# PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a,b)=(z^4+z^3+z^2+z, z^3+1)$

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Abstract. The continued fraction conjecture claims that the continued fraction defined by the $(a,b)$-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements $(a,b)$ in $\mathbb{F}_2[x]\setminus\mathbb{F}_2$. In this paper we prove the conjecture for $(a,b)=(z^4+z^3+z^2+z, z^3+1)$.

## 1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a}=(a_0(z),a_1(z),a_2(z),\ldots)$ of elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z))=\cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{a_2(z)+\cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z))=\cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{\ddots+\cfrac{1}{a_n(z)}}}}\in\mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1}+c_{-2}z^{-2}+c_{-3}z^{-3}+\cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let $a$ and $b$ be two distinct elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$. We consider the continued fraction (1.1) associated with the $(a,b)$-Thue-Morse sequence

$$\mathbf{t}=(t_0(z),t_1(z),t_2(z),\ldots)=(a,b,b,a,b,a,a,b,\ldots)$$

that we shall denote

$$\mathrm{CF}(a,b):=\mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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**Theorem 1.1.** *When* $(a,b)=(z^4+z^3+z^2+z, z^3+1)$*, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

*where, for* $\mathrm{CF}(a,b)$*,*

$$\begin{aligned}
p_0(z)&=z^{17}+z^{16}+z^{14}+z^{12}+z^8+z^7+z^3+z+1,\\
p_1(z)&=z^{21}+z^{20}+z^{19}+z^{18}+z^{16}+z^{15}+z^{14}+z^{13}+z^{12}+z^6+z^3+z,\\
p_2(z)&=z^{24}+z^{18}+z^{14}+z^{10}+z^4+z^2,\\
p_3(z)&=z^{21}+z^{20}+z^{19}+z^{18}+z^{16}+z^{15}+z^{14}+z^{13}+z^{12}+z^6+z^3+z,\\
p_4(z)&=z^{18}+z^{17}+z^{12}+z^{10}+z^7+z^3+z^2+z,
\end{aligned}$$

*and for* $\mathrm{CF}(b,a)$*,*

$$\begin{aligned}
p_0(z)&=z^{17}+z^{16}+z^{14}+z^7+z^3+z,\\
p_1(z)&=z^{21}+z^{20}+z^{19}+z^{18}+z^{16}+z^{15}+z^{14}+z^{13}+z^{12}+z^6+z^3+z,\\
p_2(z)&=z^{24}+z^{18}+z^{14}+z^{10}+z^4+z^2,\\
p_3(z)&=z^{21}+z^{20}+z^{19}+z^{18}+z^{16}+z^{15}+z^{14}+z^{13}+z^{12}+z^6+z^3+z,\\
p_4(z)&=z^{18}+z^{17}+z^{10}+z^8+z^7+z^3+z^2+z+1.
\end{aligned}$$

## 2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x)&=x^{\deg(t_{2^n-1})}\begin{pmatrix} t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x)&=x^{\deg(\bar t_{2^n-1})}\begin{pmatrix} \bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix} \bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(\bar t_0)}\begin{pmatrix} \bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the $(b,a)$-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\geq 0$

$$\begin{aligned}
M_{n+1}(x)&=W_n(x)\cdot M_n(x),\\
W_{n+1}(x)&=M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer $d_n$. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four $2\times 2$ matrices $M^e, M^o, W^e, W^o$ as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(175, 175, 175, 175, 175)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices $M^e, M^o, W^e, W^o$. The first one is about relations between them; the second, about the structure of the each matrix.

**Lemma 2.1.** *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

*Proof.* We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z\left(\phi(W^o,0,0)(x,z),\ z^{12}\cdot\phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity $m$ and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 630 and substitute it for $y$ in $Q(x,y)$. We get a series of valuation less than 630, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z\left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity $\mu$, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ to order 770 and substitute it for $y$ in $Q(x,y)$. We get a series of valuation less than 770, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. $\square$

Define

$$R^e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

**Lemma 2.2.** *For all integers $k \ge 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned} M^e[:14u] = {} & M^e[:7u] + x^u M^e[:6u] + x^{2u} M^e[:5u] + x^{4u} M^e[:3u] + x^{5u} M^e[:2u] \\ & + x^u M^e[:7u] + x^{2u} M^e[:6u] + x^{3u} M^e[:5u] + x^{5u} M^e[:3u] \\ & + x^{6u} M^e[:2u] + x^{3u} M^e[:7u] + x^{4u} M^e[:6u] + x^{5u} M^e[:5u] \\ & + x^{7u} M^e[:3u] + x^{8u} M^e[:2u] + x^{5u} M^e[:7u] + x^{6u} M^e[:6u] \\ & + x^{7u} M^e[:5u] + x^{9u} M^e[:3u] + x^{10u} M^e[:2u] + x^{7u} M^e[:7u] \\ & + x^{11u} M^e[:3u] + x^{12u} M^e[:2u] + x^{13u} M^e[:u] \end{aligned}$$

$$
\begin{aligned}
&+ (x^{7u} + x^{8u} + x^{10u} + x^{12u})R^e,\\
W^e[:14u] = {}& W^e[:7u] + x^u W^e[:6u] + x^{2u} W^e[:5u] + x^{4u} W^e[:3u] + x^{5u} W^e[:2u]\\
&+ x^u W^e[:7u] + x^{2u} W^e[:6u] + x^{3u} W^e[:5u] + x^{5u} W^e[:3u]\\
&+ x^{6u} W^e[:2u] + x^{3u} W^e[:7u] + x^{4u} W^e[:6u] + x^{5u} W^e[:5u]\\
&+ x^{7u} W^e[:3u] + x^{8u} W^e[:2u] + x^{5u} W^e[:7u] + x^{6u} W^e[:6u]\\
&+ x^{7u} W^e[:5u] + x^{9u} W^e[:3u] + x^{10u} W^e[:2u] + x^{7u} W^e[:7u]\\
&+ x^{11u} W^e[:3u] + x^{12u} W^e[:2u] + x^{13u} W^e[:u]\\
&+ (x^{7u} + x^{8u} + x^{10u} + x^{12u})R^e.
\end{aligned}
$$

*For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.*

$$
\begin{aligned}
M^o[:14u] = {}& M^o[:7u] + x^u M^o[:6u] + x^{2u} M^o[:5u] + x^{4u} M^o[:3u] + x^{5u} M^o[:2u]\\
&+ x^u M^o[:7u] + x^{2u} M^o[:6u] + x^{3u} M^o[:5u] + x^{5u} M^o[:3u]\\
&+ x^{6u} M^o[:2u] + x^{3u} M^o[:7u] + x^{4u} M^o[:6u] + x^{5u} M^o[:5u]\\
&+ x^{7u} M^o[:3u] + x^{8u} M^o[:2u] + x^{5u} M^o[:7u] + x^{6u} M^o[:6u]\\
&+ x^{7u} M^o[:5u] + x^{9u} M^o[:3u] + x^{10u} M^o[:2u] + x^{7u} M^o[:7u]\\
&+ x^{11u} M^o[:3u] + x^{12u} M^o[:2u] + x^{13u} M^o[:u]\\
&+ (x^{7u} + x^{8u} + x^{10u} + x^{12u})R^o,\\
W^o[:14u] = {}& W^o[:7u] + x^u W^o[:6u] + x^{2u} W^o[:5u] + x^{4u} W^o[:3u] + x^{5u} W^o[:2u]\\
&+ x^u W^o[:7u] + x^{2u} W^o[:6u] + x^{3u} W^o[:5u] + x^{5u} W^o[:3u]\\
&+ x^{6u} W^o[:2u] + x^{3u} W^o[:7u] + x^{4u} W^o[:6u] + x^{5u} W^o[:5u]\\
&+ x^{7u} W^o[:3u] + x^{8u} W^o[:2u] + x^{5u} W^o[:7u] + x^{6u} W^o[:6u]\\
&+ x^{7u} W^o[:5u] + x^{9u} W^o[:3u] + x^{10u} W^o[:2u] + x^{7u} W^o[:7u]\\
&+ x^{11u} W^o[:3u] + x^{12u} W^o[:2u] + x^{13u} W^o[:u]\\
&+ (x^{7u} + x^{8u} + x^{10u} + x^{12u})R^o.
\end{aligned}
$$

*Proof.* To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many $k$'s into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$
\begin{aligned}
T[:14u] = {}& T[:7u] + x^u T[:6u] + x^{2u} T[:5u] + x^{4u} T[:3u] + x^{5u} T[:2u] + x^u T[:7u]\\
&+ x^{2u} T[:6u] + x^{3u} T[:5u] + x^{5u} T[:3u] + x^{6u} T[:2u] + x^{3u} T[:7u]\\
&+ x^{4u} T[:6u] + x^{5u} T[:5u] + x^{7u} T[:3u] + x^{8u} T[:2u] + x^{5u} T[:7u]\\
&+ x^{6u} T[:6u] + x^{7u} T[:5u] + x^{9u} T[:3u] + x^{10u} T[:2u] + x^{7u} T[:7u]\\
&+ x^{11u} T[:3u] + x^{12u} T[:2u] + x^{13u} T[:u].
\end{aligned}
$$

