}+z^8+z^7+z^5+z^4+z^3+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x)=x^{\deg(t_{2^n-1})}&\begin{pmatrix} t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\cdots\times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x)=x^{\deg(\bar t_{2^n-1})}&\begin{pmatrix} \bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix} \bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\cdots\times x^{\deg(\bar t_0)}\begin{pmatrix} \bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\geq 0$

$$\begin{aligned}
M_{n+1}(x)&=W_n(x)\cdot M_n(x),\\
W_{n+1}(x)&=M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(175, 175, 175, 175, 175)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$
$$M^o = W^e \cdot M^e, \tag{2.4}$$
$$W^e = M^o \cdot W^o, \tag{2.5}$$
$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z\left(\phi(W^o,0,0)(x,z),\ z^{12}\cdot\phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 630 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 630, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z\left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ to order 770 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 770, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. $\square$

Define

$$R^e = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned}M^e[:14u] = {} & M^e[:7u] + x^u M^e[:6u] + x^{2u}M^e[:5u] + x^{5u}M^e[:2u] + x^{6u}M^e[:u]\\ & + x^u M^e[:7u] + x^{2u}M^e[:6u] + x^{3u}M^e[:5u] + x^{6u}M^e[:2u]\\ & + x^{7u}M^e[:u] + x^{3u}M^e[:7u] + x^{4u}M^e[:6u] + x^{5u}M^e[:5u]\\ & + x^{8u}M^e[:2u] + x^{9u}M^e[:u] + x^{4u}M^e[:7u] + x^{5u}M^e[:6u]\\ & + x^{6u}M^e[:5u] + x^{9u}M^e[:2u] + x^{10u}M^e[:u] + x^{5u}M^e[:7u]\\ & + x^{6u}M^e[:6u] + x^{7u}M^e[:5u] + x^{10u}M^e[:2u] + x^{11u}M^e[:u]\end{aligned}$$

$$
\begin{aligned}
&+ x^{7u}M^e[:7u] + x^{9u}M^e[:5u] + x^{10u}M^e[:4u] + x^{12u}M^e[:2u]\\
&+ x^{13u}M^e[:u] + (x^{7u} + x^{8u} + x^{10u} + x^{11u} + x^{12u})R^e,\\
W^e[:14u] = {}& W^e[:7u] + x^uW^e[:6u] + x^{2u}W^e[:5u] + x^{5u}W^e[:2u] + x^{6u}W^e[:u]\\
&+ x^uW^e[:7u] + x^{2u}W^e[:6u] + x^{3u}W^e[:5u] + x^{6u}W^e[:2u]\\
&+ x^{7u}W^e[:u] + x^{3u}W^e[:7u] + x^{4u}W^e[:6u] + x^{5u}W^e[:5u]\\
&+ x^{8u}W^e[:2u] + x^{9u}W^e[:u] + x^{4u}W^e[:7u] + x^{5u}W^e[:6u]\\
&+ x^{6u}W^e[:5u] + x^{9u}W^e[:2u] + x^{10u}W^e[:u] + x^{5u}W^e[:7u]\\
&+ x^{6u}W^e[:6u] + x^{7u}W^e[:5u] + x^{10u}W^e[:2u] + x^{11u}W^e[:u]\\
&+ x^{7u}W^e[:7u] + x^{9u}W^e[:5u] + x^{10u}W^e[:4u] + x^{12u}W^e[:2u]\\
&+ x^{13u}W^e[:u] + (x^{7u} + x^{8u} + x^{10u} + x^{11u} + x^{12u})R^e.
\end{aligned}
$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$
\begin{aligned}
M^o[:14u] = {}& M^o[:7u] + x^uM^o[:6u] + x^{2u}M^o[:5u] + x^{5u}M^o[:2u] + x^{6u}M^o[:u]\\
&+ x^uM^o[:7u] + x^{2u}M^o[:6u] + x^{3u}M^o[:5u] + x^{6u}M^o[:2u]\\
&+ x^{7u}M^o[:u] + x^{3u}M^o[:7u] + x^{4u}M^o[:6u] + x^{5u}M^o[:5u]\\
&+ x^{8u}M^o[:2u] + x^{9u}M^o[:u] + x^{4u}M^o[:7u] + x^{5u}M^o[:6u]\\
&+ x^{6u}M^o[:5u] + x^{9u}M^o[:2u] + x^{10u}M^o[:u] + x^{5u}M^o[:7u]\\
&+ x^{6u}M^o[:6u] + x^{7u}M^o[:5u] + x^{10u}M^o[:2u] + x^{11u}M^o[:u]\\
&+ x^{7u}M^o[:7u] + x^{9u}M^o[:5u] + x^{10u}M^o[:4u] + x^{12u}M^o[:2u]\\
&+ x^{13u}M^o[:u] + (x^{7u} + x^{8u} + x^{10u} + x^{11u} + x^{12u})R^o,\\
W^o[:14u] = {}& W^o[:7u] + x^uW^o[:6u] + x^{2u}W^o[:5u] + x^{5u}W^o[:2u] + x^{6u}W^o[:u]\\
&+ x^uW^o[:7u] + x^{2u}W^o[:6u] + x^{3u}W^o[:5u] + x^{6u}W^o[:2u]\\
&+ x^{7u}W^o[:u] + x^{3u}W^o[:7u] + x^{4u}W^o[:6u] + x^{5u}W^o[:5u]\\
&+ x^{8u}W^o[:2u] + x^{9u}W^o[:u] + x^{4u}W^o[:7u] + x^{5u}W^o[:6u]\\
&+ x^{6u}W^o[:5u] + x^{9u}W^o[:2u] + x^{10u}W^o[:u] + x^{5u}W^o[:7u]\\
&+ x^{6u}W^o[:6u] + x^{7u}W^o[:5u] + x^{10u}W^o[:2u] + x^{11u}W^o[:u]\\
&+ x^{7u}W^o[:7u] + x^{9u}W^o[:5u] + x^{10u}W^o[:4u] + x^{12u}W^o[:2u]\\
&+ x^{13u}W^o[:u] + (x^{7u} + x^{8u} + x^{10u} + x^{11u} + x^{12u})R^o.
\end{aligned}
$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$
\begin{aligned}
T[:14u] = {}& T[:7u] + x^uT[:6u] + x^{2u}T[:5u] + x^{5u}T[:2u] + x^{6u}T[:u] + x^uT[:7u]\\
&+ x^{2u}T[:6u] + x^{3u}T[:5u] + x^{6u}T[:2u] + x^{7u}T[:u] + x^{3u}T[:7u]\\
&+ x^{4u}T[:6u] + x^{5u}T[:5u] + x^{8u}T[:2u] + x^{9u}T[:u] + x^{4u}T[:7u]\\
&+ x^{5u}T[:6u] + x^{6u}T[:5u] + x^{9u}T[:2u] + x^{10u}T[:u] + x^{5u}T[:7u]\\
&+ x^{6u}T[:6u] + x^{7u}T[:5u] + x^{10u}T[:2u] + x^{11u}T[:u] + x^{7u}T[:7u]
\end{aligned}
$$

$$+ x^{9u}T[:5u] + x^{10u}T[:4u] + x^{12u}T[:2u] + x^{13u}T[:u].$$

The proof of the other 15 cases are similar. We break down the above identity into 7 parts:

$$\begin{aligned}
0 &= x^{7u}T[:u] + x^{6u}T[u:2u] + x^{5u}T[2u:3u] + x^{2u}T[5u:6u] + x^{u}T[6u:7u] \\
&\quad + T[7u:8u], \\
0 &= x^{8u}T[:u] + x^{5u}T[3u:4u] + x^{3u}T[5u:6u] + T[8u:9u], \\
0 &= x^{9u}T[:u] + x^{8u}T[u:2u] + x^{5u}T[4u:5u] + x^{3u}T[6u:7u] + T[9u:10u], \\
0 &= x^{10u}T[:u] + x^{4u}T[6u:7u] + T[10u:11u], \\
0 &= x^{11u}T[:u] + x^{9u}T[2u:3u] + x^{6u}T[5u:6u] + x^{5u}T[6u:7u] \\
&\quad + T[11u:12u], \\
0 &= x^{12u}T[:u] + x^{10u}T[2u:3u] + x^{9u}T[3u:4u] + x^{7u}T[5u:6u] \\
&\quad + T[12u:13u], \\
0 &= x^{13u}T[:u] + x^{12u}T[u:2u] + x^{10u}T[3u:4u] + x^{9u}T[4u:5u] \\
&\quad + x^{7u}T[6u:7u] + T[13u:14u],
\end{aligned}$$

which can be rewritten as

$$0 = T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[101w]_2] + T[[110w]_2] + T[[111w]_2], \tag{2.7}$$

$$0 = T[[w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.8}$$

$$0 = T[[w]_2] + T[[1w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1001w]_2], \tag{2.9}$$

$$0 = T[[w]_2] + T[[110w]_2] + T[[1010w]_2], \tag{2.10}$$

$$0 = T[[w]_2] + T[[10w]_2] + T[[101w]_2] + T[[110w]_2] + T[[1011w]_2], \tag{2.11}$$

$$0 = T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1100w]_2], \tag{2.12}$$

$$0 = T[[w]_2] + T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1101w]_2], \tag{2.13}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$0 = T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[101w]_2] + T[[110w]_2] + T[[111w]_2], \tag{2.14}$$

$$0 = T[[w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.15}$$

$$0 = T[[w]_2] + T[[1w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1001w]_2], \tag{2.16}$$

$$0 = T[[w]_2] + T[[110w]_2] + T[[1010w]_2], \tag{2.17}$$

$$0 = T[[w]_2] + T[[10w]_2] + T[[101w]_2] + T[[110w]_2] + T[[1011w]_2], \tag{2.18}$$

$$0 = T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1100w]_2], \tag{2.19}$$

$$0 = T[[w]_2] + T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1101w]_2], \tag{2.20}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 1330 states; its transition function and output function can be found in the annex. Let $A(s,w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i,w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.20) can be written as

$$\begin{aligned}
&0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,101)) \\
&\qquad + \tau(A(s,110)) + \tau(A(s,111)), && (2.21)\\
&0 = \tau(A(s,\epsilon)) + \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,1000)), && (2.22)\\
&0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,100)) + \tau(A(s,110)) \\
&\qquad + \tau(A(s,1001)), && (2.23)\\
&0 = \tau(A(s,\epsilon)) + \tau(A(s,110)) + \tau(A(s,1010)), && (2.24)\\
&0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,101)) + \tau(A(s,110)) \\
&\qquad + \tau(A(s,1011)), && (2.25)\\
&0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101)) \\
&\qquad + \tau(A(s,1100)), && (2.26)\\
&0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,100)) \\
&\qquad + \tau(A(s,110)) + \tau(A(s,1101)), && (2.27)
\end{aligned}$$

for all $s \in E_{2k}$, and

$$\begin{aligned}
&0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,101)) \\
&\qquad + \tau(A(s,110)) + \tau(A(s,111)), && (2.28)\\
&0 = \tau(A(s,\epsilon)) + \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,1000)), && (2.29)\\
&0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,100)) + \tau(A(s,110)) \\
&\qquad + \tau(A(s,1001)), && (2.30)\\
&0 = \tau(A(s,\epsilon)) + \tau(A(s,110)) + \tau(A(s,1010)), && (2.31)\\
&0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,101)) + \tau(A(s,110)) \\
&\qquad + \tau(A(s,1011)), && (2.32)\\
&0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101)) \\
&\qquad + \tau(A(s,1100)), && (2.33)\\
&0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,100)) \\
&\qquad + \tau(A(s,110)) + \tau(A(s,1101)), && (2.34)
\end{aligned}$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{46}), E_{46}) = (A(i, 0^{18}), E_{18})$, so that we only have to verify that identities (2.21) through (2.34) hold for $2 \leq k \leq 23$, which turns out to be true. □

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \geq 2$, and $u = 2^{2k-1}$,*

$$\begin{aligned}
M_{2k} = {} & M^e[:7u] + x^u M^e[:6u] + x^{2u} M^e[:5u] + x^{5u} M^e[:2u] + x^{6u} M^e[:u] \\
& + x^{7u} R^e,
\end{aligned}$$

$$\begin{aligned} W_{2k} = {} & W^e[:7u] + x^u W^e[:6u] + x^{2u} W^e[:5u] + x^{5u} W^e[:2u] + x^{6u} W^e[:u] \\ & + x^{7u} R^e. \end{aligned}$$

