

PROOF OF THE CONTINUED FRACTION CONJECTURE FOR

$$(a, b) = (z^2 + z, z^5 + z^3 + z^2 + z)$$

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ABSTRACT. The continued fraction conjecture claims that the continued fraction defined by the (a, b) -Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a, b) in $\mathbb{F}_2[x] \setminus \mathbb{F}_2$. In this paper we prove the conjecture for $(a, b) = (z^2 + z, z^5 + z^3 + z^2 + z)$.

1. INTRODUCTION

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \dots)$ of elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$, we define the infinite continued fraction

$$(1.1) \quad \text{CF}(\mathbf{a}(z)) = \frac{1}{a_0(z) + \frac{1}{a_1(z) + \frac{1}{a_2(z) + \frac{1}{\ddots}}}}$$

as the limit of the finite continued fractions

$$(1.2) \quad \text{CF}_n(\mathbf{a}(z)) = \frac{1}{a_0(z) + \frac{1}{a_1(z) + \frac{1}{\ddots + \frac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)).$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$(1.3) \quad c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \dots$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a, b) -Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \dots) = (a, b, b, a, b, a, a, b, \dots)$$

that we shall denote

$$(1.4) \quad \text{CF}(a, b) := \text{CF}(\mathbf{t}).$$

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Theorem 1.1. *When $(a, b) = (z^2 + z, z^5 + z^3 + z^2 + z)$, the two power series $\text{CF}(a, b)$ and $\text{CF}(b, a)$ are algebraic over $\mathbb{F}_2(z)$, with minimal polynomials of the form*

$$p_4(z)y^4 + p_3(z)y^3 + p_2(z)y^2 + p_1(z)y + p_0(z) = 0$$

where, for $\text{CF}(a, b)$,

$$\begin{aligned} p_0(z) &= z^9 + z^7 + z^6 + z^5 + z^3 + z^2 + 1, \\ p_1(z) &= z^{14} + z^{13} + z^{11} + z^6 + z^5 + z^3, \\ p_2(z) &= z^{16} + z^{14} + z^{10} + z^8 + z^6 + z^2, \\ p_3(z) &= z^{14} + z^{13} + z^{11} + z^6 + z^5 + z^3, \\ p_4(z) &= z^{12} + z^9 + z^7 + z^6 + z^5 + z^4 + z^3 + z^2 + 1, \end{aligned}$$

and for $\text{CF}(b, a)$,

$$\begin{aligned} p_0(z) &= z^9 + z^8 + z^7 + z^5 + z^3 + z^2 + 1, \\ p_1(z) &= z^{14} + z^{13} + z^{11} + z^6 + z^5 + z^3, \\ p_2(z) &= z^{16} + z^{14} + z^{10} + z^8 + z^6 + z^2, \\ p_3(z) &= z^{14} + z^{13} + z^{11} + z^6 + z^5 + z^3, \\ p_4(z) &= z^{12} + z^9 + z^8 + z^7 + z^5 + z^4 + z^3 + z^2 + 1. \end{aligned}$$

2. PROOF

Define the sequences of polynomials $(P_n(z))_{n \geq 0}$ and $(Q_n(z))_{n \geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z) \\ P_{n-1}(z) & Q_{n-1}(z) \end{pmatrix} := \begin{pmatrix} t_n(z) & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t_{n-1}(z) & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} t_0(z) & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$(2.1) \quad \text{CF}_n(a, b) = \frac{P_n(z)}{Q_n(z)} \in \mathbb{F}_2((1/z)).$$

As formula (2.1) is not efficient for calculating $\text{CF}_n(a, b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned} M_n(x) &= x^{\deg(t_{2^n-1})} \begin{pmatrix} t_{2^n-1}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times x^{\deg(t_{2^n-2})} \begin{pmatrix} t_{2^n-2}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times \\ &\quad \cdots \times x^{\deg(t_0)} \begin{pmatrix} t_0(1/x) & 1 \\ 1 & 0 \end{pmatrix}, \\ W_n(x) &= x^{\deg(\bar{t}_{2^n-1})} \begin{pmatrix} \bar{t}_{2^n-1}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times x^{\deg(\bar{t}_{2^n-2})} \begin{pmatrix} \bar{t}_{2^n-2}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times \\ &\quad \cdots \times x^{\deg(\bar{t}_0)} \begin{pmatrix} \bar{t}_0(1/x) & 1 \\ 1 & 0 \end{pmatrix}, \end{aligned}$$

where $\bar{t}$ is the (b, a) -Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n \geq 0$

$$\begin{aligned} M_{n+1}(x) &= W_n(x) \cdot M_n(x), \\ W_{n+1}(x) &= M_n(x) \cdot W_n(x). \end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\text{CF}_n(a, b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$\begin{aligned} M_n(x)_{0,1} &= x^{d_n} \tilde{P}_{2^n-1}(x), \\ M_n(x)_{0,0} &= x^{d_n} \tilde{Q}_{2^n-1}(x), \end{aligned}$$

for some positive integer d_n . Hence,

$$(2.2) \quad \text{CF}_{2^{2^n}-1}(a, b) = \frac{\tilde{P}_{2^{2^n}-1}(x)}{\tilde{Q}_{2^{2^n}-1}(x)} = \frac{M_{2^n}(x)_{0,1}}{M_{2^n}(x)_{0,0}}.$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\text{CF}(a, b)$ will be established if it is shown that both $M_{2^n}(x)_{0,1}$ and $M_{2^n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \leq i, j \leq 1$ the four sequences $(M_{2^n}(x)_{i,j})_n$, $(M_{2^{n+1}}(x)_{i,j})_n$, $(W_{2^n}(x)_{i,j})_n$, and $(W_{2^{n+1}}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \leq i, j \leq 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T, i, j)$ under certain initial conditions; the polynomials $\phi(T, i, j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2^n}(x))_n$, $(M_{2^{n+1}}(x))_n$, $(W_{2^n}(x))_n$, and $(W_{2^{n+1}}(x))_n$.