The proof of the other 15 cases are similar. We break down the above identity into 7 parts:

$$
\begin{aligned}
0 = {}& x^{7u} T[:u] + x^{5u} T[2u:3u] + x^{4u} T[3u:4u] + x^{2u} T[5u:6u]\\
&+ x^u T[6u:7u] + T[7u:8u],
\end{aligned}
$$

$$
\begin{aligned}
0 &= x^{8u}T[:u] + x^{7u}T[u:2u] + x^{6u}T[2u:3u] + x^{4u}T[4u:5u] \\
&\quad + x^{3u}T[5u:6u] + T[8u:9u], \\
0 &= x^{9u}T[:u] + x^{8u}T[u:2u] + x^{7u}T[2u:3u] + x^{6u}T[3u:4u] \\
&\quad + x^{4u}T[5u:6u] + x^{3u}T[6u:7u] + T[9u:10u], \\
0 &= x^{10u}T[:u] + x^{9u}T[u:2u] + x^{6u}T[4u:5u] + x^{5u}T[5u:6u] \\
&\quad + T[10u:11u], \\
0 &= x^{11u}T[:u] + x^{10u}T[u:2u] + x^{9u}T[2u:3u] + x^{6u}T[5u:6u] \\
&\quad + x^{5u}T[6u:7u] + T[11u:12u], \\
0 &= x^{12u}T[:u] + x^{11u}T[u:2u] + x^{7u}T[5u:6u] + T[12u:13u], \\
0 &= x^{13u}T[:u] + x^{12u}T[u:2u] + x^{11u}T[2u:3u] + x^{7u}T[6u:7u] \\
&\quad + T[13u:14u],
\end{aligned}
$$

which can be rewritten as

$$
\begin{aligned}
0 &= T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2] \\
&\quad + T[[111w]_2], \tag{2.7} \\
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[100w]_2] + T[[101w]_2] \\
&\quad + T[[1000w]_2], \tag{2.8} \\
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] \\
&\quad + T[[110w]_2] + T[[1001w]_2], \tag{2.9} \\
0 &= T[[w]_2] + T[[1w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1010w]_2], \tag{2.10} \\
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[101w]_2] + T[[110w]_2] \\
&\quad + T[[1011w]_2], \tag{2.11} \\
0 &= T[[w]_2] + T[[1w]_2] + T[[101w]_2] + T[[1100w]_2], \tag{2.12} \\
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[110w]_2] + T[[1101w]_2], \tag{2.13}
\end{aligned}
$$

for all binary word $w$ of length $2k$ and $w \neq 0^{2k}$, and

$$
\begin{aligned}
0 &= T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2] \\
&\quad + T[[111w]_2], \tag{2.14} \\
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[100w]_2] + T[[101w]_2] \\
&\quad + T[[1000w]_2], \tag{2.15} \\
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] \\
&\quad + T[[110w]_2] + T[[1001w]_2], \tag{2.16} \\
0 &= T[[w]_2] + T[[1w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1010w]_2], \tag{2.17} \\
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[101w]_2] + T[[110w]_2] \\
&\quad + T[[1011w]_2], \tag{2.18} \\
0 &= T[[w]_2] + T[[1w]_2] + T[[101w]_2] + T[[1100w]_2], \tag{2.19} \\
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[110w]_2] + T[[1101w]_2], \tag{2.20}
\end{aligned}
$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates $T$ from its minimal polynomial and its first terms. This automaton has 978 states; its transition function and output function can be found in the annex. Let $A(s,w)$ denote the state reached after reading $w$ from right to left starting from the state $s$, and $\tau$ the output function. Define

$$E_{2k} = \{A(i,w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.20) can be written as

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101)) \\ &\quad + \tau(A(s,110)) + \tau(A(s,111)), \end{aligned} \tag{2.21}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,100)) \\ &\quad + \tau(A(s,101)) + \tau(A(s,1000)), \end{aligned} \tag{2.22}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101)) \\ &\quad + \tau(A(s,110)) + \tau(A(s,1001)), \end{aligned} \tag{2.23}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,100)) + \tau(A(s,101)) \\ &\quad + \tau(A(s,1010)), \end{aligned} \tag{2.24}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,101)) \\ &\quad + \tau(A(s,110)) + \tau(A(s,1011)), \end{aligned} \tag{2.25}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,101)) + \tau(A(s,1100)), \tag{2.26}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,110)) \\ &\quad + \tau(A(s,1101)), \end{aligned} \tag{2.27}$$

for all $s \in E_{2k}$, and

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101)) \\ &\quad + \tau(A(s,110)) + \tau(A(s,111)), \end{aligned} \tag{2.28}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,100)) \\ &\quad + \tau(A(s,101)) + \tau(A(s,1000)), \end{aligned} \tag{2.29}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101)) \\ &\quad + \tau(A(s,110)) + \tau(A(s,1001)), \end{aligned} \tag{2.30}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,100)) + \tau(A(s,101)) \\ &\quad + \tau(A(s,1010)), \end{aligned} \tag{2.31}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,101)) \\ &\quad + \tau(A(s,110)) + \tau(A(s,1011)), \end{aligned} \tag{2.32}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,101)) + \tau(A(s,1100)), \tag{2.33}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,110)) \\ &\quad + \tau(A(s,1101)), \end{aligned} \tag{2.34}$$

for $s = A(i, 0^{2k})$. We find that $(A(i,0^{20}), E_{20}) = (A(i,0^{18}), E_{18})$, so that we only have to verify that identities (2.21) through (2.34) hold for $2 \leq k \leq 10$, which turns out to be true. □

In the following lemma, we express $M_{2k}$, $M_{2k+1}$, $W_{2k}$, and $W_{2k+1}$ in terms of $M^e$, $M^o$, $W^e$, and $W^o$.

**Lemma 2.3.** *For all integer $k \geq 2$, and $u = 2^{2k-1}$,*

$$\begin{aligned} M_{2k} &= M^e[:7u] + x^u M^e[:6u] + x^{2u} M^e[:5u] + x^{4u} M^e[:3u] + x^{5u} M^e[:2u] \\ &\quad + x^{7u} R^e, \\ W_{2k} &= W^e[:7u] + x^u W^e[:6u] + x^{2u} W^e[:5u] + x^{4u} W^e[:3u] + x^{5u} W^e[:2u] \\ &\quad + x^{7u} R^e. \end{aligned}$$

*For all integer $k \geq 2$, and $u = 2^{2k}$,*

$$\begin{aligned} M_{2k+1} &= M^o[:7u] + x^u M^o[:6u] + x^{2u} M^o[:5u] + x^{4u} M^o[:3u] + x^{5u} M^o[:2u] \\ &\quad + x^{7u} R^o, \\ W_{2k+1} &= W^o[:7u] + x^u W^o[:6u] + x^{2u} W^o[:5u] + x^{4u} W^o[:3u] + x^{5u} W^o[:2u] \\ &\quad + x^{7u} R^o. \end{aligned}$$

*Proof.* We call the four identities in Lemma 2.3 also by the name $M_{2k}$, $W_{2k}$, $M_{2k+1}$, and $W_{2k+1}$. For $n = 2$, the identities can be verified directly. For $n \geq 2$, we claim that

$$\begin{aligned} M_{2k} \wedge W_{2k} &\Rightarrow M_{2k+1} \wedge W_{2k+1}, \\ M_{2k+1} \wedge W_{2k+1} &\Rightarrow M_{2k+2} \wedge W_{2k+2}. \end{aligned}$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity $M_{2k+1}$ is equal to

$$\begin{aligned} W_{2k} M_{2k} = &\big(W^e[:7v] + x^v W^e[:6v] + x^{2v} W^e[:5v] + x^{4v} W^e[:3v] \\ &+ x^{5v} W^e[:2v] + x^{7u} R^e\big) \times \big(M^e[:7v] + x^v M^e[:6v] + x^{2v} M^e[:5v] \\ &+ x^{4v} M^e[:3v] + x^{5v} M^e[:2v] + x^{7u} R^e\big); \end{aligned} \tag{2.35}$$

Call this expression $lhs$. Note that both sides of identity $M_{2k+1}$ have the same term of highest degree $x^{14v} R^o$. Therefore we only need to prove that their difference is $O(x^{14v})$. Using Lemma 2.1 it can be seen that the right side of identity $M_{2k+1}$ is congruent, modulo $x^{14v}$, to

$$\begin{aligned} &W^e[:14v] M^e[:14v] + x^{2v} W^e[:12v] M^e[:12v] + x^{4v} W^e[:10v] M^e[:10v] \\ &\qquad + x^{8v} W^e[:6v] M^e[:6v] + x^{10v} W^e[:4v] M^e[:4v]. \end{aligned}$$

