PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a,b) = (z^6 + z^4 + z^3 + z, z+1)$

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Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a,b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a,b) in $\mathbb{F}_2[x] \setminus \mathbb{F}_2$. In this paper we prove the conjecture for $(a,b) = (z^6 + z^4 + z^3 + z, z+1)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \ldots)$ of elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{\ddots + \cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a,b)-Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \ldots) = (a, b, b, a, b, a, a, b, \ldots)$$

that we shall denote

$$\mathrm{CF}(a,b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When* $(a,b)=(z^6+z^4+z^3+z,z+1)$*, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

where, for $\mathrm{CF}(a,b)$*,*

$$\begin{aligned}
p_0(z)&=z^{17}+z^{16}+z^{15}+z^{13}+z^{12}+z^9+z^8+z^7+z^6+z+1,\\
p_1(z)&=z^{23}+z^{22}+z^{19}+z^{15}+z^{14}+z^{12}+z^{11}+z^7+z^2+z,\\
p_2(z)&=z^{24}+z^{22}+z^{20}+z^{18}+z^{10}+z^2,\\
p_3(z)&=z^{23}+z^{22}+z^{19}+z^{15}+z^{14}+z^{12}+z^{11}+z^7+z^2+z,\\
p_4(z)&=z^{22}+z^{18}+z^{17}+z^{16}+z^{15}+z^{13}+z^{10}+z^9+z^7+z^6+z^2+z,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$*,*

$$\begin{aligned}
p_0(z)&=z^{20}+z^{17}+z^{15}+z^{13}+z^{12}+z^9+z^7+z^6+z^4+z,\\
p_1(z)&=z^{23}+z^{22}+z^{19}+z^{15}+z^{14}+z^{12}+z^{11}+z^7+z^2+z,\\
p_2(z)&=z^{24}+z^{22}+z^{20}+z^{18}+z^{10}+z^2,\\
p_3(z)&=z^{23}+z^{22}+z^{19}+z^{15}+z^{14}+z^{12}+z^{11}+z^7+z^2+z,\\
p_4(z)&=z^{22}+z^{20}+z^{18}+z^{17}+z^{15}+z^{13}+z^{10}+z^9+z^8+z^7+z^6+z^4+z^2+z\\
&\quad+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\ge 0}$ and $(Q_n(z))_{n\ge 0}$ by

$$\begin{pmatrix}P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix}:=\begin{pmatrix}t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix}t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x)&=x^{\deg(t_{2^n-1})}\begin{pmatrix}t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix}t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(t_0)}\begin{pmatrix}t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x)&=x^{\deg(\bar t_{2^n-1})}\begin{pmatrix}\bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix}\bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(\bar t_0)}\begin{pmatrix}\bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\ge 0$

$$\begin{aligned}
M_{n+1}(x)&=W_n(x)\cdot M_n(x),\\
W_{n+1}(x)&=M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(175, 175, 175, 175, 175)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z\left(\phi(W^o,0,0)(x,z),\ z^{12}\cdot\phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 630 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 630, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z\left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ to order 770 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 770, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. $\square$

Define

$$R^e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned} M^e[:14u] = {} & M^e[:7u] + x^u M^e[:6u] + x^{3u}M^e[:4u] + x^{5u}M^e[:2u] + x^u M^e[:7u] \\ & + x^{2u}M^e[:6u] + x^{4u}M^e[:4u] + x^{6u}M^e[:2u] + x^{2u}M^e[:7u] \\ & + x^{3u}M^e[:6u] + x^{5u}M^e[:4u] + x^{7u}M^e[:2u] + x^{4u}M^e[:7u] \\ & + x^{5u}M^e[:6u] + x^{7u}M^e[:4u] + x^{9u}M^e[:2u] + x^{5u}M^e[:7u] \\ & + x^{6u}M^e[:6u] + x^{8u}M^e[:4u] + x^{10u}M^e[:2u] + x^{7u}M^e[:7u] \\ & + x^{9u}M^e[:5u] + x^{11u}M^e[:3u] + x^{12u}M^e[:2u] + x^{13u}M^e[:u] \end{aligned}$$

$$
\begin{aligned}
&+ (x^{7u} + x^{8u} + x^{9u} + x^{11u} + x^{12u})R^e,\\
W^e[:14u] &= W^e[:7u] + x^uW^e[:6u] + x^{3u}W^e[:4u] + x^{5u}W^e[:2u] + x^uW^e[:7u]\\
&+ x^{2u}W^e[:6u] + x^{4u}W^e[:4u] + x^{6u}W^e[:2u] + x^{2u}W^e[:7u]\\
&+ x^{3u}W^e[:6u] + x^{5u}W^e[:4u] + x^{7u}W^e[:2u] + x^{4u}W^e[:7u]\\
&+ x^{5u}W^e[:6u] + x^{7u}W^e[:4u] + x^{9u}W^e[:2u] + x^{5u}W^e[:7u]\\
&+ x^{6u}W^e[:6u] + x^{8u}W^e[:4u] + x^{10u}W^e[:2u] + x^{7u}W^e[:7u]\\
&+ x^{9u}W^e[:5u] + x^{11u}W^e[:3u] + x^{12u}W^e[:2u] + x^{13u}W^e[:u]\\
&+ (x^{7u} + x^{8u} + x^{9u} + x^{11u} + x^{12u})R^e.
\end{aligned}
$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$
\begin{aligned}
M^o[:14u] &= M^o[:7u] + x^uM^o[:6u] + x^{3u}M^o[:4u] + x^{5u}M^o[:2u] + x^uM^o[:7u]\\
&+ x^{2u}M^o[:6u] + x^{4u}M^o[:4u] + x^{6u}M^o[:2u] + x^{2u}M^o[:7u]\\
&+ x^{3u}M^o[:6u] + x^{5u}M^o[:4u] + x^{7u}M^o[:2u] + x^{4u}M^o[:7u]\\
&+ x^{5u}M^o[:6u] + x^{7u}M^o[:4u] + x^{9u}M^o[:2u] + x^{5u}M^o[:7u]\\
&+ x^{6u}M^o[:6u] + x^{8u}M^o[:4u] + x^{10u}M^o[:2u] + x^{7u}M^o[:7u]\\
&+ x^{9u}M^o[:5u] + x^{11u}M^o[:3u] + x^{12u}M^o[:2u] + x^{13u}M^o[:u]\\
&+ (x^{7u} + x^{8u} + x^{9u} + x^{11u} + x^{12u})R^o,\\
W^o[:14u] &= W^o[:7u] + x^uW^o[:6u] + x^{3u}W^o[:4u] + x^{5u}W^o[:2u] + x^uW^o[:7u]\\
&+ x^{2u}W^o[:6u] + x^{4u}W^o[:4u] + x^{6u}W^o[:2u] + x^{2u}W^o[:7u]\\
&+ x^{3u}W^o[:6u] + x^{5u}W^o[:4u] + x^{7u}W^o[:2u] + x^{4u}W^o[:7u]\\
&+ x^{5u}W^o[:6u] + x^{7u}W^o[:4u] + x^{9u}W^o[:2u] + x^{5u}W^o[:7u]\\
&+ x^{6u}W^o[:6u] + x^{8u}W^o[:4u] + x^{10u}W^o[:2u] + x^{7u}W^o[:7u]\\
&+ x^{9u}W^o[:5u] + x^{11u}W^o[:3u] + x^{12u}W^o[:2u] + x^{13u}W^o[:u]\\
&+ (x^{7u} + x^{8u} + x^{9u} + x^{11u} + x^{12u})R^o.
\end{aligned}
$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$
\begin{aligned}
T[:14u] = T[:7u] &+ x^uT[:6u] + x^{3u}T[:4u] + x^{5u}T[:2u] + x^uT[:7u] + x^{2u}T[:6u]\\
&+ x^{4u}T[:4u] + x^{6u}T[:2u] + x^{2u}T[:7u] + x^{3u}T[:6u] + x^{5u}T[:4u]\\
&+ x^{7u}T[:2u] + x^{4u}T[:7u] + x^{5u}T[:6u] + x^{7u}T[:4u] + x^{9u}T[:2u]\\
&+ x^{5u}T[:7u] + x^{6u}T[:6u] + x^{8u}T[:4u] + x^{10u}T[:2u] + x^{7u}T[:7u]\\
&+ x^{9u}T[:5u] + x^{11u}T[:3u] + x^{12u}T[:2u] + x^{13u}T[:u].
\end{aligned}
$$

The proof of the other 15 cases are similar. We break down the above identity into 7 parts:

$$
\begin{aligned}
0 &= x^{7u}T[:u] + x^{5u}T[2u:3u] + x^{3u}T[4u:5u] + x^uT[6u:7u] + T[7u:8u],\\
0 &= x^{8u}T[:u] + x^{7u}T[u:2u] + x^{6u}T[2u:3u] + x^{5u}T[3u:4u]
\end{aligned}
$$

$$
\begin{aligned}
&\quad + x^{4u}T[4u\!:\!5u] + x^{3u}T[5u\!:\!6u] + x^{2u}T[6u\!:\!7u] + T[8u\!:\!9u],\\
0 &= x^{8u}T[u\!:\!2u] + x^{6u}T[3u\!:\!4u] + x^{4u}T[5u\!:\!6u] + T[9u\!:\!10u],\\
0 &= x^{10u}T[:u] + x^{8u}T[2u\!:\!3u] + x^{6u}T[4u\!:\!5u] + x^{4u}T[6u\!:\!7u]\\
&\quad + T[10u\!:\!11u],\\
0 &= x^{11u}T[:u] + x^{10u}T[u\!:\!2u] + x^{9u}T[2u\!:\!3u] + x^{8u}T[3u\!:\!4u]\\
&\quad + x^{7u}T[4u\!:\!5u] + x^{6u}T[5u\!:\!6u] + x^{5u}T[6u\!:\!7u] + T[11u\!:\!12u],\\
0 &= x^{12u}T[:u] + x^{11u}T[u\!:\!2u] + x^{9u}T[3u\!:\!4u] + x^{7u}T[5u\!:\!6u]\\
&\quad + T[12u\!:\!13u],\\
0 &= x^{13u}T[:u] + x^{12u}T[u\!:\!2u] + x^{11u}T[2u\!:\!3u] + x^{9u}T[4u\!:\!5u]\\
&\quad + x^{7u}T[6u\!:\!7u] + T[13u\!:\!14u],
\end{aligned}
$$

which can be rewritten as

$$0 = T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[110w]_2] + T[[111w]_2], \tag{2.7}$$

$$
\begin{aligned}
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2]\\
&\quad + T[[101w]_2] + T[[110w]_2] + T[[1000w]_2],
\end{aligned} \tag{2.8}
$$

$$0 = T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1001w]_2], \tag{2.9}$$

$$0 = T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1010w]_2], \tag{2.10}$$

$$
\begin{aligned}
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2]\\
&\quad + T[[101w]_2] + T[[110w]_2] + T[[1011w]_2],
\end{aligned} \tag{2.11}
$$

$$0 = T[[w]_2] + T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1100w]_2], \tag{2.12}$$

$$
\begin{aligned}
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[100w]_2] + T[[110w]_2]\\
&\quad + T[[1101w]_2],
\end{aligned} \tag{2.13}
$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$0 = T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[110w]_2] + T[[111w]_2], \tag{2.14}$$

$$
\begin{aligned}
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2]\\
&\quad + T[[101w]_2] + T[[110w]_2] + T[[1000w]_2],
\end{aligned} \tag{2.15}
$$

$$0 = T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1001w]_2], \tag{2.16}$$

$$0 = T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1010w]_2], \tag{2.17}$$

$$
\begin{aligned}
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2]\\
&\quad + T[[101w]_2] + T[[110w]_2] + T[[1011w]_2],
\end{aligned} \tag{2.18}
$$

$$0 = T[[w]_2] + T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1100w]_2], \tag{2.19}$$

$$
\begin{aligned}
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[100w]_2] + T[[110w]_2]\\
&\quad + T[[1101w]_2],
\end{aligned} \tag{2.20}
$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 1297 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i,w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.20) can be written as

$$\begin{aligned}
&0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,110)) \\
&\quad + \tau(A(s,111)), &(2.21)\\
&0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\
&\quad + \tau(A(s,101)) + \tau(A(s,110)) + \tau(A(s,1000)), &(2.22)\\
&0 = \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,1001)), &(2.23)\\
&0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,110)) \\
&\quad + \tau(A(s,1010)), &(2.24)\\
&0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\
&\quad + \tau(A(s,101)) + \tau(A(s,110)) + \tau(A(s,1011)), &(2.25)\\
&0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,101)) \\
&\quad + \tau(A(s,1100)), &(2.26)\\
&0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,100)) \\
&\quad + \tau(A(s,110)) + \tau(A(s,1101)), &(2.27)
\end{aligned}$$

for all $s \in E_{2k}$, and

$$\begin{aligned}
&0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,110)) \\
&\quad + \tau(A(s,111)), &(2.28)\\
&0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\
&\quad + \tau(A(s,101)) + \tau(A(s,110)) + \tau(A(s,1000)), &(2.29)\\
&0 = \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,1001)), &(2.30)\\
&0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,110)) \\
&\quad + \tau(A(s,1010)), &(2.31)\\
&0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\
&\quad + \tau(A(s,101)) + \tau(A(s,110)) + \tau(A(s,1011)), &(2.32)\\
&0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,101)) \\
&\quad + \tau(A(s,1100)), &(2.33)\\
&0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,100)) \\
&\quad + \tau(A(s,110)) + \tau(A(s,1101)), &(2.34)
\end{aligned}$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{52}), E_{52}) = (A(i, 0^{24}), E_{24})$, so that we only have to verify that identities (2.21) through (2.34) hold for $2 \leq k \leq 26$, which turns out to be true. □

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \geq 2$, and $u = 2^{2k-1}$,*

$$M_{2k} = M^e[:7u] + x^u M^e[:6u] + x^{3u} M^e[:4u] + x^{5u} M^e[:2u] + x^{7u} R^e,$$
$$W_{2k} = W^e[:7u] + x^u W^e[:6u] + x^{3u} W^e[:4u] + x^{5u} W^e[:2u] + x^{7u} R^e.$$

For all integer $k \geq 2$, and $u = 2^{2k}$,

$$M_{2k+1} = M^o[:7u] + x^u M^o[:6u] + x^{3u} M^o[:4u] + x^{5u} M^o[:2u] + x^{7u} R^o,$$
$$W_{2k+1} = W^o[:7u] + x^u W^o[:6u] + x^{3u} W^o[:4u] + x^{5u} W^o[:2u] + x^{7u} R^o.$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \geq 2$, we claim that

