PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a,b) = (z^3+z, z^4+z^2+1)$

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Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a,b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a,b) in $\mathbb{F}_2[x]\setminus\mathbb{F}_2$. In this paper we prove the conjecture for $(a,b) = (z^3+z, z^4+z^2+1)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \ldots)$ of elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{\ddots + \cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a,b)-Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \ldots) = (a, b, b, a, b, a, a, b, \ldots)$$

that we shall denote

$$\mathrm{CF}(a,b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When $(a,b)=(z^3+z, z^4+z^2+1)$, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

where, for $\mathrm{CF}(a,b)$*,*

$$\begin{aligned}
p_0(z) &= z^{21}+z^{16}+z^{14}+z^{13}+z^{11}+z^9+z^8+z^5+z^2+z+1,\\
p_1(z) &= z^{25}+z^{24}+z^{23}+z^{22}+z^{21}+z^{17}+z^{16}+z^{13}+z^{10}+z^8+z^7+z^6+z^4\\
&\quad +z^3+z^2+z,\\
p_2(z) &= z^{28}+z^{22}+z^{20}+z^{18}+z^{16}+z^{14}+z^{10}+z^8+z^6+z^2,\\
p_3(z) &= z^{25}+z^{24}+z^{23}+z^{22}+z^{21}+z^{17}+z^{16}+z^{13}+z^{10}+z^8+z^7+z^6+z^4\\
&\quad +z^3+z^2+z,\\
p_4(z) &= z^{22}+z^{21}+z^{20}+z^{18}+z^{16}+z^{13}+z^{12}+z^{11}+z^{10}+z^9+z^8+z^6+z^5\\
&\quad +z,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$*,*

$$\begin{aligned}
p_0(z) &= z^{21}+z^{14}+z^{13}+z^{12}+z^{11}+z^9+z^5+z^4+z^2+z,\\
p_1(z) &= z^{25}+z^{24}+z^{23}+z^{22}+z^{21}+z^{17}+z^{16}+z^{13}+z^{10}+z^8+z^7+z^6+z^4\\
&\quad +z^3+z^2+z,\\
p_2(z) &= z^{28}+z^{22}+z^{20}+z^{18}+z^{16}+z^{14}+z^{10}+z^8+z^6+z^2,\\
p_3(z) &= z^{25}+z^{24}+z^{23}+z^{22}+z^{21}+z^{17}+z^{16}+z^{13}+z^{10}+z^8+z^7+z^6+z^4\\
&\quad +z^3+z^2+z,\\
p_4(z) &= z^{22}+z^{21}+z^{20}+z^{18}+z^{13}+z^{11}+z^{10}+z^9+z^6+z^5+z^4+z+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x) &= x^{\deg(t_{2^n-1})}\begin{pmatrix} t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad \cdots\times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x) &= x^{\deg(\bar t_{2^n-1})}\begin{pmatrix} \bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix} \bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad \cdots\times x^{\deg(\bar t_0)}\begin{pmatrix} \bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n \geq 0$

$$\begin{aligned} M_{n+1}(x) &= W_n(x) \cdot M_n(x), \\ W_{n+1}(x) &= M_n(x) \cdot W_n(x). \end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$\begin{aligned} M_n(x)_{0,1} &= x^{d_n} \tilde{P}_{2^n-1}(x), \\ M_n(x)_{0,0} &= x^{d_n} \tilde{Q}_{2^n-1}(x), \end{aligned}$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \leq i,j \leq 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \leq i,j \leq 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \leq i,j \leq 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(175, 175, 175, 175, 175)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \geq 0$. If for some $n \geq 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \geq 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z\left(\phi(W^o,0,0)(x,z),\ z^{12}\cdot\phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 630 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 630, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z\left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ to order 770 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 770, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. $\square$

Define

$$R^e = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

Lemma 2.2. *For all integers $k \ge 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$M^e[:14u] = M^e[:7u] + x^{8u}M^e[:6u] + x^{10u}M^e[:4u] + x^{12u}M^e[:2u] + x^{7u}R^e,$$

$$W^e[:14u] = W^e[:7u] + x^{8u}W^e[:6u] + x^{10u}W^e[:4u] + x^{12u}W^e[:2u] + x^{7u}R^e.$$

For all integers $k \ge 2$ and $u = 2^{2k}$, the following identities hold.

$$M^o[:14u] = M^o[:7u] + x^{8u}M^o[:6u] + x^{10u}M^o[:4u] + x^{12u}M^o[:2u] + x^{7u}R^o,$$

$$W^o[:14u] = W^o[:7u] + x^{8u}W^o[:6u] + x^{10u}W^o[:4u] + x^{12u}W^o[:2u] + x^{7u}R^o.$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$T[:14u] = T[:7u] + x^{8u}T[:6u] + x^{10u}T[:4u] + x^{12u}T[:2u].$$

The proof of the other 15 cases are similar. We break down the above identity into 7 parts:

$$\begin{aligned}
0 &= T[7u:8u],\\
0 &= x^{8u}T[:u] + T[8u:9u],\\
0 &= x^{8u}T[u:2u] + T[9u:10u],\\
0 &= x^{10u}T[:u] + x^{8u}T[2u:3u] + T[10u:11u],\\
0 &= x^{10u}T[u:2u] + x^{8u}T[3u:4u] + T[11u:12u],\\
0 &= x^{12u}T[:u] + x^{10u}T[2u:3u] + x^{8u}T[4u:5u] + T[12u:13u],\\
0 &= x^{12u}T[u:2u] + x^{10u}T[3u:4u] + x^{8u}T[5u:6u] + T[13u:14u],
\end{aligned}$$

which can be rewritten as

$$0 = T[[111w]_2], \tag{2.7}$$
$$0 = T[[w]_2] + T[[1000w]_2], \tag{2.8}$$
$$0 = T[[1w]_2] + T[[1001w]_2], \tag{2.9}$$
$$0 = T[[w]_2] + T[[10w]_2] + T[[1010w]_2], \tag{2.10}$$
$$0 = T[[1w]_2] + T[[11w]_2] + T[[1011w]_2], \tag{2.11}$$
$$0 = T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[1100w]_2], \tag{2.12}$$
$$0 = T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1101w]_2], \tag{2.13}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$0 = T[[111w]_2], \tag{2.14}$$
$$0 = T[[w]_2] + T[[1000w]_2], \tag{2.15}$$
$$0 = T[[1w]_2] + T[[1001w]_2], \tag{2.16}$$
$$0 = T[[w]_2] + T[[10w]_2] + T[[1010w]_2], \tag{2.17}$$
$$0 = T[[1w]_2] + T[[11w]_2] + T[[1011w]_2], \tag{2.18}$$
$$0 = T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[1100w]_2], \tag{2.19}$$
$$0 = T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1101w]_2], \tag{2.20}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 510 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.20) can be written as

$$0=\tau(A(s,111)), \tag{2.21}$$
$$0=\tau(A(s,\epsilon))+\tau(A(s,1000)), \tag{2.22}$$
$$0=\tau(A(s,1))+\tau(A(s,1001)), \tag{2.23}$$
$$0=\tau(A(s,\epsilon))+\tau(A(s,10))+\tau(A(s,1010)), \tag{2.24}$$
$$0=\tau(A(s,1))+\tau(A(s,11))+\tau(A(s,1011)), \tag{2.25}$$
$$0=\tau(A(s,\epsilon))+\tau(A(s,10))+\tau(A(s,100))+\tau(A(s,1100)), \tag{2.26}$$
$$0=\tau(A(s,1))+\tau(A(s,11))+\tau(A(s,101))+\tau(A(s,1101)), \tag{2.27}$$

for all $s\in E_{2k}$, and

$$0=\tau(A(s,111)), \tag{2.28}$$
$$0=\tau(A(s,\epsilon))+\tau(A(s,1000)), \tag{2.29}$$
$$0=\tau(A(s,1))+\tau(A(s,1001)), \tag{2.30}$$
$$0=\tau(A(s,\epsilon))+\tau(A(s,10))+\tau(A(s,1010)), \tag{2.31}$$
$$0=\tau(A(s,1))+\tau(A(s,11))+\tau(A(s,1011)), \tag{2.32}$$
$$0=\tau(A(s,\epsilon))+\tau(A(s,10))+\tau(A(s,100))+\tau(A(s,1100)), \tag{2.33}$$
$$0=\tau(A(s,1))+\tau(A(s,11))+\tau(A(s,101))+\tau(A(s,1101)), \tag{2.34}$$

for $s=A(i,0^{2k})$. We find that $(A(i,0^{28}),E_{28})=(A(i,0^{16}),E_{16})$, so that we only have to verify that identities (2.21) through (2.34) hold for $2\le k\le 14$, which turns out to be true. □

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k\ge 2$, and $u=2^{2k-1}$,*

$$M_{2k}=M^e[:7u]+x^{7u}R^e,$$
$$W_{2k}=W^e[:7u]+x^{7u}R^e.$$

For all integer $k\ge 2$, and $u=2^{2k}$,

$$M_{2k+1}=M^o[:7u]+x^{7u}R^o,$$
$$W_{2k+1}=W^o[:7u]+x^{7u}R^o.$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n=2$, the identities can be verified directly. For $n\ge 2$, we claim that

$$M_{2k}\wedge W_{2k}\Rightarrow M_{2k+1}\wedge W_{2k+1},$$
$$M_{2k+1}\wedge W_{2k+1}\Rightarrow M_{2k+2}\wedge W_{2k+2}.$$

We give the proof of

$$M_{2k}\wedge W_{2k}\Rightarrow M_{2k+1},$$

the other ones being similar. Set $u=2^{2k}$ and $v=2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$W_{2k}M_{2k}=\big(W^e[:7v]+x^{7u}R^e\big)\times\big(M^e[:7v]+x^{7u}R^e\big); \tag{2.35}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{14v}R^o$. Therefore we only need to prove that their difference is $O(x^{14v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{14v}, to

$$W^e[:14v]M^e[:14v].$$

For all $n \leq 14$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.35) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{14}, b_1, ..., b_{14}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{14})$, which completes the proof. $\square$