For all $n \leq 14$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call $rhs$. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v] / X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v] / X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions $lhs$ (2.35) and $rhs$ as polynomials in $\mathbb{F}_2[a_1, ..., a_{14}, b_1, ..., b_{14}, c][X]$. Note that it is not a problem that $a_j$ commutes with $b_k$ while $W^e$ does not commute with $M^e$, because in the expressions concerned, the products of $W^e$-terms and $M^e$-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{14})$, which completes the proof. $\square$

*Proof of Theorem 1.1.* We prove the theorem for $\mathrm{CF}(a,b)$; for $\mathrm{CF}(b,a)$, the proof is similar. By Lemma 2.3, we have For all $0 \le j,k \le 1$,

$$\begin{aligned}
\lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k},\\
\lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k},\\
\lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k},\\
\lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}.
\end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a,b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e,0,1)$ and $\phi(M^e,0,0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x,y) = \mathrm{Res}_t\left(\phi(M^e,0,1)(x,t), y^{12}\phi(M^e,0,1)(x,t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x,y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned}
q_0(x) &= x^{24} + x^{23} + x^{21} + x^{17} + x^{16} + x^{12} + x^{10} + x^8 + x^7,\\
q_1(x) &= x^{23} + x^{21} + x^{18} + x^{12} + x^{11} + x^{10} + x^9 + x^8 + x^6 + x^5 + x^4 + x^3,\\
q_2(x) &= x^{22} + x^{20} + x^{14} + x^{10} + x^6 + 1,\\
q_3(x) &= x^{23} + x^{21} + x^{18} + x^{12} + x^{11} + x^{10} + x^9 + x^8 + x^6 + x^5 + x^4 + x^3,\\
q_4(x) &= x^{23} + x^{22} + x^{21} + x^{17} + x^{14} + x^{12} + x^7 + x^6.
\end{aligned}$$

The polynomial $Q(x,y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x,y)$ of multiplicity $m$ and $R(x,y) := P(x,y)/Q(x,y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 216, and substitute it for $y$ in $R(z,y)$, we get a series with valuation smaller than 216, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b) = f(1/z)$. $\square$

## 3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e,0,0)$:

$$p_0(x) = x^{126} + x^{124} + x^{122} + x^{112} + x^{106} + x^{104} + x^{100} + x^{96} + x^{90} + x^{86}$$

$$
\begin{aligned}
&+ x^{84} + x^{82} + x^{80} + x^{78} + x^{74} + x^{72} + x^{70} + x^{66} + x^{60} + x^{58} + x^{56}\\
&+ x^{54} + x^{48} + x^{46} + x^{44} + x^{38} + x^{36},\\
p_3(x) &= x^{120} + x^{116} + x^{114} + x^{110} + x^{106} + x^{102} + x^{100} + x^{98} + x^{96} + x^{94}\\
&+ x^{92} + x^{84} + x^{82} + x^{80} + x^{76} + x^{74} + x^{72} + x^{70} + x^{58} + x^{56} + x^{48}\\
&+ x^{46} + x^{42} + x^{40} + x^{36} + x^{32} + x^{30} + x^{28} + x^{26} + x^{20},\\
p_6(x) &= x^{122} + x^{118} + x^{116} + x^{110} + x^{108} + x^{106} + x^{104} + x^{102} + x^{100} + x^{96}\\
&+ x^{92} + x^{90} + x^{88} + x^{82} + x^{80} + x^{78} + x^{70} + x^{68} + x^{66} + x^{64} + x^{62}\\
&+ x^{58} + x^{54} + x^{52} + x^{48} + x^{46} + x^{42} + x^{38} + x^{24} + x^{18} + x^{16} + x^{12}\\
&+ x^{8} + x^{6} + x^{4} + 1,\\
p_9(x) &= x^{124} + x^{122} + x^{112} + x^{104} + x^{98} + x^{92} + x^{90} + x^{86} + x^{82} + x^{76} + x^{72}\\
&+ x^{68} + x^{66} + x^{56} + x^{54} + x^{52} + x^{50} + x^{48} + x^{46} + x^{42} + x^{40} + x^{36}\\
&+ x^{34} + x^{32} + x^{28} + x^{26} + x^{22} + x^{18} + x^{14} + x^{12} + x^{8} + x^{6},\\
p_{12}(x) &= x^{124} + x^{120} + x^{108} + x^{104} + x^{100} + x^{96} + x^{92} + x^{88} + x^{84} + x^{80} + x^{68}\\
&+ x^{64} + x^{60} + x^{56} + x^{36} + x^{32} + x^{28} + x^{24} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0].

For $\phi(M^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{129} + x^{128} + x^{126} + x^{125} + x^{124} + x^{122} + x^{115} + x^{114} + x^{112} + x^{111}\\
&+ x^{110} + x^{108} + x^{105} + x^{104} + x^{102} + x^{98} + x^{97} + x^{96} + x^{94} + x^{93}\\
&+ x^{89} + x^{86} + x^{84} + x^{83} + x^{81} + x^{79} + x^{78} + x^{77} + x^{73} + x^{70} + x^{69}\\
&+ x^{68} + x^{67} + x^{66} + x^{62} + x^{61} + x^{60} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53}\\
&+ x^{52} + x^{51} + x^{48} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39},\\
p_3(x) &= x^{127} + x^{126} + x^{125} + x^{122} + x^{121} + x^{119} + x^{117} + x^{116} + x^{115} + x^{112}\\
&+ x^{111} + x^{109} + x^{107} + x^{106} + x^{105} + x^{103} + x^{101} + x^{100} + x^{99} + x^{96}\\
&+ x^{95} + x^{93} + x^{91} + x^{90} + x^{89} + x^{87} + x^{85} + x^{84} + x^{83} + x^{80} + x^{79}\\
&+ x^{77} + x^{75} + x^{74} + x^{73} + x^{70} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63}\\
&+ x^{62} + x^{61} + x^{60} + x^{56} + x^{54} + x^{51} + x^{47} + x^{46} + x^{45} + x^{42} + x^{41}\\
&+ x^{39} + x^{37} + x^{36} + x^{35} + x^{32} + x^{30} + x^{27},\\
p_6(x) &= x^{126} + x^{125} + x^{123} + x^{122} + x^{118} + x^{117} + x^{116} + x^{110} + x^{109} + x^{108}\\
&+ x^{100} + x^{99} + x^{97} + x^{95} + x^{94} + x^{88} + x^{87} + x^{86} + x^{74} + x^{73} + x^{71}\\
&+ x^{70} + x^{66} + x^{65} + x^{64} + x^{58} + x^{57} + x^{56} + x^{54} + x^{53} + x^{51} + x^{49}\\
&+ x^{48} + x^{44} + x^{43} + x^{41} + x^{40} + x^{38} + x^{37} + x^{35} + x^{34} + x^{32} + x^{31}\\
&+ x^{29} + x^{28} + x^{20} + x^{19} + x^{18},\\
p_9(x) &= x^{125} + x^{124} + x^{123} + x^{121} + x^{117} + x^{116} + x^{115} + x^{113} + x^{111} + x^{110}\\
&+ x^{108} + x^{107} + x^{106} + x^{102} + x^{100} + x^{99} + x^{98} + x^{95} + x^{91} + x^{90}\\
&+ x^{88} + x^{85} + x^{77} + x^{76} + x^{74} + x^{71} + x^{67} + x^{66} + x^{64} + x^{63} + x^{62}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{61} + x^{60} + x^{56} + x^{54} + x^{50} + x^{48} + x^{45} + x^{41} + x^{40} + x^{38} + x^{34}\\
&+ x^{32} + x^{31} + x^{30} + x^{27} + x^{21} + x^{20} + x^{18} + x^{15} + x^{13} + x^{12} + x^{11}\\
&+ x^{9},\\
p_{12}(x) &= x^{124} + x^{123} + x^{121} + x^{119} + x^{117} + x^{115} + x^{113} + x^{112} + x^{76} + x^{75}\\
&+ x^{73} + x^{71} + x^{69} + x^{67} + x^{65} + x^{64} + x^{60} + x^{59} + x^{57} + x^{55} + x^{53}\\
&+ x^{51} + x^{49} + x^{48} + x^{44} + x^{43} + x^{41} + x^{39} + x^{37} + x^{35} + x^{33} + x^{32}\\
&+ x^{28} + x^{27} + x^{25} + x^{23} + x^{21} + x^{19} + x^{17} + x^{16} + x^{12} + x^{11} + x^{9}\\
&+ x^{7} + x^{5} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 1, 0, 0, 1, 1, 1, 0, 1, 0, 0, 1].