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1} \wedge W_{2k+1},$$
$$M_{2k+1} \wedge W_{2k+1} \Rightarrow M_{2k+2} \wedge W_{2k+2}.$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$\begin{aligned} W_{2k}M_{2k} = \big(W^e[:7v] + x^v W^e[:6v] + x^{3v} W^e[:4v] + x^{5v} W^e[:2v] + x^{7u} R^e\big) \times \big(\\ M^e[:7v] + x^v M^e[:6v] + x^{3v} M^e[:4v] + x^{5v} M^e[:2v] + x^{7u} R^e\big); \end{aligned} \tag{2.35}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{14v}R^o$. Therefore we only need to prove that their difference is $O(x^{14v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{14v}, to

$$\begin{aligned} W^e[:14v]M^e[:14v] + x^{2v} W^e[:12v]M^e[:12v] + x^{6v} W^e[:8v]M^e[:8v] \\ + x^{10v} W^e[:4v]M^e[:4v]. \end{aligned}$$

For all $n \leq 14$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.35) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{14}, b_1, ..., b_{14}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{14})$, which completes the proof. $\square$

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \leq j, k \leq 1$,

$$\begin{aligned} \lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k}, \\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k}, \\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k}, \\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}. \end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a,b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e,0,1)$ and $\phi(M^e,0,0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x,y) = \mathrm{Res}_t\left(\phi(M^e,0,1)(x,t), y^{12}\phi(M^e,0,1)(x,t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x,y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned}
q_0(x) &= x^{24} + x^{23} + x^{18} + x^{17} + x^{16} + x^{15} + x^{12} + x^{11} + x^9 + x^8 + x^7,\\
q_1(x) &= x^{23} + x^{22} + x^{17} + x^{13} + x^{12} + x^{10} + x^9 + x^5 + x^2 + x,\\
q_2(x) &= x^{22} + x^{14} + x^6 + x^4 + x^2 + 1,\\
q_3(x) &= x^{23} + x^{22} + x^{17} + x^{13} + x^{12} + x^{10} + x^9 + x^5 + x^2 + x,\\
q_4(x) &= x^{23} + x^{22} + x^{18} + x^{17} + x^{15} + x^{14} + x^{11} + x^9 + x^8 + x^7 + x^6 + x^2.
\end{aligned}$$

The polynomial $Q(x,y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and $R(x,y) := P(x,y)/Q(x,y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 216, and substitute it for y in $R(z,y)$, we get a series with valuation smaller than 216, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b) = f(1/z)$. □

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e,0,0)$:

$$\begin{aligned}
p_0(x) &= x^{114} + x^{108} + x^{104} + x^{94} + x^{92} + x^{72} + x^{70} + x^{68} + x^{66} + x^{60} + x^{58}\\
&\quad + x^{54} + x^{42} + x^{38} + x^{36} + x^{34} + x^{30} + x^{28} + x^{22} + x^{20} + x^{18} + x^{14}\\
&\quad + x^{12},\\
p_3(x) &= x^{110} + x^{108} + x^{106} + x^{102} + x^{96} + x^{94} + x^{90} + x^{88} + x^{84} + x^{76} + x^{74}\\
&\quad + x^{72} + x^{70} + x^{66} + x^{64} + x^{60} + x^{58} + x^{52} + x^{46} + x^{44} + x^{32} + x^{30}\\
&\quad + x^{22} + x^{18} + x^{14} + x^8,\\
p_6(x) &= x^{108} + x^{106} + x^{100} + x^{98} + x^{94} + x^{88} + x^{86} + x^{84} + x^{82} + x^{80} + x^{78}\\
&\quad + x^{74} + x^{72} + x^{66} + x^{64} + x^{62} + x^{60} + x^{58} + x^{56} + x^{54} + x^{50} + x^{48}\\
&\quad + x^{44} + x^{42} + x^{40} + x^{38} + x^{34} + x^{26} + x^{20} + x^{18} + x^{16} + x^{12} + x^{10}
\end{aligned}$$

$$
\begin{aligned}
&+ x^8 + x^4 + 1,\\
p_9(x) &= x^{112} + x^{108} + x^{106} + x^{104} + x^{102} + x^{100} + x^{96} + x^{92} + x^{86} + x^{84}\\
&\quad + x^{78} + x^{76} + x^{74} + x^{72} + x^{68} + x^{64} + x^{58} + x^{56} + x^{54} + x^{52} + x^{50}\\
&\quad + x^{46} + x^{44} + x^{38} + x^{36} + x^{34} + x^{22} + x^{16} + x^{14} + x^8 + x^4 + x^2,\\
p_{12}(x) &= x^{112} + x^{110} + x^{106} + x^{102} + x^{98} + x^{96} + x^{80} + x^{78} + x^{74} + x^{70} + x^{66}\\
&\quad + x^{62} + x^{58} + x^{54} + x^{50} + x^{48} + x^{32} + x^{30} + x^{26} + x^{22} + x^{18} + x^{14}\\
&\quad + x^{10} + x^6 + x^2 + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0].

For $\phi(M^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{129} + x^{128} + x^{127} + x^{125} + x^{123} + x^{121} + x^{120} + x^{119} + x^{117} + x^{116}\\
&\quad + x^{112} + x^{111} + x^{110} + x^{109} + x^{107} + x^{105} + x^{96} + x^{95} + x^{94} + x^{91}\\
&\quad + x^{90} + x^{88} + x^{87} + x^{85} + x^{84} + x^{83} + x^{80} + x^{79} + x^{78} + x^{76} + x^{74}\\
&\quad + x^{72} + x^{71} + x^{67} + x^{65} + x^{64} + x^{62} + x^{60} + x^{58} + x^{57} + x^{55} + x^{53}\\
&\quad + x^{47} + x^{42} + x^{41} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{32} + x^{29} + x^{27},\\
p_3(x) &= x^{127} + x^{125} + x^{124} + x^{123} + x^{122} + x^{119} + x^{109} + x^{106} + x^{105} + x^{103}\\
&\quad + x^{93} + x^{90} + x^{89} + x^{87} + x^{79} + x^{76} + x^{75} + x^{73} + x^{71} + x^{69} + x^{68}\\
&\quad + x^{67} + x^{66} + x^{63} + x^{53} + x^{50} + x^{49} + x^{45} + x^{44} + x^{43} + x^{42} + x^{39}\\
&\quad + x^{37} + x^{34} + x^{33} + x^{31} + x^{29} + x^{26} + x^{25} + x^{20} + x^{19} + x^{17} + x^{15}\\
&\quad + x^{12} + x^{11} + x^9,\\
p_6(x) &= x^{126} + x^{125} + x^{124} + x^{122} + x^{121} + x^{120} + x^{112} + x^{111} + x^{109} + x^{108}\\
&\quad + x^{106} + x^{105} + x^{104} + x^{98} + x^{97} + x^{95} + x^{93} + x^{92} + x^{86} + x^{85} + x^{83}\\
&\quad + x^{82} + x^{78} + x^{77} + x^{76} + x^{72} + x^{71} + x^{70} + x^{68} + x^{67} + x^{66} + x^{38}\\
&\quad + x^{37} + x^{36} + x^{24} + x^{23} + x^{22} + x^{20} + x^{19} + x^{17} + x^{15} + x^{14} + x^{12}\\
&\quad + x^{11} + x^{10} + x^8 + x^7 + x^6,\\
p_9(x) &= x^{125} + x^{122} + x^{119} + x^{118} + x^{115} + x^{114} + x^{113} + x^{108} + x^{104} + x^{102}\\
&\quad + x^{100} + x^{94} + x^{91} + x^{90} + x^{86} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{76}\\
&\quad + x^{74} + x^{72} + x^{67} + x^{66} + x^{65} + x^{64} + x^{58} + x^{57} + x^{55} + x^{53} + x^{50}\\
&\quad + x^{46} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{38} + x^{32} + x^{31} + x^{29} + x^{27}\\
&\quad + x^{24} + x^{21} + x^{20} + x^{19} + x^{14} + x^{13} + x^9 + x^8 + x^7 + x^6 + x^3,\\
p_{12}(x) &= x^{124} + x^{123} + x^{121} + x^{119} + x^{117} + x^{115} + x^{113} + x^{112} + x^{108} + x^{107}\\
&\quad + x^{105} + x^{103} + x^{101} + x^{99} + x^{97} + x^{96} + x^{92} + x^{91} + x^{89} + x^{87} + x^{85}\\
&\quad + x^{83} + x^{81} + x^{80} + x^{60} + x^{59} + x^{57} + x^{55} + x^{53} + x^{51} + x^{49} + x^{48}\\
&\quad + x^{44} + x^{43} + x^{41} + x^{39} + x^{37} + x^{35} + x^{33} + x^{32} + x^{12} + x^{11} + x^9\\
&\quad + x^7 + x^5 + x^3 + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 1, 0, 0].

For $\phi(M^e, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{129} + x^{128} + x^{127} + x^{125} + x^{123} + x^{121} + x^{120} + x^{119} + x^{117} + x^{116} \\
&\quad + x^{112} + x^{111} + x^{110} + x^{109} + x^{107} + x^{105} + x^{96} + x^{95} + x^{94} + x^{91} \\
&\quad + x^{90} + x^{88} + x^{87} + x^{85} + x^{84} + x^{83} + x^{80} + x^{79} + x^{78} + x^{76} + x^{74} \\
&\quad + x^{72} + x^{71} + x^{67} + x^{65} + x^{64} + x^{62} + x^{60} + x^{58} + x^{57} + x^{55} + x^{53} \\
&\quad + x^{47} + x^{42} + x^{41} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{32} + x^{29} + x^{27}, \\
p_3(x) &= x^{127} + x^{125} + x^{124} + x^{123} + x^{122} + x^{119} + x^{109} + x^{106} + x^{105} + x^{103} \\
&\quad + x^{93} + x^{90} + x^{89} + x^{87} + x^{79} + x^{76} + x^{75} + x^{73} + x^{71} + x^{69} + x^{68} \\
&\quad + x^{67} + x^{66} + x^{63} + x^{53} + x^{50} + x^{49} + x^{45} + x^{44} + x^{43} + x^{42} + x^{39} \\
&\quad + x^{37} + x^{34} + x^{33} + x^{31} + x^{29} + x^{26} + x^{25} + x^{20} + x^{19} + x^{17} + x^{15} \\
&\quad + x^{12} + x^{11} + x^{9}, \\
p_6(x) &= x^{126} + x^{125} + x^{124} + x^{122} + x^{121} + x^{120} + x^{112} + x^{111} + x^{109} + x^{108} \\
&\quad + x^{106} + x^{105} + x^{104} + x^{98} + x^{97} + x^{95} + x^{93} + x^{92} + x^{86} + x^{85} + x^{83} \\
&\quad + x^{82} + x^{78} + x^{77} + x^{76} + x^{72} + x^{71} + x^{70} + x^{68} + x^{67} + x^{66} + x^{38} \\
&\quad + x^{37} + x^{36} + x^{24} + x^{23} + x^{22} + x^{20} + x^{19} + x^{17} + x^{15} + x^{14} + x^{12} \\
&\quad + x^{11} + x^{10} + x^{8} + x^{7} + x^{6}, \\
p_9(x) &= x^{125} + x^{122} + x^{119} + x^{118} + x^{115} + x^{114} + x^{113} + x^{108} + x^{104} + x^{102} \\
&\quad + x^{100} + x^{94} + x^{91} + x^{90} + x^{86} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{76} \\
&\quad + x^{74} + x^{72} + x^{67} + x^{66} + x^{65} + x^{64} + x^{58} + x^{57} + x^{55} + x^{53} + x^{50} \\
&\quad + x^{46} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{38} + x^{32} + x^{31} + x^{29} + x^{27} \\
&\quad + x^{24} + x^{21} + x^{20} + x^{19} + x^{14} + x^{13} + x^{9} + x^{8} + x^{7} + x^{6} + x^{3}, \\
p_{12}(x) &= x^{124} + x^{123} + x^{121} + x^{119} + x^{117} + x^{115} + x^{113} + x^{112} + x^{108} + x^{107} \\
&\quad + x^{105} + x^{103} + x^{101} + x^{99} + x^{97} + x^{96} + x^{92} + x^{91} + x^{89} + x^{87} + x^{85} \\
&\quad + x^{83} + x^{81} + x^{80} + x^{60} + x^{59} + x^{57} + x^{55} + x^{53} + x^{51} + x^{49} + x^{48} \\
&\quad + x^{44} + x^{43} + x^{41} + x^{39} + x^{37} + x^{35} + x^{33} + x^{32} + x^{12} + x^{11} + x^{9} \\
&\quad + x^{7} + x^{5} + x^{3} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 1, 0, 0].

For $\phi(M^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{144} + x^{142} + x^{140} + x^{138} + x^{132} + x^{130} + x^{124} + x^{122} + x^{114} + x^{110} \\
&\quad + x^{108} + x^{106} + x^{92} + x^{86} + x^{82} + x^{78} + x^{76} + x^{74} + x^{68} + x^{64} + x^{60} \\
&\quad + x^{58} + x^{56} + x^{50} + x^{48} + x^{44} + x^{42}, \\
p_3(x) &= x^{138} + x^{136} + x^{132} + x^{130} + x^{124} + x^{120} + x^{112} + x^{110} + x^{102} + x^{98} \\
&\quad + x^{94} + x^{92} + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{78} + x^{76} + x^{70} + x^{66} \\
&\quad + x^{64} + x^{60} + x^{56} + x^{54} + x^{50} + x^{46} + x^{44} + x^{42} + x^{40} + x^{38} + x^{36} \\
&\quad + x^{34} + x^{28} + x^{26} + x^{24} + x^{20} + x^{6},
\end{aligned}$$

$$\begin{aligned}
p_6(x) &= x^{138} + x^{134} + x^{132} + x^{122} + x^{112} + x^{110} + x^{108} + x^{104} + x^{98} + x^{96} \\
&\quad + x^{92} + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{78} + x^{72} + x^{66} + x^{64} + x^{60} \\
&\quad + x^{58} + x^{52} + x^{50} + x^{44} + x^{42} + x^{38} + x^{36} + x^{34} + x^{30} + x^{28} + x^{26} \\
&\quad + x^{24} + x^{22} + x^{20} + x^{14} + x^{12} + x^{4} + x^{2} + 1, \\
p_9(x) &= x^{136} + x^{134} + x^{122} + x^{120} + x^{116} + x^{114} + x^{108} + x^{106} + x^{104} + x^{102} \\
&\quad + x^{100} + x^{98} + x^{88} + x^{84} + x^{82} + x^{76} + x^{74} + x^{72} + x^{70} + x^{68} + x^{66} \\
&\quad + x^{64} + x^{62} + x^{54} + x^{52} + x^{46} + x^{42} + x^{40} + x^{36} + x^{32} + x^{28} + x^{18} \\
&\quad + x^{12} + x^{2}, \\
p_{12}(x) &= x^{136} + x^{120} + x^{112} + x^{104} + x^{96} + x^{80} + x^{72} + x^{56} + x^{48} + x^{32} + x^{24} \\
&\quad + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0].