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \leq j, k \leq 1$,

$$\begin{aligned} \lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k}, \\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k}, \\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k}, \\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}. \end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a, b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x, y) = \mathrm{Res}_t\left(\phi(M^e, 0, 1)(x, t), y^{12}\phi(M^e, 0, 1)(x, t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x, y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned} q_0(x) &= x^{28} + x^{27} + x^{26} + x^{23} + x^{20} + x^{19} + x^{17} + x^{15} + x^{14} + x^{12} + x^7, \\ q_1(x) &= x^{27} + x^{26} + x^{25} + x^{24} + x^{22} + x^{21} + x^{20} + x^{18} + x^{15} + x^{12} + x^{11} + x^7 \\ &\quad + x^6 + x^5 + x^4 + x^3, \\ q_2(x) &= x^{26} + x^{22} + x^{20} + x^{18} + x^{14} + x^{12} + x^{10} + x^8 + x^6 + 1, \\ q_3(x) &= x^{27} + x^{26} + x^{25} + x^{24} + x^{22} + x^{21} + x^{20} + x^{18} + x^{15} + x^{12} + x^{11} + x^7 \end{aligned}$$

$$
\begin{aligned}
&\quad + x^6 + x^5 + x^4 + x^3,\\
q_4(x) &= x^{27} + x^{23} + x^{22} + x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{12} + x^{10} + x^8\\
&\quad + x^7 + x^6.
\end{aligned}
$$

The polynomial $Q(x,y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and $R(x,y) := P(x,y)/Q(x,y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 216, and substitute it for y in $R(z,y)$, we get a series with valuation smaller than 216, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b) = f(1/z)$. □

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{150} + x^{146} + x^{142} + x^{140} + x^{138} + x^{134} + x^{130} + x^{126} + x^{124} + x^{122}\\
&\quad + x^{118} + x^{114} + x^{112} + x^{108} + x^{102} + x^{98} + x^{96} + x^{94} + x^{88} + x^{86}\\
&\quad + x^{84} + x^{82} + x^{76} + x^{66} + x^{64} + x^{56} + x^{54} + x^{52} + x^{50} + x^{44} + x^{40}\\
&\quad + x^{38} + x^{36},\\
p_3(x) &= x^{142} + x^{140} + x^{134} + x^{128} + x^{110} + x^{108} + x^{104} + x^{92} + x^{78} + x^{70}\\
&\quad + x^{62} + x^{60} + x^{56} + x^{52} + x^{48} + x^{46} + x^{44} + x^{38} + x^{32} + x^{30} + x^{28}\\
&\quad + x^{24} + x^{22} + x^{20},\\
p_6(x) &= x^{146} + x^{142} + x^{140} + x^{138} + x^{132} + x^{130} + x^{122} + x^{118} + x^{108} + x^{104}\\
&\quad + x^{94} + x^{92} + x^{90} + x^{86} + x^{84} + x^{80} + x^{78} + x^{74} + x^{66} + x^{50} + x^{46}\\
&\quad + x^{44} + x^{34} + x^{32} + x^{28} + x^{24} + x^{20} + x^{18} + x^{14} + x^{12} + x^8 + x^6 + x^4\\
&\quad + 1,\\
p_9(x) &= x^{148} + x^{146} + x^{144} + x^{140} + x^{138} + x^{136} + x^{134} + x^{132} + x^{130} + x^{128}\\
&\quad + x^{120} + x^{118} + x^{114} + x^{108} + x^{102} + x^{98} + x^{96} + x^{94} + x^{90} + x^{88}\\
&\quad + x^{76} + x^{70} + x^{68} + x^{66} + x^{58} + x^{56} + x^{50} + x^{48} + x^{46} + x^{44} + x^{42}\\
&\quad + x^{38} + x^{28} + x^{26} + x^{24} + x^{18} + x^{16} + x^{12} + x^8 + x^6,\\
p_{12}(x) &= x^{148} + x^{144} + x^{132} + x^{128} + x^{84} + x^{80} + x^{52} + x^{48} + x^{36} + x^{32} + x^{20}\\
&\quad + x^{16} + x^4 + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0].

For $\phi(M^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{147} + x^{146} + x^{144} + x^{143} + x^{141} + x^{139} + x^{134} + x^{133} + x^{129} + x^{127}\\
&\quad + x^{126} + x^{125} + x^{123} + x^{117} + x^{115} + x^{113} + x^{111} + x^{109} + x^{108}
\end{aligned}
$$

$$+ x^{105} + x^{103} + x^{96} + x^{95} + x^{92} + x^{89} + x^{88} + x^{87} + x^{82} + x^{80} + x^{79} + x^{78} + x^{75} + x^{74} + x^{73} + x^{70} + x^{66} + x^{64} + x^{62} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{51} + x^{50} + x^{48} + x^{45} + x^{43} + x^{42} + x^{40} + x^{39},$$

$$p_3(x) = x^{145} + x^{144} + x^{143} + x^{142} + x^{140} + x^{136} + x^{135} + x^{127} + x^{126} + x^{125} + x^{124} + x^{122} + x^{121} + x^{120} + x^{119} + x^{115} + x^{114} + x^{111} + x^{109} + x^{107} + x^{104} + x^{102} + x^{101} + x^{100} + x^{98} + x^{97} + x^{96} + x^{94} + x^{93} + x^{92} + x^{91} + x^{89} + x^{88} + x^{85} + x^{82} + x^{81} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{67} + x^{66} + x^{65} + x^{64} + x^{62} + x^{61} + x^{60} + x^{59} + x^{57} + x^{56} + x^{53} + x^{50} + x^{49} + x^{43} + x^{42} + x^{38} + x^{36} + x^{35} + x^{34} + x^{33} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27},$$

$$p_6(x) = x^{144} + x^{142} + x^{138} + x^{136} + x^{134} + x^{132} + x^{128} + x^{126} + x^{120} + x^{118} + x^{116} + x^{112} + x^{106} + x^{104} + x^{100} + x^{92} + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{76} + x^{66} + x^{64} + x^{62} + x^{60} + x^{56} + x^{54} + x^{44} + x^{42} + x^{38} + x^{32} + x^{30} + x^{24} + x^{20} + x^{18},$$

$$p_9(x) = x^{143} + x^{142} + x^{138} + x^{136} + x^{132} + x^{131} + x^{129} + x^{128} + x^{127} + x^{126} + x^{124} + x^{123} + x^{122} + x^{120} + x^{119} + x^{118} + x^{116} + x^{113} + x^{111} + x^{108} + x^{106} + x^{102} + x^{100} + x^{97} + x^{95} + x^{93} + x^{91} + x^{88} + x^{87} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{71} + x^{70} + x^{66} + x^{64} + x^{63} + x^{62} + x^{61} + x^{59} + x^{58} + x^{57} + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39} + x^{38} + x^{35} + x^{32} + x^{30} + x^{27} + x^{24} + x^{22} + x^{18} + x^{16} + x^{13} + x^{10} + x^{9},$$

$$p_{12}(x) = x^{142} + x^{136} + x^{118} + x^{112} + x^{110} + x^{104} + x^{94} + x^{88} + x^{86} + x^{80} + x^{78} + x^{72} + x^{70} + x^{64} + x^{54} + x^{48} + x^{30} + x^{24} + x^{6} + 1,$$

and the initial terms are [0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 1, 0, 1, 0, 1, 0].

For $\phi(M^e, 1, 0)$:

$$p_0(x) = x^{147} + x^{146} + x^{144} + x^{143} + x^{141} + x^{139} + x^{134} + x^{133} + x^{129} + x^{127} + x^{126} + x^{125} + x^{123} + x^{117} + x^{115} + x^{113} + x^{111} + x^{109} + x^{108} + x^{105} + x^{103} + x^{96} + x^{95} + x^{92} + x^{89} + x^{88} + x^{87} + x^{82} + x^{80} + x^{79} + x^{78} + x^{75} + x^{74} + x^{73} + x^{70} + x^{66} + x^{64} + x^{62} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{51} + x^{50} + x^{48} + x^{45} + x^{43} + x^{42} + x^{40} + x^{39},$$

$$p_3(x) = x^{145} + x^{144} + x^{143} + x^{142} + x^{140} + x^{136} + x^{135} + x^{127} + x^{126} + x^{125} + x^{124} + x^{122} + x^{121} + x^{120} + x^{119} + x^{115} + x^{114} + x^{111} + x^{109} + x^{107} + x^{104} + x^{102} + x^{101} + x^{100} + x^{98} + x^{97} + x^{96} + x^{94} + x^{93}$$

$$+ x^{92} + x^{91} + x^{89} + x^{88} + x^{85} + x^{82} + x^{81} + x^{75} + x^{74} + x^{73} + x^{72}$$
$$+ x^{71} + x^{67} + x^{66} + x^{65} + x^{64} + x^{62} + x^{61} + x^{60} + x^{59} + x^{57} + x^{56}$$
$$+ x^{53} + x^{50} + x^{49} + x^{43} + x^{42} + x^{38} + x^{36} + x^{35} + x^{34} + x^{33} + x^{31}$$
$$+ x^{30} + x^{29} + x^{28} + x^{27},$$

$$\begin{aligned}
p_6(x) &= x^{144} + x^{142} + x^{138} + x^{136} + x^{134} + x^{132} + x^{128} + x^{126} + x^{120} + x^{118} \\
&+ x^{116} + x^{112} + x^{106} + x^{104} + x^{100} + x^{92} + x^{90} + x^{88} + x^{86} + x^{84} \\
&+ x^{82} + x^{76} + x^{66} + x^{64} + x^{62} + x^{60} + x^{56} + x^{54} + x^{44} + x^{42} + x^{38} \\
&+ x^{32} + x^{30} + x^{24} + x^{20} + x^{18}, \\
p_9(x) &= x^{143} + x^{142} + x^{138} + x^{136} + x^{132} + x^{131} + x^{129} + x^{128} + x^{127} + x^{126} \\
&+ x^{124} + x^{123} + x^{122} + x^{120} + x^{119} + x^{118} + x^{116} + x^{113} + x^{111} \\
&+ x^{108} + x^{106} + x^{102} + x^{100} + x^{97} + x^{95} + x^{93} + x^{91} + x^{88} + x^{87} \\
&+ x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{71} \\
&+ x^{70} + x^{66} + x^{64} + x^{63} + x^{62} + x^{61} + x^{59} + x^{58} + x^{57} + x^{56} + x^{54} \\
&+ x^{53} + x^{52} + x^{51} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39} + x^{38} + x^{35} \\
&+ x^{32} + x^{30} + x^{27} + x^{24} + x^{22} + x^{18} + x^{16} + x^{13} + x^{10} + x^{9}, \\
p_{12}(x) &= x^{142} + x^{136} + x^{118} + x^{112} + x^{110} + x^{104} + x^{94} + x^{88} + x^{86} + x^{80} \\
&+ x^{78} + x^{72} + x^{70} + x^{64} + x^{54} + x^{48} + x^{30} + x^{24} + x^{6} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 1, 0, 1, 0, 1, 0].