For $\phi(M^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{129} + x^{128} + x^{126} + x^{125} + x^{124} + x^{122} + x^{115} + x^{114} + x^{112} + x^{111}\\
&+ x^{110} + x^{108} + x^{105} + x^{104} + x^{102} + x^{98} + x^{97} + x^{96} + x^{94} + x^{93}\\
&+ x^{89} + x^{86} + x^{84} + x^{83} + x^{81} + x^{79} + x^{78} + x^{77} + x^{73} + x^{70} + x^{69}\\
&+ x^{68} + x^{67} + x^{66} + x^{62} + x^{61} + x^{60} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53}\\
&+ x^{52} + x^{51} + x^{48} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39},\\
p_3(x) &= x^{127} + x^{126} + x^{125} + x^{122} + x^{121} + x^{119} + x^{117} + x^{116} + x^{115} + x^{112}\\
&+ x^{111} + x^{109} + x^{107} + x^{106} + x^{105} + x^{103} + x^{101} + x^{100} + x^{99} + x^{96}\\
&+ x^{95} + x^{93} + x^{91} + x^{90} + x^{89} + x^{87} + x^{85} + x^{84} + x^{83} + x^{80} + x^{79}\\
&+ x^{77} + x^{75} + x^{74} + x^{73} + x^{70} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63}\\
&+ x^{62} + x^{61} + x^{60} + x^{56} + x^{54} + x^{51} + x^{47} + x^{46} + x^{45} + x^{42} + x^{41}\\
&+ x^{39} + x^{37} + x^{36} + x^{35} + x^{32} + x^{30} + x^{27},\\
p_6(x) &= x^{126} + x^{125} + x^{123} + x^{122} + x^{118} + x^{117} + x^{116} + x^{110} + x^{109} + x^{108}\\
&+ x^{100} + x^{99} + x^{97} + x^{95} + x^{94} + x^{88} + x^{87} + x^{86} + x^{74} + x^{73} + x^{71}\\
&+ x^{70} + x^{66} + x^{65} + x^{64} + x^{58} + x^{57} + x^{56} + x^{54} + x^{53} + x^{51} + x^{49}\\
&+ x^{48} + x^{44} + x^{43} + x^{41} + x^{40} + x^{38} + x^{37} + x^{35} + x^{34} + x^{32} + x^{31}\\
&+ x^{29} + x^{28} + x^{20} + x^{19} + x^{18},\\
p_9(x) &= x^{125} + x^{124} + x^{123} + x^{121} + x^{117} + x^{116} + x^{115} + x^{113} + x^{111} + x^{110}\\
&+ x^{108} + x^{107} + x^{106} + x^{102} + x^{100} + x^{99} + x^{98} + x^{95} + x^{91} + x^{90}\\
&+ x^{88} + x^{85} + x^{77} + x^{76} + x^{74} + x^{71} + x^{67} + x^{66} + x^{64} + x^{63} + x^{62}\\
&+ x^{61} + x^{60} + x^{56} + x^{54} + x^{50} + x^{48} + x^{45} + x^{41} + x^{40} + x^{38} + x^{34}\\
&+ x^{32} + x^{31} + x^{30} + x^{27} + x^{21} + x^{20} + x^{18} + x^{15} + x^{13} + x^{12} + x^{11}\\
&+ x^{9},\\
p_{12}(x) &= x^{124} + x^{123} + x^{121} + x^{119} + x^{117} + x^{115} + x^{113} + x^{112} + x^{76} + x^{75}\\
&+ x^{73} + x^{71} + x^{69} + x^{67} + x^{65} + x^{64} + x^{60} + x^{59} + x^{57} + x^{55} + x^{53}\\
&+ x^{51} + x^{49} + x^{48} + x^{44} + x^{43} + x^{41} + x^{39} + x^{37} + x^{35} + x^{33} + x^{32}
\end{aligned}
$$

$$+ x^{28} + x^{27} + x^{25} + x^{23} + x^{21} + x^{19} + x^{17} + x^{16} + x^{12} + x^{11} + x^{9} + x^{7} + x^{5} + x^{3} + x + 1,$$

and the initial terms are [0, 0, 0, 0, 1, 1, 0, 0, 1, 1, 1, 0, 1, 0, 0, 1].

For $\phi(M^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{132} + x^{128} + x^{122} + x^{120} + x^{116} + x^{110} + x^{108} + x^{106} + x^{104} + x^{102} \\
&\quad + x^{96} + x^{94} + x^{90} + x^{86} + x^{80} + x^{78} + x^{76} + x^{72} + x^{68} + x^{62} + x^{56} \\
&\quad + x^{46} + x^{42}, \\
p_3(x) &= x^{126} + x^{124} + x^{112} + x^{110} + x^{106} + x^{102} + x^{96} + x^{94} + x^{92} + x^{86} \\
&\quad + x^{82} + x^{78} + x^{76} + x^{66} + x^{64} + x^{58} + x^{54} + x^{52} + x^{48} + x^{44} + x^{42} \\
&\quad + x^{40} + x^{34} + x^{26} + x^{24} + x^{22} + x^{20} + x^{18}, \\
p_6(x) &= x^{126} + x^{118} + x^{116} + x^{114} + x^{104} + x^{102} + x^{98} + x^{96} + x^{94} + x^{88} \\
&\quad + x^{86} + x^{84} + x^{80} + x^{76} + x^{68} + x^{64} + x^{60} + x^{58} + x^{52} + x^{50} + x^{48} \\
&\quad + x^{46} + x^{36} + x^{30} + x^{26} + x^{24} + x^{16} + x^{8} + x^{6} + x^{4} + x^{2} + 1, \\
p_9(x) &= x^{124} + x^{120} + x^{118} + x^{114} + x^{112} + x^{110} + x^{108} + x^{106} + x^{104} + x^{100} \\
&\quad + x^{98} + x^{90} + x^{82} + x^{80} + x^{78} + x^{76} + x^{68} + x^{56} + x^{54} + x^{52} + x^{46} \\
&\quad + x^{40} + x^{34} + x^{30} + x^{28} + x^{26} + x^{22} + x^{20} + x^{10} + x^{6}, \\
p_{12}(x) &= x^{124} + x^{122} + x^{120} + x^{116} + x^{114} + x^{112} + x^{76} + x^{74} + x^{72} + x^{68} \\
&\quad + x^{66} + x^{64} + x^{60} + x^{58} + x^{56} + x^{52} + x^{50} + x^{48} + x^{44} + x^{42} + x^{40} \\
&\quad + x^{36} + x^{34} + x^{32} + x^{28} + x^{26} + x^{24} + x^{20} + x^{18} + x^{16} + x^{12} + x^{10} \\
&\quad + x^{8} + x^{4} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{132} + x^{126} + x^{124} + x^{116} + x^{114} + x^{108} + x^{106} + x^{102} + x^{100} + x^{98} \\
&\quad + x^{90} + x^{76} + x^{74} + x^{58} + x^{56} + x^{52} + x^{50} + x^{48} + x^{46} + x^{44} + x^{42} \\
&\quad + x^{40} + x^{36}, \\
p_3(x) &= x^{126} + x^{124} + x^{122} + x^{120} + x^{118} + x^{106} + x^{104} + x^{102} + x^{100} + x^{98} \\
&\quad + x^{94} + x^{92} + x^{90} + x^{84} + x^{78} + x^{72} + x^{70} + x^{68} + x^{58} + x^{56} + x^{52} \\
&\quad + x^{50} + x^{48} + x^{44} + x^{36} + x^{32} + x^{30} + x^{28} + x^{22} + x^{20}, \\
p_6(x) &= x^{126} + x^{122} + x^{118} + x^{114} + x^{112} + x^{110} + x^{102} + x^{98} + x^{96} + x^{94} \\
&\quad + x^{92} + x^{88} + x^{86} + x^{68} + x^{66} + x^{54} + x^{52} + x^{44} + x^{42} + x^{38} + x^{28} \\
&\quad + x^{20} + x^{16} + x^{14} + x^{12} + x^{8} + x^{6} + x^{4} + x^{2} + 1, \\
p_9(x) &= x^{124} + x^{120} + x^{118} + x^{114} + x^{112} + x^{110} + x^{108} + x^{106} + x^{104} + x^{100} \\
&\quad + x^{98} + x^{90} + x^{82} + x^{80} + x^{78} + x^{76} + x^{68} + x^{56} + x^{54} + x^{52} + x^{46} \\
&\quad + x^{40} + x^{34} + x^{30} + x^{28} + x^{26} + x^{22} + x^{20} + x^{10} + x^{6}, \\
p_{12}(x) &= x^{124} + x^{122} + x^{120} + x^{116} + x^{114} + x^{112} + x^{76} + x^{74} + x^{72} + x^{68}
\end{aligned}$$

$$+ x^{66} + x^{64} + x^{60} + x^{58} + x^{56} + x^{52} + x^{50} + x^{48} + x^{44} + x^{42} + x^{40} + x^{36} + x^{34} + x^{32} + x^{28} + x^{26} + x^{24} + x^{20} + x^{18} + x^{16} + x^{12} + x^{10} + x^{8} + x^{4} + x^{2} + 1,$$

and the initial terms are [1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0].