For $\phi(W^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{144} + x^{142} + x^{138} + x^{136} + x^{134} + x^{132} + x^{130} + x^{126} + x^{118} + x^{110} \\
&\quad + x^{104} + x^{102} + x^{98} + x^{90} + x^{84} + x^{72} + x^{70} + x^{62} + x^{56} + x^{54} + x^{50} \\
&\quad + x^{46} + x^{44} + x^{42} + x^{40} + x^{38} + x^{36} + x^{34} + x^{26} + x^{24} + x^{22} + x^{16} \\
&\quad + x^{12}, \\
p_3(x) &= x^{138} + x^{136} + x^{134} + x^{128} + x^{124} + x^{122} + x^{120} + x^{118} + x^{114} + x^{102} \\
&\quad + x^{100} + x^{92} + x^{88} + x^{84} + x^{80} + x^{76} + x^{70} + x^{68} + x^{64} + x^{54} + x^{52} \\
&\quad + x^{40} + x^{36} + x^{34} + x^{28} + x^{24} + x^{22} + x^{20} + x^{16} + x^{8}, \\
p_6(x) &= x^{138} + x^{132} + x^{130} + x^{118} + x^{116} + x^{114} + x^{112} + x^{108} + x^{100} + x^{96} \\
&\quad + x^{94} + x^{88} + x^{68} + x^{64} + x^{58} + x^{46} + x^{40} + x^{36} + x^{34} + x^{32} + x^{30} \\
&\quad + x^{20} + x^{16} + x^{14} + x^{12} + x^{6} + x^{2} + 1, \\
p_9(x) &= x^{136} + x^{134} + x^{122} + x^{120} + x^{116} + x^{114} + x^{108} + x^{106} + x^{104} + x^{102} \\
&\quad + x^{100} + x^{98} + x^{88} + x^{84} + x^{82} + x^{76} + x^{74} + x^{72} + x^{70} + x^{68} + x^{66} \\
&\quad + x^{64} + x^{62} + x^{54} + x^{52} + x^{46} + x^{42} + x^{40} + x^{36} + x^{32} + x^{28} + x^{18} \\
&\quad + x^{12} + x^{2}, \\
p_{12}(x) &= x^{136} + x^{120} + x^{112} + x^{104} + x^{96} + x^{80} + x^{72} + x^{56} + x^{48} + x^{32} + x^{24} \\
&\quad + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{129} + x^{127} + x^{126} + x^{125} + x^{124} + x^{117} + x^{115} + x^{110} + x^{108} + x^{106} \\
&\quad + x^{105} + x^{104} + x^{101} + x^{98} + x^{97} + x^{93} + x^{90} + x^{88} + x^{86} + x^{85} + x^{84} \\
&\quad + x^{83} + x^{82} + x^{81} + x^{80} + x^{77} + x^{75} + x^{74} + x^{73} + x^{71} + x^{69} + x^{65} \\
&\quad + x^{62} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{52} + x^{51} + x^{47} + x^{39} + x^{38} \\
&\quad + x^{36} + x^{34} + x^{29} + x^{25} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18}, \\
p_3(x) &= x^{127} + x^{125} + x^{124} + x^{123} + x^{122} + x^{119} + x^{109} + x^{106} + x^{105} + x^{103}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{93} + x^{90} + x^{89} + x^{87} + x^{79} + x^{76} + x^{75} + x^{73} + x^{71} + x^{69} + x^{68} \\
&+ x^{67} + x^{66} + x^{63} + x^{53} + x^{50} + x^{49} + x^{45} + x^{44} + x^{43} + x^{42} + x^{39} \\
&+ x^{37} + x^{34} + x^{33} + x^{31} + x^{29} + x^{26} + x^{25} + x^{20} + x^{19} + x^{17} + x^{15} \\
&+ x^{12} + x^{11} + x^{9}, \\
p_6(x) = {}& x^{126} + x^{125} + x^{124} + x^{122} + x^{121} + x^{120} + x^{112} + x^{111} + x^{109} + x^{108} \\
&+ x^{106} + x^{105} + x^{104} + x^{98} + x^{97} + x^{95} + x^{93} + x^{92} + x^{86} + x^{85} + x^{83} \\
&+ x^{82} + x^{78} + x^{77} + x^{76} + x^{72} + x^{71} + x^{70} + x^{68} + x^{67} + x^{66} + x^{38} \\
&+ x^{37} + x^{36} + x^{24} + x^{23} + x^{22} + x^{20} + x^{19} + x^{17} + x^{15} + x^{14} + x^{12} \\
&+ x^{11} + x^{10} + x^{8} + x^{7} + x^{6}, \\
p_9(x) = {}& x^{125} + x^{122} + x^{119} + x^{118} + x^{115} + x^{114} + x^{113} + x^{108} + x^{104} + x^{102} \\
&+ x^{100} + x^{94} + x^{91} + x^{90} + x^{86} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{76} \\
&+ x^{74} + x^{72} + x^{67} + x^{66} + x^{65} + x^{64} + x^{58} + x^{57} + x^{55} + x^{53} + x^{50} \\
&+ x^{46} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{38} + x^{32} + x^{31} + x^{29} + x^{27} \\
&+ x^{24} + x^{21} + x^{20} + x^{19} + x^{14} + x^{13} + x^{9} + x^{8} + x^{7} + x^{6} + x^{3}, \\
p_{12}(x) = {}& x^{124} + x^{123} + x^{121} + x^{119} + x^{117} + x^{115} + x^{113} + x^{112} + x^{108} + x^{107} \\
&+ x^{105} + x^{103} + x^{101} + x^{99} + x^{97} + x^{96} + x^{92} + x^{91} + x^{89} + x^{87} + x^{85} \\
&+ x^{83} + x^{81} + x^{80} + x^{60} + x^{59} + x^{57} + x^{55} + x^{53} + x^{51} + x^{49} + x^{48} \\
&+ x^{44} + x^{43} + x^{41} + x^{39} + x^{37} + x^{35} + x^{33} + x^{32} + x^{12} + x^{11} + x^{9} \\
&+ x^{7} + x^{5} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 1, 1, 1, 0, 0, 0, 1, 1, 1, 0, 1, 1, 0, 1].

For $\phi(W^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) = {}& x^{129} + x^{127} + x^{126} + x^{125} + x^{124} + x^{117} + x^{115} + x^{110} + x^{108} + x^{106} \\
&+ x^{105} + x^{104} + x^{101} + x^{98} + x^{97} + x^{93} + x^{90} + x^{88} + x^{86} + x^{85} + x^{84} \\
&+ x^{83} + x^{82} + x^{81} + x^{80} + x^{77} + x^{75} + x^{74} + x^{73} + x^{71} + x^{69} + x^{65} \\
&+ x^{62} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{52} + x^{51} + x^{47} + x^{39} + x^{38} \\
&+ x^{36} + x^{34} + x^{29} + x^{25} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18}, \\
p_3(x) = {}& x^{127} + x^{125} + x^{124} + x^{123} + x^{122} + x^{119} + x^{109} + x^{106} + x^{105} + x^{103} \\
&+ x^{93} + x^{90} + x^{89} + x^{87} + x^{79} + x^{76} + x^{75} + x^{73} + x^{71} + x^{69} + x^{68} \\
&+ x^{67} + x^{66} + x^{63} + x^{53} + x^{50} + x^{49} + x^{45} + x^{44} + x^{43} + x^{42} + x^{39} \\
&+ x^{37} + x^{34} + x^{33} + x^{31} + x^{29} + x^{26} + x^{25} + x^{20} + x^{19} + x^{17} + x^{15} \\
&+ x^{12} + x^{11} + x^{9}, \\
p_6(x) = {}& x^{126} + x^{125} + x^{124} + x^{122} + x^{121} + x^{120} + x^{112} + x^{111} + x^{109} + x^{108} \\
&+ x^{106} + x^{105} + x^{104} + x^{98} + x^{97} + x^{95} + x^{93} + x^{92} + x^{86} + x^{85} + x^{83} \\
&+ x^{82} + x^{78} + x^{77} + x^{76} + x^{72} + x^{71} + x^{70} + x^{68} + x^{67} + x^{66} + x^{38} \\
&+ x^{37} + x^{36} + x^{24} + x^{23} + x^{22} + x^{20} + x^{19} + x^{17} + x^{15} + x^{14} + x^{12}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{11} + x^{10} + x^8 + x^7 + x^6,\\
p_9(x) &= x^{125} + x^{122} + x^{119} + x^{118} + x^{115} + x^{114} + x^{113} + x^{108} + x^{104} + x^{102}\\
&+ x^{100} + x^{94} + x^{91} + x^{90} + x^{86} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{76}\\
&+ x^{74} + x^{72} + x^{67} + x^{66} + x^{65} + x^{64} + x^{58} + x^{57} + x^{55} + x^{53} + x^{50}\\
&+ x^{46} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{38} + x^{32} + x^{31} + x^{29} + x^{27}\\
&+ x^{24} + x^{21} + x^{20} + x^{19} + x^{14} + x^{13} + x^9 + x^8 + x^7 + x^6 + x^3,\\
p_{12}(x) &= x^{124} + x^{123} + x^{121} + x^{119} + x^{117} + x^{115} + x^{113} + x^{112} + x^{108} + x^{107}\\
&+ x^{105} + x^{103} + x^{101} + x^{99} + x^{97} + x^{96} + x^{92} + x^{91} + x^{89} + x^{87} + x^{85}\\
&+ x^{83} + x^{81} + x^{80} + x^{60} + x^{59} + x^{57} + x^{55} + x^{53} + x^{51} + x^{49} + x^{48}\\
&+ x^{44} + x^{43} + x^{41} + x^{39} + x^{37} + x^{35} + x^{33} + x^{32} + x^{12} + x^{11} + x^9\\
&+ x^7 + x^5 + x^3 + x + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 1, 1, 1, 0, 0, 0, 1, 1, 1, 0, 1, 1, 0, 1].

For $\phi(W^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{114} + x^{112} + x^{106} + x^{100} + x^{98} + x^{96} + x^{92} + x^{90} + x^{80} + x^{78} + x^{76}\\
&+ x^{74} + x^{72} + x^{62} + x^{60} + x^{56} + x^{48} + x^{44} + x^{38} + x^{36} + x^{32} + x^{26}\\
&+ x^{24},\\
p_3(x) &= x^{108} + x^{106} + x^{104} + x^{102} + x^{98} + x^{88} + x^{86} + x^{84} + x^{82} + x^{80} + x^{78}\\
&+ x^{74} + x^{72} + x^{70} + x^{66} + x^{60} + x^{58} + x^{56} + x^{54} + x^{50} + x^{42} + x^{34}\\
&+ x^{28} + x^{26} + x^{20} + x^{18} + x^{12} + x^{10} + x^8 + x^6,\\
p_6(x) &= x^{110} + x^{100} + x^{96} + x^{92} + x^{82} + x^{78} + x^{76} + x^{72} + x^{70} + x^{64} + x^{62}\\
&+ x^{60} + x^{58} + x^{56} + x^{54} + x^{52} + x^{50} + x^{46} + x^{34} + x^{32} + x^{28} + x^{26}\\
&+ x^{24} + x^{22} + x^{18} + x^{16} + x^{14} + 1,\\
p_9(x) &= x^{112} + x^{108} + x^{106} + x^{104} + x^{102} + x^{100} + x^{96} + x^{92} + x^{86} + x^{84}\\
&+ x^{78} + x^{76} + x^{74} + x^{72} + x^{68} + x^{64} + x^{58} + x^{56} + x^{54} + x^{52} + x^{50}\\
&+ x^{46} + x^{44} + x^{38} + x^{36} + x^{34} + x^{22} + x^{16} + x^{14} + x^8 + x^4 + x^2,\\
p_{12}(x) &= x^{112} + x^{110} + x^{106} + x^{102} + x^{98} + x^{96} + x^{80} + x^{78} + x^{74} + x^{70} + x^{66}\\
&+ x^{62} + x^{58} + x^{54} + x^{50} + x^{48} + x^{32} + x^{30} + x^{26} + x^{22} + x^{18} + x^{14}\\
&+ x^{10} + x^6 + x^2 + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0].