Let us explain how the polynomials $\phi(T, i, j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \leq i, j \leq 1$, the coefficients of the polynomial $\phi(M^e, i, j)$ (resp. $\phi(M^o, i, j)$, $\phi(W^e, i, j)$, and $\phi(W^o, i, j)$) are the Padé-Hermite approximants of type

$$(175, 175, 175, 175, 175)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e, i, j)$ (resp. $\phi(M^o, i, j)$, $\phi(W^e, i, j)$, and $\phi(W^o, i, j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x, y) \in \mathbb{F}_2[x, y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \geq 0$. If for some $n \geq 0$ and $a_0, a_1, \dots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x, y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \geq 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o . The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$(2.3) \quad M^e = W^o \cdot M^o,$$

$$(2.4) \quad M^o = W^e \cdot M^e,$$

$$(2.5) \quad W^e = M^o \cdot W^o,$$

$$(2.6) \quad W^o = M^e \cdot W^e.$$

Proof. We give the proof of the identity

$$M_{0,0}^e = W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o,$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W_{0,0}^o M_{0,0}^o$ and $W_{0,1}^o M_{1,0}^o$. We know that

$$P(x, y) = \text{Res}_z (\phi(W^o, 0, 0)(x, z), z^{12} \cdot \phi(M^o, 0, 0)(x, y/z))$$

is an annihilating polynomial of $W_{0,0}^o M_{0,0}^o$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W_{0,0}^o M_{0,0}^o$, that will be called $\phi_0(x, y)$. To prove that $\phi_0(x, y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x, y)$ of multiplicity m and that $Q(x, y) := P(x, y)/\phi_0(x, y)^m$ is not an annihilating polynomial of $W_{0,0}^o M_{0,0}^o$. We verify the first point directly. For the second point, we truncate $W_{0,0}^o M_{0,0}^o$ to order 630 and substitute it for y in $Q(x, y)$. We get a series of valuation less than 630, which proves that $Q(x, y)$ is not an annihilating polynomial of $W_{0,0}^o M_{0,0}^o$. We find the minimal polynomial $\phi_1(x, y)$ of $W_{0,1}^o M_{1,0}^o$ in a similar way.

Now we prove that $\phi(M^e, 0, 0)$ is the minimal polynomial of $W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o$. We know that

$$S(x, y) = \text{Res}_z (\phi_0(x, z), \phi_1(x, y + z))$$

is an annihilating of $W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o$. We verify that $\phi(M^e, 0, 0)$ is an irreducible factor of $S(x, y)$ of multiplicity μ , and that $Q(x, y) := S(x, y)/\phi(M^e, 0, 0)^\mu$ is not an annihilating polynomial of $W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o$. To see the last point, we truncate $W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o$ to order 770 and substitute it for y in $Q(x, y)$. We get a series of valuation less than 770, and therefore $Q(x, y)$ is not an annihilating polynomial of $W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o$.

Finally, the first 16 terms of $M_{0,0}^e$ and $W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o$ coincide. As these first terms determine a unique solution of $\phi(M^e, 0, 0)$, we know that the two series are one and the same. $\square$

Define

$$R^e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad R^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned} M^e[:14u] &= M^e[:7u] + x^u M^e[:6u] + x^{2u} M^e[:5u] + x^{4u} M^e[:3u] + x^u M^e[:7u] \\ &\quad + x^{2u} M^e[:6u] + x^{3u} M^e[:5u] + x^{5u} M^e[:3u] + x^{3u} M^e[:7u] \\ &\quad + x^{4u} M^e[:6u] + x^{5u} M^e[:5u] + x^{7u} M^e[:3u] + x^{12u} M^e[:2u] \\ &\quad + (x^{7u} + x^{8u} + x^{10u}) R^e, \\ W^e[:14u] &= W^e[:7u] + x^u W^e[:6u] + x^{2u} W^e[:5u] + x^{4u} W^e[:3u] + x^u W^e[:7u] \\ &\quad + x^{2u} W^e[:6u] + x^{3u} W^e[:5u] + x^{5u} W^e[:3u] + x^{3u} W^e[:7u] \end{aligned}$$

$$\begin{aligned}
& + x^{4u}W^e[:6u] + x^{5u}W^e[:5u] + x^{7u}W^e[:3u] + x^{12u}W^e[:2u] \\
& + (x^{7u} + x^{8u} + x^{10u})R^e.
\end{aligned}$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$\begin{aligned}
M^o[:14u] &= M^o[:7u] + x^u M^o[:6u] + x^{2u} M^o[:5u] + x^{4u} M^o[:3u] + x^u M^o[:7u] \\
& + x^{2u} M^o[:6u] + x^{3u} M^o[:5u] + x^{5u} M^o[:3u] + x^{3u} M^o[:7u] \\
& + x^{4u} M^o[:6u] + x^{5u} M^o[:5u] + x^{7u} M^o[:3u] + x^{12u} M^o[:2u] \\
& + (x^{7u} + x^{8u} + x^{10u})R^o, \\
W^o[:14u] &= W^o[:7u] + x^u W^o[:6u] + x^{2u} W^o[:5u] + x^{4u} W^o[:3u] + x^u W^o[:7u] \\
& + x^{2u} W^o[:6u] + x^{3u} W^o[:5u] + x^{5u} W^o[:3u] + x^{3u} W^o[:7u] \\
& + x^{4u} W^o[:6u] + x^{5u} W^o[:5u] + x^{7u} W^o[:3u] + x^{12u} W^o[:2u] \\
& + (x^{7u} + x^{8u} + x^{10u})R^o.
\end{aligned}$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k 's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M_{0,0}^o$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$\begin{aligned}
T[:14u] &= T[:7u] + x^u T[:6u] + x^{2u} T[:5u] + x^{4u} T[:3u] + x^u T[:7u] + x^{2u} T[:6u] \\
& + x^{3u} T[:5u] + x^{5u} T[:3u] + x^{3u} T[:7u] + x^{4u} T[:6u] + x^{5u} T[:5u] \\
& + x^{7u} T[:3u] + x^{12u} T[:2u].
\end{aligned}$$