For $\phi(W^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{129} + x^{126} + x^{125} + x^{123} + x^{118} + x^{116} + x^{115} + x^{110} + x^{108} + x^{107} \\
&+ x^{105} + x^{104} + x^{102} + x^{100} + x^{98} + x^{97} + x^{96} + x^{95} + x^{92} + x^{89} \\
&+ x^{88} + x^{84} + x^{83} + x^{82} + x^{76} + x^{75} + x^{74} + x^{73} + x^{70} + x^{69} + x^{67} \\
&+ x^{66} + x^{64} + x^{63} + x^{61} + x^{59} + x^{57} + x^{56} + x^{55} + x^{53} + x^{51} + x^{50} \\
&+ x^{46} + x^{45} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39}, \\
p_3(x) &= x^{127} + x^{126} + x^{125} + x^{122} + x^{121} + x^{119} + x^{117} + x^{116} + x^{115} + x^{112} \\
&+ x^{111} + x^{109} + x^{107} + x^{106} + x^{105} + x^{103} + x^{101} + x^{100} + x^{99} + x^{96} \\
&+ x^{95} + x^{93} + x^{91} + x^{90} + x^{89} + x^{87} + x^{85} + x^{84} + x^{83} + x^{80} + x^{79} \\
&+ x^{77} + x^{75} + x^{74} + x^{73} + x^{70} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} \\
&+ x^{62} + x^{61} + x^{60} + x^{56} + x^{54} + x^{51} + x^{47} + x^{46} + x^{45} + x^{42} + x^{41} \\
&+ x^{39} + x^{37} + x^{36} + x^{35} + x^{32} + x^{30} + x^{27}, \\
p_6(x) &= x^{126} + x^{125} + x^{123} + x^{122} + x^{118} + x^{117} + x^{116} + x^{110} + x^{109} + x^{108} \\
&+ x^{100} + x^{99} + x^{97} + x^{95} + x^{94} + x^{88} + x^{87} + x^{86} + x^{74} + x^{73} + x^{71} \\
&+ x^{70} + x^{66} + x^{65} + x^{64} + x^{58} + x^{57} + x^{56} + x^{54} + x^{53} + x^{51} + x^{49} \\
&+ x^{48} + x^{44} + x^{43} + x^{41} + x^{40} + x^{38} + x^{37} + x^{35} + x^{34} + x^{32} + x^{31} \\
&+ x^{29} + x^{28} + x^{20} + x^{19} + x^{18}, \\
p_9(x) &= x^{125} + x^{124} + x^{123} + x^{121} + x^{117} + x^{116} + x^{115} + x^{113} + x^{111} + x^{110} \\
&+ x^{108} + x^{107} + x^{106} + x^{102} + x^{100} + x^{99} + x^{98} + x^{95} + x^{91} + x^{90} \\
&+ x^{88} + x^{85} + x^{77} + x^{76} + x^{74} + x^{71} + x^{67} + x^{66} + x^{64} + x^{63} + x^{62} \\
&+ x^{61} + x^{60} + x^{56} + x^{54} + x^{50} + x^{48} + x^{45} + x^{41} + x^{40} + x^{38} + x^{34} \\
&+ x^{32} + x^{31} + x^{30} + x^{27} + x^{21} + x^{20} + x^{18} + x^{15} + x^{13} + x^{12} + x^{11} \\
&+ x^{9}, \\
p_{12}(x) &= x^{124} + x^{123} + x^{121} + x^{119} + x^{117} + x^{115} + x^{113} + x^{112} + x^{76} + x^{75} \\
&+ x^{73} + x^{71} + x^{69} + x^{67} + x^{65} + x^{64} + x^{60} + x^{59} + x^{57} + x^{55} + x^{53} \\
&+ x^{51} + x^{49} + x^{48} + x^{44} + x^{43} + x^{41} + x^{39} + x^{37} + x^{35} + x^{33} + x^{32} \\
&+ x^{28} + x^{27} + x^{25} + x^{23} + x^{21} + x^{19} + x^{17} + x^{16} + x^{12} + x^{11} + x^{9} \\
&+ x^{7} + x^{5} + x^{3} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 1, 0, 1, 1, 0, 1].

For $\phi(W^e, 1, 0)$:

$$p_0(x) = x^{129} + x^{126} + x^{125} + x^{123} + x^{118} + x^{116} + x^{115} + x^{110} + x^{108} + x^{107}$$

$$
\begin{aligned}
&+ x^{105} + x^{104} + x^{102} + x^{100} + x^{98} + x^{97} + x^{96} + x^{95} + x^{92} + x^{89}\\
&+ x^{88} + x^{84} + x^{83} + x^{82} + x^{76} + x^{75} + x^{74} + x^{73} + x^{70} + x^{69} + x^{67}\\
&+ x^{66} + x^{64} + x^{63} + x^{61} + x^{59} + x^{57} + x^{56} + x^{55} + x^{53} + x^{51} + x^{50}\\
&+ x^{46} + x^{45} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39},\\
p_3(x) = {}& x^{127} + x^{126} + x^{125} + x^{122} + x^{121} + x^{119} + x^{117} + x^{116} + x^{115} + x^{112}\\
&+ x^{111} + x^{109} + x^{107} + x^{106} + x^{105} + x^{103} + x^{101} + x^{100} + x^{99} + x^{96}\\
&+ x^{95} + x^{93} + x^{91} + x^{90} + x^{89} + x^{87} + x^{85} + x^{84} + x^{83} + x^{80} + x^{79}\\
&+ x^{77} + x^{75} + x^{74} + x^{73} + x^{70} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63}\\
&+ x^{62} + x^{61} + x^{60} + x^{56} + x^{54} + x^{51} + x^{47} + x^{46} + x^{45} + x^{42} + x^{41}\\
&+ x^{39} + x^{37} + x^{36} + x^{35} + x^{32} + x^{30} + x^{27},\\
p_6(x) = {}& x^{126} + x^{125} + x^{123} + x^{122} + x^{118} + x^{117} + x^{116} + x^{110} + x^{109} + x^{108}\\
&+ x^{100} + x^{99} + x^{97} + x^{95} + x^{94} + x^{88} + x^{87} + x^{86} + x^{74} + x^{73} + x^{71}\\
&+ x^{70} + x^{66} + x^{65} + x^{64} + x^{58} + x^{57} + x^{56} + x^{54} + x^{53} + x^{51} + x^{49}\\
&+ x^{48} + x^{44} + x^{43} + x^{41} + x^{40} + x^{38} + x^{37} + x^{35} + x^{34} + x^{32} + x^{31}\\
&+ x^{29} + x^{28} + x^{20} + x^{19} + x^{18},\\
p_9(x) = {}& x^{125} + x^{124} + x^{123} + x^{121} + x^{117} + x^{116} + x^{115} + x^{113} + x^{111} + x^{110}\\
&+ x^{108} + x^{107} + x^{106} + x^{102} + x^{100} + x^{99} + x^{98} + x^{95} + x^{91} + x^{90}\\
&+ x^{88} + x^{85} + x^{77} + x^{76} + x^{74} + x^{71} + x^{67} + x^{66} + x^{64} + x^{63} + x^{62}\\
&+ x^{61} + x^{60} + x^{56} + x^{54} + x^{50} + x^{48} + x^{45} + x^{41} + x^{40} + x^{38} + x^{34}\\
&+ x^{32} + x^{31} + x^{30} + x^{27} + x^{21} + x^{20} + x^{18} + x^{15} + x^{13} + x^{12} + x^{11}\\
&+ x^{9},\\
p_{12}(x) = {}& x^{124} + x^{123} + x^{121} + x^{119} + x^{117} + x^{115} + x^{113} + x^{112} + x^{76} + x^{75}\\
&+ x^{73} + x^{71} + x^{69} + x^{67} + x^{65} + x^{64} + x^{60} + x^{59} + x^{57} + x^{55} + x^{53}\\
&+ x^{51} + x^{49} + x^{48} + x^{44} + x^{43} + x^{41} + x^{39} + x^{37} + x^{35} + x^{33} + x^{32}\\
&+ x^{28} + x^{27} + x^{25} + x^{23} + x^{21} + x^{19} + x^{17} + x^{16} + x^{12} + x^{11} + x^{9}\\
&+ x^{7} + x^{5} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 1, 0, 1, 1, 0, 1].