For $\phi(M^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{129} + x^{128} + x^{123} + x^{120} + x^{119} + x^{118} + x^{117} + x^{115} + x^{113} + x^{112}\\
&+ x^{110} + x^{109} + x^{107} + x^{106} + x^{105} + x^{103} + x^{102} + x^{98} + x^{97} + x^{95}\\
&+ x^{93} + x^{91} + x^{89} + x^{88} + x^{87} + x^{86} + x^{84} + x^{82} + x^{78} + x^{77} + x^{73}\\
&+ x^{72} + x^{70} + x^{68} + x^{66} + x^{65} + x^{64} + x^{58} + x^{57} + x^{56} + x^{53} + x^{52}\\
&+ x^{51} + x^{50} + x^{46} + x^{44} + x^{40} + x^{37} + x^{33} + x^{31} + x^{30} + x^{28} + x^{25}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{23} + x^{21} + x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{14} + x^{13} + x^{12},\\
p_3(x) &= x^{128} + x^{126} + x^{125} + x^{124} + x^{123} + x^{122} + x^{109} + x^{108} + x^{107} + x^{105}\\
&+ x^{104} + x^{98} + x^{97} + x^{96} + x^{93} + x^{90} + x^{89} + x^{88} + x^{86} + x^{85} + x^{84}\\
&+ x^{79} + x^{74} + x^{73} + x^{67} + x^{65} + x^{63} + x^{61} + x^{59} + x^{57} + x^{55} + x^{54}\\
&+ x^{53} + x^{50} + x^{49} + x^{43} + x^{41} + x^{39} + x^{36} + x^{34} + x^{33} + x^{31} + x^{30}\\
&+ x^{29} + x^{28} + x^{27} + x^{26} + x^{24} + x^{22} + x^{20} + x^{19} + x^{17} + x^{14} + x^{10}\\
&+ x^{9} + x^{8},\\
p_6(x) &= x^{128} + x^{126} + x^{125} + x^{124} + x^{112} + x^{111} + x^{107} + x^{103} + x^{100} + x^{99}\\
&+ x^{96} + x^{95} + x^{93} + x^{91} + x^{89} + x^{88} + x^{87} + x^{85} + x^{84} + x^{82} + x^{79}\\
&+ x^{75} + x^{72} + x^{71} + x^{70} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61}\\
&+ x^{57} + x^{56} + x^{55} + x^{52} + x^{49} + x^{46} + x^{44} + x^{41} + x^{40} + x^{37} + x^{35}\\
&+ x^{29} + x^{27} + x^{23} + x^{22} + x^{20} + x^{18} + x^{16} + x^{15} + x^{13} + x^{12} + x^{11}\\
&+ x^{10} + x^{9} + x^{7} + x^{6} + x^{4} + x^{3} + x + 1,\\
p_9(x) &= x^{128} + x^{127} + x^{125} + x^{124} + x^{116} + x^{115} + x^{113} + x^{111} + x^{109} + x^{108}\\
&+ x^{104} + x^{103} + x^{101} + x^{100} + x^{96} + x^{95} + x^{93} + x^{92} + x^{84} + x^{83}\\
&+ x^{81} + x^{80} + x^{72} + x^{71} + x^{69} + x^{67} + x^{65} + x^{63} + x^{61} + x^{60} + x^{56}\\
&+ x^{55} + x^{53} + x^{52} + x^{48} + x^{47} + x^{45} + x^{44} + x^{36} + x^{35} + x^{33} + x^{32}\\
&+ x^{24} + x^{23} + x^{21} + x^{19} + x^{17} + x^{15} + x^{13} + x^{12} + x^{8} + x^{7} + x^{5} + x^{4},\\
p_{12}(x) &= x^{128} + x^{127} + x^{125} + x^{123} + x^{121} + x^{119} + x^{117} + x^{116} + x^{112} + x^{111}\\
&+ x^{109} + x^{107} + x^{105} + x^{104} + x^{100} + x^{99} + x^{97} + x^{95} + x^{93} + x^{91}\\
&+ x^{89} + x^{87} + x^{85} + x^{84} + x^{72} + x^{71} + x^{69} + x^{67} + x^{65} + x^{63} + x^{61}\\
&+ x^{59} + x^{57} + x^{56} + x^{52} + x^{51} + x^{49} + x^{47} + x^{45} + x^{43} + x^{41} + x^{39}\\
&+ x^{37} + x^{36} + x^{24} + x^{23} + x^{21} + x^{19} + x^{17} + x^{15} + x^{13} + x^{11} + x^{9}\\
&+ x^{8} + x^{4} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 1, 0, 1, 0, 1, 1, 1, 1, 0, 1, 0, 0, 0, 1].

For $\phi(M^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{144} + x^{141} + x^{140} + x^{139} + x^{138} + x^{134} + x^{133} + x^{129} + x^{128} + x^{126}\\
&+ x^{124} + x^{123} + x^{119} + x^{118} + x^{116} + x^{114} + x^{112} + x^{110} + x^{109}\\
&+ x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{97} + x^{91} + x^{90} + x^{87} + x^{86}\\
&+ x^{84} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{66}\\
&+ x^{64} + x^{63} + x^{61} + x^{58} + x^{56} + x^{54} + x^{53} + x^{51} + x^{50} + x^{49} + x^{47}\\
&+ x^{46} + x^{43} + x^{42} + x^{39} + x^{37} + x^{35} + x^{34} + x^{33} + x^{31} + x^{29} + x^{28}\\
&+ x^{27},\\
p_3(x) &= x^{137} + x^{136} + x^{131} + x^{127} + x^{126} + x^{124} + x^{123} + x^{118} + x^{117} + x^{114}\\
&+ x^{113} + x^{110} + x^{109} + x^{106} + x^{105} + x^{102} + x^{101} + x^{98} + x^{97} + x^{94}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{93} + x^{90} + x^{88} + x^{86} + x^{85} + x^{83} + x^{82} + x^{80} + x^{79} + x^{77} + x^{76}\\
&+ x^{74} + x^{73} + x^{71} + x^{70} + x^{68} + x^{67} + x^{62} + x^{61} + x^{58} + x^{56} + x^{54}\\
&+ x^{53} + x^{51} + x^{50} + x^{49} + x^{47} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{33}\\
&+ x^{32} + x^{27} + x^{25} + x^{24} + x^{23} + x^{22} + x^{20} + x^{15} + x^{13} + x^{12} + x^{11}\\
&+ x^{10} + x^{9},\\
p_6(x) &= x^{138} + x^{136} + x^{134} + x^{132} + x^{126} + x^{122} + x^{118} + x^{116} + x^{112} + x^{110}\\
&+ x^{108} + x^{106} + x^{94} + x^{92} + x^{90} + x^{88} + x^{86} + x^{84} + x^{80} + x^{76} + x^{72}\\
&+ x^{70} + x^{68} + x^{66} + x^{50} + x^{48} + x^{38} + x^{34} + x^{32} + x^{26} + x^{18} + x^{14}\\
&+ x^{12} + x^{10} + x^{8} + x^{6},\\
p_9(x) &= x^{139} + x^{138} + x^{137} + x^{136} + x^{133} + x^{130} + x^{126} + x^{124} + x^{118} + x^{113}\\
&+ x^{112} + x^{111} + x^{109} + x^{107} + x^{106} + x^{105} + x^{104} + x^{103} + x^{101}\\
&+ x^{100} + x^{99} + x^{98} + x^{94} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{86} + x^{85}\\
&+ x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{71} + x^{70} + x^{68} + x^{67}\\
&+ x^{65} + x^{64} + x^{63} + x^{61} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{53} + x^{52}\\
&+ x^{51} + x^{50} + x^{46} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{38} + x^{37} + x^{34}\\
&+ x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{23} + x^{22} + x^{20} + x^{19} + x^{17}\\
&+ x^{16} + x^{15} + x^{13} + x^{7} + x^{4} + x^{3},\\
p_{12}(x) &= x^{140} + x^{124} + x^{104} + x^{100} + x^{96} + x^{72} + x^{68} + x^{64} + x^{56} + x^{52} + x^{48}\\
&+ x^{24} + x^{20} + x^{16} + x^{12} + x^{8} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 0, 1, 1, 0, 0].

For $\phi(M^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{118} + x^{117} + x^{116} + x^{114} + x^{113} + x^{112} + x^{109} + x^{106} + x^{105} + x^{104}\\
&+ x^{103} + x^{100} + x^{98} + x^{94} + x^{92} + x^{90} + x^{87} + x^{85} + x^{84} + x^{83} + x^{80}\\
&+ x^{79} + x^{78} + x^{76} + x^{74} + x^{65} + x^{63} + x^{61} + x^{59} + x^{58} + x^{57} + x^{55}\\
&+ x^{54} + x^{52} + x^{49} + x^{46} + x^{42} + x^{40} + x^{39} + x^{38} + x^{37} + x^{35} + x^{34}\\
&+ x^{33} + x^{31} + x^{29} + x^{28} + x^{25} + x^{22} + x^{21} + x^{18},\\
p_3(x) &= x^{117} + x^{116} + x^{111} + x^{109} + x^{108} + x^{107} + x^{106} + x^{105} + x^{101} + x^{100}\\
&+ x^{95} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{85} + x^{84} + x^{79} + x^{77} + x^{76}\\
&+ x^{75} + x^{74} + x^{73} + x^{61} + x^{60} + x^{55} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49}\\
&+ x^{45} + x^{44} + x^{39} + x^{35} + x^{34} + x^{33} + x^{31} + x^{27} + x^{26} + x^{25} + x^{23}\\
&+ x^{19} + x^{18} + x^{17} + x^{15} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9},\\
p_6(x) &= x^{118} + x^{116} + x^{114} + x^{112} + x^{110} + x^{108} + x^{106} + x^{102} + x^{96} + x^{94}\\
&+ x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{80} + x^{76} + x^{72} + x^{70} + x^{60} + x^{58}\\
&+ x^{52} + x^{50} + x^{44} + x^{42} + x^{36} + x^{34} + x^{30} + x^{26} + x^{22} + x^{18} + x^{16}\\
&+ x^{12} + x^{10} + x^{8} + x^{6},
\end{aligned}
$$

$$\begin{aligned}
p_9(x) &= x^{119} + x^{118} + x^{117} + x^{116} + x^{113} + x^{111} + x^{109} + x^{108} + x^{107} + x^{103}\\
&\quad + x^{100} + x^{99} + x^{87} + x^{86} + x^{85} + x^{84} + x^{81} + x^{79} + x^{77} + x^{76} + x^{75}\\
&\quad + x^{70} + x^{69} + x^{67} + x^{65} + x^{63} + x^{61} + x^{60} + x^{59} + x^{55} + x^{52} + x^{51}\\
&\quad + x^{39} + x^{38} + x^{37} + x^{36} + x^{33} + x^{31} + x^{29} + x^{28} + x^{27} + x^{22} + x^{21}\\
&\quad + x^{19} + x^{17} + x^{15} + x^{13} + x^{12} + x^{11} + x^{7} + x^{4} + x^{3},\\
p_{12}(x) &= x^{120} + x^{112} + x^{108} + x^{100} + x^{96} + x^{88} + x^{80} + x^{76} + x^{72} + x^{68} + x^{60}\\
&\quad + x^{52} + x^{48} + x^{40} + x^{32} + x^{28} + x^{24} + x^{20} + x^{12} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [0, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 1, 1, 0].

For $\phi(M^o, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{129} + x^{124} + x^{121} + x^{120} + x^{119} + x^{116} + x^{115} + x^{114} + x^{110} + x^{107}\\
&\quad + x^{105} + x^{102} + x^{101} + x^{98} + x^{97} + x^{96} + x^{95} + x^{92} + x^{91} + x^{90} + x^{89}\\
&\quad + x^{88} + x^{86} + x^{85} + x^{82} + x^{81} + x^{79} + x^{74} + x^{72} + x^{71} + x^{70} + x^{68}\\
&\quad + x^{66} + x^{65} + x^{64} + x^{63} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{52} + x^{51}\\
&\quad + x^{46} + x^{45} + x^{42} + x^{39} + x^{37} + x^{36} + x^{35} + x^{33},\\
p_3(x) &= x^{128} + x^{124} + x^{122} + x^{121} + x^{119} + x^{118} + x^{109} + x^{106} + x^{105} + x^{103}\\
&\quad + x^{102} + x^{98} + x^{97} + x^{96} + x^{93} + x^{92} + x^{91} + x^{89} + x^{87} + x^{85} + x^{84}\\
&\quad + x^{79} + x^{78} + x^{77} + x^{76} + x^{72} + x^{70} + x^{69} + x^{62} + x^{61} + x^{59} + x^{57}\\
&\quad + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{45} + x^{38} + x^{35}\\
&\quad + x^{33} + x^{32} + x^{29} + x^{23} + x^{22} + x^{21} + x^{19} + x^{18} + x^{16} + x^{13} + x^{12},\\
p_6(x) &= x^{128} + x^{124} + x^{123} + x^{122} + x^{112} + x^{111} + x^{108} + x^{106} + x^{103} + x^{100}\\
&\quad + x^{99} + x^{98} + x^{97} + x^{96} + x^{92} + x^{91} + x^{89} + x^{88} + x^{87} + x^{86} + x^{84}\\
&\quad + x^{83} + x^{80} + x^{78} + x^{75} + x^{72} + x^{71} + x^{69} + x^{68} + x^{67} + x^{63} + x^{62}\\
&\quad + x^{61} + x^{57} + x^{56} + x^{55} + x^{52} + x^{49} + x^{46} + x^{44} + x^{41} + x^{40} + x^{38}\\
&\quad + x^{33} + x^{31} + x^{25} + x^{19} + x^{18} + x^{17} + x^{15} + x^{14} + x^{11} + x^{8} + x^{7} + x^{5}\\
&\quad + x^{3} + x + 1,\\
p_9(x) &= x^{128} + x^{127} + x^{125} + x^{124} + x^{116} + x^{115} + x^{113} + x^{111} + x^{109} + x^{108}\\
&\quad + x^{104} + x^{103} + x^{101} + x^{100} + x^{96} + x^{95} + x^{93} + x^{92} + x^{84} + x^{83}\\
&\quad + x^{81} + x^{80} + x^{72} + x^{71} + x^{69} + x^{67} + x^{65} + x^{63} + x^{61} + x^{60} + x^{56}\\
&\quad + x^{55} + x^{53} + x^{52} + x^{48} + x^{47} + x^{45} + x^{44} + x^{36} + x^{35} + x^{33} + x^{32}\\
&\quad + x^{24} + x^{23} + x^{21} + x^{19} + x^{17} + x^{15} + x^{13} + x^{12} + x^{8} + x^{7} + x^{5} + x^{4},\\
p_{12}(x) &= x^{128} + x^{127} + x^{125} + x^{123} + x^{121} + x^{119} + x^{117} + x^{116} + x^{112} + x^{111}\\
&\quad + x^{109} + x^{107} + x^{105} + x^{104} + x^{100} + x^{99} + x^{97} + x^{95} + x^{93} + x^{91}\\
&\quad + x^{89} + x^{87} + x^{85} + x^{84} + x^{72} + x^{71} + x^{69} + x^{67} + x^{65} + x^{63} + x^{61}\\
&\quad + x^{59} + x^{57} + x^{56} + x^{52} + x^{51} + x^{49} + x^{47} + x^{45} + x^{43} + x^{41} + x^{39}\\
&\quad + x^{37} + x^{36} + x^{24} + x^{23} + x^{21} + x^{19} + x^{17} + x^{15} + x^{13} + x^{11} + x^{9}
\end{aligned}$$

$$+ x^8 + x^4 + x^3 + x + 1,$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 1, 1, 0, 0, 1].