For $\phi(M^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{144} + x^{142} + x^{140} + x^{136} + x^{132} + x^{128} + x^{124} + x^{114} + x^{110} + x^{108} \\
&+ x^{106} + x^{104} + x^{102} + x^{100} + x^{96} + x^{92} + x^{90} + x^{88} + x^{84} + x^{80} \\
&+ x^{78} + x^{76} + x^{74} + x^{66} + x^{64} + x^{52} + x^{50} + x^{46} + x^{42}, \\
p_3(x) &= x^{138} + x^{136} + x^{134} + x^{132} + x^{128} + x^{126} + x^{124} + x^{120} + x^{118} + x^{116} \\
&+ x^{114} + x^{112} + x^{94} + x^{90} + x^{84} + x^{78} + x^{70} + x^{68} + x^{64} + x^{58} + x^{52} \\
&+ x^{50} + x^{36} + x^{32} + x^{28} + x^{18}, \\
p_6(x) &= x^{138} + x^{132} + x^{128} + x^{126} + x^{124} + x^{120} + x^{112} + x^{110} + x^{108} + x^{106} \\
&+ x^{98} + x^{94} + x^{92} + x^{86} + x^{82} + x^{80} + x^{72} + x^{68} + x^{64} + x^{54} + x^{52} \\
&+ x^{50} + x^{46} + x^{44} + x^{38} + x^{36} + x^{34} + x^{30} + x^{26} + x^{22} + x^{20} + x^{18} \\
&+ x^{16} + x^{6} + x^{4} + 1, \\
p_9(x) &= x^{136} + x^{134} + x^{132} + x^{124} + x^{118} + x^{116} + x^{114} + x^{112} + x^{108} + x^{106} \\
&+ x^{104} + x^{98} + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{80} + x^{78} + x^{72} + x^{58} \\
&+ x^{56} + x^{44} + x^{38} + x^{36} + x^{34} + x^{26} + x^{22} + x^{14} + x^{12} + x^{8} + x^{6}, \\
p_{12}(x) &= x^{136} + x^{132} + x^{128} + x^{120} + x^{116} + x^{112} + x^{104} + x^{100} + x^{96} + x^{56} \\
&+ x^{52} + x^{48} + x^{24} + x^{20} + x^{16} + x^{8} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0].

For $\phi(W^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{144} + x^{142} + x^{136} + x^{134} + x^{130} + x^{126} + x^{116} + x^{106} + x^{104} + x^{102} \\
&\quad + x^{94} + x^{90} + x^{86} + x^{82} + x^{80} + x^{72} + x^{70} + x^{66} + x^{64} + x^{60} + x^{48} \\
&\quad + x^{46} + x^{40} + x^{38} + x^{36}, \\
p_3(x) &= x^{138} + x^{136} + x^{130} + x^{126} + x^{122} + x^{118} + x^{114} + x^{112} + x^{110} + x^{106} \\
&\quad + x^{102} + x^{98} + x^{94} + x^{92} + x^{90} + x^{88} + x^{86} + x^{80} + x^{78} + x^{72} + x^{68} \\
&\quad + x^{64} + x^{56} + x^{52} + x^{50} + x^{48} + x^{44} + x^{42} + x^{40} + x^{38} + x^{36} + x^{34} \\
&\quad + x^{30} + x^{24} + x^{22} + x^{20}, \\
p_6(x) &= x^{138} + x^{134} + x^{132} + x^{130} + x^{126} + x^{124} + x^{122} + x^{120} + x^{112} + x^{102} \\
&\quad + x^{94} + x^{92} + x^{88} + x^{82} + x^{80} + x^{70} + x^{58} + x^{54} + x^{50} + x^{48} + x^{46} \\
&\quad + x^{42} + x^{36} + x^{32} + x^{28} + x^{26} + x^{20} + x^{16} + x^{12} + x^{6} + x^{4} + 1, \\
p_9(x) &= x^{136} + x^{134} + x^{132} + x^{124} + x^{118} + x^{116} + x^{114} + x^{112} + x^{108} + x^{106} \\
&\quad + x^{104} + x^{98} + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{80} + x^{78} + x^{72} + x^{58} \\
&\quad + x^{56} + x^{44} + x^{38} + x^{36} + x^{34} + x^{26} + x^{22} + x^{14} + x^{12} + x^{8} + x^{6}, \\
p_{12}(x) &= x^{136} + x^{132} + x^{128} + x^{120} + x^{116} + x^{112} + x^{104} + x^{100} + x^{96} + x^{56} \\
&\quad + x^{52} + x^{48} + x^{24} + x^{20} + x^{16} + x^{8} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0].

For $\phi(W^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{147} + x^{145} + x^{144} + x^{142} + x^{141} + x^{140} + x^{139} + x^{138} + x^{137} + x^{136} \\
&\quad + x^{134} + x^{130} + x^{129} + x^{126} + x^{123} + x^{120} + x^{118} + x^{116} + x^{113} \\
&\quad + x^{112} + x^{111} + x^{110} + x^{108} + x^{106} + x^{105} + x^{104} + x^{99} + x^{95} + x^{94} \\
&\quad + x^{91} + x^{90} + x^{89} + x^{86} + x^{84} + x^{81} + x^{79} + x^{78} + x^{76} + x^{74} + x^{72} \\
&\quad + x^{66} + x^{64} + x^{61} + x^{60} + x^{57} + x^{55} + x^{54} + x^{49} + x^{48} + x^{45} + x^{44} \\
&\quad + x^{43} + x^{42} + x^{40} + x^{39}, \\
p_3(x) &= x^{145} + x^{144} + x^{143} + x^{142} + x^{140} + x^{136} + x^{135} + x^{127} + x^{126} + x^{125} \\
&\quad + x^{124} + x^{122} + x^{121} + x^{120} + x^{119} + x^{115} + x^{114} + x^{111} + x^{109} \\
&\quad + x^{107} + x^{104} + x^{102} + x^{101} + x^{100} + x^{98} + x^{97} + x^{96} + x^{94} + x^{93} \\
&\quad + x^{92} + x^{91} + x^{89} + x^{88} + x^{85} + x^{82} + x^{81} + x^{75} + x^{74} + x^{73} + x^{72} \\
&\quad + x^{71} + x^{67} + x^{66} + x^{65} + x^{64} + x^{62} + x^{61} + x^{60} + x^{59} + x^{57} + x^{56} \\
&\quad + x^{53} + x^{50} + x^{49} + x^{43} + x^{42} + x^{38} + x^{36} + x^{35} + x^{34} + x^{33} + x^{31} \\
&\quad + x^{30} + x^{29} + x^{28} + x^{27}, \\
p_6(x) &= x^{144} + x^{142} + x^{138} + x^{136} + x^{134} + x^{132} + x^{128} + x^{126} + x^{120} + x^{118} \\
&\quad + x^{116} + x^{112} + x^{106} + x^{104} + x^{100} + x^{92} + x^{90} + x^{88} + x^{86} + x^{84} \\
&\quad + x^{82} + x^{76} + x^{66} + x^{64} + x^{62} + x^{60} + x^{56} + x^{54} + x^{44} + x^{42} + x^{38} \\
&\quad + x^{32} + x^{30} + x^{24} + x^{20} + x^{18},
\end{aligned}$$

$$
\begin{aligned}
p_9(x) &= x^{143} + x^{142} + x^{138} + x^{136} + x^{132} + x^{131} + x^{129} + x^{128} + x^{127} + x^{126} \\
&\quad + x^{124} + x^{123} + x^{122} + x^{120} + x^{119} + x^{118} + x^{116} + x^{113} + x^{111} \\
&\quad + x^{108} + x^{106} + x^{102} + x^{100} + x^{97} + x^{95} + x^{93} + x^{91} + x^{88} + x^{87} \\
&\quad + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{71} \\
&\quad + x^{70} + x^{66} + x^{64} + x^{63} + x^{62} + x^{61} + x^{59} + x^{58} + x^{57} + x^{56} + x^{54} \\
&\quad + x^{53} + x^{52} + x^{51} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39} + x^{38} + x^{35} \\
&\quad + x^{32} + x^{30} + x^{27} + x^{24} + x^{22} + x^{18} + x^{16} + x^{13} + x^{10} + x^{9}, \\
p_{12}(x) &= x^{142} + x^{136} + x^{118} + x^{112} + x^{110} + x^{104} + x^{94} + x^{88} + x^{86} + x^{80} \\
&\quad + x^{78} + x^{72} + x^{70} + x^{64} + x^{54} + x^{48} + x^{30} + x^{24} + x^{6} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 1, 1, 0].

For $\phi(W^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{147} + x^{145} + x^{144} + x^{142} + x^{141} + x^{140} + x^{139} + x^{138} + x^{137} + x^{136} \\
&\quad + x^{134} + x^{130} + x^{129} + x^{126} + x^{123} + x^{120} + x^{118} + x^{116} + x^{113} \\
&\quad + x^{112} + x^{111} + x^{110} + x^{108} + x^{106} + x^{105} + x^{104} + x^{99} + x^{95} + x^{94} \\
&\quad + x^{91} + x^{90} + x^{89} + x^{86} + x^{84} + x^{81} + x^{79} + x^{78} + x^{76} + x^{74} + x^{72} \\
&\quad + x^{66} + x^{64} + x^{61} + x^{60} + x^{57} + x^{55} + x^{54} + x^{49} + x^{48} + x^{45} + x^{44} \\
&\quad + x^{43} + x^{42} + x^{40} + x^{39}, \\
p_3(x) &= x^{145} + x^{144} + x^{143} + x^{142} + x^{140} + x^{136} + x^{135} + x^{127} + x^{126} + x^{125} \\
&\quad + x^{124} + x^{122} + x^{121} + x^{120} + x^{119} + x^{115} + x^{114} + x^{111} + x^{109} \\
&\quad + x^{107} + x^{104} + x^{102} + x^{101} + x^{100} + x^{98} + x^{97} + x^{96} + x^{94} + x^{93} \\
&\quad + x^{92} + x^{91} + x^{89} + x^{88} + x^{85} + x^{82} + x^{81} + x^{75} + x^{74} + x^{73} + x^{72} \\
&\quad + x^{71} + x^{67} + x^{66} + x^{65} + x^{64} + x^{62} + x^{61} + x^{60} + x^{59} + x^{57} + x^{56} \\
&\quad + x^{53} + x^{50} + x^{49} + x^{43} + x^{42} + x^{38} + x^{36} + x^{35} + x^{34} + x^{33} + x^{31} \\
&\quad + x^{30} + x^{29} + x^{28} + x^{27}, \\
p_6(x) &= x^{144} + x^{142} + x^{138} + x^{136} + x^{134} + x^{132} + x^{128} + x^{126} + x^{120} + x^{118} \\
&\quad + x^{116} + x^{112} + x^{106} + x^{104} + x^{100} + x^{92} + x^{90} + x^{88} + x^{86} + x^{84} \\
&\quad + x^{82} + x^{76} + x^{66} + x^{64} + x^{62} + x^{60} + x^{56} + x^{54} + x^{44} + x^{42} + x^{38} \\
&\quad + x^{32} + x^{30} + x^{24} + x^{20} + x^{18}, \\
p_9(x) &= x^{143} + x^{142} + x^{138} + x^{136} + x^{132} + x^{131} + x^{129} + x^{128} + x^{127} + x^{126} \\
&\quad + x^{124} + x^{123} + x^{122} + x^{120} + x^{119} + x^{118} + x^{116} + x^{113} + x^{111} \\
&\quad + x^{108} + x^{106} + x^{102} + x^{100} + x^{97} + x^{95} + x^{93} + x^{91} + x^{88} + x^{87} \\
&\quad + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{71} \\
&\quad + x^{70} + x^{66} + x^{64} + x^{63} + x^{62} + x^{61} + x^{59} + x^{58} + x^{57} + x^{56} + x^{54} \\
&\quad + x^{53} + x^{52} + x^{51} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39} + x^{38} + x^{35} \\
&\quad + x^{32} + x^{30} + x^{27} + x^{24} + x^{22} + x^{18} + x^{16} + x^{13} + x^{10} + x^{9},
\end{aligned}
$$