The proof of the other 15 cases are similar. We break down the above identity into 7 parts:

$$\begin{aligned}
0 &= x^{7u}T[:u] + x^{4u}T[3u:4u] + x^{2u}T[5u:6u] + x^u T[6u:7u] + T[7u:8u], \\
0 &= x^{7u}T[u:2u] + x^{5u}T[3u:4u] + x^{4u}T[4u:5u] + x^{3u}T[5u:6u] \\
& + T[8u:9u], \\
0 &= x^{7u}T[2u:3u] + x^{5u}T[4u:5u] + x^{4u}T[5u:6u] + x^{3u}T[6u:7u] \\
& + T[9u:10u], \\
0 &= T[10u:11u], \\
0 &= T[11u:12u], \\
0 &= x^{12u}T[:u] + T[12u:13u], \\
0 &= x^{12u}T[u:2u] + T[13u:14u],
\end{aligned}$$

which can be rewritten as

$$(2.7) \quad 0 = T[[w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2] + T[[111w]_2],$$

$$(2.8) \quad 0 = T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1000w]_2],$$

$$(2.9) \quad 0 = T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[110w]_2] + T[[1001w]_2],$$

$$(2.10) \quad 0 = T[[1010w]_2],$$

$$(2.11) \quad 0 = T[[1011w]_2],$$

$$(2.12) \quad 0 = T[[w]_2] + T[[1100w]_2],$$

$$(2.13) \quad 0 = T[[1w]_2] + T[[1101w]_2],$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$(2.14) \quad 0 = T[[w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2] + T[[111w]_2],$$

$$(2.15) \quad 0 = T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1000w]_2],$$

$$(2.16) \quad 0 = T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[110w]_2] + T[[1001w]_2],$$

$$(2.17) \quad 0 = T[[1010w]_2],$$

$$(2.18) \quad 0 = T[[1011w]_2],$$

$$(2.19) \quad 0 = T[[w]_2] + T[[1100w]_2],$$

$$(2.20) \quad 0 = T[[1w]_2] + T[[1101w]_2],$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 86 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s , and τ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.20) can be written as

$$(2.21) \quad \begin{aligned} 0 &= \tau(A(s, \epsilon)) + \tau(A(s, 11)) + \tau(A(s, 101)) + \tau(A(s, 110)) \\ &\quad + \tau(A(s, 111)), \end{aligned}$$

$$(2.22) \quad \begin{aligned} 0 &= \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 101)) \\ &\quad + \tau(A(s, 1000)), \end{aligned}$$

$$(2.23) \quad \begin{aligned} 0 &= \tau(A(s, 10)) + \tau(A(s, 100)) + \tau(A(s, 101)) + \tau(A(s, 110)) \\ &\quad + \tau(A(s, 1001)), \end{aligned}$$

$$(2.24) \quad 0 = \tau(A(s, 1010)),$$

$$(2.25) \quad 0 = \tau(A(s, 1011)),$$

$$(2.26) \quad 0 = \tau(A(s, \epsilon)) + \tau(A(s, 1100)),$$

$$(2.27) \quad 0 = \tau(A(s, 1)) + \tau(A(s, 1101)),$$

for all $s \in E_{2k}$, and

$$(2.28) \quad \begin{aligned} 0 &= \tau(A(s, \epsilon)) + \tau(A(s, 11)) + \tau(A(s, 101)) + \tau(A(s, 110)) \\ &\quad + \tau(A(s, 111)), \end{aligned}$$

$$(2.29) \quad \begin{aligned} 0 &= \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 101)) \\ &\quad + \tau(A(s, 1000)), \end{aligned}$$

$$(2.30) \quad \begin{aligned} 0 &= \tau(A(s, 10)) + \tau(A(s, 100)) + \tau(A(s, 101)) + \tau(A(s, 110)) \\ &\quad + \tau(A(s, 1001)), \end{aligned}$$

$$(2.31) \quad 0 = \tau(A(s, 1010)),$$

$$(2.32) \quad 0 = \tau(A(s, 1011)),$$

$$(2.33) \quad 0 = \tau(A(s, \epsilon)) + \tau(A(s, 1100)),$$

$$(2.34) \quad 0 = \tau(A(s, 1)) + \tau(A(s, 1101)),$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{12}), E_{12}) = (A(i, 0^{10}), E_{10})$, so that we only have to verify that identities (2.21) through (2.34) hold for $2 \leq k \leq 6$, which turns out to be true. $\square$

In the following lemma, we express M_{2k} , M_{2k+1} , W_{2k} , and W_{2k+1} in terms of M^e , M^o , W^e , and W^o .

Lemma 2.3. *For all integer $k \geq 2$, and $u = 2^{2k-1}$,*

$$M_{2k} = M^e[:7u] + x^u M^e[:6u] + x^{2u} M^e[:5u] + x^{4u} M^e[:3u] + x^{7u} R^e,$$

$$W_{2k} = W^e[:7u] + x^u W^e[:6u] + x^{2u} W^e[:5u] + x^{4u} W^e[:3u] + x^{7u} R^e.$$

For all integer $k \geq 2$, and $u = 2^{2k}$,

$$M_{2k+1} = M^o[:7u] + x^u M^o[:6u] + x^{2u} M^o[:5u] + x^{4u} M^o[:3u] + x^{7u} R^o,$$

$$W_{2k+1} = W^o[:7u] + x^u W^o[:6u] + x^{2u} W^o[:5u] + x^{4u} W^o[:3u] + x^{7u} R^o.$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k} , W_{2k} , M_{2k+1} , and W_{2k+1} . For $n = 2$, the identities can be verified directly. For $n \geq 2$, we claim that

$$\begin{aligned} M_{2k} \wedge W_{2k} &\Rightarrow M_{2k+1} \wedge W_{2k+1}, \\ M_{2k+1} \wedge W_{2k+1} &\Rightarrow M_{2k+2} \wedge W_{2k+2}. \end{aligned}$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$(2.35) \quad \begin{aligned} &W_{2k} M_{2k} = (W^e[:7v] + x^v W^e[:6v] + x^{2v} W^e[:5v] + x^{4v} W^e[:3v] + x^{7v} R^e) \times (\\ &M^e[:7v] + x^v M^e[:6v] + x^{2v} M^e[:5v] + x^{4v} M^e[:3v] + x^{7v} R^e); \end{aligned}$$

Call this expression *lhs*. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{14v} R^o$. Therefore we only need to prove that their difference is $O(x^{14v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{14v} , to

$$\begin{aligned} &W^e[:14v] M^e[:14v] + x^{2v} W^e[:12v] M^e[:12v] + x^{4v} W^e[:10v] M^e[:10v] \\ &+ x^{8v} W^e[:6v] M^e[:6v]. \end{aligned}$$

For all $n \leq 14$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call *rhs*. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v] / X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v] / X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions *lhs* (2.35) and *rhs* as polynomials in $\mathbb{F}_2[a_1, \dots, a_{14}, b_1, \dots, b_{14}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e , because in the expressions concerned, the products of W^e -terms and M^e -terms are always in the

same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{14})$, which completes the proof. $\square$

Proof of Theorem 1.1. We prove the theorem for $\text{CF}(a, b)$; for $\text{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \leq j, k \leq 1$,

$$\begin{aligned}\lim_{n \rightarrow \infty} M_{2n, j, k} &= M_{j, k}^e, \\ \lim_{n \rightarrow \infty} M_{2n+1, j, k} &= M_{j, k}^o, \\ \lim_{n \rightarrow \infty} W_{2n, j, k} &= W_{j, k}^e, \\ \lim_{n \rightarrow \infty} W_{2n+1, j, k} &= W_{j, k}^o.\end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\text{CF}(a, b) = \frac{M_{0,1}^e(x)}{M_{0,0}^e(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M_{0,1}^e$ and $M_{0,0}^e$. Therefore

$$P(x, y) = \text{Res}_t (\phi(M^e, 0, 1)(x, t), y^{12} \phi(M^e, 0, 1)(x, t/y))$$

is an annihilating polynomial of $f(x) = M_{0,1}^e/M_{0,0}^e$.