For all integer $k \geq 2$, and $u = 2^{2k}$,

$$\begin{aligned} M_{2k+1} = {} & M^o[:7u] + x^u M^o[:6u] + x^{2u} M^o[:5u] + x^{5u} M^o[:2u] + x^{6u} M^o[:u] \\ & + x^{7u} R^o, \\ W_{2k+1} = {} & W^o[:7u] + x^u W^o[:6u] + x^{2u} W^o[:5u] + x^{5u} W^o[:2u] + x^{6u} W^o[:u] \\ & + x^{7u} R^o. \end{aligned}$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \geq 2$, we claim that

$$\begin{aligned} M_{2k} \wedge W_{2k} &\Rightarrow M_{2k+1} \wedge W_{2k+1}, \\ M_{2k+1} \wedge W_{2k+1} &\Rightarrow M_{2k+2} \wedge W_{2k+2}. \end{aligned}$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$\begin{aligned} W_{2k} M_{2k} = {} & \big(W^e[:7v] + x^v W^e[:6v] + x^{2v} W^e[:5v] + x^{5v} W^e[:2v] + x^{6v} W^e[:v] \\ & + x^{7u} R^e\big) \times \big(M^e[:7v] + x^v M^e[:6v] + x^{2v} M^e[:5v] + x^{5v} M^e[:2v] \\ & + x^{6v} M^e[:v] + x^{7u} R^e\big); \end{aligned} \tag{2.35}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{14v} R^o$. Therefore we only need to prove that their difference is $O(x^{14v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{14v}, to

$$\begin{aligned} & W^e[:14v] M^e[:14v] + x^{2v} W^e[:12v] M^e[:12v] + x^{4v} W^e[:10v] M^e[:10v] \\ & \qquad + x^{10v} W^e[:4v] M^e[:4v] + x^{12v} W^e[:2v] M^e[:2v]. \end{aligned}$$

For all $n \leq 14$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v] / X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v] / X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.35) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{14}, b_1, ..., b_{14}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{14})$, which completes the proof. $\square$

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \leq j, k \leq 1$,

$$\lim_{n\to\infty} M_{2n,j,k} = M^e_{j,k},$$
$$\lim_{n\to\infty} M_{2n+1,j,k} = M^o_{j,k},$$
$$\lim_{n\to\infty} W_{2n,j,k} = W^e_{j,k},$$
$$\lim_{n\to\infty} W_{2n+1,j,k} = W^o_{j,k}.$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a, b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x, y) = \mathrm{Res}_t \left(\phi(M^e, 0, 1)(x, t), y^{12}\phi(M^e, 0, 1)(x, t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x, y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$q_0(x) = x^{22} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{12} + x^{10} + x^8 + x^7,$$
$$q_1(x) = x^{18} + x^{15} + x^{13} + x^{12} + x^{11} + x^6 + x^4 + x^3,$$
$$q_2(x) = x^{20} + x^{16} + x^{10} + x^6 + x^4 + 1,$$
$$q_3(x) = x^{18} + x^{15} + x^{13} + x^{12} + x^{11} + x^6 + x^4 + x^3,$$
$$q_4(x) = x^{22} + x^{19} + x^{18} + x^{17} + x^{15} + x^{14} + x^7 + x^6.$$

The polynomial $Q(x, y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x, y)$ of multiplicity m and $R(x, y) := P(x, y)/Q(x, y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 216, and substitute it for y in $R(z, y)$, we get a series with valuation smaller than 216, which proves that $R(z, y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z, y)$ is the minimal polynomial of $\mathrm{CF}(a, b) = f(1/z)$. □

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e, 0, 0)$:

$$p_0(x) = x^{120} + x^{116} + x^{114} + x^{112} + x^{100} + x^{94} + x^{86} + x^{80} + x^{76} + x^{74} + x^{68} + x^{66} + x^{62} + x^{60} + x^{58} + x^{46} + x^{44} + x^{38} + x^{36},$$
$$p_3(x) = x^{108} + x^{104} + x^{96} + x^{94} + x^{92} + x^{82} + x^{80} + x^{74} + x^{70} + x^{66} + x^{64}$$

$$
\begin{aligned}
&+ x^{60} + x^{46} + x^{44} + x^{40} + x^{38} + x^{34} + x^{28} + x^{24} + x^{20},\\
p_6(x) &= x^{112} + x^{108} + x^{106} + x^{98} + x^{96} + x^{90} + x^{86} + x^{82} + x^{78} + x^{72} + x^{68}\\
&+ x^{64} + x^{60} + x^{58} + x^{54} + x^{52} + x^{50} + x^{48} + x^{44} + x^{42} + x^{40} + x^{34}\\
&+ x^{28} + x^{26} + x^{22} + x^{20} + x^{16} + x^{14} + x^{10} + x^{6} + x^{4} + 1,\\
p_9(x) &= x^{104} + x^{98} + x^{96} + x^{88} + x^{86} + x^{80} + x^{76} + x^{74} + x^{72} + x^{70} + x^{64}\\
&+ x^{58} + x^{56} + x^{48} + x^{46} + x^{36} + x^{30} + x^{24} + x^{22} + x^{16} + x^{12} + x^{10}\\
&+ x^{8} + x^{6},\\
p_{12}(x) &= x^{108} + x^{104} + x^{92} + x^{88} + x^{84} + x^{80} + x^{76} + x^{72} + x^{52} + x^{48} + x^{36}\\
&+ x^{32} + x^{20} + x^{16} + x^{12} + x^{8} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0].

For $\phi(M^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{117} + x^{114} + x^{113} + x^{112} + x^{109} + x^{108} + x^{107} + x^{104} + x^{103} + x^{100}\\
&+ x^{99} + x^{97} + x^{96} + x^{95} + x^{94} + x^{89} + x^{86} + x^{82} + x^{80} + x^{77} + x^{75}\\
&+ x^{73} + x^{70} + x^{69} + x^{63} + x^{62} + x^{52} + x^{51} + x^{48} + x^{46} + x^{43} + x^{40}\\
&+ x^{39},\\
p_3(x) &= x^{87} + x^{83} + x^{77} + x^{73} + x^{63} + x^{61} + x^{59} + x^{57} + x^{53} + x^{49} + x^{45}\\
&+ x^{41} + x^{39} + x^{37} + x^{35} + x^{33} + x^{31} + x^{27},\\
p_6(x) &= x^{93} + x^{90} + x^{89} + x^{88} + x^{87} + x^{84} + x^{82} + x^{81} + x^{80} + x^{75} + x^{73}\\
&+ x^{72} + x^{71} + x^{68} + x^{66} + x^{64} + x^{62} + x^{61} + x^{60} + x^{55} + x^{53} + x^{52}\\
&+ x^{51} + x^{48} + x^{46} + x^{45} + x^{44} + x^{41} + x^{39} + x^{38} + x^{37} + x^{34} + x^{32}\\
&+ x^{31} + x^{30} + x^{27} + x^{24} + x^{23} + x^{22} + x^{21} + x^{19} + x^{18},\\
p_9(x) &= x^{99} + x^{95} + x^{93} + x^{89} + x^{87} + x^{85} + x^{81} + x^{77} + x^{69} + x^{65} + x^{63}\\
&+ x^{59} + x^{51} + x^{49} + x^{31} + x^{27} + x^{25} + x^{23} + x^{21} + x^{19} + x^{15} + x^{13}\\
&+ x^{11} + x^{9},\\
p_{12}(x) &= x^{105} + x^{102} + x^{101} + x^{100} + x^{93} + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{81}\\
&+ x^{80} + x^{73} + x^{70} + x^{69} + x^{68} + x^{61} + x^{58} + x^{57} + x^{56} + x^{53} + x^{50}\\
&+ x^{49} + x^{48} + x^{45} + x^{42} + x^{41} + x^{40} + x^{37} + x^{34} + x^{33} + x^{32} + x^{29}\\
&+ x^{26} + x^{25} + x^{24} + x^{21} + x^{18} + x^{17} + x^{16} + x^{13} + x^{10} + x^{8} + x^{6} + x^{4}\\
&+ x^{2} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 1, 1, 1, 1, 0, 0].

For $\phi(M^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{117} + x^{114} + x^{113} + x^{112} + x^{109} + x^{108} + x^{107} + x^{104} + x^{103} + x^{100}\\
&+ x^{99} + x^{97} + x^{96} + x^{95} + x^{94} + x^{89} + x^{86} + x^{82} + x^{80} + x^{77} + x^{75}\\
&+ x^{73} + x^{70} + x^{69} + x^{63} + x^{62} + x^{52} + x^{51} + x^{48} + x^{46} + x^{43} + x^{40}\\
&+ x^{39},
\end{aligned}
$$

$$
\begin{aligned}
p_3(x) &= x^{87} + x^{83} + x^{77} + x^{73} + x^{63} + x^{61} + x^{59} + x^{57} + x^{53} + x^{49} + x^{45} \\
&\quad + x^{41} + x^{39} + x^{37} + x^{35} + x^{33} + x^{31} + x^{27}, \\
p_6(x) &= x^{93} + x^{90} + x^{89} + x^{88} + x^{87} + x^{84} + x^{82} + x^{81} + x^{80} + x^{75} + x^{73} \\
&\quad + x^{72} + x^{71} + x^{68} + x^{66} + x^{64} + x^{62} + x^{61} + x^{60} + x^{55} + x^{53} + x^{52} \\
&\quad + x^{51} + x^{48} + x^{46} + x^{45} + x^{44} + x^{41} + x^{39} + x^{38} + x^{37} + x^{34} + x^{32} \\
&\quad + x^{31} + x^{30} + x^{27} + x^{24} + x^{23} + x^{22} + x^{21} + x^{19} + x^{18}, \\
p_9(x) &= x^{99} + x^{95} + x^{93} + x^{89} + x^{87} + x^{85} + x^{81} + x^{77} + x^{69} + x^{65} + x^{63} \\
&\quad + x^{59} + x^{51} + x^{49} + x^{31} + x^{27} + x^{25} + x^{23} + x^{21} + x^{19} + x^{15} + x^{13} \\
&\quad + x^{11} + x^{9}, \\
p_{12}(x) &= x^{105} + x^{102} + x^{101} + x^{100} + x^{93} + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{81} \\
&\quad + x^{80} + x^{73} + x^{70} + x^{69} + x^{68} + x^{61} + x^{58} + x^{57} + x^{56} + x^{53} + x^{50} \\
&\quad + x^{49} + x^{48} + x^{45} + x^{42} + x^{41} + x^{40} + x^{37} + x^{34} + x^{33} + x^{32} + x^{29} \\
&\quad + x^{26} + x^{25} + x^{24} + x^{21} + x^{18} + x^{17} + x^{16} + x^{13} + x^{10} + x^{8} + x^{6} + x^{4} \\
&\quad + x^{2} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 1, 1, 1, 1, 0, 0].

For $\phi(M^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{114} + x^{110} + x^{106} + x^{100} + x^{94} + x^{92} + x^{78} + x^{76} + x^{74} + x^{72} + x^{54} \\
&\quad + x^{52} + x^{42}, \\
p_3(x) &= x^{102} + x^{98} + x^{96} + x^{88} + x^{84} + x^{82} + x^{78} + x^{76} + x^{74} + x^{72} + x^{70} \\
&\quad + x^{66} + x^{64} + x^{62} + x^{60} + x^{52} + x^{48} + x^{46} + x^{42} + x^{36} + x^{34} + x^{32} \\
&\quad + x^{26} + x^{22} + x^{20} + x^{18}, \\
p_6(x) &= x^{106} + x^{102} + x^{92} + x^{88} + x^{82} + x^{80} + x^{78} + x^{76} + x^{72} + x^{70} + x^{68} \\
&\quad + x^{66} + x^{62} + x^{58} + x^{52} + x^{48} + x^{46} + x^{44} + x^{36} + x^{34} + x^{32} + x^{28} \\
&\quad + x^{22} + x^{20} + x^{18} + x^{10} + x^{2} + 1, \\
p_9(x) &= x^{98} + x^{90} + x^{88} + x^{86} + x^{80} + x^{78} + x^{76} + x^{72} + x^{70} + x^{68} + x^{64} \\
&\quad + x^{56} + x^{54} + x^{52} + x^{48} + x^{46} + x^{44} + x^{42} + x^{40} + x^{34} + x^{28} + x^{26} \\
&\quad + x^{16} + x^{12} + x^{10} + x^{6}, \\
p_{12}(x) &= x^{102} + x^{98} + x^{96} + x^{86} + x^{82} + x^{80} + x^{70} + x^{66} + x^{64} + x^{6} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0].