$$\begin{aligned}p_{12}(x) &= x^{142}+x^{136}+x^{118}+x^{112}+x^{110}+x^{104}+x^{94}+x^{88}+x^{86}+x^{80}\\ &\quad +x^{78}+x^{72}+x^{70}+x^{64}+x^{54}+x^{48}+x^{30}+x^{24}+x^{6}+1,\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 1, 1, 0].

For $\phi(W^e, 1, 1)$:

$$\begin{aligned}p_0(x) &= x^{150}+x^{148}+x^{144}+x^{140}+x^{138}+x^{134}+x^{130}+x^{128}+x^{124}+x^{120}\\ &\quad +x^{116}+x^{104}+x^{102}+x^{100}+x^{98}+x^{96}+x^{92}+x^{90}+x^{86}+x^{82}\\ &\quad +x^{80}+x^{76}+x^{74}+x^{72}+x^{64}+x^{62}+x^{56}+x^{46}+x^{42},\\ p_3(x) &= x^{146}+x^{144}+x^{138}+x^{132}+x^{130}+x^{124}+x^{116}+x^{114}+x^{104}+x^{100}\\ &\quad +x^{98}+x^{92}+x^{90}+x^{88}+x^{80}+x^{68}+x^{66}+x^{64}+x^{60}+x^{58}+x^{52}\\ &\quad +x^{50}+x^{48}+x^{44}+x^{42}+x^{28}+x^{26}+x^{18},\\ p_6(x) &= x^{134}+x^{128}+x^{122}+x^{118}+x^{116}+x^{114}+x^{112}+x^{110}+x^{100}+x^{98}\\ &\quad +x^{96}+x^{94}+x^{86}+x^{76}+x^{74}+x^{72}+x^{70}+x^{64}+x^{62}+x^{58}+x^{56}\\ &\quad +x^{52}+x^{48}+x^{44}+x^{42}+x^{38}+x^{34}+x^{32}+x^{30}+x^{26}+x^{22}+x^{14}\\ &\quad +x^{8}+x^{6}+x^{4}+1,\\ p_9(x) &= x^{148}+x^{146}+x^{144}+x^{140}+x^{138}+x^{136}+x^{134}+x^{132}+x^{130}+x^{128}\\ &\quad +x^{120}+x^{118}+x^{114}+x^{108}+x^{102}+x^{98}+x^{96}+x^{94}+x^{90}+x^{88}\\ &\quad +x^{76}+x^{70}+x^{68}+x^{66}+x^{58}+x^{56}+x^{50}+x^{48}+x^{46}+x^{44}+x^{42}\\ &\quad +x^{38}+x^{28}+x^{26}+x^{24}+x^{18}+x^{16}+x^{12}+x^{8}+x^{6},\\ p_{12}(x) &= x^{148}+x^{144}+x^{132}+x^{128}+x^{84}+x^{80}+x^{52}+x^{48}+x^{36}+x^{32}+x^{20}\\ &\quad +x^{16}+x^{4}+1,\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0].

For $\phi(M^o, 0, 0)$:

$$\begin{aligned}p_0(x) &= x^{147}+x^{146}+x^{145}+x^{144}+x^{142}+x^{141}+x^{140}+x^{138}+x^{137}+x^{135}\\ &\quad +x^{134}+x^{133}+x^{131}+x^{130}+x^{128}+x^{127}+x^{125}+x^{123}+x^{119}\\ &\quad +x^{118}+x^{117}+x^{115}+x^{114}+x^{113}+x^{112}+x^{111}+x^{107}+x^{105}\\ &\quad +x^{101}+x^{99}+x^{92}+x^{91}+x^{90}+x^{84}+x^{83}+x^{81}+x^{80}+x^{78}+x^{77}\\ &\quad +x^{73}+x^{72}+x^{70}+x^{69}+x^{68}+x^{65}+x^{64}+x^{62}+x^{61}+x^{60}+x^{59}\\ &\quad +x^{57}+x^{56}+x^{55}+x^{54}+x^{53}+x^{50}+x^{49}+x^{48}+x^{44}+x^{43}+x^{42}\\ &\quad +x^{38}+x^{36},\\ p_3(x) &= x^{146}+x^{145}+x^{144}+x^{142}+x^{141}+x^{140}+x^{139}+x^{130}+x^{127}+x^{126}\\ &\quad +x^{124}+x^{123}+x^{122}+x^{116}+x^{115}+x^{113}+x^{112}+x^{111}+x^{109}\\ &\quad +x^{107}+x^{106}+x^{103}+x^{99}+x^{98}+x^{95}+x^{92}+x^{90}+x^{89}+x^{87}+x^{86}\\ &\quad +x^{85}+x^{82}+x^{80}+x^{76}+x^{75}+x^{74}+x^{72}+x^{67}+x^{66}+x^{63}+x^{57}\\ &\quad +x^{56}+x^{55}+x^{53}+x^{52}+x^{48}+x^{46}+x^{43}+x^{41}+x^{40}+x^{37}+x^{32}\\ &\quad +x^{31}+x^{30}+x^{26}+x^{24},\end{aligned}$$

$$\begin{aligned} p_6(x) &= x^{146}+x^{145}+x^{144}+x^{142}+x^{139}+x^{136}+x^{135}+x^{134}+x^{133}+x^{132}\\ &\quad +x^{131}+x^{130}+x^{127}+x^{124}+x^{121}+x^{117}+x^{116}+x^{115}+x^{109}\\ &\quad +x^{108}+x^{102}+x^{98}+x^{97}+x^{96}+x^{95}+x^{93}+x^{89}+x^{85}+x^{80}+x^{77}\\ &\quad +x^{76}+x^{75}+x^{73}+x^{72}+x^{71}+x^{68}+x^{67}+x^{66}+x^{64}+x^{63}+x^{62}\\ &\quad +x^{61}+x^{59}+x^{56}+x^{55}+x^{48}+x^{45}+x^{42}+x^{40}+x^{39}+x^{37}+x^{36}\\ &\quad +x^{35}+x^{33}+x^{32}+x^{28}+x^{26}+x^{25}+x^{24}+x^{23}+x^{22}+x^{20}+x^{19}\\ &\quad +x^{16}+x^{14}+x^{12}+x^{6}+1,\\ p_9(x) &= x^{146}+x^{142}+x^{140}+x^{138}+x^{136}+x^{134}+x^{132}+x^{130}+x^{128}+x^{124}\\ &\quad +x^{114}+x^{110}+x^{108}+x^{106}+x^{104}+x^{102}+x^{100}+x^{96}+x^{94}+x^{90}\\ &\quad +x^{88}+x^{86}+x^{84}+x^{80}+x^{78}+x^{74}+x^{72}+x^{70}+x^{68}+x^{66}+x^{64}\\ &\quad +x^{60}+x^{34}+x^{30}+x^{28}+x^{26}+x^{24}+x^{22}+x^{20}+x^{18}+x^{16}+x^{12},\\ p_{12}(x) &= x^{146}+x^{140}+x^{138}+x^{134}+x^{132}+x^{130}+x^{128}+x^{124}+x^{118}+x^{114}\\ &\quad +x^{112}+x^{108}+x^{106}+x^{102}+x^{100}+x^{96}+x^{90}+x^{84}+x^{74}+x^{68}\\ &\quad +x^{66}+x^{60}+x^{54}+x^{48}+x^{34}+x^{28}+x^{26}+x^{22}+x^{20}+x^{18}+x^{16}\\ &\quad +x^{12}+x^{6}+1, \end{aligned}$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 1, 1, 0, 1, 0, 1, 0, 0, 1].