For $\phi(W^o, 0, 0)$:

$$\begin{aligned}
p_0(x) = {} & x^{129} + x^{128} + x^{123} + x^{120} + x^{119} + x^{118} + x^{117} + x^{115} + x^{113} + x^{112} \\
& + x^{110} + x^{109} + x^{107} + x^{106} + x^{105} + x^{103} + x^{102} + x^{98} + x^{97} + x^{95} \\
& + x^{93} + x^{91} + x^{89} + x^{88} + x^{87} + x^{86} + x^{84} + x^{82} + x^{78} + x^{77} + x^{73} \\
& + x^{72} + x^{70} + x^{68} + x^{66} + x^{65} + x^{64} + x^{58} + x^{57} + x^{56} + x^{53} + x^{52} \\
& + x^{51} + x^{50} + x^{46} + x^{44} + x^{40} + x^{37} + x^{33} + x^{31} + x^{30} + x^{28} + x^{25} \\
& + x^{23} + x^{21} + x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{14} + x^{13} + x^{12}, \\
p_3(x) = {} & x^{128} + x^{126} + x^{125} + x^{124} + x^{123} + x^{122} + x^{109} + x^{108} + x^{107} + x^{105} \\
& + x^{104} + x^{98} + x^{97} + x^{96} + x^{93} + x^{90} + x^{89} + x^{88} + x^{86} + x^{85} + x^{84} \\
& + x^{79} + x^{74} + x^{73} + x^{67} + x^{65} + x^{63} + x^{61} + x^{59} + x^{57} + x^{55} + x^{54} \\
& + x^{53} + x^{50} + x^{49} + x^{43} + x^{41} + x^{39} + x^{36} + x^{34} + x^{33} + x^{31} + x^{30} \\
& + x^{29} + x^{28} + x^{27} + x^{26} + x^{24} + x^{22} + x^{20} + x^{19} + x^{17} + x^{14} + x^{10} \\
& + x^9 + x^8, \\
p_6(x) = {} & x^{128} + x^{126} + x^{125} + x^{124} + x^{112} + x^{111} + x^{107} + x^{103} + x^{100} + x^{99} \\
& + x^{96} + x^{95} + x^{93} + x^{91} + x^{89} + x^{88} + x^{87} + x^{85} + x^{84} + x^{82} + x^{79} \\
& + x^{75} + x^{72} + x^{71} + x^{70} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} \\
& + x^{57} + x^{56} + x^{55} + x^{52} + x^{49} + x^{46} + x^{44} + x^{41} + x^{40} + x^{37} + x^{35} \\
& + x^{29} + x^{27} + x^{23} + x^{22} + x^{20} + x^{18} + x^{16} + x^{15} + x^{13} + x^{12} + x^{11} \\
& + x^{10} + x^9 + x^7 + x^6 + x^4 + x^3 + x + 1, \\
p_9(x) = {} & x^{128} + x^{127} + x^{125} + x^{124} + x^{116} + x^{115} + x^{113} + x^{111} + x^{109} + x^{108} \\
& + x^{104} + x^{103} + x^{101} + x^{100} + x^{96} + x^{95} + x^{93} + x^{92} + x^{84} + x^{83} \\
& + x^{81} + x^{80} + x^{72} + x^{71} + x^{69} + x^{67} + x^{65} + x^{63} + x^{61} + x^{60} + x^{56} \\
& + x^{55} + x^{53} + x^{52} + x^{48} + x^{47} + x^{45} + x^{44} + x^{36} + x^{35} + x^{33} + x^{32} \\
& + x^{24} + x^{23} + x^{21} + x^{19} + x^{17} + x^{15} + x^{13} + x^{12} + x^8 + x^7 + x^5 + x^4, \\
p_{12}(x) = {} & x^{128} + x^{127} + x^{125} + x^{123} + x^{121} + x^{119} + x^{117} + x^{116} + x^{112} + x^{111} \\
& + x^{109} + x^{107} + x^{105} + x^{104} + x^{100} + x^{99} + x^{97} + x^{95} + x^{93} + x^{91} \\
& + x^{89} + x^{87} + x^{85} + x^{84} + x^{72} + x^{71} + x^{69} + x^{67} + x^{65} + x^{63} + x^{61} \\
& + x^{59} + x^{57} + x^{56} + x^{52} + x^{51} + x^{49} + x^{47} + x^{45} + x^{43} + x^{41} + x^{39} \\
& + x^{37} + x^{36} + x^{24} + x^{23} + x^{21} + x^{19} + x^{17} + x^{15} + x^{13} + x^{11} + x^9 \\
& + x^8 + x^4 + x^3 + x + 1,
\end{aligned}$$

and the initial terms are [1, 0, 1, 0, 1, 0, 1, 1, 1, 1, 0, 1, 0, 0, 0, 1].

For $\phi(W^o, 0, 1)$:

$$p_0(x) = x^{118} + x^{117} + x^{116} + x^{114} + x^{113} + x^{112} + x^{109} + x^{106} + x^{105} + x^{104}$$

$$
\begin{aligned}
&+ x^{103} + x^{100} + x^{98} + x^{94} + x^{92} + x^{90} + x^{87} + x^{85} + x^{84} + x^{83} + x^{80}\\
&+ x^{79} + x^{78} + x^{76} + x^{74} + x^{65} + x^{63} + x^{61} + x^{59} + x^{58} + x^{57} + x^{55}\\
&+ x^{54} + x^{52} + x^{49} + x^{46} + x^{42} + x^{40} + x^{39} + x^{38} + x^{37} + x^{35} + x^{34}\\
&+ x^{33} + x^{31} + x^{29} + x^{28} + x^{25} + x^{22} + x^{21} + x^{18},\\
p_3(x) &= x^{117} + x^{116} + x^{111} + x^{109} + x^{108} + x^{107} + x^{106} + x^{105} + x^{101} + x^{100}\\
&+ x^{95} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{85} + x^{84} + x^{79} + x^{77} + x^{76}\\
&+ x^{75} + x^{74} + x^{73} + x^{61} + x^{60} + x^{55} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49}\\
&+ x^{45} + x^{44} + x^{39} + x^{35} + x^{34} + x^{33} + x^{31} + x^{27} + x^{26} + x^{25} + x^{23}\\
&+ x^{19} + x^{18} + x^{17} + x^{15} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9},\\
p_6(x) &= x^{118} + x^{116} + x^{114} + x^{112} + x^{110} + x^{108} + x^{106} + x^{102} + x^{96} + x^{94}\\
&+ x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{80} + x^{76} + x^{72} + x^{70} + x^{60} + x^{58}\\
&+ x^{52} + x^{50} + x^{44} + x^{42} + x^{36} + x^{34} + x^{30} + x^{26} + x^{22} + x^{18} + x^{16}\\
&+ x^{12} + x^{10} + x^{8} + x^{6},\\
p_9(x) &= x^{119} + x^{118} + x^{117} + x^{116} + x^{113} + x^{111} + x^{109} + x^{108} + x^{107} + x^{103}\\
&+ x^{100} + x^{99} + x^{87} + x^{86} + x^{85} + x^{84} + x^{81} + x^{79} + x^{77} + x^{76} + x^{75}\\
&+ x^{70} + x^{69} + x^{67} + x^{65} + x^{63} + x^{61} + x^{60} + x^{59} + x^{55} + x^{52} + x^{51}\\
&+ x^{39} + x^{38} + x^{37} + x^{36} + x^{33} + x^{31} + x^{29} + x^{28} + x^{27} + x^{22} + x^{21}\\
&+ x^{19} + x^{17} + x^{15} + x^{13} + x^{12} + x^{11} + x^{7} + x^{4} + x^{3},\\
p_{12}(x) &= x^{120} + x^{112} + x^{108} + x^{100} + x^{96} + x^{88} + x^{80} + x^{76} + x^{72} + x^{68} + x^{60}\\
&+ x^{52} + x^{48} + x^{40} + x^{32} + x^{28} + x^{24} + x^{20} + x^{12} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 1, 1, 0].

For $\phi(W^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{144} + x^{141} + x^{140} + x^{139} + x^{138} + x^{134} + x^{133} + x^{129} + x^{128} + x^{126}\\
&+ x^{124} + x^{123} + x^{119} + x^{118} + x^{116} + x^{114} + x^{112} + x^{110} + x^{109}\\
&+ x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{97} + x^{91} + x^{90} + x^{87} + x^{86}\\
&+ x^{84} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{66}\\
&+ x^{64} + x^{63} + x^{61} + x^{58} + x^{56} + x^{54} + x^{53} + x^{51} + x^{50} + x^{49} + x^{47}\\
&+ x^{46} + x^{43} + x^{42} + x^{39} + x^{37} + x^{35} + x^{34} + x^{33} + x^{31} + x^{29} + x^{28}\\
&+ x^{27},\\
p_3(x) &= x^{137} + x^{136} + x^{131} + x^{127} + x^{126} + x^{124} + x^{123} + x^{118} + x^{117} + x^{114}\\
&+ x^{113} + x^{110} + x^{109} + x^{106} + x^{105} + x^{102} + x^{101} + x^{98} + x^{97} + x^{94}\\
&+ x^{93} + x^{90} + x^{88} + x^{86} + x^{85} + x^{83} + x^{82} + x^{80} + x^{79} + x^{77} + x^{76}\\
&+ x^{74} + x^{73} + x^{71} + x^{70} + x^{68} + x^{67} + x^{62} + x^{61} + x^{58} + x^{56} + x^{54}\\
&+ x^{53} + x^{51} + x^{50} + x^{49} + x^{47} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{33}\\
&+ x^{32} + x^{27} + x^{25} + x^{24} + x^{23} + x^{22} + x^{20} + x^{15} + x^{13} + x^{12} + x^{11}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{10}+x^{9},\\
p_6(x)&=x^{138}+x^{136}+x^{134}+x^{132}+x^{126}+x^{122}+x^{118}+x^{116}+x^{112}+x^{110}\\
&+x^{108}+x^{106}+x^{94}+x^{92}+x^{90}+x^{88}+x^{86}+x^{84}+x^{80}+x^{76}+x^{72}\\
&+x^{70}+x^{68}+x^{66}+x^{50}+x^{48}+x^{38}+x^{34}+x^{32}+x^{26}+x^{18}+x^{14}\\
&+x^{12}+x^{10}+x^{8}+x^{6},\\
p_9(x)&=x^{139}+x^{138}+x^{137}+x^{136}+x^{133}+x^{130}+x^{126}+x^{124}+x^{118}+x^{113}\\
&+x^{112}+x^{111}+x^{109}+x^{107}+x^{106}+x^{105}+x^{104}+x^{103}+x^{101}\\
&+x^{100}+x^{99}+x^{98}+x^{94}+x^{92}+x^{91}+x^{90}+x^{89}+x^{88}+x^{86}+x^{85}\\
&+x^{82}+x^{81}+x^{80}+x^{79}+x^{78}+x^{77}+x^{76}+x^{71}+x^{70}+x^{68}+x^{67}\\
&+x^{65}+x^{64}+x^{63}+x^{61}+x^{59}+x^{58}+x^{57}+x^{56}+x^{55}+x^{53}+x^{52}\\
&+x^{51}+x^{50}+x^{46}+x^{44}+x^{43}+x^{42}+x^{41}+x^{40}+x^{38}+x^{37}+x^{34}\\
&+x^{33}+x^{32}+x^{31}+x^{30}+x^{29}+x^{28}+x^{23}+x^{22}+x^{20}+x^{19}+x^{17}\\
&+x^{16}+x^{15}+x^{13}+x^{7}+x^{4}+x^{3},\\
p_{12}(x)&=x^{140}+x^{124}+x^{104}+x^{100}+x^{96}+x^{72}+x^{68}+x^{64}+x^{56}+x^{52}+x^{48}\\
&+x^{24}+x^{20}+x^{16}+x^{12}+x^{8}+x^{4}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 0, 1, 1, 0, 0].

For $\phi(W^o,1,1)$:

$$
\begin{aligned}
p_0(x)&=x^{129}+x^{124}+x^{121}+x^{120}+x^{119}+x^{116}+x^{115}+x^{114}+x^{110}+x^{107}\\
&+x^{105}+x^{102}+x^{101}+x^{98}+x^{97}+x^{96}+x^{95}+x^{92}+x^{91}+x^{90}+x^{89}\\
&+x^{88}+x^{86}+x^{85}+x^{82}+x^{81}+x^{79}+x^{74}+x^{72}+x^{71}+x^{70}+x^{68}\\
&+x^{66}+x^{65}+x^{64}+x^{63}+x^{58}+x^{57}+x^{56}+x^{55}+x^{54}+x^{52}+x^{51}\\
&+x^{46}+x^{45}+x^{42}+x^{39}+x^{37}+x^{36}+x^{35}+x^{33},\\
p_3(x)&=x^{128}+x^{124}+x^{122}+x^{121}+x^{119}+x^{118}+x^{109}+x^{106}+x^{105}+x^{103}\\
&+x^{102}+x^{98}+x^{97}+x^{96}+x^{93}+x^{92}+x^{91}+x^{89}+x^{87}+x^{85}+x^{84}\\
&+x^{79}+x^{78}+x^{77}+x^{76}+x^{72}+x^{70}+x^{69}+x^{62}+x^{61}+x^{59}+x^{57}\\
&+x^{55}+x^{54}+x^{53}+x^{52}+x^{51}+x^{49}+x^{48}+x^{47}+x^{45}+x^{38}+x^{35}\\
&+x^{33}+x^{32}+x^{29}+x^{23}+x^{22}+x^{21}+x^{19}+x^{18}+x^{16}+x^{13}+x^{12},\\
p_6(x)&=x^{128}+x^{124}+x^{123}+x^{122}+x^{112}+x^{111}+x^{108}+x^{106}+x^{103}+x^{100}\\
&+x^{99}+x^{98}+x^{97}+x^{96}+x^{92}+x^{91}+x^{89}+x^{88}+x^{87}+x^{86}+x^{84}\\
&+x^{83}+x^{80}+x^{78}+x^{75}+x^{72}+x^{71}+x^{69}+x^{68}+x^{67}+x^{63}+x^{62}\\
&+x^{61}+x^{57}+x^{56}+x^{55}+x^{52}+x^{49}+x^{46}+x^{44}+x^{41}+x^{40}+x^{38}\\
&+x^{33}+x^{31}+x^{25}+x^{19}+x^{18}+x^{17}+x^{15}+x^{14}+x^{11}+x^{8}+x^{7}+x^{5}\\
&+x^{3}+x+1,\\
p_9(x)&=x^{128}+x^{127}+x^{125}+x^{124}+x^{116}+x^{115}+x^{113}+x^{111}+x^{109}+x^{108}\\
&+x^{104}+x^{103}+x^{101}+x^{100}+x^{96}+x^{95}+x^{93}+x^{92}+x^{84}+x^{83}
\end{aligned}
$$