For $\phi(W^e, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{114} + x^{110} + x^{106} + x^{102} + x^{100} + x^{98} + x^{96} + x^{94} + x^{92} + x^{90} + x^{80} \\
&\quad + x^{72} + x^{62} + x^{60} + x^{58} + x^{54} + x^{52} + x^{46} + x^{44} + x^{42} + x^{36}, \\
p_3(x) &= x^{102} + x^{98} + x^{96} + x^{90} + x^{88} + x^{86} + x^{84} + x^{78} + x^{76} + x^{72} + x^{70} \\
&\quad + x^{64} + x^{60} + x^{58} + x^{52} + x^{50} + x^{48} + x^{46} + x^{38} + x^{36} + x^{34} + x^{32} \\
&\quad + x^{30} + x^{26} + x^{22} + x^{20},
\end{aligned}
$$

$$
\begin{aligned}
p_6(x) &= x^{106} + x^{102} + x^{94} + x^{92} + x^{86} + x^{82} + x^{80} + x^{78} + x^{74} + x^{70} + x^{66} \\
&\quad + x^{64} + x^{58} + x^{50} + x^{46} + x^{40} + x^{36} + x^{34} + x^{30} + x^{26} + x^{24} + x^{20} \\
&\quad + x^{16} + x^{14} + x^{12} + x^{10} + x^2 + 1, \\
p_9(x) &= x^{98} + x^{90} + x^{88} + x^{86} + x^{80} + x^{78} + x^{76} + x^{72} + x^{70} + x^{68} + x^{64} \\
&\quad + x^{56} + x^{54} + x^{52} + x^{48} + x^{46} + x^{44} + x^{42} + x^{40} + x^{34} + x^{28} + x^{26} \\
&\quad + x^{16} + x^{12} + x^{10} + x^6, \\
p_{12}(x) &= x^{102} + x^{98} + x^{96} + x^{86} + x^{82} + x^{80} + x^{70} + x^{66} + x^{64} + x^6 + x^2 + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0].

For $\phi(W^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{117} + x^{114} + x^{113} + x^{112} + x^{109} + x^{108} + x^{107} + x^{104} + x^{103} + x^{101} \\
&\quad + x^{100} + x^{99} + x^{98} + x^{94} + x^{93} + x^{92} + x^{89} + x^{87} + x^{86} + x^{84} + x^{78} \\
&\quad + x^{75} + x^{74} + x^{72} + x^{69} + x^{68} + x^{64} + x^{63} + x^{62} + x^{61} + x^{58} + x^{57} \\
&\quad + x^{56} + x^{55} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} \\
&\quad + x^{44} + x^{43} + x^{40} + x^{39}, \\
p_3(x) &= x^{87} + x^{83} + x^{77} + x^{73} + x^{63} + x^{61} + x^{59} + x^{57} + x^{53} + x^{49} + x^{45} \\
&\quad + x^{41} + x^{39} + x^{37} + x^{35} + x^{33} + x^{31} + x^{27}, \\
p_6(x) &= x^{93} + x^{90} + x^{89} + x^{88} + x^{87} + x^{84} + x^{82} + x^{81} + x^{80} + x^{75} + x^{73} \\
&\quad + x^{72} + x^{71} + x^{68} + x^{66} + x^{64} + x^{62} + x^{61} + x^{60} + x^{55} + x^{53} + x^{52} \\
&\quad + x^{51} + x^{48} + x^{46} + x^{45} + x^{44} + x^{41} + x^{39} + x^{38} + x^{37} + x^{34} + x^{32} \\
&\quad + x^{31} + x^{30} + x^{27} + x^{24} + x^{23} + x^{22} + x^{21} + x^{19} + x^{18}, \\
p_9(x) &= x^{99} + x^{95} + x^{93} + x^{89} + x^{87} + x^{85} + x^{81} + x^{77} + x^{69} + x^{65} + x^{63} \\
&\quad + x^{59} + x^{51} + x^{49} + x^{31} + x^{27} + x^{25} + x^{23} + x^{21} + x^{19} + x^{15} + x^{13} \\
&\quad + x^{11} + x^9, \\
p_{12}(x) &= x^{105} + x^{102} + x^{101} + x^{100} + x^{93} + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{81} \\
&\quad + x^{80} + x^{73} + x^{70} + x^{69} + x^{68} + x^{61} + x^{58} + x^{57} + x^{56} + x^{53} + x^{50} \\
&\quad + x^{49} + x^{48} + x^{45} + x^{42} + x^{41} + x^{40} + x^{37} + x^{34} + x^{33} + x^{32} + x^{29} \\
&\quad + x^{26} + x^{25} + x^{24} + x^{21} + x^{18} + x^{17} + x^{16} + x^{13} + x^{10} + x^8 + x^6 + x^4 \\
&\quad + x^2 + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 1, 1, 0, 1, 0, 1, 1, 0, 1, 0, 1].

For $\phi(W^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{117} + x^{114} + x^{113} + x^{112} + x^{109} + x^{108} + x^{107} + x^{104} + x^{103} + x^{101} \\
&\quad + x^{100} + x^{99} + x^{98} + x^{94} + x^{93} + x^{92} + x^{89} + x^{87} + x^{86} + x^{84} + x^{78} \\
&\quad + x^{75} + x^{74} + x^{72} + x^{69} + x^{68} + x^{64} + x^{63} + x^{62} + x^{61} + x^{58} + x^{57} \\
&\quad + x^{56} + x^{55} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} \\
&\quad + x^{44} + x^{43} + x^{40} + x^{39},
\end{aligned}
$$

$$\begin{aligned}
p_3(x) &= x^{87}+x^{83}+x^{77}+x^{73}+x^{63}+x^{61}+x^{59}+x^{57}+x^{53}+x^{49}+x^{45}\\
&\quad +x^{41}+x^{39}+x^{37}+x^{35}+x^{33}+x^{31}+x^{27},\\
p_6(x) &= x^{93}+x^{90}+x^{89}+x^{88}+x^{87}+x^{84}+x^{82}+x^{81}+x^{80}+x^{75}+x^{73}\\
&\quad +x^{72}+x^{71}+x^{68}+x^{66}+x^{64}+x^{62}+x^{61}+x^{60}+x^{55}+x^{53}+x^{52}\\
&\quad +x^{51}+x^{48}+x^{46}+x^{45}+x^{44}+x^{41}+x^{39}+x^{38}+x^{37}+x^{34}+x^{32}\\
&\quad +x^{31}+x^{30}+x^{27}+x^{24}+x^{23}+x^{22}+x^{21}+x^{19}+x^{18},\\
p_9(x) &= x^{99}+x^{95}+x^{93}+x^{89}+x^{87}+x^{85}+x^{81}+x^{77}+x^{69}+x^{65}+x^{63}\\
&\quad +x^{59}+x^{51}+x^{49}+x^{31}+x^{27}+x^{25}+x^{23}+x^{21}+x^{19}+x^{15}+x^{13}\\
&\quad +x^{11}+x^{9},\\
p_{12}(x) &= x^{105}+x^{102}+x^{101}+x^{100}+x^{93}+x^{90}+x^{88}+x^{86}+x^{84}+x^{82}+x^{81}\\
&\quad +x^{80}+x^{73}+x^{70}+x^{69}+x^{68}+x^{61}+x^{58}+x^{57}+x^{56}+x^{53}+x^{50}\\
&\quad +x^{49}+x^{48}+x^{45}+x^{42}+x^{41}+x^{40}+x^{37}+x^{34}+x^{33}+x^{32}+x^{29}\\
&\quad +x^{26}+x^{25}+x^{24}+x^{21}+x^{18}+x^{17}+x^{16}+x^{13}+x^{10}+x^{8}+x^{6}+x^{4}\\
&\quad +x^{2}+x+1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 1, 1, 0, 1, 0, 1, 1, 0, 1, 0, 1].

For $\phi(W^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{120}+x^{116}+x^{114}+x^{112}+x^{108}+x^{104}+x^{100}+x^{96}+x^{94}+x^{90}\\
&\quad +x^{86}+x^{84}+x^{78}+x^{76}+x^{68}+x^{66}+x^{62}+x^{58}+x^{56}+x^{54}+x^{52}\\
&\quad +x^{50}+x^{48}+x^{44}+x^{42},\\
p_3(x) &= x^{108}+x^{104}+x^{94}+x^{90}+x^{88}+x^{84}+x^{80}+x^{74}+x^{70}+x^{66}+x^{64}\\
&\quad +x^{60}+x^{56}+x^{52}+x^{50}+x^{48}+x^{46}+x^{42}+x^{40}+x^{38}+x^{34}+x^{32}\\
&\quad +x^{26}+x^{18},\\
p_6(x) &= x^{112}+x^{108}+x^{106}+x^{100}+x^{98}+x^{96}+x^{92}+x^{86}+x^{84}+x^{78}+x^{76}\\
&\quad +x^{72}+x^{68}+x^{64}+x^{58}+x^{54}+x^{48}+x^{40}+x^{34}+x^{22}+x^{18}+x^{16}\\
&\quad +x^{14}+x^{12}+x^{10}+x^{6}+x^{4}+1,\\
p_9(x) &= x^{104}+x^{98}+x^{96}+x^{88}+x^{86}+x^{80}+x^{76}+x^{74}+x^{72}+x^{70}+x^{64}\\
&\quad +x^{58}+x^{56}+x^{48}+x^{46}+x^{36}+x^{30}+x^{24}+x^{22}+x^{16}+x^{12}+x^{10}\\
&\quad +x^{8}+x^{6},\\
p_{12}(x) &= x^{108}+x^{104}+x^{92}+x^{88}+x^{84}+x^{80}+x^{76}+x^{72}+x^{52}+x^{48}+x^{36}\\
&\quad +x^{32}+x^{20}+x^{16}+x^{12}+x^{8}+x^{4}+1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0].

For $\phi(M^o, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{117}+x^{114}+x^{113}+x^{112}+x^{111}+x^{109}+x^{108}+x^{107}+x^{105}+x^{102}\\
&\quad +x^{99}+x^{96}+x^{93}+x^{91}+x^{85}+x^{82}+x^{81}+x^{79}+x^{76}+x^{75}+x^{72}\\
&\quad +x^{67}+x^{65}+x^{64}+x^{60}+x^{59}+x^{57}+x^{56}+x^{55}+x^{50}+x^{49}+x^{44}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{37} + x^{36},\\
p_3(x) &= x^{101} + x^{98} + x^{97} + x^{96} + x^{95} + x^{93} + x^{92} + x^{91} + x^{89} + x^{86} + x^{85}\\
&+ x^{83} + x^{82} + x^{81} + x^{79} + x^{77} + x^{76} + x^{74} + x^{73} + x^{72} + x^{69} + x^{67}\\
&+ x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{53}\\
&+ x^{49} + x^{47} + x^{44} + x^{43} + x^{39} + x^{38} + x^{36} + x^{32} + x^{31} + x^{30} + x^{29}\\
&+ x^{28} + x^{27} + x^{25} + x^{24},\\
p_6(x) &= x^{109} + x^{106} + x^{104} + x^{103} + x^{102} + x^{101} + x^{99} + x^{98} + x^{97} + x^{96}\\
&+ x^{95} + x^{93} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{83} + x^{80}\\
&+ x^{79} + x^{77} + x^{76} + x^{75} + x^{72} + x^{71} + x^{67} + x^{66} + x^{65} + x^{64} + x^{62}\\
&+ x^{60} + x^{59} + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{47} + x^{42}\\
&+ x^{41} + x^{40} + x^{37} + x^{36} + x^{32} + x^{31} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24}\\
&+ x^{22} + x^{21} + x^{20} + x^{17} + x^{16} + x^{15} + x^{12} + x^{10} + x^{8} + x^{6} + x^{4} + x^{2}\\
&+ x + 1,\\
p_9(x) &= x^{97} + x^{94} + x^{93} + x^{92} + x^{65} + x^{62} + x^{61} + x^{60} + x^{49} + x^{46} + x^{45}\\
&+ x^{44} + x^{33} + x^{30} + x^{29} + x^{28} + x^{17} + x^{14} + x^{13} + x^{12},\\
p_{12}(x) &= x^{105} + x^{102} + x^{101} + x^{100} + x^{97} + x^{94} + x^{92} + x^{90} + x^{88} + x^{86} + x^{84}\\
&+ x^{82} + x^{81} + x^{80} + x^{73} + x^{70} + x^{69} + x^{68} + x^{65} + x^{62} + x^{60} + x^{58}\\
&+ x^{57} + x^{56} + x^{53} + x^{50} + x^{48} + x^{46} + x^{44} + x^{42} + x^{41} + x^{40} + x^{37}\\
&+ x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{25} + x^{24} + x^{21} + x^{18} + x^{16} + x^{14}\\
&+ x^{12} + x^{10} + x^{8} + x^{6} + x^{4} + x^{2} + x + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, 1, 1].