For $\phi(W^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) = {}& x^{126} + x^{114} + x^{110} + x^{106} + x^{100} + x^{98} + x^{94} + x^{90} + x^{78} + x^{76} + x^{70}\\
&+ x^{68} + x^{66} + x^{62} + x^{60} + x^{54} + x^{52} + x^{50} + x^{48} + x^{44} + x^{42},\\
p_3(x) = {}& x^{122} + x^{120} + x^{118} + x^{116} + x^{112} + x^{104} + x^{96} + x^{94} + x^{92} + x^{90}\\
&+ x^{86} + x^{84} + x^{82} + x^{76} + x^{74} + x^{72} + x^{70} + x^{68} + x^{64} + x^{62} + x^{60}\\
&+ x^{58} + x^{56} + x^{54} + x^{46} + x^{40} + x^{38} + x^{34} + x^{32} + x^{30} + x^{26} + x^{24}\\
&+ x^{22} + x^{18},\\
p_6(x) = {}& x^{120} + x^{114} + x^{100} + x^{98} + x^{96} + x^{86} + x^{84} + x^{82} + x^{78} + x^{72} + x^{70}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{66} + x^{60} + x^{58} + x^{56} + x^{52} + x^{46} + x^{34} + x^{32} + x^{22} + x^{20} + x^{16}\\
&+ x^{8} + x^{6} + x^{4} + 1,\\
p_9(x) &= x^{124} + x^{122} + x^{112} + x^{104} + x^{98} + x^{92} + x^{90} + x^{86} + x^{82} + x^{76} + x^{72}\\
&+ x^{68} + x^{66} + x^{56} + x^{54} + x^{52} + x^{50} + x^{48} + x^{46} + x^{42} + x^{40} + x^{36}\\
&+ x^{34} + x^{32} + x^{28} + x^{26} + x^{22} + x^{18} + x^{14} + x^{12} + x^{8} + x^{6},\\
p_{12}(x) &= x^{124} + x^{120} + x^{108} + x^{104} + x^{100} + x^{96} + x^{92} + x^{88} + x^{84} + x^{80} + x^{68}\\
&+ x^{64} + x^{60} + x^{56} + x^{36} + x^{32} + x^{28} + x^{24} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0].

For $\phi(M^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{129} + x^{128} + x^{127} + x^{126} + x^{123} + x^{122} + x^{121} + x^{118} + x^{117} + x^{114}\\
&+ x^{111} + x^{110} + x^{109} + x^{106} + x^{103} + x^{102} + x^{95} + x^{89} + x^{88} + x^{87}\\
&+ x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{78} + x^{77} + x^{76} + x^{74} + x^{71} + x^{68}\\
&+ x^{67} + x^{66} + x^{64} + x^{63} + x^{58} + x^{55} + x^{54} + x^{53} + x^{51} + x^{49} + x^{48}\\
&+ x^{47} + x^{46} + x^{45} + x^{44} + x^{41} + x^{40} + x^{38} + x^{37} + x^{36},\\
p_3(x) &= x^{128} + x^{126} + x^{124} + x^{122} + x^{118} + x^{115} + x^{112} + x^{108} + x^{106} + x^{104}\\
&+ x^{103} + x^{100} + x^{97} + x^{91} + x^{88} + x^{87} + x^{84} + x^{82} + x^{80} + x^{78} + x^{77}\\
&+ x^{74} + x^{72} + x^{70} + x^{69} + x^{68} + x^{64} + x^{63} + x^{61} + x^{60} + x^{59} + x^{54}\\
&+ x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{42} + x^{40} + x^{39} + x^{35}\\
&+ x^{29} + x^{28} + x^{26} + x^{25} + x^{24},\\
p_6(x) &= x^{128} + x^{126} + x^{124} + x^{123} + x^{121} + x^{120} + x^{119} + x^{117} + x^{114} + x^{113}\\
&+ x^{111} + x^{106} + x^{104} + x^{103} + x^{100} + x^{99} + x^{98} + x^{97} + x^{93} + x^{86}\\
&+ x^{78} + x^{77} + x^{73} + x^{72} + x^{67} + x^{65} + x^{57} + x^{56} + x^{55} + x^{54} + x^{52}\\
&+ x^{48} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35}\\
&+ x^{33} + x^{32} + x^{31} + x^{27} + x^{26} + x^{24} + x^{20} + x^{19} + x^{18} + x^{17} + x^{16}\\
&+ x^{14} + x^{13} + x^{11} + x^{9} + x^{7} + x^{5} + x^{3} + x + 1,\\
p_9(x) &= x^{128} + x^{127} + x^{125} + x^{123} + x^{121} + x^{119} + x^{117} + x^{116} + x^{108} + x^{107}\\
&+ x^{105} + x^{103} + x^{101} + x^{100} + x^{92} + x^{91} + x^{89} + x^{87} + x^{85} + x^{84}\\
&+ x^{80} + x^{79} + x^{77} + x^{76} + x^{64} + x^{63} + x^{61} + x^{59} + x^{57} + x^{55} + x^{53}\\
&+ x^{52} + x^{48} + x^{47} + x^{45} + x^{44} + x^{32} + x^{31} + x^{29} + x^{27} + x^{25} + x^{23}\\
&+ x^{21} + x^{20} + x^{16} + x^{15} + x^{13} + x^{12},\\
p_{12}(x) &= x^{128} + x^{127} + x^{125} + x^{124} + x^{116} + x^{115} + x^{113} + x^{112} + x^{108} + x^{107}\\
&+ x^{105} + x^{103} + x^{101} + x^{100} + x^{92} + x^{91} + x^{89} + x^{87} + x^{85} + x^{84}\\
&+ x^{80} + x^{79} + x^{77} + x^{75} + x^{73} + x^{71} + x^{69} + x^{67} + x^{65} + x^{63} + x^{61}\\
&+ x^{60} + x^{52} + x^{51} + x^{49} + x^{47} + x^{45} + x^{43} + x^{41} + x^{39} + x^{37} + x^{35}\\
&+ x^{33} + x^{31} + x^{29} + x^{28} + x^{20} + x^{19} + x^{17} + x^{15} + x^{13} + x^{11} + x^{9}
\end{aligned}
$$

$$+ x^7 + x^5 + x^3 + x + 1,$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1].

For $\phi(M^o, 0, 1)$:

$$\begin{aligned}
p_0(x) = {} & x^{136} + x^{133} + x^{132} + x^{131} + x^{129} + x^{127} + x^{126} + x^{123} + x^{119} + x^{118} \\
& + x^{117} + x^{115} + x^{113} + x^{112} + x^{109} + x^{108} + x^{106} + x^{102} + x^{98} + x^{97} \\
& + x^{95} + x^{93} + x^{91} + x^{89} + x^{88} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{79} \\
& + x^{74} + x^{73} + x^{69} + x^{66} + x^{65} + x^{64} + x^{62} + x^{61} + x^{60} + x^{59} + x^{57} \\
& + x^{56} + x^{55} + x^{53} + x^{51} + x^{50} + x^{48} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39}, \\
p_3(x) = {} & x^{129} + x^{125} + x^{124} + x^{122} + x^{119} + x^{118} + x^{115} + x^{114} + x^{111} + x^{110} \\
& + x^{107} + x^{106} + x^{103} + x^{102} + x^{99} + x^{98} + x^{95} + x^{94} + x^{91} + x^{90} \\
& + x^{87} + x^{86} + x^{83} + x^{82} + x^{79} + x^{78} + x^{75} + x^{74} + x^{73} + x^{71} + x^{70} \\
& + x^{69} + x^{68} + x^{67} + x^{65} + x^{61} + x^{60} + x^{58} + x^{57} + x^{55} + x^{54} + x^{53} \\
& + x^{52} + x^{51} + x^{49} + x^{45} + x^{44} + x^{42} + x^{39} + x^{38} + x^{35} + x^{34} + x^{33} \\
& + x^{31} + x^{30} + x^{29} + x^{28} + x^{27}, \\
p_6(x) = {} & x^{130} + x^{120} + x^{118} + x^{116} + x^{114} + x^{112} + x^{110} + x^{108} + x^{104} + x^{100} \\
& + x^{98} + x^{94} + x^{92} + x^{90} + x^{88} + x^{86} + x^{78} + x^{68} + x^{66} + x^{64} + x^{62} \\
& + x^{60} + x^{56} + x^{54} + x^{52} + x^{42} + x^{40} + x^{36} + x^{34} + x^{28} + x^{24} + x^{22} \\
& + x^{20} + x^{18}, \\
p_9(x) = {} & x^{131} + x^{127} + x^{126} + x^{124} + x^{123} + x^{120} + x^{119} + x^{118} + x^{117} + x^{116} \\
& + x^{115} + x^{114} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} + x^{107} \\
& + x^{106} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97} \\
& + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} \\
& + x^{85} + x^{84} + x^{82} + x^{81} + x^{80} + x^{77} + x^{74} + x^{73} + x^{67} + x^{63} + x^{62} \\
& + x^{60} + x^{59} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{50} + x^{49} + x^{48} + x^{45} \\
& + x^{42} + x^{41} + x^{35} + x^{31} + x^{30} + x^{28} + x^{27} + x^{24} + x^{23} + x^{22} + x^{21} \\
& + x^{20} + x^{18} + x^{17} + x^{16} + x^{13} + x^{10} + x^{9}, \\
p_{12}(x) = {} & x^{132} + x^{124} + x^{120} + x^{108} + x^{104} + x^{100} + x^{96} + x^{92} + x^{88} + x^{80} \\
& + x^{64} + x^{60} + x^{56} + x^{52} + x^{32} + x^{28} + x^{24} + x^{20} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 1, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0].