$$\begin{aligned}
&+ x^{81} + x^{80} + x^{72} + x^{71} + x^{69} + x^{67} + x^{65} + x^{63} + x^{61} + x^{60} + x^{56}\\
&+ x^{55} + x^{53} + x^{52} + x^{48} + x^{47} + x^{45} + x^{44} + x^{36} + x^{35} + x^{33} + x^{32}\\
&+ x^{24} + x^{23} + x^{21} + x^{19} + x^{17} + x^{15} + x^{13} + x^{12} + x^{8} + x^{7} + x^{5} + x^{4},\\
p_{12}(x) = {} & x^{128} + x^{127} + x^{125} + x^{123} + x^{121} + x^{119} + x^{117} + x^{116} + x^{112} + x^{111}\\
&+ x^{109} + x^{107} + x^{105} + x^{104} + x^{100} + x^{99} + x^{97} + x^{95} + x^{93} + x^{91}\\
&+ x^{89} + x^{87} + x^{85} + x^{84} + x^{72} + x^{71} + x^{69} + x^{67} + x^{65} + x^{63} + x^{61}\\
&+ x^{59} + x^{57} + x^{56} + x^{52} + x^{51} + x^{49} + x^{47} + x^{45} + x^{43} + x^{41} + x^{39}\\
&+ x^{37} + x^{36} + x^{24} + x^{23} + x^{21} + x^{19} + x^{17} + x^{15} + x^{13} + x^{11} + x^{9}\\
&+ x^{8} + x^{4} + x^{3} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 1, 1, 0, 0, 1].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	325	[471, 10]	650	[58, 120]	975	[1057, 285]
1	[3, 4]	326	[409, 165]	651	[769, 457]	976	[268, 75]
2	[5, 6]	327	[158, 10]	652	[770, 475]	977	[1058, 207]
3	[7, 8]	328	[167, 472]	653	[771, 725]	978	[1059, 499]
4	[9, 10]	329	[473, 54]	654	[772, 366]	979	[1060, 814]
5	[11, 12]	330	[474, 167]	655	[36, 160]	980	[1061, 298]
6	[13, 14]	331	[475, 155]	656	[773, 61]	981	[999, 1062]
7	[15, 16]	332	[476, 121]	657	[774, 357]	982	[1063, 90]
8	[17, 18]	333	[477, 59]	658	[775, 370]	983	[1064, 1025]
9	[19, 20]	334	[478, 196]	659	[776, 177]	984	[282, 1065]
10	[21, 22]	335	[353, 58]	660	[777, 519]	985	[84, 86]
11	[23, 24]	336	[196, 478]	661	[778, 643]	986	[1066, 264]
12	[25, 26]	337	[479, 480]	662	[491, 599]	987	[1067, 217]
13	[27, 28]	338	[481, 380]	663	[779, 334]	988	[1068, 295]
14	[29, 30]	339	[482, 483]	664	[780, 781]	989	[566, 1069]
15	[31, 32]	340	[484, 384]	665	[782, 783]	990	[1070, 860]
16	[8, 33]	341	[202, 378]	666	[532, 784]	991	[266, 1071]
17	[34, 35]	342	[485, 8]	667	[785, 209]	992	[1072, 864]
18	[36, 37]	343	[393, 392]	668	[786, 556]	993	[1073, 474]
19	[38, 39]	344	[486, 380]	669	[787, 788]	994	[1074, 14]
20	[40, 41]	345	[487, 181]	670	[789, 550]	995	[13, 916]
21	[42, 43]	346	[488, 41]	671	[790, 92]	996	[1075, 425]
22	[44, 45]	347	[489, 262]	672	[791, 19]	997	[42, 423]
23	[46, 47]	348	[490, 491]	673	[259, 792]	998	[1076, 989]
24	[48, 49]	349	[492, 248]	674	[793, 295]	999	[1077, 737]
25	[50, 51]	350	[297, 493]	675	[553, 794]	1000	[1078, 32]
26	[52, 53]	351	[494, 495]	676	[27, 795]	1001	[1079, 981]

27	[54, 55]	352	[209, 252]	677	[540, 558]	1002	[1080, 127]
28	[56, 57]	353	[29, 112]	678	[796, 797]	1003	[1081, 152]
29	[58, 59]	354	[496, 497]	679	[796, 798]	1004	[1082, 725]
30	[60, 61]	355	[498, 499]	680	[799, 800]	1005	[427, 429]
31	[62, 63]	356	[500, 501]	681	[801, 227]	1006	[1083, 449]
32	[64, 65]	357	[502, 239]	682	[802, 522]	1007	[1084, 361]
33	[16, 66]	358	[334, 503]	683	[234, 803]	1008	[1085, 1086]
34	[67, 68]	359	[504, 301]	684	[332, 804]	1009	[1087, 193]
35	[69, 70]	360	[505, 506]	685	[803, 794]	1010	[1088, 47]
36	[71, 72]	361	[507, 508]	686	[805, 495]	1011	[1089, 124]
37	[73, 74]	362	[238, 509]	687	[806, 655]	1012	[1090, 737]
38	[75, 76]	363	[75, 510]	688	[807, 222]	1013	[351, 705]
39	[77, 78]	364	[98, 231]	689	[808, 609]	1014	[1091, 1086]
40	[79, 80]	365	[511, 316]	690	[809, 810]	1015	[1092, 147]
41	[81, 82]	366	[512, 513]	691	[811, 812]	1016	[385, 447]
42	[83, 84]	367	[5, 90]	692	[813, 557]	1017	[1093, 119]
43	[83, 85]	368	[84, 514]	693	[814, 330]	1018	[439, 120]
44	[86, 87]	369	[515, 516]	694	[815, 816]	1019	[1094, 397]
45	[4, 16]	370	[517, 332]	695	[817, 534]	1020	[1095, 735]
46	[88, 89]	371	[518, 519]	696	[818, 819]	1021	[1096, 731]
47	[90, 91]	372	[520, 521]	697	[820, 325]	1022	[1097, 674]
48	[92, 93]	373	[522, 108]	698	[821, 525]	1023	[179, 483]
49	[94, 68]	374	[523, 524]	699	[822, 823]	1024	[1098, 348]
50	[95, 96]	375	[525, 335]	700	[824, 282]	1025	[959, 171]
51	[97, 98]	376	[526, 527]	701	[804, 648]	1026	[1099, 348]
52	[99, 100]	377	[528, 529]	702	[825, 84]	1027	[197, 198]
53	[101, 102]	378	[530, 327]	703	[826, 827]	1028	[1100, 380]
54	[103, 104]	379	[531, 532]	704	[828, 213]	1029	[662, 443]
55	[105, 106]	380	[533, 534]	705	[20, 829]	1030	[46, 962]
56	[69, 107]	381	[535, 325]	706	[289, 830]	1031	[438, 1101]
57	[108, 109]	382	[536, 537]	707	[831, 832]	1032	[1102, 168]
58	[110, 111]	383	[538, 112]	708	[833, 546]	1033	[1103, 1104]
59	[112, 113]	384	[539, 540]	709	[331, 282]	1034	[1105, 381]
60	[114, 115]	385	[541, 542]	710	[579, 636]	1035	[1106, 698]
61	[116, 117]	386	[269, 543]	711	[77, 213]	1036	[1107, 1086]
62	[118, 119]	387	[76, 544]	712	[242, 624]	1037	[1108, 674]
63	[120, 121]	388	[545, 27]	713	[232, 834]	1038	[1109, 981]
64	[122, 123]	389	[216, 546]	714	[326, 835]	1039	[1110, 696]
65	[124, 125]	390	[547, 548]	715	[562, 249]	1040	[1111, 1112]
66	[32, 126]	391	[549, 550]	716	[836, 617]	1041	[1113, 475]
67	[127, 128]	392	[109, 551]	717	[837, 555]	1042	[411, 188]
68	[48, 129]	393	[552, 301]	718	[617, 838]	1043	[1114, 725]
69	[130, 131]	394	[553, 554]	719	[839, 516]	1044	[1115, 768]
70	[132, 133]	395	[244, 96]	720	[840, 80]	1045	[1116, 370]

71	[134, 135]	396	[72, 555]	721	[841, 309]	1046	[1117, 1104]
72	[136, 43]	397	[556, 557]	722	[842, 299]	1047	[1118, 163]
73	[137, 138]	398	[277, 558]	723	[580, 506]	1048	[1119, 9]
74	[139, 140]	399	[559, 560]	724	[30, 113]	1049	[1120, 989]
75	[141, 142]	400	[561, 562]	725	[843, 844]	1050	[122, 676]
76	[143, 33]	401	[563, 564]	726	[845, 290]	1051	[1121, 144]
77	[144, 142]	402	[542, 105]	727	[70, 260]	1052	[1122, 905]
78	[145, 144]	403	[249, 565]	728	[615, 23]	1053	[436, 416]
79	[146, 147]	404	[539, 277]	729	[846, 847]	1054	[1123, 361]
80	[148, 149]	405	[566, 567]	730	[848, 796]	1055	[141, 410]
81	[150, 151]	406	[495, 504]	731	[536, 849]	1056	[711, 422]
82	[152, 153]	407	[568, 569]	732	[850, 72]	1057	[1124, 664]
83	[154, 155]	408	[253, 509]	733	[532, 851]	1058	[1125, 381]
84	[156, 123]	409	[570, 532]	734	[852, 853]	1059	[1126, 965]
85	[157, 158]	410	[552, 571]	735	[854, 855]	1060	[1127, 163]
86	[159, 160]	411	[25, 572]	736	[856, 548]	1061	[1128, 731]
87	[158, 161]	412	[573, 110]	737	[857, 858]	1062	[1129, 139]
88	[162, 163]	413	[574, 575]	738	[859, 860]	1063	[1130, 168]
89	[164, 165]	414	[229, 576]	739	[861, 583]	1064	[1131, 179]
90	[166, 167]	415	[577, 578]	740	[862, 863]	1065	[159, 37]
91	[168, 169]	416	[519, 579]	741	[864, 865]	1066	[1132, 1133]
92	[170, 170]	417	[501, 580]	742	[866, 616]	1067	[976, 362]
93	[49, 171]	418	[581, 94]	743	[857, 867]	1068	[1134, 1135]
94	[172, 48]	419	[582, 583]	744	[868, 869]	1069	[1136, 136]
95	[173, 174]	420	[584, 585]	745	[870, 549]	1070	[1137, 348]
96	[150, 175]	421	[586, 271]	746	[520, 581]	1071	[404, 149]
97	[176, 177]	422	[331, 587]	747	[871, 294]	1072	[1138, 350]
98	[1, 44]	423	[579, 588]	748	[872, 797]	1073	[1139, 108]
99	[178, 179]	424	[589, 491]	749	[873, 874]	1074	[213, 317]
100	[180, 181]	425	[590, 591]	750	[830, 111]	1075	[1140, 100]
101	[182, 183]	426	[592, 524]	751	[632, 875]	1076	[1141, 1029]
102	[184, 185]	427	[593, 594]	752	[869, 876]	1077	[1142, 19]
103	[186, 12]	428	[595, 346]	753	[877, 116]	1078	[1143, 646]
104	[187, 188]	429	[596, 597]	754	[878, 879]	1079	[1144, 345]
105	[189, 123]	430	[598, 599]	755	[880, 855]	1080	[1145, 823]
106	[190, 14]	431	[272, 600]	756	[881, 882]	1081	[338, 264]
107	[191, 192]	432	[601, 602]	757	[883, 234]	1082	[1146, 814]
108	[193, 194]	433	[89, 603]	758	[884, 885]	1083	[20, 1147]
109	[45, 43]	434	[604, 605]	759	[886, 310]	1084	[1148, 603]
110	[195, 196]	435	[606, 543]	760	[887, 76]	1085	[1149, 604]
111	[59, 121]	436	[304, 607]	761	[512, 888]	1086	[1150, 777]
112	[197, 121]	437	[608, 609]	762	[889, 640]	1087	[1151, 330]
113	[59, 198]	438	[610, 101]	763	[890, 615]	1088	[1150, 518]
114	[199, 200]	439	[63, 210]	764	[339, 292]	1089	[571, 250]

115	[201, 175]	440	[611, 612]	765	[891, 20]	1090	[1152, 572]
116	[202, 138]	441	[613, 614]	766	[832, 810]	1091	[1153, 827]
117	[203, 204]	442	[615, 309]	767	[892, 240]	1092	[1154, 647]
118	[205, 98]	443	[293, 616]	768	[267, 893]	1093	[1155, 1016]
119	[206, 207]	444	[229, 560]	769	[894, 4]	1094	[501, 505]
120	[208, 209]	445	[617, 497]	770	[313, 796]	1095	[1156, 885]
121	[210, 208]	446	[618, 18]	771	[895, 64]	1096	[585, 74]
122	[211, 212]	447	[324, 619]	772	[896, 542]	1097	[1157, 864]
123	[213, 214]	448	[620, 116]	773	[881, 897]	1098	[1158, 812]
124	[215, 216]	449	[295, 621]	774	[898, 899]	1099	[1159, 101]
125	[217, 218]	450	[590, 262]	775	[900, 863]	1100	[1160, 612]
126	[63, 219]	451	[609, 622]	776	[901, 333]	1101	[313, 872]
127	[220, 221]	452	[623, 624]	777	[902, 391]	1102	[1161, 502]
128	[222, 223]	453	[625, 29]	778	[903, 370]	1103	[1162, 851]
129	[24, 92]	454	[534, 626]	779	[904, 905]	1104	[500, 1163]
130	[224, 225]	455	[627, 221]	780	[906, 698]	1105	[1164, 291]
131	[226, 227]	456	[628, 607]	781	[907, 389]	1106	[1165, 238]
132	[228, 229]	457	[629, 630]	782	[908, 163]	1107	[1166, 332]
133	[207, 230]	458	[631, 632]	783	[687, 387]	1108	[1167, 897]
134	[231, 232]	459	[264, 292]	784	[696, 454]	1109	[1168, 1025]
135	[233, 234]	460	[633, 210]	785	[909, 364]	1110	[1169, 638]
136	[235, 236]	461	[634, 235]	786	[910, 409]	1111	[1170, 83]
137	[237, 238]	462	[521, 67]	787	[911, 698]	1112	[246, 1171]
138	[239, 240]	463	[635, 636]	788	[470, 368]	1113	[609, 1172]
139	[241, 27]	464	[561, 248]	789	[912, 152]	1114	[1173, 308]
140	[242, 243]	465	[637, 638]	790	[913, 417]	1115	[1174, 1175]
141	[21, 232]	466	[639, 584]	791	[914, 40]	1116	[1176, 788]
142	[244, 245]	467	[526, 640]	792	[674, 741]	1117	[214, 77]
143	[246, 247]	468	[505, 641]	793	[915, 179]	1118	[1177, 255]
144	[248, 249]	469	[265, 293]	794	[448, 713]	1119	[259, 1056]
145	[250, 244]	470	[642, 643]	795	[190, 916]	1120	[1178, 847]
146	[251, 252]	471	[109, 81]	796	[917, 39]	1121	[495, 552]
147	[237, 253]	472	[107, 260]	797	[382, 420]	1122	[1179, 242]
148	[254, 255]	473	[644, 105]	798	[44, 136]	1123	[85, 86]
149	[256, 257]	474	[645, 549]	799	[918, 919]	1124	[1180, 899]
150	[258, 259]	475	[583, 584]	800	[444, 129]	1125	[1181, 239]
151	[260, 261]	476	[289, 110]	801	[920, 674]	1126	[1182, 888]
152	[262, 263]	477	[646, 647]	802	[921, 737]	1127	[1183, 241]
153	[264, 265]	478	[333, 648]	803	[735, 133]	1128	[1184, 1172]
154	[74, 225]	479	[649, 650]	804	[427, 406]	1129	[632, 1185]
155	[74, 266]	480	[651, 652]	805	[922, 674]	1130	[1186, 290]
156	[267, 268]	481	[653, 501]	806	[923, 400]	1131	[1187, 655]
157	[230, 269]	482	[654, 64]	807	[924, 370]	1132	[302, 244]
158	[270, 271]	483	[655, 623]	808	[925, 384]	1133	[523, 1188]