For $\phi(M^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{120} + x^{111} + x^{109} + x^{107} + x^{106} + x^{105} + x^{104} + x^{103} + x^{100} + x^{99}\\
&+ x^{98} + x^{97} + x^{96} + x^{95} + x^{92} + x^{91} + x^{90} + x^{89} + x^{87} + x^{85} + x^{82}\\
&+ x^{81} + x^{79} + x^{78} + x^{77} + x^{74} + x^{73} + x^{71} + x^{70} + x^{68} + x^{67} + x^{66}\\
&+ x^{64} + x^{63} + x^{62} + x^{56} + x^{54} + x^{53} + x^{50} + x^{49} + x^{47} + x^{45} + x^{44}\\
&+ x^{43} + x^{42} + x^{41} + x^{39},\\
p_3(x) &= x^{90} + x^{87} + x^{86} + x^{85} + x^{84} + x^{81} + x^{79} + x^{77} + x^{75} + x^{73} + x^{71}\\
&+ x^{69} + x^{67} + x^{66} + x^{65} + x^{62} + x^{60} + x^{59} + x^{42} + x^{39} + x^{38} + x^{37}\\
&+ x^{36} + x^{34} + x^{33} + x^{30} + x^{28} + x^{27},\\
p_6(x) &= x^{96} + x^{90} + x^{88} + x^{86} + x^{84} + x^{80} + x^{78} + x^{76} + x^{72} + x^{70} + x^{64}\\
&+ x^{60} + x^{58} + x^{56} + x^{52} + x^{50} + x^{48} + x^{42} + x^{38} + x^{36} + x^{34} + x^{24}\\
&+ x^{20} + x^{18},\\
p_9(x) &= x^{102} + x^{99} + x^{98} + x^{97} + x^{94} + x^{91} + x^{90} + x^{89} + x^{70} + x^{67} + x^{66}\\
&+ x^{65} + x^{62} + x^{59} + x^{58} + x^{57} + x^{54} + x^{51} + x^{50} + x^{49} + x^{46} + x^{43}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{42} + x^{41} + x^{38} + x^{35} + x^{34} + x^{33} + x^{30} + x^{27} + x^{26} + x^{25} + x^{22}\\
&+ x^{19} + x^{18} + x^{17} + x^{14} + x^{11} + x^{10} + x^{9},\\
p_{12}(x) &= x^{108} + x^{96} + x^{84} + x^{80} + x^{76} + x^{64} + x^{60} + x^{52} + x^{44} + x^{36} + x^{28}\\
&+ x^{20} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0].

For $\phi(M^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{124} + x^{115} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{107} + x^{104} + x^{100}\\
&+ x^{97} + x^{95} + x^{92} + x^{91} + x^{89} + x^{88} + x^{87} + x^{86} + x^{84} + x^{83} + x^{81}\\
&+ x^{80} + x^{79} + x^{73} + x^{72} + x^{70} + x^{69} + x^{58} + x^{57} + x^{55} + x^{53} + x^{51}\\
&+ x^{50} + x^{48} + x^{47} + x^{42} + x^{41} + x^{39},\\
p_3(x) &= x^{94} + x^{91} + x^{90} + x^{89} + x^{88} + x^{85} + x^{83} + x^{82} + x^{81} + x^{78} + x^{76}\\
&+ x^{75} + x^{74} + x^{71} + x^{69} + x^{68} + x^{67} + x^{64} + x^{62} + x^{61} + x^{60} + x^{57}\\
&+ x^{55} + x^{53} + x^{51} + x^{49} + x^{47} + x^{46} + x^{45} + x^{42} + x^{40} + x^{39} + x^{38}\\
&+ x^{35} + x^{33} + x^{32} + x^{31} + x^{30} + x^{28} + x^{27},\\
p_6(x) &= x^{100} + x^{94} + x^{92} + x^{90} + x^{84} + x^{80} + x^{78} + x^{72} + x^{68} + x^{66} + x^{64}\\
&+ x^{60} + x^{58} + x^{32} + x^{26} + x^{22} + x^{20} + x^{18},\\
p_9(x) &= x^{106} + x^{103} + x^{102} + x^{101} + x^{98} + x^{95} + x^{93} + x^{91} + x^{90} + x^{89} + x^{74}\\
&+ x^{71} + x^{70} + x^{69} + x^{66} + x^{63} + x^{61} + x^{59} + x^{57} + x^{55} + x^{54} + x^{53}\\
&+ x^{50} + x^{47} + x^{45} + x^{43} + x^{41} + x^{39} + x^{38} + x^{37} + x^{34} + x^{31} + x^{29}\\
&+ x^{27} + x^{25} + x^{23} + x^{22} + x^{21} + x^{18} + x^{15} + x^{13} + x^{11} + x^{10} + x^{9},\\
p_{12}(x) &= x^{112} + x^{92} + x^{64} + x^{60} + x^{44} + x^{28} + x^{16} + x^{12} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 1, 1, 0, 1, 0, 1, 0, 1, 0, 0, 1].

For $\phi(M^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{117} + x^{114} + x^{113} + x^{112} + x^{111} + x^{109} + x^{108} + x^{107} + x^{100} + x^{99}\\
&+ x^{98} + x^{97} + x^{95} + x^{92} + x^{91} + x^{83} + x^{80} + x^{79} + x^{74} + x^{73} + x^{71}\\
&+ x^{70} + x^{66} + x^{63} + x^{62} + x^{58} + x^{56} + x^{55} + x^{54} + x^{52} + x^{48} + x^{46}\\
&+ x^{45} + x^{43} + x^{42},\\
p_3(x) &= x^{101} + x^{98} + x^{97} + x^{96} + x^{95} + x^{93} + x^{92} + x^{91} + x^{85} + x^{84} + x^{83}\\
&+ x^{80} + x^{77} + x^{73} + x^{71} + x^{70} + x^{69} + x^{67} + x^{66} + x^{64} + x^{61} + x^{60}\\
&+ x^{59} + x^{58} + x^{54} + x^{53} + x^{52} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{43}\\
&+ x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{36} + x^{32} + x^{30} + x^{29} + x^{27} + x^{26},\\
p_6(x) &= x^{109} + x^{106} + x^{104} + x^{103} + x^{102} + x^{101} + x^{99} + x^{98} + x^{96} + x^{95}\\
&+ x^{94} + x^{93} + x^{92} + x^{91} + x^{89} + x^{83} + x^{82} + x^{79} + x^{78} + x^{77} + x^{74}\\
&+ x^{72} + x^{69} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{62} + x^{61} + x^{60} + x^{59}\\
&+ x^{58} + x^{55} + x^{53} + x^{52} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{42}+x^{40}+x^{39}+x^{38}+x^{36}+x^{34}+x^{30}+x^{29}+x^{28}+x^{24}+x^{23}\\
&+x^{20}+x^{19}+x^{18}+x^{16}+x^{15}+x^{14}+x^{13}+x^{10}+x^{8}+x^{6}+x^{4}+x^{2}\\
&+x+1,\\
p_9(x)&=x^{97}+x^{94}+x^{93}+x^{92}+x^{65}+x^{62}+x^{61}+x^{60}+x^{49}+x^{46}+x^{45}\\
&+x^{44}+x^{33}+x^{30}+x^{29}+x^{28}+x^{17}+x^{14}+x^{13}+x^{12},\\
p_{12}(x)&=x^{105}+x^{102}+x^{101}+x^{100}+x^{97}+x^{94}+x^{92}+x^{90}+x^{88}+x^{86}+x^{84}\\
&+x^{82}+x^{81}+x^{80}+x^{73}+x^{70}+x^{69}+x^{68}+x^{65}+x^{62}+x^{60}+x^{58}\\
&+x^{57}+x^{56}+x^{53}+x^{50}+x^{48}+x^{46}+x^{44}+x^{42}+x^{41}+x^{40}+x^{37}\\
&+x^{34}+x^{32}+x^{30}+x^{28}+x^{26}+x^{25}+x^{24}+x^{21}+x^{18}+x^{16}+x^{14}\\
&+x^{12}+x^{10}+x^{8}+x^{6}+x^{4}+x^{2}+x+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 1, 0, 0, 1].
For $\phi(W^o, 0, 0)$:

$$
\begin{aligned}
p_0(x)&=x^{117}+x^{114}+x^{113}+x^{112}+x^{111}+x^{109}+x^{108}+x^{107}+x^{105}+x^{102}\\
&+x^{99}+x^{96}+x^{93}+x^{91}+x^{85}+x^{82}+x^{81}+x^{79}+x^{76}+x^{75}+x^{72}\\
&+x^{67}+x^{65}+x^{64}+x^{60}+x^{59}+x^{57}+x^{56}+x^{55}+x^{50}+x^{49}+x^{44}\\
&+x^{43}+x^{42}+x^{41}+x^{40}+x^{39}+x^{37}+x^{36},\\
p_3(x)&=x^{101}+x^{98}+x^{97}+x^{96}+x^{95}+x^{93}+x^{92}+x^{91}+x^{89}+x^{86}+x^{85}\\
&+x^{83}+x^{82}+x^{81}+x^{79}+x^{77}+x^{76}+x^{74}+x^{73}+x^{72}+x^{69}+x^{67}\\
&+x^{66}+x^{65}+x^{64}+x^{63}+x^{62}+x^{61}+x^{60}+x^{59}+x^{58}+x^{57}+x^{53}\\
&+x^{49}+x^{47}+x^{44}+x^{43}+x^{39}+x^{38}+x^{36}+x^{32}+x^{31}+x^{30}+x^{29}\\
&+x^{28}+x^{27}+x^{25}+x^{24},\\
p_6(x)&=x^{109}+x^{106}+x^{104}+x^{103}+x^{102}+x^{101}+x^{99}+x^{98}+x^{97}+x^{96}\\
&+x^{95}+x^{93}+x^{91}+x^{90}+x^{89}+x^{88}+x^{87}+x^{86}+x^{85}+x^{83}+x^{80}\\
&+x^{79}+x^{77}+x^{76}+x^{75}+x^{72}+x^{71}+x^{67}+x^{66}+x^{65}+x^{64}+x^{62}\\
&+x^{60}+x^{59}+x^{56}+x^{54}+x^{53}+x^{52}+x^{51}+x^{50}+x^{49}+x^{47}+x^{42}\\
&+x^{41}+x^{40}+x^{37}+x^{36}+x^{32}+x^{31}+x^{28}+x^{27}+x^{26}+x^{25}+x^{24}\\
&+x^{22}+x^{21}+x^{20}+x^{17}+x^{16}+x^{15}+x^{12}+x^{10}+x^{8}+x^{6}+x^{4}+x^{2}\\
&+x+1,\\
p_9(x)&=x^{97}+x^{94}+x^{93}+x^{92}+x^{65}+x^{62}+x^{61}+x^{60}+x^{49}+x^{46}+x^{45}\\
&+x^{44}+x^{33}+x^{30}+x^{29}+x^{28}+x^{17}+x^{14}+x^{13}+x^{12},\\
p_{12}(x)&=x^{105}+x^{102}+x^{101}+x^{100}+x^{97}+x^{94}+x^{92}+x^{90}+x^{88}+x^{86}+x^{84}\\
&+x^{82}+x^{81}+x^{80}+x^{73}+x^{70}+x^{69}+x^{68}+x^{65}+x^{62}+x^{60}+x^{58}\\
&+x^{57}+x^{56}+x^{53}+x^{50}+x^{48}+x^{46}+x^{44}+x^{42}+x^{41}+x^{40}+x^{37}\\
&+x^{34}+x^{32}+x^{30}+x^{28}+x^{26}+x^{25}+x^{24}+x^{21}+x^{18}+x^{16}+x^{14}\\
&+x^{12}+x^{10}+x^{8}+x^{6}+x^{4}+x^{2}+x+1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, 1, 1].

For $\phi(W^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{124} + x^{115} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{107} + x^{104} + x^{100}\\
&\quad + x^{97} + x^{95} + x^{92} + x^{91} + x^{89} + x^{88} + x^{87} + x^{86} + x^{84} + x^{83} + x^{81}\\
&\quad + x^{80} + x^{79} + x^{73} + x^{72} + x^{70} + x^{69} + x^{58} + x^{57} + x^{55} + x^{53} + x^{51}\\
&\quad + x^{50} + x^{48} + x^{47} + x^{42} + x^{41} + x^{39},\\
p_3(x) &= x^{94} + x^{91} + x^{90} + x^{89} + x^{88} + x^{85} + x^{83} + x^{82} + x^{81} + x^{78} + x^{76}\\
&\quad + x^{75} + x^{74} + x^{71} + x^{69} + x^{68} + x^{67} + x^{64} + x^{62} + x^{61} + x^{60} + x^{57}\\
&\quad + x^{55} + x^{53} + x^{51} + x^{49} + x^{47} + x^{46} + x^{45} + x^{42} + x^{40} + x^{39} + x^{38}\\
&\quad + x^{35} + x^{33} + x^{32} + x^{31} + x^{30} + x^{28} + x^{27},\\
p_6(x) &= x^{100} + x^{94} + x^{92} + x^{90} + x^{84} + x^{80} + x^{78} + x^{72} + x^{68} + x^{66} + x^{64}\\
&\quad + x^{60} + x^{58} + x^{32} + x^{26} + x^{22} + x^{20} + x^{18},\\
p_9(x) &= x^{106} + x^{103} + x^{102} + x^{101} + x^{98} + x^{95} + x^{93} + x^{91} + x^{90} + x^{89} + x^{74}\\
&\quad + x^{71} + x^{70} + x^{69} + x^{66} + x^{63} + x^{61} + x^{59} + x^{57} + x^{55} + x^{54} + x^{53}\\
&\quad + x^{50} + x^{47} + x^{45} + x^{43} + x^{41} + x^{39} + x^{38} + x^{37} + x^{34} + x^{31} + x^{29}\\
&\quad + x^{27} + x^{25} + x^{23} + x^{22} + x^{21} + x^{18} + x^{15} + x^{13} + x^{11} + x^{10} + x^{9},\\
p_{12}(x) &= x^{112} + x^{92} + x^{64} + x^{60} + x^{44} + x^{28} + x^{16} + x^{12} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 1, 1, 0, 1, 0, 1, 0, 1, 0, 0, 1].