For $\phi(M^o, 1, 0)$:

$$\begin{aligned}
p_0(x) = {} & x^{126} + x^{125} + x^{124} + x^{120} + x^{117} + x^{116} + x^{114} + x^{113} + x^{110} + x^{109} \\
& + x^{107} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97} \\
& + x^{96} + x^{94} + x^{93} + x^{90} + x^{86} + x^{82} + x^{81} + x^{80} + x^{76} + x^{75} + x^{73} \\
& + x^{70} + x^{69} + x^{68} + x^{66} + x^{64} + x^{63} + x^{53} + x^{51} + x^{50} + x^{49} + x^{48} \\
& + x^{46} + x^{43} + x^{42} + x^{41} + x^{39},
\end{aligned}$$

$$\begin{aligned}
p_3(x) &= x^{125} + x^{121} + x^{120} + x^{118} + x^{115} + x^{114} + x^{113} + x^{111} + x^{110} + x^{108} \\
&\quad + x^{107} + x^{105} + x^{104} + x^{102} + x^{99} + x^{98} + x^{97} + x^{95} + x^{94} + x^{92} \\
&\quad + x^{91} + x^{89} + x^{88} + x^{86} + x^{83} + x^{82} + x^{81} + x^{79} + x^{78} + x^{76} + x^{75} \\
&\quad + x^{73} + x^{72} + x^{70} + x^{69} + x^{67} + x^{66} + x^{64} + x^{63} + x^{61} + x^{60} + x^{58} \\
&\quad + x^{57} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{45} + x^{41} + x^{40} + x^{38} + x^{35} \\
&\quad + x^{34} + x^{33} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27}, \\
p_6(x) &= x^{126} + x^{116} + x^{112} + x^{108} + x^{106} + x^{102} + x^{100} + x^{98} + x^{96} + x^{84} \\
&\quad + x^{82} + x^{80} + x^{78} + x^{76} + x^{72} + x^{70} + x^{68} + x^{66} + x^{62} + x^{58} + x^{54} \\
&\quad + x^{52} + x^{50} + x^{44} + x^{42} + x^{40} + x^{38} + x^{36} + x^{34} + x^{32} + x^{30} + x^{28} \\
&\quad + x^{24} + x^{22} + x^{20} + x^{18}, \\
p_9(x) &= x^{127} + x^{123} + x^{122} + x^{120} + x^{119} + x^{116} + x^{114} + x^{113} + x^{112} + x^{111} \\
&\quad + x^{109} + x^{107} + x^{105} + x^{104} + x^{103} + x^{100} + x^{98} + x^{97} + x^{96} + x^{95} \\
&\quad + x^{93} + x^{91} + x^{89} + x^{88} + x^{87} + x^{84} + x^{82} + x^{81} + x^{80} + x^{77} + x^{74} \\
&\quad + x^{73} + x^{63} + x^{59} + x^{58} + x^{56} + x^{55} + x^{52} + x^{50} + x^{49} + x^{48} + x^{45} \\
&\quad + x^{42} + x^{41} + x^{31} + x^{27} + x^{26} + x^{24} + x^{23} + x^{20} + x^{18} + x^{17} + x^{16} \\
&\quad + x^{13} + x^{10} + x^{9}, \\
p_{12}(x) &= x^{128} + x^{120} + x^{112} + x^{108} + x^{104} + x^{100} + x^{92} + x^{88} + x^{84} + x^{80} \\
&\quad + x^{76} + x^{68} + x^{56} + x^{44} + x^{36} + x^{24} + x^{12} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 1, 0, 1, 1, 1].

For $\phi(M^o, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{129} + x^{127} + x^{124} + x^{121} + x^{117} + x^{116} + x^{114} + x^{112} + x^{110} + x^{104} \\
&\quad + x^{102} + x^{99} + x^{98} + x^{96} + x^{95} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{86} \\
&\quad + x^{85} + x^{82} + x^{81} + x^{79} + x^{77} + x^{74} + x^{73} + x^{70} + x^{69} + x^{66} + x^{61} \\
&\quad + x^{60} + x^{58} + x^{57} + x^{54} + x^{53} + x^{52} + x^{51} + x^{46} + x^{44} + x^{43} + x^{42}, \\
p_3(x) &= x^{128} + x^{125} + x^{122} + x^{117} + x^{114} + x^{112} + x^{107} + x^{106} + x^{105} + x^{103} \\
&\quad + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97} + x^{92} + x^{89} + x^{87} + x^{86} + x^{85} \\
&\quad + x^{84} + x^{83} + x^{80} + x^{78} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{70} + x^{69} \\
&\quad + x^{67} + x^{66} + x^{64} + x^{63} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{53} \\
&\quad + x^{52} + x^{47} + x^{44} + x^{42} + x^{40} + x^{39} + x^{38} + x^{37} + x^{34} + x^{30} + x^{28} \\
&\quad + x^{27} + x^{26}, \\
p_6(x) &= x^{128} + x^{125} + x^{123} + x^{122} + x^{119} + x^{118} + x^{115} + x^{114} + x^{110} + x^{108} \\
&\quad + x^{107} + x^{106} + x^{105} + x^{104} + x^{101} + x^{100} + x^{98} + x^{95} + x^{94} + x^{93} \\
&\quad + x^{88} + x^{87} + x^{84} + x^{83} + x^{81} + x^{80} + x^{78} + x^{77} + x^{74} + x^{70} + x^{69} \\
&\quad + x^{68} + x^{67} + x^{66} + x^{63} + x^{61} + x^{59} + x^{58} + x^{57} + x^{51} + x^{49} + x^{48} \\
&\quad + x^{47} + x^{43} + x^{42} + x^{41} + x^{39} + x^{36} + x^{34} + x^{33} + x^{29} + x^{28} + x^{27}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{26} + x^{23} + x^{21} + x^{20} + x^{17} + x^{15} + x^{14} + x^{12} + x^{11} + x^{9} + x^{7} + x^{5}\\
&+ x^{3} + x + 1,\\
p_9(x) = {}& x^{128} + x^{127} + x^{125} + x^{123} + x^{121} + x^{119} + x^{117} + x^{116} + x^{108} + x^{107}\\
&+ x^{105} + x^{103} + x^{101} + x^{100} + x^{92} + x^{91} + x^{89} + x^{87} + x^{85} + x^{84}\\
&+ x^{80} + x^{79} + x^{77} + x^{76} + x^{64} + x^{63} + x^{61} + x^{59} + x^{57} + x^{55} + x^{53}\\
&+ x^{52} + x^{48} + x^{47} + x^{45} + x^{44} + x^{32} + x^{31} + x^{29} + x^{27} + x^{25} + x^{23}\\
&+ x^{21} + x^{20} + x^{16} + x^{15} + x^{13} + x^{12},\\
p_{12}(x) = {}& x^{128} + x^{127} + x^{125} + x^{124} + x^{116} + x^{115} + x^{113} + x^{112} + x^{108} + x^{107}\\
&+ x^{105} + x^{103} + x^{101} + x^{100} + x^{92} + x^{91} + x^{89} + x^{87} + x^{85} + x^{84}\\
&+ x^{80} + x^{79} + x^{77} + x^{75} + x^{73} + x^{71} + x^{69} + x^{67} + x^{65} + x^{63} + x^{61}\\
&+ x^{60} + x^{52} + x^{51} + x^{49} + x^{47} + x^{45} + x^{43} + x^{41} + x^{39} + x^{37} + x^{35}\\
&+ x^{33} + x^{31} + x^{29} + x^{28} + x^{20} + x^{19} + x^{17} + x^{15} + x^{13} + x^{11} + x^{9}\\
&+ x^{7} + x^{5} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 0, 0, 1].