159	[272, 273]	484	[656, 336]	809	[926, 451]	1134	[1189, 109]
160	[102, 274]	485	[657, 236]	810	[477, 120]	1135	[620, 1190]
161	[269, 275]	486	[658, 281]	811	[927, 350]	1136	[643, 83]
162	[276, 277]	487	[521, 94]	812	[928, 175]	1137	[1191, 1013]
163	[278, 279]	488	[659, 596]	813	[929, 433]	1138	[1192, 216]
164	[280, 281]	489	[660, 361]	814	[367, 33]	1139	[1136, 45]
165	[282, 283]	490	[661, 46]	815	[930, 32]	1140	[1193, 1135]
166	[284, 285]	491	[662, 170]	816	[432, 387]	1141	[1194, 1133]
167	[286, 287]	492	[663, 664]	817	[931, 390]	1142	[1195, 480]
168	[288, 289]	493	[665, 666]	818	[932, 119]	1143	[1196, 438]
169	[290, 291]	494	[667, 376]	819	[766, 58]	1144	[1197, 1133]
170	[292, 293]	495	[668, 185]	820	[933, 756]	1145	[1198, 696]
171	[93, 294]	496	[669, 47]	821	[934, 119]	1146	[1199, 474]
172	[295, 296]	497	[172, 181]	822	[935, 400]	1147	[467, 447]
173	[297, 298]	498	[670, 51]	823	[672, 188]	1148	[125, 144]
174	[299, 300]	499	[402, 468]	824	[936, 433]	1149	[1200, 1104]
175	[301, 302]	500	[671, 35]	825	[370, 433]	1150	[1201, 366]
176	[303, 196]	501	[672, 157]	826	[937, 438]	1151	[37, 475]
177	[304, 305]	502	[458, 55]	827	[478, 478]	1152	[1202, 965]
178	[306, 104]	503	[119, 673]	828	[938, 357]	1153	[1203, 1112]
179	[307, 308]	504	[674, 373]	829	[174, 939]	1154	[1204, 364]
180	[309, 24]	505	[675, 10]	830	[438, 451]	1155	[1205, 203]
181	[310, 311]	506	[156, 676]	831	[940, 427]	1156	[1206, 184]
182	[312, 313]	507	[677, 189]	832	[941, 196]	1157	[1207, 381]
183	[225, 314]	508	[678, 679]	833	[942, 373]	1158	[1208, 381]
184	[315, 316]	509	[189, 676]	834	[152, 943]	1159	[1209, 184]
185	[317, 77]	510	[160, 475]	835	[32, 944]	1160	[1210, 168]
186	[318, 296]	511	[680, 425]	836	[945, 127]	1161	[1211, 390]
187	[319, 270]	512	[681, 438]	837	[52, 152]	1162	[454, 14]
188	[320, 321]	513	[682, 683]	838	[127, 946]	1163	[1212, 944]
189	[322, 222]	514	[433, 684]	839	[947, 480]	1164	[907, 144]
190	[323, 324]	515	[685, 40]	840	[948, 203]	1165	[1213, 989]
191	[325, 326]	516	[191, 363]	841	[949, 128]	1166	[1214, 768]
192	[327, 328]	517	[686, 427]	842	[950, 746]	1167	[1215, 184]
193	[329, 241]	518	[687, 55]	843	[951, 60]	1168	[1216, 1135]
194	[330, 331]	519	[124, 145]	844	[941, 478]	1169	[1217, 124]
195	[332, 333]	520	[688, 127]	845	[952, 457]	1170	[693, 944]
196	[196, 196]	521	[180, 48]	846	[953, 46]	1171	[1218, 706]
197	[334, 335]	522	[689, 183]	847	[129, 180]	1172	[712, 363]
198	[111, 336]	523	[690, 384]	848	[954, 153]	1173	[1219, 441]
199	[337, 273]	524	[461, 416]	849	[955, 939]	1174	[169, 358]
200	[338, 339]	525	[691, 451]	850	[956, 664]	1175	[154, 679]
201	[340, 341]	526	[692, 54]	851	[395, 392]	1176	[1220, 390]
202	[342, 343]	527	[31, 693]	852	[957, 361]	1177	[1221, 756]

203	[344, 345]	528	[694, 54]	853	[182, 420]	1178	[1222, 731]
204	[346, 217]	529	[396, 679]	854	[958, 127]	1179	[371, 684]
205	[347, 348]	530	[56, 167]	855	[959, 180]	1180	[1223, 1104]
206	[349, 350]	531	[695, 696]	856	[960, 664]	1181	[57, 391]
207	[351, 41]	532	[355, 167]	857	[961, 179]	1182	[1224, 1112]
208	[121, 352]	533	[697, 698]	858	[669, 962]	1183	[1225, 981]
209	[352, 59]	534	[699, 123]	859	[963, 177]	1184	[741, 449]
210	[353, 352]	535	[700, 409]	860	[413, 149]	1185	[675, 161]
211	[354, 35]	536	[701, 177]	861	[964, 965]	1186	[1226, 441]
212	[355, 57]	537	[668, 447]	862	[966, 457]	1187	[1227, 696]
213	[356, 165]	538	[428, 406]	863	[699, 676]	1188	[954, 943]
214	[357, 358]	539	[702, 8]	864	[738, 403]	1189	[1228, 449]
215	[359, 51]	540	[703, 433]	865	[381, 378]	1190	[1229, 728]
216	[360, 140]	541	[704, 189]	866	[967, 128]	1191	[1230, 193]
217	[361, 44]	542	[472, 394]	867	[443, 443]	1192	[1231, 965]
218	[362, 363]	543	[374, 705]	868	[968, 193]	1193	[1190, 1232]
219	[119, 364]	544	[357, 706]	869	[928, 151]	1194	[1233, 997]
220	[365, 366]	545	[707, 190]	870	[969, 480]	1195	[1234, 646]
221	[367, 368]	546	[51, 175]	871	[970, 483]	1196	[1235, 313]
222	[369, 145]	547	[708, 135]	872	[203, 174]	1197	[1236, 218]
223	[370, 371]	548	[709, 363]	873	[971, 390]	1198	[1237, 783]
224	[372, 373]	549	[710, 43]	874	[440, 714]	1199	[551, 286]
225	[374, 41]	550	[40, 705]	875	[725, 693]	1200	[1163, 505]
226	[375, 376]	551	[194, 139]	876	[193, 149]	1201	[1238, 556]
227	[377, 378]	552	[711, 140]	877	[972, 203]	1202	[1239, 819]
228	[379, 380]	553	[712, 192]	878	[973, 45]	1203	[514, 87]
229	[181, 49]	554	[185, 402]	879	[137, 378]	1204	[1240, 853]
230	[381, 138]	555	[135, 713]	880	[974, 919]	1205	[1241, 594]
231	[348, 152]	556	[678, 155]	881	[975, 746]	1206	[1242, 626]
232	[382, 183]	557	[409, 714]	882	[976, 191]	1207	[1243, 530]
233	[383, 384]	558	[8, 368]	883	[977, 735]	1208	[510, 1244]
234	[385, 185]	559	[665, 417]	884	[978, 746]	1209	[78, 317]
235	[386, 142]	560	[442, 129]	885	[415, 713]	1210	[1245, 1050]
236	[54, 387]	561	[715, 135]	886	[979, 483]	1211	[1246, 230]
237	[388, 189]	562	[716, 174]	887	[168, 357]	1212	[326, 256]
238	[389, 142]	563	[717, 131]	888	[120, 198]	1213	[1247, 521]
239	[390, 157]	564	[710, 423]	889	[980, 981]	1214	[106, 327]
240	[391, 392]	565	[174, 718]	890	[982, 46]	1215	[1248, 28]
241	[393, 394]	566	[719, 350]	891	[983, 174]	1216	[1249, 553]
242	[395, 394]	567	[689, 420]	892	[984, 944]	1217	[1250, 300]
243	[160, 396]	568	[720, 427]	893	[985, 684]	1218	[76, 1244]
244	[397, 39]	569	[198, 58]	894	[986, 425]	1219	[591, 263]
245	[194, 398]	570	[721, 400]	895	[987, 124]	1220	[1062, 876]
246	[399, 400]	571	[185, 397]	896	[988, 989]	1221	[1251, 867]

247	[401, 125]	572	[136, 423]	897	[488, 705]	1222	[1252, 274]
248	[402, 39]	573	[722, 406]	898	[990, 60]	1223	[1253, 619]
249	[135, 403]	574	[723, 189]	899	[476, 198]	1224	[6, 1055]
250	[404, 194]	575	[471, 161]	900	[991, 441]	1225	[1254, 617]
251	[405, 406]	576	[380, 666]	901	[992, 61]	1226	[1255, 73]
252	[352, 120]	577	[724, 376]	902	[234, 553]	1227	[1256, 323]
253	[407, 357]	578	[1, 416]	903	[993, 89]	1228	[81, 286]
254	[408, 409]	579	[391, 394]	904	[994, 331]	1229	[232, 1053]
255	[389, 410]	580	[725, 32]	905	[601, 995]	1230	[1257, 65]
256	[411, 157]	581	[726, 128]	906	[996, 343]	1231	[1258, 832]
257	[144, 410]	582	[727, 51]	907	[596, 997]	1232	[917, 468]
258	[412, 376]	583	[709, 192]	908	[998, 575]	1233	[752, 728]
259	[413, 194]	584	[51, 151]	909	[999, 869]	1234	[1259, 1112]
260	[131, 183]	585	[133, 728]	910	[1000, 326]	1235	[1260, 203]
261	[133, 414]	586	[729, 373]	911	[1001, 529]	1236	[1261, 153]
262	[415, 403]	587	[167, 391]	912	[1002, 800]	1237	[1262, 474]
263	[361, 416]	588	[145, 389]	913	[651, 323]	1238	[1263, 756]
264	[129, 171]	589	[730, 731]	914	[1003, 81]	1239	[1264, 905]
265	[380, 417]	590	[732, 200]	915	[1004, 632]	1240	[1265, 200]
266	[138, 362]	591	[733, 397]	916	[832, 1005]	1241	[1266, 193]
267	[418, 12]	592	[734, 735]	917	[327, 816]	1242	[188, 158]
268	[419, 160]	593	[736, 737]	918	[565, 321]	1243	[1267, 158]
269	[362, 192]	594	[738, 713]	919	[258, 1006]	1244	[464, 125]
270	[398, 140]	595	[739, 200]	920	[1007, 115]	1245	[1268, 474]
271	[131, 420]	596	[377, 138]	921	[1008, 270]	1246	[378, 362]
272	[421, 40]	597	[348, 53]	922	[1009, 522]	1247	[1269, 919]
273	[398, 422]	598	[740, 47]	923	[1010, 93]	1248	[1270, 391]
274	[45, 423]	599	[459, 443]	924	[1011, 518]	1249	[204, 718]
275	[138, 191]	600	[729, 741]	925	[219, 208]	1250	[149, 139]
276	[424, 425]	601	[742, 431]	926	[1012, 1013]	1251	[1271, 1135]
277	[396, 155]	602	[166, 57]	927	[1014, 564]	1252	[1272, 718]
278	[426, 427]	603	[474, 57]	928	[1015, 1016]	1253	[1273, 684]
279	[428, 429]	604	[198, 352]	929	[1017, 1018]	1254	[1274, 919]
280	[430, 431]	605	[60, 683]	930	[278, 650]	1255	[1275, 139]
281	[419, 37]	606	[743, 133]	931	[1019, 320]	1256	[1276, 441]
282	[432, 55]	607	[472, 392]	932	[1020, 341]	1257	[1119, 158]
283	[433, 434]	608	[744, 735]	933	[1021, 615]	1258	[1277, 768]
284	[435, 135]	609	[148, 194]	934	[1022, 259]	1259	[1278, 579]
285	[360, 422]	610	[745, 746]	935	[1023, 311]	1260	[1279, 299]
286	[436, 44]	611	[747, 12]	936	[629, 604]	1261	[225, 1071]
287	[139, 422]	612	[369, 125]	937	[1024, 537]	1262	[1280, 1147]
288	[437, 438]	613	[748, 131]	938	[821, 334]	1263	[1281, 229]
289	[439, 59]	614	[38, 468]	939	[1025, 1023]	1264	[835, 257]
290	[440, 165]	615	[487, 48]	940	[1026, 596]	1265	[1282, 614]

291	[441, 37]	616	[49, 180]	941	[604, 1027]	1266	[587, 1065]
292	[442, 49]	617	[443, 170]	942	[1028, 1029]	1267	[98, 21]
293	[171, 48]	618	[749, 725]	943	[491, 1030]	1268	[1283, 865]
294	[170, 443]	619	[475, 679]	944	[646, 1031]	1269	[1284, 21]
295	[444, 49]	620	[750, 147]	945	[1032, 268]	1270	[116, 1232]
296	[445, 180]	621	[179, 751]	946	[268, 887]	1271	[82, 287]
297	[446, 46]	622	[735, 752]	947	[1033, 844]	1272	[260, 879]
298	[181, 129]	623	[696, 190]	948	[1034, 567]	1273	[16, 1042]
299	[201, 151]	624	[143, 368]	949	[1035, 502]	1274	[1285, 269]
300	[184, 447]	625	[753, 60]	950	[1036, 107]	1275	[655, 242]
301	[448, 403]	626	[390, 188]	951	[1037, 578]	1276	[1286, 102]
302	[373, 449]	627	[754, 124]	952	[1038, 235]	1277	[893, 75]
303	[450, 451]	628	[755, 756]	953	[1039, 247]	1278	[1287, 358]
304	[452, 409]	629	[757, 60]	954	[584, 73]	1279	[1288, 184]
305	[453, 454]	630	[758, 364]	955	[301, 250]	1280	[1289, 362]
306	[455, 168]	631	[759, 35]	956	[1040, 289]	1281	[1290, 1133]
307	[456, 457]	632	[401, 145]	957	[1041, 1042]	1282	[1291, 1086]
308	[458, 387]	633	[682, 61]	958	[1043, 212]	1283	[447, 397]
309	[459, 170]	634	[760, 54]	959	[1025, 310]	1284	[53, 153]
310	[445, 171]	635	[761, 190]	960	[1044, 569]	1285	[1292, 728]
311	[171, 181]	636	[356, 714]	961	[1045, 602]	1286	[416, 45]
312	[460, 147]	637	[762, 366]	962	[643, 825]	1287	[105, 530]
313	[461, 44]	638	[164, 714]	963	[630, 1027]	1288	[652, 324]
314	[373, 462]	639	[763, 133]	964	[1046, 29]	1289	[222, 1175]
315	[463, 8]	640	[9, 161]	965	[1047, 1048]	1290	[1293, 552]
316	[386, 410]	641	[157, 9]	966	[1049, 508]	1291	[1294, 1018]
317	[464, 145]	642	[764, 431]	967	[1047, 1050]	1292	[249, 320]
318	[465, 417]	643	[187, 157]	968	[1051, 256]	1293	[955, 718]
319	[466, 131]	644	[453, 190]	969	[1052, 63]	1294	[1295, 905]
320	[467, 185]	645	[765, 40]	970	[780, 874]	1295	[1296, 1185]
321	[397, 468]	646	[766, 352]	971	[22, 1053]	1296	[1218, 358]
322	[469, 431]	647	[195, 478]	972	[1054, 346]		
323	[470, 33]	648	[121, 58]	973	[90, 1055]		
324	[441, 160]	649	[767, 768]	974	[1006, 1056]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	217	1	434	1	651	1	868	0	1085	1
1	0	218	0	435	1	652	0	869	0	1086	1
2	1	219	1	436	1	653	0	870	1	1087	0
3	0	220	0	437	0	654	1	871	0	1088	1
4	1	221	1	438	1	655	0	872	1	1089	0
5	1	222	1	439	0	656	0	873	1	1090	0
6	0	223	0	440	0	657	1	874	0	1091	0