Define

$$Q(x, y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned}q_0(x) &= x^{16} + x^{14} + x^{13} + x^{11} + x^{10} + x^9 + x^7, \\ q_1(x) &= x^{13} + x^{11} + x^{10} + x^5 + x^3 + x^2, \\ q_2(x) &= x^{14} + x^{10} + x^8 + x^6 + x^2 + 1, \\ q_3(x) &= x^{13} + x^{11} + x^{10} + x^5 + x^3 + x^2, \\ q_4(x) &= x^{16} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^9 + x^7 + x^4.\end{aligned}$$

The polynomial $Q(x, y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x, y)$ of multiplicity m and $R(x, y) := P(x, y)/Q(x, y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 216, and substitute it for y in $R(z, y)$, we get a series with valuation smaller than 216, which proves that $R(z, y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z, y)$ is the minimal polynomial of $\text{CF}(a, b) = f(1/z)$. $\square$

3. ANNEXE

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e, 0, 0)$:

$$p_0(x) = x^{96} + x^{92} + x^{90} + x^{86} + x^{84} + x^{80} + x^{78} + x^{74} + x^{72} + x^{58} + x^{56}$$

$$\begin{aligned}
& + x^{54} + x^{48} + x^{46} + x^{40} + x^{38} + x^{34} + x^{30} + x^{24}, \\
p_3(x) &= x^{86} + x^{72} + x^{70} + x^{68} + x^{64} + x^{60} + x^{56} + x^{52} + x^{50} + x^{48} + x^{40} \\
& + x^{38} + x^{36} + x^{32} + x^{28} + x^{24} + x^{22} + x^{20} + x^{18} + x^{16}, \\
p_6(x) &= x^{88} + x^{86} + x^{78} + x^{74} + x^{72} + x^{70} + x^{68} + x^{66} + x^{60} + x^{58} + x^{54} \\
& + x^{52} + x^{50} + x^{36} + x^{34} + x^{32} + x^{28} + x^{26} + x^{24} + x^{20} + x^{18} + x^{14} \\
& + x^{10} + x^8 + x^6 + 1, \\
p_9(x) &= x^{82} + x^{80} + x^{78} + x^{76} + x^{58} + x^{56} + x^{54} + x^{52} + x^{42} + x^{40} + x^{38} \\
& + x^{36} + x^{26} + x^{24} + x^{22} + x^{20} + x^{18} + x^{16} + x^{14} + x^{12} + x^{10} + x^8 \\
& + x^6 + x^4, \\
p_{12}(x) &= x^{84} + x^{80} + x^{76} + x^{72} + x^{60} + x^{56} + x^{52} + x^{48} + x^{44} + x^{40} + x^{36} \\
& + x^{32} + x^{28} + x^{24} + x^4 + 1,
\end{aligned}$$

and the initial terms are $[1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0]$.

For $\phi(M^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{87} + x^{85} + x^{84} + x^{80} + x^{79} + x^{76} + x^{73} + x^{71} + x^{70} + x^{67} + x^{65} \\
& + x^{62} + x^{61} + x^{59} + x^{58} + x^{55} + x^{54} + x^{50} + x^{47} + x^{46} + x^{45} + x^{42} \\
& + x^{39} + x^{37} + x^{35} + x^{34} + x^{33}, \\
p_3(x) &= x^{66} + x^{64} + x^{62} + x^{56} + x^{54} + x^{50} + x^{34} + x^{32} + x^{30} + x^{24} + x^{22} \\
& + x^{18}, \\
p_6(x) &= x^{69} + x^{67} + x^{66} + x^{65} + x^{62} + x^{61} + x^{60} + x^{53} + x^{51} + x^{50} + x^{49} \\
& + x^{46} + x^{45} + x^{44} + x^{37} + x^{35} + x^{34} + x^{33} + x^{30} + x^{29} + x^{28} + x^{21} \\
& + x^{19} + x^{18} + x^{17} + x^{14} + x^{13} + x^{12}, \\
p_9(x) &= x^{72} + x^{70} + x^{68} + x^{66} + x^{64} + x^{62} + x^{52} + x^{50} + x^{48} + x^{46} + x^{36} \\
& + x^{34} + x^{32} + x^{30} + x^{20} + x^{18} + x^{16} + x^{14} + x^8 + x^6, \\
p_{12}(x) &= x^{75} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{63} + x^{61} + x^{60} + x^{59} + x^{57} \\
& + x^{56} + x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44} + x^{43} \\
& + x^{41} + x^{40} + x^{39} + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{31} + x^{29} + x^{28} \\
& + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{19} + x^{17} + x^{16} + x^{15} + x^{13} \\
& + x^{12} + x^3 + x + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 1, 1, 0]$.

For $\phi(M^e, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{87} + x^{85} + x^{84} + x^{80} + x^{79} + x^{76} + x^{73} + x^{71} + x^{70} + x^{67} + x^{65} \\
& + x^{62} + x^{61} + x^{59} + x^{58} + x^{55} + x^{54} + x^{50} + x^{47} + x^{46} + x^{45} + x^{42} \\
& + x^{39} + x^{37} + x^{35} + x^{34} + x^{33}, \\
p_3(x) &= x^{66} + x^{64} + x^{62} + x^{56} + x^{54} + x^{50} + x^{34} + x^{32} + x^{30} + x^{24} + x^{22} \\
& + x^{18},
\end{aligned}$$

$$\begin{aligned}
p_6(x) &= x^{69} + x^{67} + x^{66} + x^{65} + x^{62} + x^{61} + x^{60} + x^{53} + x^{51} + x^{50} + x^{49} \\
&\quad + x^{46} + x^{45} + x^{44} + x^{37} + x^{35} + x^{34} + x^{33} + x^{30} + x^{29} + x^{28} + x^{21} \\
&\quad + x^{19} + x^{18} + x^{17} + x^{14} + x^{13} + x^{12}, \\
p_9(x) &= x^{72} + x^{70} + x^{68} + x^{66} + x^{64} + x^{62} + x^{52} + x^{50} + x^{48} + x^{46} + x^{36} \\
&\quad + x^{34} + x^{32} + x^{30} + x^{20} + x^{18} + x^{16} + x^{14} + x^8 + x^6, \\
p_{12}(x) &= x^{75} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{63} + x^{61} + x^{60} + x^{59} + x^{57} \\
&\quad + x^{56} + x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44} + x^{43} \\
&\quad + x^{41} + x^{40} + x^{39} + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{31} + x^{29} + x^{28} \\
&\quad + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{19} + x^{17} + x^{16} + x^{15} + x^{13} \\
&\quad + x^{12} + x^3 + x + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 1, 1, 0]$.