For $\phi(W^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{120} + x^{111} + x^{109} + x^{107} + x^{106} + x^{105} + x^{104} + x^{103} + x^{100} + x^{99}\\
&\quad + x^{98} + x^{97} + x^{96} + x^{95} + x^{92} + x^{91} + x^{90} + x^{89} + x^{87} + x^{85} + x^{82}\\
&\quad + x^{81} + x^{79} + x^{78} + x^{77} + x^{74} + x^{73} + x^{71} + x^{70} + x^{68} + x^{67} + x^{66}\\
&\quad + x^{64} + x^{63} + x^{62} + x^{56} + x^{54} + x^{53} + x^{50} + x^{49} + x^{47} + x^{45} + x^{44}\\
&\quad + x^{43} + x^{42} + x^{41} + x^{39},\\
p_3(x) &= x^{90} + x^{87} + x^{86} + x^{85} + x^{84} + x^{81} + x^{79} + x^{77} + x^{75} + x^{73} + x^{71}\\
&\quad + x^{69} + x^{67} + x^{66} + x^{65} + x^{62} + x^{60} + x^{59} + x^{42} + x^{39} + x^{38} + x^{37}\\
&\quad + x^{36} + x^{34} + x^{33} + x^{30} + x^{28} + x^{27},\\
p_6(x) &= x^{96} + x^{90} + x^{88} + x^{86} + x^{84} + x^{80} + x^{78} + x^{76} + x^{72} + x^{70} + x^{64}\\
&\quad + x^{60} + x^{58} + x^{56} + x^{52} + x^{50} + x^{48} + x^{42} + x^{38} + x^{36} + x^{34} + x^{24}\\
&\quad + x^{20} + x^{18},\\
p_9(x) &= x^{102} + x^{99} + x^{98} + x^{97} + x^{94} + x^{91} + x^{90} + x^{89} + x^{70} + x^{67} + x^{66}\\
&\quad + x^{65} + x^{62} + x^{59} + x^{58} + x^{57} + x^{54} + x^{51} + x^{50} + x^{49} + x^{46} + x^{43}\\
&\quad + x^{42} + x^{41} + x^{38} + x^{35} + x^{34} + x^{33} + x^{30} + x^{27} + x^{26} + x^{25} + x^{22}\\
&\quad + x^{19} + x^{18} + x^{17} + x^{14} + x^{11} + x^{10} + x^{9},\\
p_{12}(x) &= x^{108} + x^{96} + x^{84} + x^{80} + x^{76} + x^{64} + x^{60} + x^{52} + x^{44} + x^{36} + x^{28}
\end{aligned}$$

$$+ x^{20} + x^4 + 1,$$

and the initial terms are [0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0].

For $\phi(W^o, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{117} + x^{114} + x^{113} + x^{112} + x^{111} + x^{109} + x^{108} + x^{107} + x^{100} + x^{99}\\
&\quad + x^{98} + x^{97} + x^{95} + x^{92} + x^{91} + x^{83} + x^{80} + x^{79} + x^{74} + x^{73} + x^{71}\\
&\quad + x^{70} + x^{66} + x^{63} + x^{62} + x^{58} + x^{56} + x^{55} + x^{54} + x^{52} + x^{48} + x^{46}\\
&\quad + x^{45} + x^{43} + x^{42},\\
p_3(x) &= x^{101} + x^{98} + x^{97} + x^{96} + x^{95} + x^{93} + x^{92} + x^{91} + x^{85} + x^{84} + x^{83}\\
&\quad + x^{80} + x^{77} + x^{73} + x^{71} + x^{70} + x^{69} + x^{67} + x^{66} + x^{64} + x^{61} + x^{60}\\
&\quad + x^{59} + x^{58} + x^{54} + x^{53} + x^{52} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{43}\\
&\quad + x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{36} + x^{32} + x^{30} + x^{29} + x^{27} + x^{26},\\
p_6(x) &= x^{109} + x^{106} + x^{104} + x^{103} + x^{102} + x^{101} + x^{99} + x^{98} + x^{96} + x^{95}\\
&\quad + x^{94} + x^{93} + x^{92} + x^{91} + x^{89} + x^{83} + x^{82} + x^{79} + x^{78} + x^{77} + x^{74}\\
&\quad + x^{72} + x^{69} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{62} + x^{61} + x^{60} + x^{59}\\
&\quad + x^{58} + x^{55} + x^{53} + x^{52} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44}\\
&\quad + x^{42} + x^{40} + x^{39} + x^{38} + x^{36} + x^{34} + x^{30} + x^{29} + x^{28} + x^{24} + x^{23}\\
&\quad + x^{20} + x^{19} + x^{18} + x^{16} + x^{15} + x^{14} + x^{13} + x^{10} + x^{8} + x^{6} + x^{4} + x^{2}\\
&\quad + x + 1,\\
p_9(x) &= x^{97} + x^{94} + x^{93} + x^{92} + x^{65} + x^{62} + x^{61} + x^{60} + x^{49} + x^{46} + x^{45}\\
&\quad + x^{44} + x^{33} + x^{30} + x^{29} + x^{28} + x^{17} + x^{14} + x^{13} + x^{12},\\
p_{12}(x) &= x^{105} + x^{102} + x^{101} + x^{100} + x^{97} + x^{94} + x^{92} + x^{90} + x^{88} + x^{86} + x^{84}\\
&\quad + x^{82} + x^{81} + x^{80} + x^{73} + x^{70} + x^{69} + x^{68} + x^{65} + x^{62} + x^{60} + x^{58}\\
&\quad + x^{57} + x^{56} + x^{53} + x^{50} + x^{48} + x^{46} + x^{44} + x^{42} + x^{41} + x^{40} + x^{37}\\
&\quad + x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{25} + x^{24} + x^{21} + x^{18} + x^{16} + x^{14}\\
&\quad + x^{12} + x^{10} + x^{8} + x^{6} + x^{4} + x^{2} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 1, 0, 0, 1].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	333	[429, 520]	666	[476, 119]	999	[1104, 478]
1	[3, 4]	334	[521, 350]	667	[516, 188]	1000	[1063, 427]
2	[5, 6]	335	[447, 154]	668	[818, 424]	1001	[1105, 370]
3	[7, 8]	336	[113, 175]	669	[858, 173]	1002	[1087, 186]
4	[9, 10]	337	[522, 523]	670	[859, 803]	1003	[1106, 356]
5	[11, 12]	338	[365, 132]	671	[780, 853]	1004	[874, 852]
6	[13, 14]	339	[165, 421]	672	[860, 769]	1005	[1107, 500]

7	[15, 16]	340	[368, 167]	673	[861, 138]	1006	[1108, 795]
8	[17, 18]	341	[399, 372]	674	[764, 34]	1007	[1109, 772]
9	[19, 20]	342	[413, 397]	675	[862, 416]	1008	[894, 843]
10	[21, 22]	343	[380, 40]	676	[493, 766]	1009	[1110, 430]
11	[23, 24]	344	[499, 136]	677	[470, 863]	1010	[1111, 808]
12	[25, 26]	345	[464, 524]	678	[429, 864]	1011	[424, 350]
13	[27, 28]	346	[525, 526]	679	[369, 812]	1012	[1112, 188]
14	[29, 17]	347	[527, 318]	680	[865, 431]	1013	[174, 114]
15	[30, 31]	348	[528, 529]	681	[485, 755]	1014	[1113, 401]
16	[32, 33]	349	[530, 531]	682	[866, 380]	1015	[1114, 415]
17	[8, 34]	350	[532, 533]	683	[486, 13]	1016	[1115, 184]
18	[10, 35]	351	[223, 534]	684	[376, 519]	1017	[1116, 455]
19	[36, 37]	352	[535, 536]	685	[111, 133]	1018	[1117, 501]
20	[38, 39]	353	[537, 313]	686	[383, 181]	1019	[1118, 180]
21	[40, 41]	354	[83, 538]	687	[867, 116]	1020	[378, 353]
22	[42, 43]	355	[539, 540]	688	[868, 498]	1021	[1119, 1055]
23	[44, 45]	356	[541, 202]	689	[869, 8]	1022	[518, 377]
24	[46, 47]	357	[94, 4]	690	[870, 835]	1023	[1120, 40]
25	[48, 49]	358	[254, 542]	691	[9, 484]	1024	[439, 461]
26	[50, 51]	359	[543, 543]	692	[120, 51]	1025	[1121, 13]
27	[52, 53]	360	[209, 544]	693	[145, 132]	1026	[355, 437]
28	[54, 55]	361	[545, 546]	694	[871, 398]	1027	[1122, 361]
29	[56, 57]	362	[547, 548]	695	[834, 365]	1028	[1123, 418]
30	[58, 59]	363	[204, 549]	696	[872, 852]	1029	[505, 389]
31	[60, 61]	364	[550, 551]	697	[873, 523]	1030	[1124, 374]
32	[62, 63]	365	[552, 553]	698	[837, 874]	1031	[809, 779]
33	[64, 65]	366	[554, 555]	699	[875, 821]	1032	[1125, 351]
34	[16, 66]	367	[556, 557]	700	[859, 495]	1033	[1126, 405]
35	[20, 67]	368	[558, 559]	701	[876, 489]	1034	[1127, 125]
36	[68, 69]	369	[560, 561]	702	[467, 398]	1035	[1128, 467]
37	[70, 71]	370	[552, 562]	703	[494, 803]	1036	[1129, 176]
38	[72, 73]	371	[563, 564]	704	[877, 433]	1037	[1130, 51]
39	[74, 75]	372	[565, 566]	705	[504, 474]	1038	[758, 401]
40	[76, 77]	373	[567, 219]	706	[878, 766]	1039	[1131, 43]
41	[78, 79]	374	[277, 568]	707	[109, 163]	1040	[375, 515]
42	[80, 81]	375	[569, 570]	708	[879, 393]	1041	[138, 416]
43	[82, 83]	376	[571, 572]	709	[443, 372]	1042	[1132, 493]
44	[84, 85]	377	[106, 69]	710	[880, 420]	1043	[881, 13]
45	[86, 87]	378	[573, 574]	711	[761, 127]	1044	[188, 520]
46	[88, 89]	379	[565, 575]	712	[399, 444]	1045	[1133, 797]
47	[60, 90]	380	[576, 577]	713	[881, 115]	1046	[1134, 806]
48	[91, 92]	381	[208, 578]	714	[851, 468]	1047	[1135, 822]
49	[93, 94]	382	[579, 100]	715	[353, 396]	1048	[477, 478]
50	[69, 95]	383	[580, 581]	716	[882, 756]	1049	[1136, 443]