7	0	224	0	441	1	658	1	875	1	1092	0
8	0	225	0	442	0	659	1	876	0	1093	1
9	1	226	0	443	1	660	0	877	0	1094	0
10	1	227	1	444	1	661	0	878	1	1095	1
11	1	228	0	445	1	662	1	879	0	1096	1
12	1	229	1	446	1	663	1	880	0	1097	0
13	0	230	0	447	1	664	0	881	0	1098	1
14	0	231	0	448	1	665	0	882	0	1099	0
15	0	232	0	449	1	666	1	883	0	1100	1
16	0	233	1	450	1	667	1	884	0	1101	0
17	0	234	0	451	1	668	0	885	0	1102	0
18	0	235	0	452	1	669	0	886	1	1103	0
19	1	236	0	453	0	670	1	887	0	1104	1
20	0	237	0	454	0	671	1	888	0	1105	1
21	1	238	0	455	1	672	0	889	0	1106	1
22	1	239	1	456	0	673	1	890	0	1107	1
23	1	240	0	457	0	674	0	891	0	1108	1
24	0	241	1	458	1	675	0	892	0	1109	1
25	1	242	0	459	0	676	0	893	1	1110	1
26	1	243	1	460	1	677	1	894	1	1111	1
27	0	244	0	461	0	678	0	895	0	1112	0
28	0	245	0	462	0	679	0	896	1	1113	1
29	0	246	0	463	1	680	0	897	1	1114	1
30	0	247	1	464	0	681	1	898	1	1115	0
31	0	248	1	465	0	682	1	899	0	1116	1
32	0	249	1	466	0	683	0	900	1	1117	1
33	0	250	0	467	1	684	0	901	1	1118	0
34	0	251	0	468	0	685	1	902	0	1119	1
35	0	252	1	469	1	686	0	903	0	1120	0
36	0	253	1	470	1	687	1	904	1	1121	0
37	0	254	1	471	1	688	0	905	0	1122	1
38	1	255	0	472	1	689	1	906	0	1123	0
39	1	256	1	473	0	690	0	907	1	1124	0
40	0	257	1	474	0	691	0	908	0	1125	0
41	1	258	1	475	1	692	1	909	1	1126	0
42	1	259	1	476	0	693	1	910	0	1127	1
43	1	260	0	477	1	694	1	911	0	1128	0
44	1	261	1	478	1	695	0	912	1	1129	1
45	1	262	0	479	0	696	1	913	1	1130	0
46	1	263	1	480	1	697	1	914	0	1131	0
47	1	264	0	481	0	698	0	915	0	1132	1
48	0	265	1	482	1	699	0	916	1	1133	0
49	1	266	1	483	0	700	0	917	0	1134	1
50	1	267	1	484	0	701	0	918	0	1135	1

51	1	268	0	485	1	702	0	919	1	1136	0
52	1	269	0	486	1	703	1	920	1	1137	1
53	1	270	0	487	0	704	0	921	1	1138	1
54	0	271	0	488	1	705	0	922	0	1139	0
55	1	272	1	489	0	706	0	923	1	1140	0
56	0	273	0	490	0	707	0	924	0	1141	0
57	0	274	1	491	1	708	1	925	1	1142	1
58	0	275	1	492	1	709	1	926	0	1143	0
59	1	276	1	493	0	710	0	927	0	1144	0
60	0	277	0	494	1	711	1	928	0	1145	1
61	1	278	1	495	0	712	0	929	1	1146	0
62	0	279	1	496	0	713	0	930	1	1147	1
63	0	280	0	497	1	714	1	931	0	1148	1
64	0	281	0	498	1	715	0	932	1	1149	1
65	0	282	0	499	1	716	0	933	1	1150	1
66	0	283	0	500	1	717	1	934	0	1151	0
67	0	284	1	501	0	718	1	935	0	1152	0
68	0	285	0	502	1	719	0	936	0	1153	0
69	0	286	1	503	1	720	1	937	1	1154	0
70	0	287	1	504	0	721	1	938	0	1155	1
71	0	288	0	505	0	722	1	939	0	1156	1
72	0	289	0	506	1	723	1	940	0	1157	0
73	0	290	0	507	1	724	0	941	1	1158	1
74	1	291	1	508	0	725	1	942	1	1159	0
75	1	292	0	509	1	726	1	943	1	1160	1
76	0	293	1	510	1	727	0	944	1	1161	0
77	1	294	0	511	0	728	0	945	0	1162	0
78	0	295	1	512	1	729	1	946	0	1163	1
79	0	296	1	513	1	730	1	947	0	1164	1
80	1	297	1	514	0	731	0	948	1	1165	1
81	1	298	1	515	1	732	1	949	1	1166	1
82	0	299	0	516	1	733	1	950	1	1167	1
83	1	300	1	517	0	734	1	951	1	1168	1
84	1	301	1	518	1	735	1	952	1	1169	1
85	0	302	1	519	0	736	0	953	1	1170	1
86	1	303	1	520	0	737	1	954	1	1171	0
87	0	304	1	521	0	738	0	955	1	1172	0
88	1	305	0	522	1	739	1	956	1	1173	1
89	0	306	1	523	0	740	0	957	1	1174	0
90	1	307	0	524	0	741	0	958	1	1175	1
91	0	308	1	525	0	742	0	959	0	1176	1
92	0	309	0	526	1	743	1	960	0	1177	0
93	1	310	1	527	0	744	0	961	1	1178	0
94	1	311	1	528	1	745	1	962	0	1179	1

95	1	312	1	529	0	746	0	963	0	1180	0
96	1	313	0	530	0	747	0	964	1	1181	0
97	1	314	1	531	0	748	1	965	0	1182	0
98	0	315	1	532	1	749	1	966	0	1183	1
99	1	316	0	533	1	750	1	967	0	1184	0
100	0	317	0	534	0	751	1	968	0	1185	0
101	1	318	0	535	0	752	0	969	1	1186	0
102	1	319	0	536	0	753	0	970	0	1187	0
103	0	320	1	537	0	754	1	971	1	1188	1
104	0	321	0	538	1	755	0	972	0	1189	1
105	1	322	1	539	0	756	0	973	1	1190	0
106	1	323	1	540	1	757	0	974	0	1191	1
107	1	324	1	541	0	758	0	975	0	1192	1
108	0	325	1	542	1	759	1	976	0	1193	0
109	1	326	1	543	0	760	0	977	0	1194	0
110	0	327	0	544	1	761	1	978	0	1195	1
111	1	328	1	545	0	762	0	979	1	1196	0
112	1	329	0	546	1	763	0	980	0	1197	0
113	1	330	0	547	1	764	1	981	1	1198	1
114	0	331	1	548	1	765	0	982	0	1199	0
115	0	332	0	549	0	766	1	983	0	1200	1
116	1	333	1	550	0	767	0	984	0	1201	1
117	1	334	1	551	0	768	1	985	1	1202	0
118	0	335	0	552	1	769	1	986	1	1203	0
119	1	336	0	553	0	770	0	987	0	1204	0
120	0	337	0	554	0	771	0	988	1	1205	1
121	0	338	0	555	1	772	1	989	0	1206	1
122	0	339	1	556	0	773	0	990	1	1207	0
123	1	340	0	557	1	774	1	991	1	1208	1
124	0	341	1	558	0	775	1	992	1	1209	0
125	1	342	1	559	0	776	1	993	0	1210	1
126	0	343	1	560	0	777	0	994	1	1211	0
127	0	344	1	561	0	778	0	995	0	1212	1
128	1	345	0	562	0	779	1	996	0	1213	1
129	0	346	1	563	1	780	0	997	1	1214	1
130	0	347	0	564	0	781	1	998	0	1215	1
131	0	348	0	565	0	782	0	999	1	1216	1
132	0	349	1	566	0	783	1	1000	0	1217	1
133	1	350	1	567	1	784	1	1001	0	1218	0
134	0	351	1	568	1	785	1	1002	1	1219	1
135	1	352	1	569	1	786	0	1003	0	1220	1
136	0	353	0	570	1	787	0	1004	0	1221	0
137	0	354	0	571	0	788	1	1005	0	1222	0
138	1	355	1	572	0	789	1	1006	0	1223	0

139	1	356	1	573	1	790	1	1007	1	1224	0
140	0	357	1	574	1	791	0	1008	1	1225	1
141	1	358	1	575	1	792	0	1009	0	1226	0
142	0	359	0	576	1	793	0	1010	1	1227	0
143	0	360	0	577	0	794	1	1011	0	1228	1
144	1	361	1	578	0	795	1	1012	0	1229	0
145	0	362	0	579	0	796	0	1013	1	1230	1
146	0	363	1	580	1	797	0	1014	0	1231	1
147	0	364	0	581	1	798	1	1015	0	1232	0
148	1	365	0	582	0	799	0	1016	0	1233	0
149	1	366	1	583	1	800	1	1017	1	1234	1
150	1	367	1	584	1	801	1	1018	0	1235	0
151	0	368	1	585	1	802	1	1019	0	1236	0
152	0	369	1	586	1	803	1	1020	1	1237	1
153	0	370	0	587	1	804	0	1021	1	1238	1
154	1	371	1	588	0	805	0	1022	0	1239	0
155	1	372	0	589	1	806	1	1023	0	1240	0
156	1	373	1	590	1	807	0	1024	1	1241	1
157	0	374	0	591	1	808	1	1025	0	1242	1
158	0	375	0	592	1	809	0	1026	0	1243	0
159	1	376	1	593	0	810	1	1027	1	1244	0
160	1	377	1	594	0	811	0	1028	1	1245	1
161	0	378	0	595	1	812	0	1029	1	1246	0
162	1	379	0	596	1	813	1	1030	1	1247	1
163	1	380	1	597	0	814	1	1031	1	1248	1
164	0	381	0	598	0	815	1	1032	0	1249	1
165	0	382	0	599	0	816	0	1033	0	1250	1
166	1	383	1	600	1	817	0	1034	1	1251	0
167	1	384	0	601	0	818	1	1035	1	1252	0
168	0	385	0	602	1	819	1	1036	1	1253	0
169	0	386	0	603	0	820	1	1037	1	1254	1
170	0	387	0	604	1	821	0	1038	1	1255	0
171	1	388	0	605	0	822	0	1039	1	1256	0
172	1	389	0	606	1	823	0	1040	1	1257	1
173	1	390	1	607	1	824	0	1041	1	1258	1
174	0	391	0	608	0	825	0	1042	1	1259	1
175	1	392	1	609	1	826	1	1043	1	1260	0
176	1	393	1	610	1	827	1	1044	0	1261	0
177	1	394	0	611	0	828	0	1045	1	1262	1
178	1	395	0	612	1	829	0	1046	1	1263	1
179	0	396	0	613	1	830	1	1047	0	1264	0
180	0	397	0	614	1	831	0	1048	1	1265	0
181	1	398	0	615	0	832	1	1049	0	1266	1
182	1	399	0	616	1	833	1	1050	0	1267	0

183	0	400	0	617	1	834	0	1051	0	1268	1
184	1	401	1	618	1	835	0	1052	1	1269	1
185	0	402	1	619	1	836	0	1053	1	1270	1
186	0	403	1	620	1	837	1	1054	0	1271	0
187	0	404	0	621	0	838	0	1055	1	1272	0
188	1	405	0	622	1	839	0	1056	1	1273	0
189	1	406	0	623	1	840	1	1057	0	1274	1
190	1	407	1	624	0	841	1	1058	0	1275	0
191	1	408	1	625	0	842	1	1059	0	1276	0
192	0	409	1	626	1	843	1	1060	1	1277	1
193	0	410	1	627	1	844	1	1061	0	1278	1
194	0	411	1	628	0	845	1	1062	1	1279	0
195	0	412	1	629	0	846	1	1063	0	1280	1
196	0	413	1	630	0	847	0	1064	0	1281	1
197	1	414	1	631	1	848	1	1065	1	1282	0
198	1	415	0	632	1	849	1	1066	1	1283	1
199	0	416	0	633	1	850	1	1067	0	1284	1
200	0	417	0	634	0	851	0	1068	1	1285	1
201	0	418	1	635	1	852	1	1069	0	1286	0
202	1	419	0	636	1	853	1	1070	1	1287	1
203	1	420	1	637	0	854	1	1071	0	1288	0
204	1	421	1	638	0	855	0	1072	1	1289	1
205	0	422	1	639	0	856	0	1073	0	1290	1
206	1	423	0	640	1	857	1	1074	1	1291	0
207	1	424	1	641	0	858	0	1075	0	1292	1
208	0	425	1	642	1	859	0	1076	0	1293	1
209	1	426	1	643	0	860	1	1077	1	1294	0
210	0	427	0	644	0	861	1	1078	0	1295	0
211	0	428	1	645	0	862	0	1079	0	1296	0
212	1	429	1	646	1	863	0	1080	1		
213	1	430	0	647	0	864	0	1081	0		
214	1	431	1	648	0	865	0	1082	0		
215	0	432	0	649	0	866	0	1083	0		
216	0	433	0	650	0	867	1	1084	1		

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