For $\phi(M^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{78} + x^{68} + x^{66} + x^{58} + x^{54} + x^{48} + x^{46} + x^{44} + x^{42}, \\
p_3(x) &= x^{68} + x^{64} + x^{62} + x^{60} + x^{54} + x^{48} + x^{44} + x^{36} + x^{32} + x^{30} + x^{28} \\
&\quad + x^{22} + x^{16} + x^{12}, \\
p_6(x) &= x^{70} + x^{68} + x^{66} + x^{58} + x^{56} + x^{54} + x^{52} + x^{48} + x^{46} + x^{44} + x^{38} \\
&\quad + x^{36} + x^{32} + x^{26} + x^{24} + x^{22} + x^{20} + x^{16} + x^{14} + x^{12} + x^2 + 1, \\
p_9(x) &= x^{64} + x^{62} + x^{58} + x^{52} + x^{48} + x^{46} + x^{42} + x^{36} + x^{32} + x^{30} + x^{26} \\
&\quad + x^{20} + x^{16} + x^{14} + x^{10} + x^4, \\
p_{12}(x) &= x^{66} + x^{60} + x^{56} + x^{52} + x^{48} + x^{44} + x^{40} + x^{36} + x^{32} + x^{28} + x^{24} \\
&\quad + x^{20} + x^{16} + x^{12} + x^8 + x^4 + x^2 + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0]$.

For $\phi(W^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{78} + x^{70} + x^{68} + x^{66} + x^{64} + x^{60} + x^{58} + x^{52} + x^{50} + x^{44} + x^{40} \\
&\quad + x^{38} + x^{36} + x^{34} + x^{32} + x^{26} + x^{24}, \\
p_3(x) &= x^{68} + x^{64} + x^{62} + x^{56} + x^{54} + x^{52} + x^{48} + x^{36} + x^{32} + x^{30} + x^{24} \\
&\quad + x^{22} + x^{20} + x^{16}, \\
p_6(x) &= x^{70} + x^{68} + x^{66} + x^{62} + x^{60} + x^{58} + x^{56} + x^{54} + x^{52} + x^{50} + x^{42} \\
&\quad + x^{40} + x^{38} + x^{36} + x^{32} + x^{30} + x^{28} + x^{26} + x^{24} + x^{22} + x^{20} + x^{18} \\
&\quad + x^{10} + x^8 + x^2 + 1, \\
p_9(x) &= x^{64} + x^{62} + x^{58} + x^{52} + x^{48} + x^{46} + x^{42} + x^{36} + x^{32} + x^{30} + x^{26} \\
&\quad + x^{20} + x^{16} + x^{14} + x^{10} + x^4, \\
p_{12}(x) &= x^{66} + x^{60} + x^{56} + x^{52} + x^{48} + x^{44} + x^{40} + x^{36} + x^{32} + x^{28} + x^{24} \\
&\quad + x^{20} + x^{16} + x^{12} + x^8 + x^4 + x^2 + 1,
\end{aligned}$$

and the initial terms are $[1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0]$.

For $\phi(W^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{87} + x^{85} + x^{84} + x^{80} + x^{79} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{70} \\
&\quad + x^{69} + x^{68} + x^{67} + x^{63} + x^{61} + x^{60} + x^{59} + x^{58} + x^{53} + x^{46} + x^{45} \\
&\quad + x^{40} + x^{37} + x^{36} + x^{35} + x^{33}, \\
p_3(x) &= x^{66} + x^{64} + x^{62} + x^{56} + x^{54} + x^{50} + x^{34} + x^{32} + x^{30} + x^{24} + x^{22} \\
&\quad + x^{18}, \\
p_6(x) &= x^{69} + x^{67} + x^{66} + x^{65} + x^{62} + x^{61} + x^{60} + x^{53} + x^{51} + x^{50} + x^{49} \\
&\quad + x^{46} + x^{45} + x^{44} + x^{37} + x^{35} + x^{34} + x^{33} + x^{30} + x^{29} + x^{28} + x^{21} \\
&\quad + x^{19} + x^{18} + x^{17} + x^{14} + x^{13} + x^{12}, \\
p_9(x) &= x^{72} + x^{70} + x^{68} + x^{66} + x^{64} + x^{62} + x^{52} + x^{50} + x^{48} + x^{46} + x^{36} \\
&\quad + x^{34} + x^{32} + x^{30} + x^{20} + x^{18} + x^{16} + x^{14} + x^8 + x^6, \\
p_{12}(x) &= x^{75} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{63} + x^{61} + x^{60} + x^{59} + x^{57} \\
&\quad + x^{56} + x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44} + x^{43} \\
&\quad + x^{41} + x^{40} + x^{39} + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{31} + x^{29} + x^{28} \\
&\quad + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{19} + x^{17} + x^{16} + x^{15} + x^{13} \\
&\quad + x^{12} + x^3 + x + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 0, 0, 0, 1, 0, 1, 1, 1, 0, 0, 1, 0, 1, 1]$.