51	[96, 97]	384	[582, 583]	717	[883, 424]	1050	[870, 53]
52	[98, 99]	385	[556, 322]	718	[884, 496]	1051	[1137, 399]
53	[100, 101]	386	[584, 585]	719	[840, 162]	1052	[1138, 379]
54	[102, 103]	387	[307, 579]	720	[844, 831]	1053	[766, 411]
55	[104, 105]	388	[586, 587]	721	[885, 857]	1054	[1139, 1140]
56	[106, 107]	389	[104, 588]	722	[179, 30]	1055	[619, 1141]
57	[15, 108]	390	[589, 590]	723	[127, 409]	1056	[1142, 1143]
58	[109, 110]	391	[259, 591]	724	[886, 437]	1057	[969, 228]
59	[111, 112]	392	[27, 315]	725	[129, 511]	1058	[326, 1144]
60	[113, 114]	393	[199, 592]	726	[887, 50]	1059	[1145, 1146]
61	[115, 116]	394	[317, 70]	727	[888, 396]	1060	[921, 65]
62	[117, 118]	395	[29, 593]	728	[411, 431]	1061	[1147, 1148]
63	[119, 120]	396	[536, 594]	729	[889, 795]	1062	[1149, 914]
64	[121, 122]	397	[313, 595]	730	[890, 391]	1063	[1150, 1151]
65	[123, 124]	398	[596, 565]	731	[379, 413]	1064	[937, 683]
66	[31, 125]	399	[597, 598]	732	[509, 387]	1065	[1152, 1153]
67	[37, 50]	400	[599, 68]	733	[891, 33]	1066	[1154, 1155]
68	[119, 50]	401	[221, 292]	734	[892, 10]	1067	[1156, 1157]
69	[126, 127]	402	[600, 601]	735	[463, 760]	1068	[207, 605]
70	[128, 129]	403	[602, 603]	736	[893, 843]	1069	[1158, 281]
71	[130, 47]	404	[222, 604]	737	[894, 895]	1070	[1159, 1160]
72	[131, 132]	405	[605, 606]	738	[896, 498]	1071	[717, 942]
73	[133, 134]	406	[607, 608]	739	[897, 764]	1072	[714, 90]
74	[135, 136]	407	[90, 609]	740	[40, 817]	1073	[1161, 1162]
75	[137, 138]	408	[265, 610]	741	[889, 525]	1074	[1163, 913]
76	[139, 140]	409	[231, 611]	742	[898, 146]	1075	[1164, 751]
77	[141, 142]	410	[612, 613]	743	[899, 189]	1076	[1165, 670]
78	[143, 144]	411	[24, 614]	744	[135, 500]	1077	[1166, 326]
79	[145, 146]	412	[615, 616]	745	[900, 845]	1078	[1167, 1164]
80	[147, 148]	413	[617, 618]	746	[779, 8]	1079	[1168, 58]
81	[149, 150]	414	[619, 85]	747	[796, 7]	1080	[1162, 627]
82	[151, 45]	415	[570, 620]	748	[445, 451]	1081	[1169, 1166]
83	[152, 153]	416	[621, 622]	749	[901, 129]	1082	[1170, 1171]
84	[154, 155]	417	[623, 624]	750	[362, 153]	1083	[1172, 332]
85	[31, 156]	418	[625, 626]	751	[773, 513]	1084	[1173, 1174]
86	[157, 158]	419	[289, 627]	752	[173, 370]	1085	[725, 919]
87	[159, 160]	420	[628, 78]	753	[828, 902]	1086	[1175, 1149]
88	[161, 162]	421	[281, 629]	754	[902, 30]	1087	[1176, 1177]
89	[163, 164]	422	[630, 299]	755	[903, 904]	1088	[1178, 1179]
90	[165, 154]	423	[631, 632]	756	[343, 905]	1089	[1180, 690]
91	[166, 167]	424	[633, 244]	757	[5, 264]	1090	[529, 633]
92	[168, 169]	425	[250, 580]	758	[906, 907]	1091	[719, 657]
93	[170, 56]	426	[634, 635]	759	[322, 644]	1092	[1181, 74]
94	[171, 172]	427	[636, 631]	760	[232, 665]	1093	[1182, 639]

95	[50, 173]	428	[637, 638]	761	[908, 909]	1094	[269, 1144]
96	[174, 175]	429	[639, 640]	762	[17, 910]	1095	[1183, 1184]
97	[176, 177]	430	[625, 641]	763	[606, 82]	1096	[678, 591]
98	[178, 35]	431	[93, 3]	764	[636, 911]	1097	[3, 29]
99	[179, 180]	432	[642, 628]	765	[260, 254]	1098	[1185, 1186]
100	[181, 182]	433	[103, 643]	766	[671, 343]	1099	[107, 973]
101	[183, 184]	434	[557, 644]	767	[21, 912]	1100	[1187, 6]
102	[185, 186]	435	[277, 645]	768	[913, 286]	1101	[28, 295]
103	[187, 188]	436	[646, 647]	769	[596, 914]	1102	[1188, 1189]
104	[189, 125]	437	[597, 638]	770	[915, 84]	1103	[1190, 1191]
105	[190, 191]	438	[648, 345]	771	[916, 917]	1104	[245, 645]
106	[192, 193]	439	[649, 650]	772	[279, 705]	1105	[1192, 1193]
107	[194, 195]	440	[582, 651]	773	[918, 291]	1106	[917, 960]
108	[196, 197]	441	[652, 653]	774	[532, 919]	1107	[1194, 216]
109	[198, 199]	442	[654, 655]	775	[94, 29]	1108	[1195, 932]
110	[200, 201]	443	[541, 656]	776	[315, 527]	1109	[1029, 720]
111	[202, 203]	444	[537, 657]	777	[920, 921]	1110	[1196, 93]
112	[204, 205]	445	[227, 658]	778	[81, 922]	1111	[1197, 1198]
113	[206, 207]	446	[74, 654]	779	[692, 923]	1112	[247, 664]
114	[208, 209]	447	[659, 279]	780	[924, 227]	1113	[546, 948]
115	[210, 211]	448	[660, 661]	781	[593, 910]	1114	[1199, 621]
116	[212, 213]	449	[24, 576]	782	[925, 926]	1115	[6, 293]
117	[214, 215]	450	[662, 663]	783	[683, 927]	1116	[1200, 217]
118	[103, 216]	451	[86, 664]	784	[928, 929]	1117	[1201, 1044]
119	[217, 218]	452	[633, 327]	785	[263, 930]	1118	[1202, 1203]
120	[219, 220]	453	[665, 341]	786	[632, 312]	1119	[338, 712]
121	[221, 222]	454	[666, 667]	787	[265, 931]	1120	[1204, 1205]
122	[223, 224]	455	[212, 668]	788	[932, 933]	1121	[624, 531]
123	[225, 226]	456	[669, 236]	789	[672, 905]	1122	[1206, 530]
124	[227, 228]	457	[670, 534]	790	[85, 617]	1123	[986, 3]
125	[59, 229]	458	[671, 672]	791	[934, 208]	1124	[1207, 1208]
126	[230, 231]	459	[673, 674]	792	[935, 619]	1125	[611, 754]
127	[232, 233]	460	[675, 676]	793	[936, 557]	1126	[1209, 1210]
128	[234, 235]	461	[579, 677]	794	[301, 930]	1127	[1211, 1212]
129	[236, 237]	462	[294, 560]	795	[735, 937]	1128	[1213, 596]
130	[238, 239]	463	[258, 678]	796	[938, 939]	1129	[723, 315]
131	[240, 241]	464	[263, 679]	797	[605, 545]	1130	[1214, 1215]
132	[242, 243]	465	[680, 681]	798	[798, 798]	1131	[616, 339]
133	[244, 245]	466	[217, 682]	799	[285, 940]	1132	[702, 974]
134	[246, 247]	467	[683, 684]	800	[911, 632]	1133	[1216, 1217]
135	[248, 249]	468	[209, 685]	801	[941, 903]	1134	[1218, 1219]
136	[250, 251]	469	[222, 686]	802	[578, 544]	1135	[977, 922]
137	[252, 253]	470	[687, 688]	803	[199, 942]	1136	[1220, 1221]
138	[254, 255]	471	[617, 667]	804	[943, 944]	1137	[1222, 537]

139	[256, 257]	472	[682, 671]	805	[945, 607]	1138	[1223, 912]
140	[258, 259]	473	[689, 104]	806	[25, 946]	1139	[1224, 450]
141	[260, 261]	474	[690, 691]	807	[318, 947]	1140	[463, 787]
142	[262, 263]	475	[692, 693]	808	[82, 948]	1141	[1116, 395]
143	[264, 265]	476	[694, 695]	809	[949, 636]	1142	[1225, 787]
144	[266, 267]	477	[696, 697]	810	[332, 950]	1143	[883, 819]
145	[95, 27]	478	[242, 698]	811	[951, 952]	1144	[851, 398]
146	[268, 269]	479	[699, 700]	812	[903, 953]	1145	[1226, 790]
147	[270, 106]	480	[701, 702]	813	[954, 955]	1146	[1074, 769]
148	[81, 271]	481	[580, 703]	814	[956, 957]	1147	[1227, 172]
149	[272, 273]	482	[233, 341]	815	[958, 959]	1148	[1228, 455]
150	[246, 86]	483	[704, 705]	816	[231, 328]	1149	[39, 767]
151	[274, 21]	484	[85, 666]	817	[200, 960]	1150	[1229, 195]
152	[275, 276]	485	[706, 707]	818	[961, 532]	1151	[1103, 845]
153	[244, 277]	486	[708, 709]	819	[20, 962]	1152	[1230, 832]
154	[59, 278]	487	[710, 711]	820	[963, 964]	1153	[188, 864]
155	[279, 280]	488	[210, 712]	821	[302, 633]	1154	[1231, 367]
156	[281, 282]	489	[713, 714]	822	[662, 683]	1155	[1114, 138]
157	[283, 284]	490	[203, 715]	823	[965, 966]	1156	[1232, 168]
158	[285, 286]	491	[716, 717]	824	[798, 543]	1157	[51, 501]
159	[287, 288]	492	[288, 637]	825	[70, 967]	1158	[1233, 127]
160	[289, 290]	493	[621, 625]	826	[261, 542]	1159	[1234, 177]
161	[291, 292]	494	[718, 719]	827	[968, 969]	1160	[1134, 458]
162	[264, 293]	495	[295, 318]	828	[970, 971]	1161	[1235, 824]
163	[78, 294]	496	[570, 242]	829	[972, 735]	1162	[892, 484]
164	[295, 296]	497	[292, 604]	830	[308, 677]	1163	[1236, 816]
165	[297, 298]	498	[628, 720]	831	[69, 973]	1164	[1055, 177]
166	[299, 300]	499	[721, 722]	832	[96, 974]	1165	[1237, 895]
167	[262, 301]	500	[584, 608]	833	[639, 605]	1166	[1128, 851]
168	[302, 303]	501	[95, 723]	834	[975, 268]	1167	[1238, 363]
169	[278, 304]	502	[724, 725]	835	[732, 921]	1168	[1239, 111]
170	[259, 15]	503	[726, 25]	836	[976, 977]	1169	[1240, 57]
171	[305, 306]	504	[727, 728]	837	[978, 262]	1170	[1241, 175]
172	[307, 308]	505	[294, 729]	838	[979, 289]	1171	[505, 797]
173	[218, 309]	506	[730, 731]	839	[980, 981]	1172	[1242, 429]
174	[310, 311]	507	[218, 671]	840	[982, 983]	1173	[1243, 461]
175	[76, 312]	508	[732, 64]	841	[984, 266]	1174	[801, 167]
176	[313, 314]	509	[64, 250]	842	[985, 986]	1175	[1244, 114]
177	[315, 295]	510	[733, 734]	843	[345, 300]	1176	[1245, 170]
178	[316, 317]	511	[735, 662]	844	[987, 920]	1177	[48, 196]
179	[318, 319]	512	[736, 737]	845	[219, 988]	1178	[1246, 39]
180	[90, 320]	513	[202, 738]	846	[983, 308]	1179	[770, 832]
181	[100, 321]	514	[739, 740]	847	[576, 989]	1180	[1247, 56]
182	[322, 323]	515	[530, 741]	848	[952, 328]	1181	[1248, 112]