For $\phi(M^o, 0, 1)$:

$$\begin{aligned} p_0(x) &= x^{152}+x^{149}+x^{148}+x^{147}+x^{146}+x^{143}+x^{132}+x^{130}+x^{129}+x^{128}\\ &\quad +x^{125}+x^{123}+x^{121}+x^{118}+x^{117}+x^{116}+x^{114}+x^{112}+x^{111}\\ &\quad +x^{110}+x^{109}+x^{107}+x^{105}+x^{104}+x^{103}+x^{102}+x^{101}+x^{100}\\ &\quad +x^{99}+x^{98}+x^{97}+x^{95}+x^{93}+x^{88}+x^{87}+x^{79}+x^{78}+x^{77}+x^{73}\\ &\quad +x^{68}+x^{67}+x^{66}+x^{61}+x^{60}+x^{59}+x^{58}+x^{57}+x^{54}+x^{50}+x^{49}\\ &\quad +x^{48}+x^{46}+x^{45}+x^{43}+x^{42}+x^{40}+x^{39},\\ p_3(x) &= x^{145}+x^{144}+x^{143}+x^{142}+x^{141}+x^{139}+x^{138}+x^{137}+x^{136}+x^{135}\\ &\quad +x^{133}+x^{132}+x^{131}+x^{130}+x^{128}+x^{124}+x^{122}+x^{118}+x^{116}\\ &\quad +x^{113}+x^{111}+x^{109}+x^{107}+x^{105}+x^{103}+x^{101}+x^{99}+x^{96}+x^{94}\\ &\quad +x^{90}+x^{89}+x^{87}+x^{86}+x^{85}+x^{84}+x^{83}+x^{81}+x^{80}+x^{79}+x^{78}\\ &\quad +x^{77}+x^{75}+x^{74}+x^{70}+x^{68}+x^{64}+x^{62}+x^{58}+x^{57}+x^{55}+x^{54}\\ &\quad +x^{53}+x^{52}+x^{51}+x^{49}+x^{48}+x^{47}+x^{46}+x^{45}+x^{43}+x^{42}+x^{41}\\ &\quad +x^{40}+x^{39}+x^{37}+x^{36}+x^{35}+x^{34}+x^{33}+x^{31}+x^{30}+x^{29}+x^{28}\\ &\quad +x^{27},\\ p_6(x) &= x^{146}+x^{144}+x^{138}+x^{134}+x^{128}+x^{126}+x^{122}+x^{120}+x^{118}+x^{116}\\ &\quad +x^{114}+x^{102}+x^{100}+x^{98}+x^{96}+x^{94}+x^{82}+x^{80}+x^{74}+x^{70}+x^{66}\\ &\quad +x^{62}+x^{56}+x^{54}+x^{46}+x^{44}+x^{36}+x^{34}+x^{30}+x^{28}+x^{20}+x^{18},\\ p_9(x) &= x^{147}+x^{146}+x^{143}+x^{140}+x^{138}+x^{137}+x^{136}+x^{135}+x^{133}+x^{132} \end{aligned}$$

$$
\begin{aligned}
&+ x^{131} + x^{130} + x^{128} + x^{125} + x^{122} + x^{121} + x^{115} + x^{114} + x^{111}\\
&+ x^{108} + x^{106} + x^{105} + x^{104} + x^{103} + x^{101} + x^{100} + x^{96} + x^{95} + x^{93}\\
&+ x^{92} + x^{88} + x^{87} + x^{85} + x^{84} + x^{80} + x^{79} + x^{77} + x^{76} + x^{72} + x^{71}\\
&+ x^{69} + x^{68} + x^{67} + x^{66} + x^{64} + x^{61} + x^{58} + x^{57} + x^{35} + x^{34} + x^{31}\\
&+ x^{28} + x^{26} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{19} + x^{18} + x^{16} + x^{13}\\
&+ x^{10} + x^{9},\\
p_{12}(x) &= x^{148} + x^{136} + x^{128} + x^{124} + x^{120} + x^{116} + x^{112} + x^{104} + x^{100} + x^{96}\\
&+ x^{92} + x^{84} + x^{76} + x^{60} + x^{56} + x^{48} + x^{36} + x^{24} + x^{16} + x^{12} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1].

For $\phi(M^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{154} + x^{153} + x^{152} + x^{149} + x^{147} + x^{146} + x^{145} + x^{136} + x^{134} + x^{133}\\
&+ x^{130} + x^{128} + x^{127} + x^{126} + x^{124} + x^{122} + x^{119} + x^{117} + x^{116}\\
&+ x^{113} + x^{108} + x^{107} + x^{106} + x^{104} + x^{101} + x^{100} + x^{96} + x^{93} + x^{92}\\
&+ x^{90} + x^{89} + x^{88} + x^{87} + x^{85} + x^{84} + x^{83} + x^{80} + x^{79} + x^{78} + x^{74}\\
&+ x^{71} + x^{70} + x^{69} + x^{61} + x^{60} + x^{58} + x^{57} + x^{56} + x^{55} + x^{51} + x^{48}\\
&+ x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{40} + x^{39},\\
p_3(x) &= x^{153} + x^{152} + x^{151} + x^{150} + x^{149} + x^{147} + x^{146} + x^{145} + x^{144} + x^{143}\\
&+ x^{141} + x^{140} + x^{139} + x^{138} + x^{137} + x^{135} + x^{134} + x^{133} + x^{132}\\
&+ x^{131} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117} + x^{115} + x^{114} + x^{113}\\
&+ x^{112} + x^{111} + x^{109} + x^{108} + x^{107} + x^{106} + x^{105} + x^{103} + x^{102}\\
&+ x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{91} + x^{90} + x^{89}\\
&+ x^{88} + x^{87} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{79} + x^{78} + x^{77} + x^{76}\\
&+ x^{75} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55}\\
&+ x^{53} + x^{52} + x^{51} + x^{50} + x^{48} + x^{44} + x^{42} + x^{41} + x^{40} + x^{39} + x^{37}\\
&+ x^{36} + x^{35} + x^{34} + x^{33} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27},\\
p_6(x) &= x^{154} + x^{152} + x^{146} + x^{142} + x^{138} + x^{134} + x^{130} + x^{124} + x^{122} + x^{114}\\
&+ x^{112} + x^{106} + x^{100} + x^{96} + x^{88} + x^{86} + x^{74} + x^{72} + x^{66} + x^{62} + x^{56}\\
&+ x^{52} + x^{44} + x^{42} + x^{30} + x^{28} + x^{20} + x^{18},\\
p_9(x) &= x^{155} + x^{154} + x^{151} + x^{148} + x^{146} + x^{145} + x^{144} + x^{143} + x^{141} + x^{140}\\
&+ x^{136} + x^{135} + x^{133} + x^{132} + x^{131} + x^{130} + x^{128} + x^{125} + x^{123}\\
&+ x^{121} + x^{119} + x^{116} + x^{114} + x^{113} + x^{112} + x^{111} + x^{109} + x^{108}\\
&+ x^{107} + x^{106} + x^{104} + x^{101} + x^{99} + x^{97} + x^{95} + x^{92} + x^{91} + x^{89}\\
&+ x^{88} + x^{85} + x^{83} + x^{81} + x^{79} + x^{76} + x^{74} + x^{73} + x^{72} + x^{71} + x^{69}\\
&+ x^{68} + x^{67} + x^{66} + x^{64} + x^{61} + x^{58} + x^{57} + x^{43} + x^{42} + x^{39} + x^{36}\\
&+ x^{34} + x^{33} + x^{32} + x^{31} + x^{29} + x^{28} + x^{24} + x^{23} + x^{21} + x^{20} + x^{19}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{18} + x^{16} + x^{13} + x^{10} + x^{9},\\
p_{12}(x) &= x^{156} + x^{144} + x^{140} + x^{136} + x^{120} + x^{104} + x^{96} + x^{92} + x^{64} + x^{60}\\
&+ x^{56} + x^{48} + x^{44} + x^{32} + x^{28} + x^{24} + x^{12} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0].
For $\phi(M^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{147} + x^{142} + x^{141} + x^{139} + x^{136} + x^{132} + x^{131} + x^{130} + x^{127} + x^{124}\\
&+ x^{122} + x^{116} + x^{115} + x^{114} + x^{113} + x^{110} + x^{109} + x^{108} + x^{106}\\
&+ x^{105} + x^{104} + x^{101} + x^{100} + x^{98} + x^{92} + x^{87} + x^{86} + x^{83} + x^{82}\\
&+ x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{72} + x^{71} + x^{70} + x^{69} + x^{67} + x^{65}\\
&+ x^{64} + x^{59} + x^{58} + x^{57} + x^{54} + x^{53} + x^{52} + x^{51} + x^{48} + x^{47} + x^{42},\\
p_3(x) &= x^{146} + x^{145} + x^{142} + x^{141} + x^{139} + x^{134} + x^{132} + x^{130} + x^{127} + x^{124}\\
&+ x^{123} + x^{120} + x^{115} + x^{114} + x^{113} + x^{111} + x^{110} + x^{109} + x^{107}\\
&+ x^{106} + x^{104} + x^{103} + x^{100} + x^{99} + x^{96} + x^{95} + x^{94} + x^{90} + x^{89}\\
&+ x^{87} + x^{85} + x^{84} + x^{78} + x^{76} + x^{75} + x^{72} + x^{68} + x^{67} + x^{63} + x^{62}\\
&+ x^{60} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{50} + x^{43} + x^{42} + x^{41} + x^{37}\\
&+ x^{34} + x^{32} + x^{31} + x^{30} + x^{26},\\
p_6(x) &= x^{146} + x^{145} + x^{142} + x^{139} + x^{136} + x^{135} + x^{133} + x^{132} + x^{131} + x^{128}\\
&+ x^{127} + x^{121} + x^{117} + x^{115} + x^{110} + x^{109} + x^{108} + x^{98} + x^{97} + x^{95}\\
&+ x^{94} + x^{93} + x^{92} + x^{90} + x^{89} + x^{86} + x^{85} + x^{84} + x^{82} + x^{80} + x^{78}\\
&+ x^{77} + x^{75} + x^{74} + x^{73} + x^{71} + x^{67} + x^{66} + x^{63} + x^{61} + x^{59} + x^{58}\\
&+ x^{55} + x^{52} + x^{45} + x^{44} + x^{42} + x^{40} + x^{39} + x^{37} + x^{35} + x^{34} + x^{33}\\
&+ x^{32} + x^{30} + x^{25} + x^{23} + x^{20} + x^{19} + x^{14} + x^{6} + 1,\\
p_9(x) &= x^{146} + x^{142} + x^{140} + x^{138} + x^{136} + x^{134} + x^{132} + x^{130} + x^{128} + x^{124}\\
&+ x^{114} + x^{110} + x^{108} + x^{106} + x^{104} + x^{102} + x^{100} + x^{96} + x^{94} + x^{90}\\
&+ x^{88} + x^{86} + x^{84} + x^{80} + x^{78} + x^{74} + x^{72} + x^{70} + x^{68} + x^{66} + x^{64}\\
&+ x^{60} + x^{34} + x^{30} + x^{28} + x^{26} + x^{24} + x^{22} + x^{20} + x^{18} + x^{16} + x^{12},\\
p_{12}(x) &= x^{146} + x^{140} + x^{138} + x^{134} + x^{132} + x^{130} + x^{128} + x^{124} + x^{118} + x^{114}\\
&+ x^{112} + x^{108} + x^{106} + x^{102} + x^{100} + x^{96} + x^{90} + x^{84} + x^{74} + x^{68}\\
&+ x^{66} + x^{60} + x^{54} + x^{48} + x^{34} + x^{28} + x^{26} + x^{22} + x^{20} + x^{18} + x^{16}\\
&+ x^{12} + x^{6} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 1].
For $\phi(W^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{147} + x^{146} + x^{145} + x^{144} + x^{142} + x^{141} + x^{140} + x^{138} + x^{137} + x^{135}\\
&+ x^{134} + x^{133} + x^{131} + x^{130} + x^{128} + x^{127} + x^{125} + x^{123} + x^{119}\\
&+ x^{118} + x^{117} + x^{115} + x^{114} + x^{113} + x^{112} + x^{111} + x^{107} + x^{105}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{101} + x^{99} + x^{92} + x^{91} + x^{90} + x^{84} + x^{83} + x^{81} + x^{80} + x^{78} + x^{77} \\
&+ x^{73} + x^{72} + x^{70} + x^{69} + x^{68} + x^{65} + x^{64} + x^{62} + x^{61} + x^{60} + x^{59} \\
&+ x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{50} + x^{49} + x^{48} + x^{44} + x^{43} + x^{42} \\
&+ x^{38} + x^{36}, \\
p_3(x) &= x^{146} + x^{145} + x^{144} + x^{142} + x^{141} + x^{140} + x^{139} + x^{130} + x^{127} + x^{126} \\
&+ x^{124} + x^{123} + x^{122} + x^{116} + x^{115} + x^{113} + x^{112} + x^{111} + x^{109} \\
&+ x^{107} + x^{106} + x^{103} + x^{99} + x^{98} + x^{95} + x^{92} + x^{90} + x^{89} + x^{87} + x^{86} \\
&+ x^{85} + x^{82} + x^{80} + x^{76} + x^{75} + x^{74} + x^{72} + x^{67} + x^{66} + x^{63} + x^{57} \\
&+ x^{56} + x^{55} + x^{53} + x^{52} + x^{48} + x^{46} + x^{43} + x^{41} + x^{40} + x^{37} + x^{32} \\
&+ x^{31} + x^{30} + x^{26} + x^{24}, \\
p_6(x) &= x^{146} + x^{145} + x^{144} + x^{142} + x^{139} + x^{136} + x^{135} + x^{134} + x^{133} + x^{132} \\
&+ x^{131} + x^{130} + x^{127} + x^{124} + x^{121} + x^{117} + x^{116} + x^{115} + x^{109} \\
&+ x^{108} + x^{102} + x^{98} + x^{97} + x^{96} + x^{95} + x^{93} + x^{89} + x^{85} + x^{80} + x^{77} \\
&+ x^{76} + x^{75} + x^{73} + x^{72} + x^{71} + x^{68} + x^{67} + x^{66} + x^{64} + x^{63} + x^{62} \\
&+ x^{61} + x^{59} + x^{56} + x^{55} + x^{48} + x^{45} + x^{42} + x^{40} + x^{39} + x^{37} + x^{36} \\
&+ x^{35} + x^{33} + x^{32} + x^{28} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{20} + x^{19} \\
&+ x^{16} + x^{14} + x^{12} + x^{6} + 1, \\
p_9(x) &= x^{146} + x^{142} + x^{140} + x^{138} + x^{136} + x^{134} + x^{132} + x^{130} + x^{128} + x^{124} \\
&+ x^{114} + x^{110} + x^{108} + x^{106} + x^{104} + x^{102} + x^{100} + x^{96} + x^{94} + x^{90} \\
&+ x^{88} + x^{86} + x^{84} + x^{80} + x^{78} + x^{74} + x^{72} + x^{70} + x^{68} + x^{66} + x^{64} \\
&+ x^{60} + x^{34} + x^{30} + x^{28} + x^{26} + x^{24} + x^{22} + x^{20} + x^{18} + x^{16} + x^{12}, \\
p_{12}(x) &= x^{146} + x^{140} + x^{138} + x^{134} + x^{132} + x^{130} + x^{128} + x^{124} + x^{118} + x^{114} \\
&+ x^{112} + x^{108} + x^{106} + x^{102} + x^{100} + x^{96} + x^{90} + x^{84} + x^{74} + x^{68} \\
&+ x^{66} + x^{60} + x^{54} + x^{48} + x^{34} + x^{28} + x^{26} + x^{22} + x^{20} + x^{18} + x^{16} \\
&+ x^{12} + x^{6} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 1, 1, 0, 1, 0, 1, 0, 0, 1].