For $\phi(W^e, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{87} + x^{85} + x^{84} + x^{80} + x^{79} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{70} \\
&\quad + x^{69} + x^{68} + x^{67} + x^{63} + x^{61} + x^{60} + x^{59} + x^{58} + x^{53} + x^{46} + x^{45} \\
&\quad + x^{40} + x^{37} + x^{36} + x^{35} + x^{33}, \\
p_3(x) &= x^{66} + x^{64} + x^{62} + x^{56} + x^{54} + x^{50} + x^{34} + x^{32} + x^{30} + x^{24} + x^{22} \\
&\quad + x^{18}, \\
p_6(x) &= x^{69} + x^{67} + x^{66} + x^{65} + x^{62} + x^{61} + x^{60} + x^{53} + x^{51} + x^{50} + x^{49} \\
&\quad + x^{46} + x^{45} + x^{44} + x^{37} + x^{35} + x^{34} + x^{33} + x^{30} + x^{29} + x^{28} + x^{21} \\
&\quad + x^{19} + x^{18} + x^{17} + x^{14} + x^{13} + x^{12}, \\
p_9(x) &= x^{72} + x^{70} + x^{68} + x^{66} + x^{64} + x^{62} + x^{52} + x^{50} + x^{48} + x^{46} + x^{36} \\
&\quad + x^{34} + x^{32} + x^{30} + x^{20} + x^{18} + x^{16} + x^{14} + x^8 + x^6, \\
p_{12}(x) &= x^{75} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{63} + x^{61} + x^{60} + x^{59} + x^{57} \\
&\quad + x^{56} + x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44} + x^{43} \\
&\quad + x^{41} + x^{40} + x^{39} + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{31} + x^{29} + x^{28} \\
&\quad + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{19} + x^{17} + x^{16} + x^{15} + x^{13} \\
&\quad + x^{12} + x^3 + x + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 0, 0, 0, 1, 0, 1, 1, 1, 0, 0, 1, 0, 1, 1]$.

For $\phi(W^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{96} + x^{92} + x^{90} + x^{88} + x^{86} + x^{78} + x^{76} + x^{70} + x^{60} + x^{58} + x^{56} \\
&\quad + x^{54} + x^{52} + x^{50} + x^{48} + x^{44} + x^{42}, \\
p_3(x) &= x^{86} + x^{78} + x^{64} + x^{62} + x^{60} + x^{50} + x^{44} + x^{32} + x^{30} + x^{28} + x^{22} \\
&\quad + x^{18} + x^{14} + x^{12}, \\
p_6(x) &= x^{88} + x^{86} + x^{80} + x^{76} + x^{74} + x^{72} + x^{66} + x^{62} + x^{58} + x^{56} + x^{54} \\
&\quad + x^{52} + x^{50} + x^{40} + x^{38} + x^{34} + x^{32} + x^{30} + x^{26} + x^{20} + x^{18} + x^{16} \\
&\quad + x^{12} + x^{10} + x^6 + 1, \\
p_9(x) &= x^{82} + x^{80} + x^{78} + x^{76} + x^{58} + x^{56} + x^{54} + x^{52} + x^{42} + x^{40} + x^{38} \\
&\quad + x^{36} + x^{26} + x^{24} + x^{22} + x^{20} + x^{18} + x^{16} + x^{14} + x^{12} + x^{10} + x^8 \\
&\quad + x^6 + x^4, \\
p_{12}(x) &= x^{84} + x^{80} + x^{76} + x^{72} + x^{60} + x^{56} + x^{52} + x^{48} + x^{44} + x^{40} + x^{36} \\
&\quad + x^{32} + x^{28} + x^{24} + x^4 + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0]$.

For $\phi(M^o, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{87} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{79} + x^{78} + x^{74} + x^{70} + x^{69} \\
&\quad + x^{68} + x^{67} + x^{65} + x^{63} + x^{62} + x^{61} + x^{58} + x^{56} + x^{54} + x^{53} + x^{51} \\
&\quad + x^{46} + x^{41} + x^{39} + x^{38} + x^{36} + x^{34} + x^{33} + x^{31} + x^{27} + x^{25} + x^{24}, \\
p_3(x) &= x^{75} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{67} + x^{66} + x^{62} + x^{59} + x^{58} \\
&\quad + x^{57} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{43} + x^{41} + x^{40} + x^{39} + x^{37} \\
&\quad + x^{36} + x^{35} + x^{34} + x^{30} + x^{27} + x^{26} + x^{25} + x^{21} + x^{20} + x^{19} + x^{17} \\
&\quad + x^{16}, \\
p_6(x) &= x^{79} + x^{77} + x^{76} + x^{75} + x^{73} + x^{72} + x^{70} + x^{69} + x^{68} + x^{67} + x^{62} \\
&\quad + x^{61} + x^{59} + x^{58} + x^{57} + x^{55} + x^{52} + x^{51} + x^{50} + x^{48} + x^{47} + x^{38} \\
&\quad + x^{37} + x^{36} + x^{33} + x^{32} + x^{30} + x^{29} + x^{27} + x^{26} + x^{25} + x^{23} + x^{20} \\
&\quad + x^{19} + x^{18} + x^{16} + x^{13} + x^{12} + x^{11} + x^9 + x^8 + x^3 + x + 1, \\
p_9(x) &= x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} + x^{59} + x^{57} \\
&\quad + x^{56} + x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44} + x^{43} \\
&\quad + x^{41} + x^{40} + x^{39} + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{31} + x^{29} + x^{28} \\
&\quad + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{19} + x^{17} + x^{16} + x^{15} + x^{13} \\
&\quad + x^{12} + x^{11} + x^9 + x^8, \\
p_{12}(x) &= x^{75} + x^{73} + x^{72} + x^{67} + x^{65} + x^{64} + x^{11} + x^9 + x^8 + x^3 + x + 1,
\end{aligned}$$

and the initial terms are $[1, 0, 0, 0, 1, 0, 1, 1, 0, 0, 1, 0, 1, 0, 1, 0]$.

For $\phi(M^o, 0, 1)$:

$$p_0(x) = x^{84} + x^{78} + x^{77} + x^{76} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{68} + x^{66}$$

$$\begin{aligned}
& + x^{63} + x^{60} + x^{59} + x^{57} + x^{55} + x^{53} + x^{51} + x^{48} + x^{44} + x^{43} + x^{42} \\
& + x^{41} + x^{40} + x^{39} + x^{35} + x^{33}, \\
p_3(x) &= x^{63} + x^{61} + x^{60} + x^{57} + x^{54} + x^{52} + x^{51} + x^{50} + x^{31} + x^{29} + x^{28} \\
& + x^{25} + x^{22} + x^{20} + x^{19} + x^{18}, \\
p_6(x) &= x^{66} + x^{62} + x^{60} + x^{50} + x^{46} + x^{44} + x^{34} + x^{30} + x^{28} + x^{18} + x^{14} \\
& + x^{12}, \\
p_9(x) &= x^{69} + x^{67} + x^{66} + x^{65} + x^{63} + x^{62} + x^{61} + x^{59} + x^{58} + x^{57} + x^{55} \\
& + x^{54} + x^{53} + x^{51} + x^{50} + x^{49} + x^{47} + x^{46} + x^{45} + x^{43} + x^{42} + x^{41} \\
& + x^{39} + x^{38} + x^{37} + x^{35} + x^{34} + x^{33} + x^{31} + x^{30} + x^{29} + x^{27} + x^{26} \\
& + x^{25} + x^{23} + x^{22} + x^{21} + x^{19} + x^{18} + x^{17} + x^{15} + x^{14} + x^{13} + x^{11} \\
& + x^{10} + x^9 + x^7 + x^6, \\
p_{12}(x) &= x^{72} + x^{64} + x^8 + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 1, 1, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1]$.