183	[324, 325]	516	[742, 528]	849	[990, 991]	1182	[1249, 381]
184	[268, 326]	517	[743, 744]	850	[650, 950]	1183	[1250, 119]
185	[327, 277]	518	[745, 746]	851	[288, 597]	1184	[1054, 176]
186	[328, 258]	519	[83, 747]	852	[560, 992]	1185	[1251, 481]
187	[329, 330]	520	[654, 748]	853	[203, 993]	1186	[1252, 348]
188	[278, 331]	521	[749, 750]	854	[991, 631]	1187	[1253, 47]
189	[332, 333]	522	[65, 251]	855	[251, 703]	1188	[1254, 759]
190	[334, 335]	523	[58, 751]	856	[690, 994]	1189	[1252, 442]
191	[326, 336]	524	[236, 752]	857	[932, 995]	1190	[1255, 427]
192	[337, 338]	525	[341, 753]	858	[996, 552]	1191	[1059, 124]
193	[266, 339]	526	[328, 754]	859	[997, 998]	1192	[1256, 153]
194	[233, 340]	527	[42, 755]	860	[999, 1000]	1193	[13, 116]
195	[341, 342]	528	[411, 756]	861	[620, 698]	1194	[455, 466]
196	[343, 344]	529	[757, 186]	862	[1001, 1002]	1195	[1257, 819]
197	[345, 346]	530	[758, 759]	863	[536, 70]	1196	[1258, 7]
198	[347, 348]	531	[144, 760]	864	[528, 302]	1197	[1259, 41]
199	[197, 349]	532	[761, 408]	865	[1003, 1004]	1198	[877, 866]
200	[350, 133]	533	[30, 189]	866	[631, 76]	1199	[1260, 755]
201	[32, 351]	534	[426, 762]	867	[1005, 1006]	1200	[1261, 866]
202	[352, 353]	535	[763, 369]	868	[1007, 1008]	1201	[1262, 193]
203	[148, 354]	536	[764, 765]	869	[1009, 260]	1202	[1263, 874]
204	[355, 356]	537	[766, 430]	870	[1010, 1011]	1203	[46, 810]
205	[357, 358]	538	[45, 767]	871	[1012, 1013]	1204	[1264, 150]
206	[359, 360]	539	[768, 769]	872	[1014, 1015]	1205	[398, 443]
207	[49, 361]	540	[163, 393]	873	[1016, 936]	1206	[1265, 144]
208	[362, 363]	541	[770, 507]	874	[16, 1017]	1207	[1266, 505]
209	[364, 365]	542	[164, 462]	875	[614, 989]	1208	[192, 140]
210	[366, 367]	543	[771, 772]	876	[1018, 704]	1209	[1267, 433]
211	[368, 369]	544	[773, 774]	877	[1019, 1020]	1210	[829, 129]
212	[51, 370]	545	[361, 775]	878	[1021, 1022]	1211	[1268, 140]
213	[361, 349]	546	[776, 492]	879	[1023, 1024]	1212	[515, 496]
214	[371, 372]	547	[777, 778]	880	[1025, 1026]	1213	[1269, 468]
215	[373, 119]	548	[779, 764]	881	[1027, 212]	1214	[1270, 154]
216	[186, 374]	549	[780, 150]	882	[1028, 1029]	1215	[170, 354]
217	[375, 357]	550	[781, 405]	883	[1030, 1031]	1216	[1271, 158]
218	[376, 377]	551	[132, 488]	884	[1032, 1033]	1217	[1126, 55]
219	[378, 379]	552	[180, 189]	885	[1034, 1035]	1218	[1272, 863]
220	[380, 381]	553	[460, 440]	886	[720, 79]	1219	[1070, 863]
221	[382, 383]	554	[782, 783]	887	[1036, 96]	1220	[1273, 49]
222	[384, 385]	555	[43, 170]	888	[1037, 1038]	1221	[374, 144]
223	[386, 382]	556	[508, 121]	889	[1039, 1040]	1222	[1274, 853]
224	[122, 387]	557	[784, 785]	890	[677, 101]	1223	[1095, 160]
225	[388, 389]	558	[786, 467]	891	[1041, 995]	1224	[1275, 1276]
226	[390, 391]	559	[473, 505]	892	[1042, 1043]	1225	[1277, 1278]

227	[132, 392]	560	[144, 787]	893	[1044, 946]	1226	[1279, 1280]
228	[151, 39]	561	[757, 788]	894	[1045, 1046]	1227	[1281, 622]
229	[110, 393]	562	[110, 164]	895	[545, 82]	1228	[1282, 1283]
230	[394, 395]	563	[789, 449]	896	[1047, 1048]	1229	[1284, 1285]
231	[396, 397]	564	[39, 790]	897	[1049, 1050]	1230	[914, 566]
232	[398, 399]	565	[468, 443]	898	[1051, 210]	1231	[594, 967]
233	[400, 401]	566	[7, 764]	899	[1052, 1053]	1232	[1286, 1287]
234	[402, 197]	567	[791, 380]	900	[1008, 651]	1233	[587, 263]
235	[354, 57]	568	[143, 463]	901	[548, 752]	1234	[1288, 1223]
236	[403, 404]	569	[792, 413]	902	[713, 60]	1235	[1289, 637]
237	[54, 405]	570	[133, 490]	903	[1054, 1055]	1236	[656, 738]
238	[406, 407]	571	[793, 513]	904	[436, 356]	1237	[309, 672]
239	[408, 409]	572	[150, 445]	905	[853, 445]	1238	[697, 665]
240	[410, 411]	573	[794, 795]	906	[1056, 452]	1239	[1290, 204]
241	[412, 33]	574	[197, 775]	907	[788, 143]	1240	[1291, 1292]
242	[413, 414]	575	[171, 404]	908	[1057, 160]	1241	[1151, 282]
243	[415, 416]	576	[796, 779]	909	[393, 856]	1242	[572, 654]
244	[417, 418]	577	[12, 430]	910	[514, 357]	1243	[296, 947]
245	[176, 419]	578	[474, 797]	911	[415, 1058]	1244	[751, 229]
246	[420, 381]	579	[425, 384]	912	[409, 524]	1245	[1293, 1294]
247	[421, 168]	580	[122, 469]	913	[1059, 822]	1246	[657, 595]
248	[422, 423]	581	[425, 382]	914	[823, 407]	1247	[1295, 692]
249	[424, 111]	582	[182, 509]	915	[1060, 806]	1248	[1296, 246]
250	[387, 425]	583	[798, 799]	916	[1061, 864]	1249	[1297, 200]
251	[426, 427]	584	[799, 799]	917	[512, 774]	1250	[948, 747]
252	[428, 424]	585	[387, 386]	918	[1062, 495]	1251	[1298, 682]
253	[187, 429]	586	[800, 136]	919	[805, 458]	1252	[1299, 713]
254	[430, 431]	587	[801, 369]	920	[1063, 762]	1253	[591, 108]
255	[432, 433]	588	[8, 765]	921	[842, 895]	1254	[1300, 953]
256	[434, 435]	589	[802, 803]	922	[896, 847]	1255	[61, 609]
257	[49, 197]	590	[804, 785]	923	[480, 367]	1256	[293, 931]
258	[436, 437]	591	[805, 806]	924	[1064, 776]	1257	[957, 202]
259	[438, 196]	592	[348, 807]	925	[1065, 134]	1258	[4, 17]
260	[439, 440]	593	[190, 493]	926	[503, 493]	1259	[1301, 1302]
261	[441, 442]	594	[369, 808]	927	[866, 420]	1260	[754, 259]
262	[443, 444]	595	[430, 756]	928	[1066, 419]	1261	[306, 576]
263	[445, 446]	596	[809, 7]	929	[390, 778]	1262	[973, 723]
264	[447, 421]	597	[130, 810]	930	[128, 194]	1263	[647, 994]
265	[12, 411]	598	[474, 389]	931	[421, 155]	1264	[340, 342]
266	[448, 449]	599	[811, 812]	932	[1067, 874]	1265	[1303, 232]
267	[115, 14]	600	[813, 133]	933	[152, 363]	1266	[229, 304]
268	[450, 451]	601	[814, 158]	934	[1068, 474]	1267	[622, 626]
269	[10, 452]	602	[815, 807]	935	[1069, 31]	1268	[1160, 562]
270	[453, 194]	603	[433, 380]	936	[1070, 471]	1269	[1304, 541]

271	[148, 56]	604	[784, 816]	937	[432, 866]	1270	[75, 748]
272	[454, 455]	605	[381, 817]	938	[1071, 442]	1271	[663, 927]
273	[450, 446]	606	[818, 819]	939	[464, 482]	1272	[1292, 988]
274	[456, 409]	607	[820, 821]	940	[181, 121]	1273	[1148, 588]
275	[457, 458]	608	[123, 822]	941	[1072, 902]	1274	[1285, 662]
276	[150, 450]	609	[823, 523]	942	[838, 1055]	1275	[1305, 463]
277	[459, 163]	610	[157, 824]	943	[1073, 14]	1276	[403, 172]
278	[460, 461]	611	[395, 825]	944	[1074, 360]	1277	[1306, 810]
279	[393, 462]	612	[826, 365]	945	[1075, 762]	1278	[1055, 419]
280	[55, 423]	613	[180, 31]	946	[1076, 391]	1279	[1307, 835]
281	[374, 463]	614	[45, 790]	947	[755, 148]	1280	[1228, 395]
282	[127, 464]	615	[827, 526]	948	[819, 350]	1281	[167, 812]
283	[465, 466]	616	[788, 374]	949	[1077, 415]	1282	[1308, 788]
284	[467, 468]	617	[155, 169]	950	[438, 49]	1283	[1067, 141]
285	[469, 386]	618	[395, 466]	951	[1078, 156]	1284	[1309, 440]
286	[470, 471]	619	[814, 824]	952	[1079, 902]	1285	[1108, 525]
287	[472, 191]	620	[828, 179]	953	[902, 180]	1286	[1310, 354]
288	[473, 474]	621	[829, 194]	954	[1080, 163]	1287	[366, 481]
289	[354, 475]	622	[755, 170]	955	[120, 173]	1288	[1311, 53]
290	[476, 37]	623	[830, 831]	956	[1081, 1058]	1289	[405, 423]
291	[477, 184]	624	[379, 396]	957	[43, 148]	1290	[1312, 357]
292	[183, 478]	625	[506, 832]	958	[1082, 475]	1291	[1313, 377]
293	[479, 435]	626	[126, 408]	959	[1079, 179]	1292	[874, 142]
294	[480, 481]	627	[155, 833]	960	[196, 361]	1293	[1314, 515]
295	[409, 482]	628	[834, 145]	961	[1083, 30]	1294	[448, 418]
296	[55, 118]	629	[52, 835]	962	[139, 193]	1295	[1315, 145]
297	[483, 484]	630	[836, 760]	963	[1084, 358]	1296	[1316, 421]
298	[396, 414]	631	[837, 141]	964	[873, 407]	1297	[1317, 196]
299	[485, 43]	632	[37, 120]	965	[1085, 785]	1298	[484, 35]
300	[486, 115]	633	[838, 176]	966	[820, 772]	1299	[1318, 851]
301	[487, 467]	634	[839, 142]	967	[191, 12]	1300	[442, 807]
302	[146, 488]	635	[840, 435]	968	[1086, 817]	1301	[1319, 519]
303	[348, 489]	636	[168, 833]	969	[1087, 788]	1302	[129, 195]
304	[112, 490]	637	[191, 766]	970	[1088, 120]	1303	[1320, 353]
305	[491, 492]	638	[364, 145]	971	[149, 853]	1304	[1321, 141]
306	[493, 12]	639	[412, 351]	972	[1089, 164]	1305	[1322, 1323]
307	[385, 181]	640	[159, 776]	973	[1076, 778]	1306	[226, 693]
308	[494, 495]	641	[194, 511]	974	[459, 110]	1307	[1324, 67]
309	[357, 496]	642	[841, 143]	975	[1090, 10]	1308	[99, 549]
310	[497, 498]	643	[417, 449]	976	[1091, 507]	1309	[1325, 714]
311	[365, 146]	644	[842, 843]	977	[515, 358]	1310	[1326, 1327]
312	[352, 379]	645	[844, 845]	978	[1092, 445]	1311	[303, 327]
313	[499, 500]	646	[846, 847]	979	[1093, 155]	1312	[89, 254]
314	[441, 348]	647	[408, 464]	980	[1094, 404]	1313	[944, 247]

315	[173, 501]	648	[848, 525]	981	[1095, 776]	1314	[108, 66]
316	[502, 444]	649	[849, 825]	982	[1096, 124]	1315	[1015, 242]
317	[503, 191]	650	[433, 420]	983	[804, 816]	1316	[1186, 302]
318	[504, 505]	651	[771, 821]	984	[1097, 115]	1317	[640, 545]
319	[506, 507]	652	[850, 179]	985	[1098, 767]	1318	[249, 209]
320	[468, 399]	653	[38, 45]	986	[868, 847]	1319	[738, 715]
321	[384, 383]	654	[112, 134]	987	[1099, 471]	1320	[1141, 617]
322	[383, 508]	655	[487, 851]	988	[860, 360]	1321	[79, 560]
323	[121, 509]	656	[141, 852]	989	[479, 162]	1322	[1328, 408]
324	[510, 511]	657	[853, 450]	990	[1100, 490]	1323	[350, 112]
325	[512, 513]	658	[776, 783]	991	[373, 37]	1324	[160, 783]
326	[514, 515]	659	[854, 55]	992	[146, 392]	1325	[795, 526]
327	[186, 143]	660	[855, 774]	993	[400, 759]	1326	[1329, 779]
328	[516, 429]	661	[154, 168]	994	[56, 475]	1327	[420, 40]
329	[517, 464]	662	[164, 856]	995	[819, 111]	1328	[729, 992]
330	[189, 156]	663	[525, 857]	996	[1101, 110]	1329	[1033, 213]
331	[137, 415]	664	[381, 41]	997	[1102, 520]		
332	[518, 519]	665	[353, 413]	998	[1103, 831]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	222	1	444	0	666	1	888	1	1110	0
1	0	223	1	445	1	667	0	889	1	1111	1
2	0	224	1	446	0	668	0	890	1	1112	1
3	0	225	0	447	0	669	0	891	1	1113	0
4	0	226	1	448	1	670	1	892	1	1114	0
5	0	227	1	449	0	671	0	893	1	1115	0
6	0	228	0	450	0	672	1	894	1	1116	0
7	0	229	1	451	1	673	1	895	0	1117	0
8	1	230	0	452	0	674	0	896	1	1118	1
9	0	231	0	453	0	675	1	897	1	1119	1
10	1	232	0	454	1	676	0	898	0	1120	1
11	0	233	0	455	1	677	1	899	1	1121	1
12	1	234	1	456	0	678	0	900	1	1122	0
13	0	235	1	457	1	679	1	901	1	1123	0
14	0	236	1	458	0	680	1	902	0	1124	1
15	0	237	0	459	1	681	1	903	1	1125	0
16	1	238	1	460	1	682	0	904	0	1126	1
17	1	239	1	461	0	683	1	905	1	1127	1
18	1	240	1	462	0	684	0	906	1	1128	0
19	0	241	0	463	0	685	0	907	1	1129	0
20	1	242	1	464	1	686	1	908	1	1130	1
21	1	243	0	465	1	687	1	909	1	1131	1
22	0	244	0	466	0	688	0	910	1	1132	1