For $\phi(W^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{154} + x^{153} + x^{152} + x^{149} + x^{147} + x^{146} + x^{145} + x^{136} + x^{134} + x^{133} \\
&+ x^{130} + x^{128} + x^{127} + x^{126} + x^{124} + x^{122} + x^{119} + x^{117} + x^{116} \\
&+ x^{113} + x^{108} + x^{107} + x^{106} + x^{104} + x^{101} + x^{100} + x^{96} + x^{93} + x^{92} \\
&+ x^{90} + x^{89} + x^{88} + x^{87} + x^{85} + x^{84} + x^{83} + x^{80} + x^{79} + x^{78} + x^{74} \\
&+ x^{71} + x^{70} + x^{69} + x^{61} + x^{60} + x^{58} + x^{57} + x^{56} + x^{55} + x^{51} + x^{48} \\
&+ x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{40} + x^{39}, \\
p_3(x) &= x^{153} + x^{152} + x^{151} + x^{150} + x^{149} + x^{147} + x^{146} + x^{145} + x^{144} + x^{143} \\
&+ x^{141} + x^{140} + x^{139} + x^{138} + x^{137} + x^{135} + x^{134} + x^{133} + x^{132}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{131} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117} + x^{115} + x^{114} + x^{113}\\
&+ x^{112} + x^{111} + x^{109} + x^{108} + x^{107} + x^{106} + x^{105} + x^{103} + x^{102}\\
&+ x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{91} + x^{90} + x^{89}\\
&+ x^{88} + x^{87} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{79} + x^{78} + x^{77} + x^{76}\\
&+ x^{75} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55}\\
&+ x^{53} + x^{52} + x^{51} + x^{50} + x^{48} + x^{44} + x^{42} + x^{41} + x^{40} + x^{39} + x^{37}\\
&+ x^{36} + x^{35} + x^{34} + x^{33} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27},\\
p_6(x) = {}& x^{154} + x^{152} + x^{146} + x^{142} + x^{138} + x^{134} + x^{130} + x^{124} + x^{122} + x^{114}\\
&+ x^{112} + x^{106} + x^{100} + x^{96} + x^{88} + x^{86} + x^{74} + x^{72} + x^{66} + x^{62} + x^{56}\\
&+ x^{52} + x^{44} + x^{42} + x^{30} + x^{28} + x^{20} + x^{18},\\
p_9(x) = {}& x^{155} + x^{154} + x^{151} + x^{148} + x^{146} + x^{145} + x^{144} + x^{143} + x^{141} + x^{140}\\
&+ x^{136} + x^{135} + x^{133} + x^{132} + x^{131} + x^{130} + x^{128} + x^{125} + x^{123}\\
&+ x^{121} + x^{119} + x^{116} + x^{114} + x^{113} + x^{112} + x^{111} + x^{109} + x^{108}\\
&+ x^{107} + x^{106} + x^{104} + x^{101} + x^{99} + x^{97} + x^{95} + x^{92} + x^{91} + x^{89}\\
&+ x^{88} + x^{85} + x^{83} + x^{81} + x^{79} + x^{76} + x^{74} + x^{73} + x^{72} + x^{71} + x^{69}\\
&+ x^{68} + x^{67} + x^{66} + x^{64} + x^{61} + x^{58} + x^{57} + x^{43} + x^{42} + x^{39} + x^{36}\\
&+ x^{34} + x^{33} + x^{32} + x^{31} + x^{29} + x^{28} + x^{24} + x^{23} + x^{21} + x^{20} + x^{19}\\
&+ x^{18} + x^{16} + x^{13} + x^{10} + x^{9},\\
p_{12}(x) = {}& x^{156} + x^{144} + x^{140} + x^{136} + x^{120} + x^{104} + x^{96} + x^{92} + x^{64} + x^{60}\\
&+ x^{56} + x^{48} + x^{44} + x^{32} + x^{28} + x^{24} + x^{12} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0].

For $\phi(W^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) = {}& x^{152} + x^{149} + x^{148} + x^{147} + x^{146} + x^{143} + x^{132} + x^{130} + x^{129} + x^{128}\\
&+ x^{125} + x^{123} + x^{121} + x^{118} + x^{117} + x^{116} + x^{114} + x^{112} + x^{111}\\
&+ x^{110} + x^{109} + x^{107} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100}\\
&+ x^{99} + x^{98} + x^{97} + x^{95} + x^{93} + x^{88} + x^{87} + x^{79} + x^{78} + x^{77} + x^{73}\\
&+ x^{68} + x^{67} + x^{66} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{54} + x^{50} + x^{49}\\
&+ x^{48} + x^{46} + x^{45} + x^{43} + x^{42} + x^{40} + x^{39},\\
p_3(x) = {}& x^{145} + x^{144} + x^{143} + x^{142} + x^{141} + x^{139} + x^{138} + x^{137} + x^{136} + x^{135}\\
&+ x^{133} + x^{132} + x^{131} + x^{130} + x^{128} + x^{124} + x^{122} + x^{118} + x^{116}\\
&+ x^{113} + x^{111} + x^{109} + x^{107} + x^{105} + x^{103} + x^{101} + x^{99} + x^{96} + x^{94}\\
&+ x^{90} + x^{89} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{79} + x^{78}\\
&+ x^{77} + x^{75} + x^{74} + x^{70} + x^{68} + x^{64} + x^{62} + x^{58} + x^{57} + x^{55} + x^{54}\\
&+ x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{43} + x^{42} + x^{41}\\
&+ x^{40} + x^{39} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{31} + x^{30} + x^{29} + x^{28}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{27},\\
p_6(x) &= x^{146} + x^{144} + x^{138} + x^{134} + x^{128} + x^{126} + x^{122} + x^{120} + x^{118} + x^{116}\\
&+ x^{114} + x^{102} + x^{100} + x^{98} + x^{96} + x^{94} + x^{82} + x^{80} + x^{74} + x^{70} + x^{66}\\
&+ x^{62} + x^{56} + x^{54} + x^{46} + x^{44} + x^{36} + x^{34} + x^{30} + x^{28} + x^{20} + x^{18},\\
p_9(x) &= x^{147} + x^{146} + x^{143} + x^{140} + x^{138} + x^{137} + x^{136} + x^{135} + x^{133} + x^{132}\\
&+ x^{131} + x^{130} + x^{128} + x^{125} + x^{122} + x^{121} + x^{115} + x^{114} + x^{111}\\
&+ x^{108} + x^{106} + x^{105} + x^{104} + x^{103} + x^{101} + x^{100} + x^{96} + x^{95} + x^{93}\\
&+ x^{92} + x^{88} + x^{87} + x^{85} + x^{84} + x^{80} + x^{79} + x^{77} + x^{76} + x^{72} + x^{71}\\
&+ x^{69} + x^{68} + x^{67} + x^{66} + x^{64} + x^{61} + x^{58} + x^{57} + x^{35} + x^{34} + x^{31}\\
&+ x^{28} + x^{26} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{19} + x^{18} + x^{16} + x^{13}\\
&+ x^{10} + x^{9},\\
p_{12}(x) &= x^{148} + x^{136} + x^{128} + x^{124} + x^{120} + x^{116} + x^{112} + x^{104} + x^{100} + x^{96}\\
&+ x^{92} + x^{84} + x^{76} + x^{60} + x^{56} + x^{48} + x^{36} + x^{24} + x^{16} + x^{12} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1].