For $\phi(M^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{96} + x^{90} + x^{89} + x^{85} + x^{84} + x^{83} + x^{81} + x^{74} + x^{73} + x^{68} + x^{65} \\
& + x^{64} + x^{61} + x^{60} + x^{59} + x^{58} + x^{56} + x^{52} + x^{51} + x^{50} + x^{48} + x^{47} \\
& + x^{44} + x^{43} + x^{41} + x^{40} + x^{39} + x^{34} + x^{33}, \\
p_3(x) &= x^{75} + x^{73} + x^{72} + x^{69} + x^{67} + x^{66} + x^{65} + x^{62} + x^{60} + x^{58} + x^{57} \\
& + x^{56} + x^{55} + x^{52} + x^{51} + x^{50} + x^{43} + x^{41} + x^{40} + x^{37} + x^{35} + x^{34} \\
& + x^{33} + x^{30} + x^{28} + x^{26} + x^{25} + x^{24} + x^{23} + x^{20} + x^{19} + x^{18}, \\
p_6(x) &= x^{78} + x^{74} + x^{72} + x^{70} + x^{64} + x^{60} + x^{58} + x^{56} + x^{54} + x^{48} + x^{44} \\
& + x^{42} + x^{40} + x^{38} + x^{32} + x^{28} + x^{26} + x^{24} + x^{22} + x^{16} + x^{14} + x^{12}, \\
p_9(x) &= x^{81} + x^{79} + x^{78} + x^{77} + x^{75} + x^{74} + x^{69} + x^{67} + x^{66} + x^{65} + x^{63} \\
& + x^{62} + x^{61} + x^{59} + x^{58} + x^{57} + x^{55} + x^{54} + x^{53} + x^{51} + x^{50} + x^{49} \\
& + x^{47} + x^{46} + x^{45} + x^{43} + x^{42} + x^{41} + x^{39} + x^{38} + x^{37} + x^{35} + x^{34} \\
& + x^{33} + x^{31} + x^{30} + x^{29} + x^{27} + x^{26} + x^{25} + x^{23} + x^{22} + x^{21} + x^{19} \\
& + x^{18} + x^9 + x^7 + x^6, \\
p_{12}(x) &= x^{84} + x^{72} + x^{68} + x^{64} + x^{20} + x^8 + x^4 + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 0, 0, 0, 1, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0]$.

For $\phi(M^o, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{87} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{78} + x^{76} + x^{73} + x^{72} + x^{71} \\
& + x^{70} + x^{69} + x^{66} + x^{62} + x^{61} + x^{60} + x^{59} + x^{50} + x^{49} + x^{48} + x^{47} \\
& + x^{46} + x^{45} + x^{42}, \\
p_3(x) &= x^{75} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{66} + x^{64} + x^{61} + x^{60} + x^{59} \\
& + x^{58} + x^{54} + x^{43} + x^{41} + x^{40} + x^{39} + x^{37} + x^{36} + x^{34} + x^{32} + x^{29}
\end{aligned}$$

$$\begin{aligned}
& + x^{28} + x^{27} + x^{26} + x^{22}, \\
p_6(x) &= x^{79} + x^{77} + x^{76} + x^{75} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{66} + x^{65} \\
& + x^{63} + x^{61} + x^{58} + x^{56} + x^{49} + x^{48} + x^{46} + x^{43} + x^{41} + x^{40} + x^{39} \\
& + x^{38} + x^{37} + x^{35} + x^{34} + x^{32} + x^{31} + x^{29} + x^{26} + x^{24} + x^{17} + x^{16} \\
& + x^{15} + x^{14} + x^{13} + x^{12} + x^3 + x + 1, \\
p_9(x) &= x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} + x^{59} + x^{57} \\
& + x^{56} + x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44} + x^{43} \\
& + x^{41} + x^{40} + x^{39} + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{31} + x^{29} + x^{28} \\
& + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{19} + x^{17} + x^{16} + x^{15} + x^{13} \\
& + x^{12} + x^{11} + x^9 + x^8, \\
p_{12}(x) &= x^{75} + x^{73} + x^{72} + x^{67} + x^{65} + x^{64} + x^{11} + x^9 + x^8 + x^3 + x + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 0, 1, 0]$.

For $\phi(W^o, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{87} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{79} + x^{78} + x^{74} + x^{70} + x^{69} \\
& + x^{68} + x^{67} + x^{65} + x^{63} + x^{62} + x^{61} + x^{58} + x^{56} + x^{54} + x^{53} + x^{51} \\
& + x^{46} + x^{41} + x^{39} + x^{38} + x^{36} + x^{34} + x^{33} + x^{31} + x^{27} + x^{25} + x^{24}, \\
p_3(x) &= x^{75} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{67} + x^{66} + x^{62} + x^{59} + x^{58} \\
& + x^{57} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{43} + x^{41} + x^{40} + x^{39} + x^{37} \\
& + x^{36} + x^{35} + x^{34} + x^{30} + x^{27} + x^{26} + x^{25} + x^{21} + x^{20} + x^{19} + x^{17} \\
& + x^{16}, \\
p_6(x) &= x^{79} + x^{77} + x^{76} + x^{75} + x^{73} + x^{72} + x^{70} + x^{69} + x^{68} + x^{67} + x^{62} \\
& + x^{61} + x^{59} + x^{58} + x^{57} + x^{55} + x^{52} + x^{51} + x^{50} + x^{48} + x^{47} + x^{38} \\
& + x^{37} + x^{36} + x^{33} + x^{32} + x^{30} + x^{29} + x^{27} + x^{26} + x^{25} + x^{23} + x^{20} \\
& + x^{19} + x^{18} + x^{16} + x^{13} + x^{12} + x^{11} + x^9 + x^8 + x^3 + x + 1, \\
p_9(x) &= x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} + x^{59} + x^{57} \\
& + x^{56} + x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44} + x^{43} \\
& + x^{41} + x^{40} + x^{39} + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{31} + x^{29} + x^{28} \\
& + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{19} + x^{17} + x^{16} + x^{15} + x^{13} \\
& + x^{12} + x^{11} + x^9 + x^8, \\
p_{12}(x) &= x^{75} + x^{73} + x^{72} + x^{67} + x^{65} + x^{64} + x^{11} + x^9 + x^8 + x^3 + x + 1,
\end{aligned}$$

and the initial terms are $[1, 0, 0, 0, 1, 0, 1, 1, 0, 0, 1, 0, 1, 0, 1, 0]$.