23	0	245	1	467	1	689	0	911	0	1133	1
24	0	246	0	468	1	690	1	912	0	1134	1
25	1	247	1	469	1	691	0	913	1	1135	1
26	0	248	0	470	1	692	1	914	1	1136	1
27	0	249	0	471	1	693	0	915	0	1137	0
28	0	250	0	472	0	694	1	916	0	1138	1
29	0	251	1	473	0	695	0	917	1	1139	1
30	0	252	1	474	1	696	0	918	0	1140	0
31	0	253	1	475	1	697	0	919	0	1141	0
32	1	254	1	476	1	698	0	920	0	1142	1
33	0	255	0	477	0	699	1	921	0	1143	1
34	1	256	1	478	1	700	1	922	1	1144	0
35	1	257	0	479	1	701	0	923	0	1145	1
36	0	258	0	480	0	702	1	924	0	1146	0
37	1	259	0	481	1	703	0	925	1	1147	0
38	1	260	0	482	0	704	1	926	0	1148	1
39	0	261	1	483	1	705	1	927	0	1149	0
40	1	262	0	484	0	706	1	928	1	1150	0
41	0	263	1	485	1	707	0	929	1	1151	1
42	0	264	0	486	1	708	1	930	1	1152	1
43	0	265	1	487	1	709	0	931	1	1153	1
44	0	266	1	488	1	710	1	932	1	1154	1
45	1	267	1	489	0	711	1	933	1	1155	0
46	0	268	0	490	1	712	1	934	0	1156	1
47	0	269	1	491	0	713	0	935	0	1157	1
48	1	270	0	492	0	714	0	936	0	1158	0
49	0	271	1	493	0	715	0	937	0	1159	0
50	0	272	1	494	0	716	0	938	1	1160	1
51	1	273	0	495	0	717	1	939	1	1161	0
52	0	274	0	496	0	718	0	940	1	1162	1
53	1	275	1	497	1	719	0	941	0	1163	0
54	0	276	0	498	0	720	0	942	0	1164	0
55	1	277	1	499	1	721	1	943	0	1165	0
56	0	278	1	500	1	722	1	944	0	1166	0
57	0	279	1	501	0	723	0	945	0	1167	0
58	0	280	1	502	0	724	0	946	0	1168	0
59	0	281	1	503	0	725	1	947	1	1169	0
60	0	282	0	504	1	726	0	948	1	1170	1
61	1	283	1	505	0	727	1	949	0	1171	0
62	1	284	1	506	1	728	0	950	0	1172	0
63	0	285	1	507	0	729	1	951	0	1173	1
64	0	286	1	508	0	730	1	952	0	1174	0
65	0	287	0	509	0	731	1	953	0	1175	0
66	0	288	0	510	1	732	0	954	1	1176	1

67	1	289	1	511	0	733	1	955	1	1177	1
68	0	290	1	512	1	734	1	956	0	1178	1
69	0	291	0	513	0	735	0	957	0	1179	0
70	1	292	1	514	1	736	1	958	1	1180	0
71	1	293	1	515	1	737	1	959	0	1181	0
72	1	294	0	516	0	738	1	960	1	1182	0
73	0	295	0	517	1	739	1	961	0	1183	1
74	0	296	1	518	1	740	1	962	1	1184	1
75	1	297	1	519	1	741	1	963	1	1185	0
76	1	298	0	520	1	742	0	964	0	1186	0
77	0	299	1	521	1	743	1	965	1	1187	0
78	0	300	1	522	0	744	0	966	1	1188	1
79	0	301	1	523	0	745	1	967	1	1189	0
80	0	302	0	524	1	746	1	968	0	1190	1
81	1	303	0	525	1	747	1	969	1	1191	1
82	0	304	1	526	0	748	1	970	1	1192	1
83	1	305	0	527	0	749	1	971	1	1193	0
84	0	306	0	528	0	750	0	972	0	1194	1
85	0	307	0	529	0	751	0	973	0	1195	0
86	1	308	0	530	1	752	0	974	1	1196	0
87	0	309	0	531	1	753	1	975	0	1197	1
88	0	310	1	532	1	754	0	976	0	1198	1
89	0	311	1	533	0	755	1	977	1	1199	0
90	1	312	0	534	1	756	1	978	0	1200	0
91	1	313	1	535	0	757	0	979	0	1201	0
92	0	314	1	536	0	758	1	980	1	1202	1
93	0	315	0	537	0	759	1	981	1	1203	0
94	0	316	0	538	1	760	0	982	0	1204	1
95	0	317	0	539	1	761	1	983	0	1205	0
96	1	318	1	540	0	762	1	984	0	1206	0
97	1	319	1	541	0	763	0	985	0	1207	1
98	0	320	1	542	0	764	0	986	0	1208	0
99	1	321	1	543	0	765	0	987	0	1209	1
100	1	322	1	544	0	766	0	988	1	1210	0
101	1	323	0	545	0	767	1	989	1	1211	1
102	0	324	1	546	0	768	1	990	0	1212	1
103	1	325	1	547	0	769	0	991	0	1213	0
104	1	326	1	548	1	770	0	992	0	1214	1
105	1	327	0	549	0	771	0	993	0	1215	0
106	0	328	0	550	1	772	1	994	0	1216	1
107	0	329	1	551	1	773	0	995	1	1217	1
108	1	330	1	552	1	774	1	996	0	1218	1
109	0	331	1	553	1	775	0	997	1	1219	0
110	1	332	1	554	1	776	0	998	1	1220	1

111	0	333	0	555	0	777	0	999	1	1221	1
112	1	334	1	556	0	778	1	1000	0	1222	0
113	0	335	0	557	1	779	1	1001	1	1223	1
114	0	336	0	558	1	780	0	1002	1	1224	1
115	1	337	0	559	0	781	1	1003	1	1225	1
116	1	338	1	560	1	782	1	1004	1	1226	1
117	1	339	1	561	0	783	1	1005	1	1227	0
118	1	340	1	562	1	784	1	1006	0	1228	1
119	0	341	1	563	1	785	1	1007	0	1229	0
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122	1	344	1	566	0	788	1	1010	1	1232	1
123	0	345	1	567	0	789	1	1011	0	1233	0
124	1	346	1	568	0	790	0	1012	1	1234	0
125	0	347	0	569	0	791	0	1013	1	1235	0
126	0	348	0	570	0	792	0	1014	0	1236	0
127	0	349	1	571	0	793	0	1015	0	1237	0
128	1	350	1	572	0	794	1	1016	0	1238	0
129	1	351	1	573	1	795	0	1017	0	1239	0
130	1	352	0	574	1	796	1	1018	0	1240	0
131	1	353	0	575	0	797	0	1019	1	1241	1
132	1	354	1	576	1	798	0	1020	1	1242	0
133	0	355	1	577	1	799	1	1021	1	1243	1
134	0	356	0	578	1	800	0	1022	1	1244	0
135	0	357	0	579	0	801	0	1023	1	1245	1
136	0	358	1	580	1	802	1	1024	0	1246	1
137	1	359	0	581	0	803	1	1025	1	1247	0
138	1	360	1	582	1	804	0	1026	1	1248	0
139	1	361	0	583	0	805	0	1027	0	1249	0
140	0	362	0	584	1	806	1	1028	0	1250	1
141	0	363	1	585	0	807	1	1029	0	1251	0
142	0	364	1	586	0	808	0	1030	1	1252	0
143	0	365	1	587	0	809	0	1031	0	1253	0
144	1	366	1	588	1	810	1	1032	0	1254	1
145	0	367	0	589	1	811	0	1033	1	1255	1
146	0	368	1	590	0	812	1	1034	1	1256	1
147	0	369	1	591	0	813	1	1035	0	1257	0
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149	1	371	1	593	1	815	1	1037	1	1259	1
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152	1	374	1	596	0	818	0	1040	0	1262	0
153	0	375	0	597	1	819	1	1041	1	1263	1
154	0	376	0	598	1	820	1	1042	1	1264	1

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162	0	384	1	606	0	828	1	1050	1	1272	1
163	0	385	0	607	1	829	0	1051	0	1273	1
164	0	386	1	608	0	830	0	1052	1	1274	0
165	1	387	0	609	1	831	0	1053	0	1275	1
166	1	388	0	610	1	832	1	1054	1	1276	1
167	0	389	1	611	0	833	0	1055	0	1277	1
168	0	390	1	612	1	834	0	1056	1	1278	0
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171	0	393	1	615	0	837	0	1059	1	1281	0
172	0	394	0	616	1	838	0	1060	0	1282	1
173	0	395	0	617	1	839	1	1061	0	1283	1
174	1	396	0	618	0	840	0	1062	0	1284	0
175	1	397	1	619	0	841	0	1063	0	1285	0
176	1	398	0	620	1	842	0	1064	0	1286	1
177	0	399	1	621	0	843	1	1065	1	1287	1
178	0	400	0	622	1	844	0	1066	1	1288	0
179	1	401	0	623	0	845	1	1067	1	1289	0
180	1	402	1	624	1	846	0	1068	0	1290	0
181	1	403	1	625	1	847	1	1069	0	1291	0
182	1	404	1	626	0	848	0	1070	0	1292	1
183	1	405	0	627	1	849	0	1071	1	1293	1
184	0	406	1	628	0	850	1	1072	0	1294	1
185	0	407	1	629	0	851	0	1073	0	1295	0
186	0	408	1	630	0	852	1	1074	0	1296	0
187	1	409	0	631	0	853	1	1075	0	1297	0
188	1	410	1	632	1	854	0	1076	0	1298	0
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190	1	412	0	634	1	856	1	1078	0	1300	1
191	1	413	1	635	0	857	1	1079	0	1301	1
192	0	414	0	636	0	858	0	1080	1	1302	1
193	1	415	0	637	1	859	1	1081	0	1303	0
194	0	416	0	638	1	860	1	1082	1	1304	0
195	1	417	0	639	0	861	1	1083	0	1305	1
196	1	418	1	640	0	862	1	1084	1	1306	1
197	1	419	1	641	0	863	0	1085	1	1307	1
198	0	420	0	642	0	864	0	1086	0	1308	1

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200	1	422	0	644	0	866	0	1088	1	1310	1
201	1	423	0	645	0	867	1	1089	0	1311	0
202	0	424	0	646	0	868	0	1090	0	1312	0
203	1	425	0	647	1	869	0	1091	0	1313	0
204	1	426	1	648	0	870	1	1092	0	1314	1
205	0	427	0	649	0	871	1	1093	0	1315	0
206	0	428	1	650	1	872	0	1094	1	1316	0
207	0	429	0	651	0	873	0	1095	1	1317	0
208	0	430	1	652	1	874	1	1096	0	1318	0
209	1	431	0	653	1	875	1	1097	0	1319	1
210	1	432	0	654	1	876	0	1098	0	1320	0
211	1	433	1	655	1	877	1	1099	0	1321	0
212	1	434	1	656	0	878	1	1100	0	1322	1
213	0	435	1	657	1	879	1	1101	0	1323	1
214	1	436	0	658	0	880	1	1102	1	1324	1
215	0	437	1	659	0	881	0	1103	1	1325	0
216	0	438	0	660	1	882	0	1104	1	1326	1
217	0	439	0	661	0	883	1	1105	1	1327	0
218	0	440	1	662	0	884	0	1106	1	1328	1
219	1	441	1	663	1	885	1	1107	1	1329	1
220	1	442	1	664	0	886	0	1108	0		
221	0	443	0	665	0	887	0	1109	0		

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