For $\phi(W^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{147} + x^{142} + x^{141} + x^{139} + x^{136} + x^{132} + x^{131} + x^{130} + x^{127} + x^{124}\\
&+ x^{122} + x^{116} + x^{115} + x^{114} + x^{113} + x^{110} + x^{109} + x^{108} + x^{106}\\
&+ x^{105} + x^{104} + x^{101} + x^{100} + x^{98} + x^{92} + x^{87} + x^{86} + x^{83} + x^{82}\\
&+ x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{72} + x^{71} + x^{70} + x^{69} + x^{67} + x^{65}\\
&+ x^{64} + x^{59} + x^{58} + x^{57} + x^{54} + x^{53} + x^{52} + x^{51} + x^{48} + x^{47} + x^{42},\\
p_3(x) &= x^{146} + x^{145} + x^{142} + x^{141} + x^{139} + x^{134} + x^{132} + x^{130} + x^{127} + x^{124}\\
&+ x^{123} + x^{120} + x^{115} + x^{114} + x^{113} + x^{111} + x^{110} + x^{109} + x^{107}\\
&+ x^{106} + x^{104} + x^{103} + x^{100} + x^{99} + x^{96} + x^{95} + x^{94} + x^{90} + x^{89}\\
&+ x^{87} + x^{85} + x^{84} + x^{78} + x^{76} + x^{75} + x^{72} + x^{68} + x^{67} + x^{63} + x^{62}\\
&+ x^{60} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{50} + x^{43} + x^{42} + x^{41} + x^{37}\\
&+ x^{34} + x^{32} + x^{31} + x^{30} + x^{26},\\
p_6(x) &= x^{146} + x^{145} + x^{142} + x^{139} + x^{136} + x^{135} + x^{133} + x^{132} + x^{131} + x^{128}\\
&+ x^{127} + x^{121} + x^{117} + x^{115} + x^{110} + x^{109} + x^{108} + x^{98} + x^{97} + x^{95}\\
&+ x^{94} + x^{93} + x^{92} + x^{90} + x^{89} + x^{86} + x^{85} + x^{84} + x^{82} + x^{80} + x^{78}\\
&+ x^{77} + x^{75} + x^{74} + x^{73} + x^{71} + x^{67} + x^{66} + x^{63} + x^{61} + x^{59} + x^{58}\\
&+ x^{55} + x^{52} + x^{45} + x^{44} + x^{42} + x^{40} + x^{39} + x^{37} + x^{35} + x^{34} + x^{33}\\
&+ x^{32} + x^{30} + x^{25} + x^{23} + x^{20} + x^{19} + x^{14} + x^{6} + 1,\\
p_9(x) &= x^{146} + x^{142} + x^{140} + x^{138} + x^{136} + x^{134} + x^{132} + x^{130} + x^{128} + x^{124}\\
&+ x^{114} + x^{110} + x^{108} + x^{106} + x^{104} + x^{102} + x^{100} + x^{96} + x^{94} + x^{90}\\
&+ x^{88} + x^{86} + x^{84} + x^{80} + x^{78} + x^{74} + x^{72} + x^{70} + x^{68} + x^{66} + x^{64}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{60} + x^{34} + x^{30} + x^{28} + x^{26} + x^{24} + x^{22} + x^{20} + x^{18} + x^{16} + x^{12}, \\
p_{12}(x) = \; & x^{146} + x^{140} + x^{138} + x^{134} + x^{132} + x^{130} + x^{128} + x^{124} + x^{118} + x^{114} \\
&+ x^{112} + x^{108} + x^{106} + x^{102} + x^{100} + x^{96} + x^{90} + x^{84} + x^{74} + x^{68} \\
&+ x^{66} + x^{60} + x^{54} + x^{48} + x^{34} + x^{28} + x^{26} + x^{22} + x^{20} + x^{18} + x^{16} \\
&+ x^{12} + x^{6} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 1].

Below is the transition function and output function of an 2-automaton that generates $T = M^{o}_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	103	[152, 127]	206	[41, 160]	309	[98, 228]	412	[441, 442]
1	[3, 4]	104	[166, 33]	207	[141, 165]	310	[376, 377]	413	[229, 443]
2	[5, 6]	105	[167, 168]	208	[280, 146]	311	[346, 378]	414	[444, 60]
3	[7, 8]	106	[169, 170]	209	[288, 9]	312	[16, 379]	415	[368, 445]
4	[9, 10]	107	[171, 172]	210	[149, 123]	313	[380, 274]	416	[446, 447]
5	[11, 12]	108	[173, 174]	211	[8, 298]	314	[381, 382]	417	[448, 449]
6	[13, 14]	109	[175, 176]	212	[138, 31]	315	[383, 384]	418	[450, 451]
7	[15, 16]	110	[110, 110]	213	[299, 51]	316	[385, 386]	419	[452, 453]
8	[17, 18]	111	[177, 178]	214	[163, 300]	317	[387, 388]	420	[261, 454]
9	[19, 20]	112	[57, 179]	215	[301, 2]	318	[389, 390]	421	[455, 456]
10	[21, 22]	113	[180, 181]	216	[120, 302]	319	[391, 392]	422	[457, 458]
11	[23, 24]	114	[182, 183]	217	[152, 303]	320	[205, 381]	423	[459, 450]
12	[25, 26]	115	[184, 185]	218	[146, 128]	321	[393, 394]	424	[286, 166]
13	[27, 28]	116	[186, 187]	219	[141, 53]	322	[395, 396]	425	[46, 298]
14	[29, 15]	117	[188, 189]	220	[122, 282]	323	[397, 235]	426	[148, 165]
15	[30, 31]	118	[190, 191]	221	[304, 166]	324	[398, 399]	427	[460, 30]
16	[32, 33]	119	[192, 193]	222	[305, 49]	325	[168, 400]	428	[149, 282]
17	[34, 35]	120	[6, 194]	223	[153, 306]	326	[238, 401]	429	[48, 417]
18	[36, 37]	121	[61, 195]	224	[126, 303]	327	[289, 114]	430	[412, 403]
19	[38, 9]	122	[196, 197]	225	[161, 33]	328	[314, 108]	431	[461, 161]
20	[39, 40]	123	[198, 199]	226	[38, 307]	329	[281, 111]	432	[415, 280]
21	[41, 42]	124	[200, 201]	227	[142, 123]	330	[314, 308]	433	[34, 410]
22	[43, 31]	125	[202, 203]	228	[107, 308]	331	[128, 128]	434	[462, 325]
23	[44, 45]	126	[204, 205]	229	[145, 33]	332	[45, 300]	435	[118, 463]
24	[46, 47]	127	[206, 207]	230	[309, 163]	333	[288, 105]	436	[464, 320]
25	[48, 49]	128	[178, 208]	231	[288, 307]	334	[316, 296]	437	[41, 113]
26	[50, 51]	129	[209, 210]	232	[310, 40]	335	[147, 164]	438	[117, 300]
27	[52, 49]	130	[211, 212]	233	[305, 150]	336	[290, 282]	439	[301, 465]
28	[53, 54]	131	[213, 214]	234	[10, 311]	337	[402, 150]	440	[464, 37]
29	[46, 55]	132	[215, 216]	235	[39, 312]	338	[135, 403]	441	[122, 14]
30	[56, 57]	133	[217, 218]	236	[30, 139]	339	[404, 164]	442	[286, 145]
31	[58, 59]	134	[83, 219]	237	[313, 283]	340	[405, 42]	443	[308, 146]