For $\phi(W^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{96} + x^{90} + x^{89} + x^{85} + x^{84} + x^{83} + x^{81} + x^{74} + x^{73} + x^{68} + x^{65} \\
& + x^{64} + x^{61} + x^{60} + x^{59} + x^{58} + x^{56} + x^{52} + x^{51} + x^{50} + x^{48} + x^{47} \\
& + x^{44} + x^{43} + x^{41} + x^{40} + x^{39} + x^{34} + x^{33},
\end{aligned}$$

$$\begin{aligned}
p_3(x) &= x^{75} + x^{73} + x^{72} + x^{69} + x^{67} + x^{66} + x^{65} + x^{62} + x^{60} + x^{58} + x^{57} \\
&\quad + x^{56} + x^{55} + x^{52} + x^{51} + x^{50} + x^{43} + x^{41} + x^{40} + x^{37} + x^{35} + x^{34} \\
&\quad + x^{33} + x^{30} + x^{28} + x^{26} + x^{25} + x^{24} + x^{23} + x^{20} + x^{19} + x^{18}, \\
p_6(x) &= x^{78} + x^{74} + x^{72} + x^{70} + x^{64} + x^{60} + x^{58} + x^{56} + x^{54} + x^{48} + x^{44} \\
&\quad + x^{42} + x^{40} + x^{38} + x^{32} + x^{28} + x^{26} + x^{24} + x^{22} + x^{16} + x^{14} + x^{12}, \\
p_9(x) &= x^{81} + x^{79} + x^{78} + x^{77} + x^{75} + x^{74} + x^{69} + x^{67} + x^{66} + x^{65} + x^{63} \\
&\quad + x^{62} + x^{61} + x^{59} + x^{58} + x^{57} + x^{55} + x^{54} + x^{53} + x^{51} + x^{50} + x^{49} \\
&\quad + x^{47} + x^{46} + x^{45} + x^{43} + x^{42} + x^{41} + x^{39} + x^{38} + x^{37} + x^{35} + x^{34} \\
&\quad + x^{33} + x^{31} + x^{30} + x^{29} + x^{27} + x^{26} + x^{25} + x^{23} + x^{22} + x^{21} + x^{19} \\
&\quad + x^{18} + x^9 + x^7 + x^6, \\
p_{12}(x) &= x^{84} + x^{72} + x^{68} + x^{64} + x^{20} + x^8 + x^4 + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 0, 0, 0, 1, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0]$.

For $\phi(W^\circ, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{84} + x^{78} + x^{77} + x^{76} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{68} + x^{66} \\
&\quad + x^{63} + x^{60} + x^{59} + x^{57} + x^{55} + x^{53} + x^{51} + x^{48} + x^{44} + x^{43} + x^{42} \\
&\quad + x^{41} + x^{40} + x^{39} + x^{35} + x^{33}, \\
p_3(x) &= x^{63} + x^{61} + x^{60} + x^{57} + x^{54} + x^{52} + x^{51} + x^{50} + x^{31} + x^{29} + x^{28} \\
&\quad + x^{25} + x^{22} + x^{20} + x^{19} + x^{18}, \\
p_6(x) &= x^{66} + x^{62} + x^{60} + x^{50} + x^{46} + x^{44} + x^{34} + x^{30} + x^{28} + x^{18} + x^{14} \\
&\quad + x^{12}, \\
p_9(x) &= x^{69} + x^{67} + x^{66} + x^{65} + x^{63} + x^{62} + x^{61} + x^{59} + x^{58} + x^{57} + x^{55} \\
&\quad + x^{54} + x^{53} + x^{51} + x^{50} + x^{49} + x^{47} + x^{46} + x^{45} + x^{43} + x^{42} + x^{41} \\
&\quad + x^{39} + x^{38} + x^{37} + x^{35} + x^{34} + x^{33} + x^{31} + x^{30} + x^{29} + x^{27} + x^{26} \\
&\quad + x^{25} + x^{23} + x^{22} + x^{21} + x^{19} + x^{18} + x^{17} + x^{15} + x^{14} + x^{13} + x^{11} \\
&\quad + x^{10} + x^9 + x^7 + x^6, \\
p_{12}(x) &= x^{72} + x^{64} + x^8 + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 1, 1, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1]$.

For $\phi(W^\circ, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{87} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{78} + x^{76} + x^{73} + x^{72} + x^{71} \\
&\quad + x^{70} + x^{69} + x^{66} + x^{62} + x^{61} + x^{60} + x^{59} + x^{50} + x^{49} + x^{48} + x^{47} \\
&\quad + x^{46} + x^{45} + x^{42}, \\
p_3(x) &= x^{75} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{66} + x^{64} + x^{61} + x^{60} + x^{59} \\
&\quad + x^{58} + x^{54} + x^{43} + x^{41} + x^{40} + x^{39} + x^{37} + x^{36} + x^{34} + x^{32} + x^{29} \\
&\quad + x^{28} + x^{27} + x^{26} + x^{22}, \\
p_6(x) &= x^{79} + x^{77} + x^{76} + x^{75} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{66} + x^{65}
\end{aligned}$$

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	13	1	26	1	39	0	52	1	65	0	78	1
1	0	14	0	27	1	40	1	53	1	66	1	79	1
2	0	15	1	28	1	41	1	54	0	67	0	80	1
3	0	16	0	29	0	42	0	55	0	68	1	81	0
4	0	17	0	30	0	43	0	56	1	69	1	82	1
5	0	18	0	31	0	44	0	57	0	70	1	83	1

6	0	19	0	32	0	45	1	58	1	71	0	84	1
7	0	20	0	33	0	46	0	59	0	72	1	85	0
8	0	21	1	34	0	47	0	60	0	73	0		
9	0	22	1	35	1	48	1	61	1	74	1		
10	0	23	0	36	0	49	1	62	1	75	0		
11	1	24	1	37	1	50	0	63	1	76	1		
12	0	25	0	38	1	51	1	64	1	77	1		

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