32	[60, 61]	135	[220, 221]	238	[12, 42]	341	[145, 112]	444	[30, 32]
33	[62, 63]	136	[222, 223]	239	[314, 280]	342	[166, 326]	445	[466, 280]
34	[64, 65]	137	[224, 225]	240	[155, 33]	343	[406, 30]	446	[467, 9]
35	[66, 67]	138	[226, 227]	241	[315, 45]	344	[407, 42]	447	[407, 130]
36	[68, 69]	139	[228, 229]	242	[149, 14]	345	[38, 289]	448	[156, 468]
37	[70, 71]	140	[230, 99]	243	[147, 283]	346	[136, 296]	449	[420, 302]
38	[72, 73]	141	[231, 232]	244	[316, 312]	347	[131, 408]	450	[152, 134]
39	[74, 75]	142	[233, 234]	245	[305, 317]	348	[409, 295]	451	[308, 111]
40	[76, 77]	143	[235, 236]	246	[165, 54]	349	[297, 9]	452	[422, 469]
41	[78, 79]	144	[237, 238]	247	[318, 319]	350	[315, 129]	453	[470, 14]
42	[80, 81]	145	[239, 240]	248	[158, 320]	351	[316, 40]	454	[471, 409]
43	[82, 83]	146	[172, 173]	249	[159, 113]	352	[294, 410]	455	[404, 141]
44	[84, 85]	147	[241, 242]	250	[291, 321]	353	[165, 278]	456	[472, 298]
45	[86, 87]	148	[243, 244]	251	[108, 110]	354	[321, 295]	457	[406, 163]
46	[88, 89]	149	[245, 246]	252	[322, 303]	355	[117, 139]	458	[473, 298]
47	[90, 91]	150	[247, 248]	253	[41, 130]	356	[136, 312]	459	[279, 308]
48	[26, 92]	151	[249, 250]	254	[307, 114]	357	[164, 53]	460	[474, 475]
49	[93, 94]	152	[251, 252]	255	[297, 105]	358	[32, 326]	461	[476, 477]
50	[95, 96]	153	[253, 254]	256	[314, 281]	359	[161, 326]	462	[478, 479]
51	[20, 97]	154	[255, 62]	257	[300, 109]	360	[411, 163]	463	[480, 481]
52	[28, 98]	155	[256, 257]	258	[115, 51]	361	[145, 109]	464	[482, 483]
53	[99, 100]	156	[258, 259]	259	[141, 114]	362	[139, 112]	465	[484, 68]
54	[101, 102]	157	[260, 261]	260	[323, 41]	363	[13, 143]	466	[485, 328]
55	[103, 104]	158	[262, 263]	261	[13, 282]	364	[154, 166]	467	[486, 487]
56	[38, 105]	159	[264, 265]	262	[285, 130]	365	[145, 326]	468	[488, 489]
57	[106, 14]	160	[266, 267]	263	[124, 321]	366	[114, 295]	469	[490, 371]
58	[107, 108]	161	[268, 269]	264	[324, 35]	367	[105, 53]	470	[491, 492]
59	[32, 109]	162	[73, 270]	265	[50, 325]	368	[108, 146]	471	[493, 494]
60	[110, 111]	163	[271, 175]	266	[8, 287]	369	[307, 162]	472	[495, 496]
61	[32, 112]	164	[272, 273]	267	[284, 300]	370	[297, 307]	473	[497, 498]
62	[12, 113]	165	[274, 275]	268	[280, 110]	371	[412, 116]	474	[129, 31]
63	[105, 114]	166	[276, 277]	269	[155, 326]	372	[286, 161]	475	[415, 281]
64	[115, 116]	167	[140, 148]	270	[129, 300]	373	[146, 110]	476	[404, 148]
65	[117, 31]	168	[39, 137]	271	[140, 283]	374	[140, 164]	477	[407, 160]
66	[118, 119]	169	[131, 35]	272	[147, 141]	375	[128, 110]	478	[159, 130]
67	[120, 121]	170	[53, 278]	273	[310, 137]	376	[131, 410]	479	[461, 155]
68	[122, 123]	171	[279, 280]	274	[106, 123]	377	[127, 413]	480	[292, 499]
69	[124, 125]	172	[107, 281]	275	[129, 32]	378	[289, 162]	481	[470, 282]
70	[126, 127]	173	[280, 111]	276	[280, 128]	379	[31, 112]	482	[285, 160]
71	[111, 128]	174	[281, 146]	277	[300, 112]	380	[414, 129]	483	[421, 10]
72	[44, 129]	175	[106, 282]	278	[201, 327]	381	[108, 111]	484	[11, 405]
73	[12, 130]	176	[283, 53]	279	[195, 177]	382	[415, 308]	485	[279, 281]
74	[131, 132]	177	[107, 280]	280	[328, 329]	383	[416, 161]	486	[500, 45]
75	[112, 133]	178	[110, 128]	281	[330, 331]	384	[281, 110]	487	[407, 113]

76	[126, 134]	179	[105, 162]	282	[24, 332]	385	[305, 417]	488	[323, 407]
77	[31, 109]	180	[136, 40]	283	[333, 334]	386	[109, 418]	489	[122, 143]
78	[34, 132]	181	[284, 32]	284	[335, 336]	387	[318, 419]	490	[324, 408]
79	[135, 116]	182	[12, 160]	285	[337, 338]	388	[420, 121]	491	[52, 417]
80	[136, 137]	183	[163, 31]	286	[339, 340]	389	[409, 54]	492	[409, 311]
81	[138, 139]	184	[285, 113]	287	[341, 342]	390	[421, 125]	493	[404, 283]
82	[140, 141]	185	[286, 155]	288	[343, 344]	391	[422, 159]	494	[473, 55]
83	[142, 143]	186	[46, 287]	289	[345, 346]	392	[293, 123]	495	[48, 317]
84	[144, 145]	187	[9, 114]	290	[347, 348]	393	[142, 14]	496	[501, 320]
85	[110, 146]	188	[288, 289]	291	[349, 186]	394	[117, 32]	497	[34, 408]
86	[147, 148]	189	[290, 14]	292	[350, 351]	395	[108, 128]	498	[502, 302]
87	[149, 143]	190	[50, 116]	293	[352, 353]	396	[423, 127]	499	[503, 504]
88	[48, 150]	191	[291, 10]	294	[354, 355]	397	[309, 30]	500	[505, 506]
89	[151, 121]	192	[292, 285]	295	[356, 357]	398	[114, 54]	501	[489, 507]
90	[152, 153]	193	[293, 143]	296	[358, 359]	399	[163, 32]	502	[508, 509]
91	[139, 109]	194	[291, 125]	297	[360, 182]	400	[148, 53]	503	[402, 317]
92	[154, 155]	195	[111, 110]	298	[361, 362]	401	[283, 165]	504	[462, 51]
93	[156, 157]	196	[294, 132]	299	[363, 364]	402	[348, 424]	505	[45, 31]
94	[158, 37]	197	[10, 295]	300	[85, 365]	403	[425, 426]	506	[281, 128]
95	[159, 160]	198	[39, 296]	301	[366, 367]	404	[427, 428]	507	[421, 409]
96	[154, 161]	199	[45, 32]	302	[59, 368]	405	[429, 430]	508	[159, 42]
97	[9, 162]	200	[297, 289]	303	[227, 369]	406	[431, 432]	509	[471, 10]
98	[154, 145]	201	[8, 47]	304	[370, 253]	407	[433, 434]		
99	[106, 143]	202	[142, 282]	305	[371, 372]	408	[435, 436]		
100	[163, 139]	203	[43, 139]	306	[365, 373]	409	[437, 438]		
101	[8, 55]	204	[279, 108]	307	[374, 356]	410	[439, 440]		
102	[164, 165]	205	[126, 153]	308	[208, 375]	411	[399, 58]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	73	0	146	0	219	1	292	1	365	1	438	1
1	0	74	0	147	1	220	1	293	1	366	0	439	0
2	0	75	0	148	1	221	0	294	1	367	0	440	1
3	0	76	0	149	1	222	1	295	1	368	1	441	1
4	0	77	0	150	1	223	1	296	0	369	0	442	1
5	0	78	1	151	0	224	0	297	0	370	0	443	1
6	0	79	1	152	1	225	1	298	1	371	1	444	0
7	0	80	1	153	1	226	0	299	0	372	1	445	0
8	1	81	0	154	0	227	1	300	0	373	0	446	1
9	0	82	0	155	1	228	0	301	0	374	0	447	1
10	1	83	1	156	1	229	1	302	0	375	0	448	1
11	0	84	0	157	0	230	0	303	1	376	0	449	0
12	0	85	0	158	1	231	1	304	0	377	1	450	1
13	0	86	1	159	0	232	0	305	1	378	0	451	1

14	0	87	1	160	1	233	1	306	1	379	0	452	1
15	0	88	0	161	1	234	1	307	0	380	0	453	0
16	0	89	0	162	0	235	0	308	1	381	1	454	1
17	1	90	1	163	0	236	0	309	0	382	1	455	1
18	1	91	0	164	1	237	0	310	0	383	1	456	0
19	0	92	0	165	0	238	0	311	1	384	1	457	1
20	0	93	1	166	1	239	1	312	0	385	1	458	1
21	1	94	1	167	0	240	1	313	0	386	0	459	0
22	0	95	0	168	0	241	1	314	1	387	1	460	1
23	0	96	0	169	0	242	1	315	1	388	0	461	1
24	0	97	0	170	0	243	1	316	1	389	1	462	0
25	0	98	0	171	0	244	1	317	1	390	1	463	1
26	0	99	0	172	0	245	1	318	1	391	1	464	1
27	0	100	0	173	1	246	0	319	1	392	1	465	0
28	0	101	1	174	1	247	1	320	0	393	1	466	0
29	0	102	1	175	0	248	1	321	1	394	1	467	1
30	0	103	1	176	1	249	0	322	1	395	1	468	0
31	0	104	1	177	0	250	0	323	0	396	0	469	0
32	0	105	0	178	0	251	1	324	0	397	0	470	0
33	0	106	0	179	0	252	1	325	0	398	0	471	1
34	1	107	0	180	1	253	1	326	0	399	0	472	0
35	0	108	1	181	1	254	0	327	0	400	1	473	1
36	1	109	0	182	0	255	0	328	1	401	1	474	1
37	0	110	0	183	0	256	1	329	1	402	1	475	1
38	0	111	0	184	1	257	0	330	1	403	0	476	1
39	0	112	0	185	1	258	1	331	0	404	1	477	1
40	0	113	1	186	0	259	1	332	1	405	0	478	0
41	1	114	0	187	0	260	0	333	1	406	1	479	1
42	1	115	1	188	1	261	0	334	1	407	1	480	1
43	0	116	0	189	0	262	1	335	1	408	0	481	0
44	0	117	1	190	0	263	0	336	0	409	1	482	1
45	1	118	0	191	0	264	0	337	1	410	0	483	1
46	0	119	1	192	1	265	0	338	1	411	0	484	0
47	1	120	0	193	1	266	1	339	1	412	1	485	0
48	0	121	0	194	0	267	1	340	0	413	1	486	1
49	1	122	1	195	0	268	1	341	1	414	0	487	1
50	0	123	0	196	1	269	1	342	1	415	1	488	0
51	0	124	0	197	1	270	1	343	1	416	1	489	1
52	0	125	1	198	0	271	0	344	1	417	1	490	0
53	0	126	0	199	1	272	1	345	0	418	1	491	0
54	1	127	1	200	0	273	0	346	1	419	1	492	1
55	1	128	0	201	1	274	0	347	0	420	0	493	1
56	0	129	1	202	1	275	1	348	1	421	1	494	1
57	0	130	1	203	0	276	1	349	0	422	1	495	0

58	0	131	0	204	0	277	0	350	1	423	0	496	1
59	0	132	0	205	0	278	1	351	1	424	1	497	1
60	0	133	1	206	1	279	0	352	1	425	0	498	0
61	0	134	1	207	1	280	1	353	0	426	1	499	1
62	0	135	1	208	1	281	1	354	1	427	1	500	1
63	0	136	1	209	1	282	0	355	1	428	1	501	1
64	1	137	0	210	1	283	1	356	1	429	0	502	0
65	1	138	0	211	1	284	1	357	1	430	1	503	1
66	0	139	0	212	0	285	1	358	0	431	1	504	0
67	0	140	0	213	0	286	1	359	1	432	1	505	1
68	1	141	1	214	0	287	1	360	0	433	1	506	1
69	0	142	1	215	0	288	1	361	1	434	0	507	1
70	0	143	0	216	0	289	0	362	0	435	0	508	0
71	0	144	0	217	1	290	0	363	0	436	1	509	1
72	0	145	1	218	0	291	0	364	0	437	1		

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