For $\phi(W^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) = {}& x^{129} + x^{128} + x^{127} + x^{126} + x^{123} + x^{122} + x^{121} + x^{118} + x^{117} + x^{114}\\
&+ x^{111} + x^{110} + x^{109} + x^{106} + x^{103} + x^{102} + x^{95} + x^{89} + x^{88} + x^{87}\\
&+ x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{78} + x^{77} + x^{76} + x^{74} + x^{71} + x^{68}\\
&+ x^{67} + x^{66} + x^{64} + x^{63} + x^{58} + x^{55} + x^{54} + x^{53} + x^{51} + x^{49} + x^{48}\\
&+ x^{47} + x^{46} + x^{45} + x^{44} + x^{41} + x^{40} + x^{38} + x^{37} + x^{36},\\
p_3(x) = {}& x^{128} + x^{126} + x^{124} + x^{122} + x^{118} + x^{115} + x^{112} + x^{108} + x^{106} + x^{104}\\
&+ x^{103} + x^{100} + x^{97} + x^{91} + x^{88} + x^{87} + x^{84} + x^{82} + x^{80} + x^{78} + x^{77}\\
&+ x^{74} + x^{72} + x^{70} + x^{69} + x^{68} + x^{64} + x^{63} + x^{61} + x^{60} + x^{59} + x^{54}\\
&+ x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{42} + x^{40} + x^{39} + x^{35}\\
&+ x^{29} + x^{28} + x^{26} + x^{25} + x^{24},\\
p_6(x) = {}& x^{128} + x^{126} + x^{124} + x^{123} + x^{121} + x^{120} + x^{119} + x^{117} + x^{114} + x^{113}\\
&+ x^{111} + x^{106} + x^{104} + x^{103} + x^{100} + x^{99} + x^{98} + x^{97} + x^{93} + x^{86}\\
&+ x^{78} + x^{77} + x^{73} + x^{72} + x^{67} + x^{65} + x^{57} + x^{56} + x^{55} + x^{54} + x^{52}\\
&+ x^{48} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35}\\
&+ x^{33} + x^{32} + x^{31} + x^{27} + x^{26} + x^{24} + x^{20} + x^{19} + x^{18} + x^{17} + x^{16}\\
&+ x^{14} + x^{13} + x^{11} + x^{9} + x^{7} + x^{5} + x^{3} + x + 1,\\
p_9(x) = {}& x^{128} + x^{127} + x^{125} + x^{123} + x^{121} + x^{119} + x^{117} + x^{116} + x^{108} + x^{107}\\
&+ x^{105} + x^{103} + x^{101} + x^{100} + x^{92} + x^{91} + x^{89} + x^{87} + x^{85} + x^{84}\\
&+ x^{80} + x^{79} + x^{77} + x^{76} + x^{64} + x^{63} + x^{61} + x^{59} + x^{57} + x^{55} + x^{53}\\
&+ x^{52} + x^{48} + x^{47} + x^{45} + x^{44} + x^{32} + x^{31} + x^{29} + x^{27} + x^{25} + x^{23}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{21} + x^{20} + x^{16} + x^{15} + x^{13} + x^{12},\\
p_{12}(x) &= x^{128} + x^{127} + x^{125} + x^{124} + x^{116} + x^{115} + x^{113} + x^{112} + x^{108} + x^{107}\\
&+ x^{105} + x^{103} + x^{101} + x^{100} + x^{92} + x^{91} + x^{89} + x^{87} + x^{85} + x^{84}\\
&+ x^{80} + x^{79} + x^{77} + x^{75} + x^{73} + x^{71} + x^{69} + x^{67} + x^{65} + x^{63} + x^{61}\\
&+ x^{60} + x^{52} + x^{51} + x^{49} + x^{47} + x^{45} + x^{43} + x^{41} + x^{39} + x^{37} + x^{35}\\
&+ x^{33} + x^{31} + x^{29} + x^{28} + x^{20} + x^{19} + x^{17} + x^{15} + x^{13} + x^{11} + x^{9}\\
&+ x^{7} + x^{5} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1].

For $\phi(W^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{125} + x^{124} + x^{120} + x^{117} + x^{116} + x^{114} + x^{113} + x^{110} + x^{109}\\
&+ x^{107} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97}\\
&+ x^{96} + x^{94} + x^{93} + x^{90} + x^{86} + x^{82} + x^{81} + x^{80} + x^{76} + x^{75} + x^{73}\\
&+ x^{70} + x^{69} + x^{68} + x^{66} + x^{64} + x^{63} + x^{53} + x^{51} + x^{50} + x^{49} + x^{48}\\
&+ x^{46} + x^{43} + x^{42} + x^{41} + x^{39},\\
p_3(x) &= x^{125} + x^{121} + x^{120} + x^{118} + x^{115} + x^{114} + x^{113} + x^{111} + x^{110} + x^{108}\\
&+ x^{107} + x^{105} + x^{104} + x^{102} + x^{99} + x^{98} + x^{97} + x^{95} + x^{94} + x^{92}\\
&+ x^{91} + x^{89} + x^{88} + x^{86} + x^{83} + x^{82} + x^{81} + x^{79} + x^{78} + x^{76} + x^{75}\\
&+ x^{73} + x^{72} + x^{70} + x^{69} + x^{67} + x^{66} + x^{64} + x^{63} + x^{61} + x^{60} + x^{58}\\
&+ x^{57} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{45} + x^{41} + x^{40} + x^{38} + x^{35}\\
&+ x^{34} + x^{33} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27},\\
p_6(x) &= x^{126} + x^{116} + x^{112} + x^{108} + x^{106} + x^{102} + x^{100} + x^{98} + x^{96} + x^{84}\\
&+ x^{82} + x^{80} + x^{78} + x^{76} + x^{72} + x^{70} + x^{68} + x^{66} + x^{62} + x^{58} + x^{54}\\
&+ x^{52} + x^{50} + x^{44} + x^{42} + x^{40} + x^{38} + x^{36} + x^{34} + x^{32} + x^{30} + x^{28}\\
&+ x^{24} + x^{22} + x^{20} + x^{18},\\
p_9(x) &= x^{127} + x^{123} + x^{122} + x^{120} + x^{119} + x^{116} + x^{114} + x^{113} + x^{112} + x^{111}\\
&+ x^{109} + x^{107} + x^{105} + x^{104} + x^{103} + x^{100} + x^{98} + x^{97} + x^{96} + x^{95}\\
&+ x^{93} + x^{91} + x^{89} + x^{88} + x^{87} + x^{84} + x^{82} + x^{81} + x^{80} + x^{77} + x^{74}\\
&+ x^{73} + x^{63} + x^{59} + x^{58} + x^{56} + x^{55} + x^{52} + x^{50} + x^{49} + x^{48} + x^{45}\\
&+ x^{42} + x^{41} + x^{31} + x^{27} + x^{26} + x^{24} + x^{23} + x^{20} + x^{18} + x^{17} + x^{16}\\
&+ x^{13} + x^{10} + x^{9},\\
p_{12}(x) &= x^{128} + x^{120} + x^{112} + x^{108} + x^{104} + x^{100} + x^{92} + x^{88} + x^{84} + x^{80}\\
&+ x^{76} + x^{68} + x^{56} + x^{44} + x^{36} + x^{24} + x^{12} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 1, 0, 1, 1, 1].

For $\phi(W^o, 1, 0)$:

$$
p_0(x) = x^{136} + x^{133} + x^{132} + x^{131} + x^{129} + x^{127} + x^{126} + x^{123} + x^{119} + x^{118}
$$

$$
\begin{aligned}
&+ x^{117} + x^{115} + x^{113} + x^{112} + x^{109} + x^{108} + x^{106} + x^{102} + x^{98} + x^{97}\\
&+ x^{95} + x^{93} + x^{91} + x^{89} + x^{88} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{79}\\
&+ x^{74} + x^{73} + x^{69} + x^{66} + x^{65} + x^{64} + x^{62} + x^{61} + x^{60} + x^{59} + x^{57}\\
&+ x^{56} + x^{55} + x^{53} + x^{51} + x^{50} + x^{48} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39},\\
p_3(x) &= x^{129} + x^{125} + x^{124} + x^{122} + x^{119} + x^{118} + x^{115} + x^{114} + x^{111} + x^{110}\\
&+ x^{107} + x^{106} + x^{103} + x^{102} + x^{99} + x^{98} + x^{95} + x^{94} + x^{91} + x^{90}\\
&+ x^{87} + x^{86} + x^{83} + x^{82} + x^{79} + x^{78} + x^{75} + x^{74} + x^{73} + x^{71} + x^{70}\\
&+ x^{69} + x^{68} + x^{67} + x^{65} + x^{61} + x^{60} + x^{58} + x^{57} + x^{55} + x^{54} + x^{53}\\
&+ x^{52} + x^{51} + x^{49} + x^{45} + x^{44} + x^{42} + x^{39} + x^{38} + x^{35} + x^{34} + x^{33}\\
&+ x^{31} + x^{30} + x^{29} + x^{28} + x^{27},\\
p_6(x) &= x^{130} + x^{120} + x^{118} + x^{116} + x^{114} + x^{112} + x^{110} + x^{108} + x^{104} + x^{100}\\
&+ x^{98} + x^{94} + x^{92} + x^{90} + x^{88} + x^{86} + x^{78} + x^{68} + x^{66} + x^{64} + x^{62}\\
&+ x^{60} + x^{56} + x^{54} + x^{52} + x^{42} + x^{40} + x^{36} + x^{34} + x^{28} + x^{24} + x^{22}\\
&+ x^{20} + x^{18},\\
p_9(x) &= x^{131} + x^{127} + x^{126} + x^{124} + x^{123} + x^{120} + x^{119} + x^{118} + x^{117} + x^{116}\\
&+ x^{115} + x^{114} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} + x^{107}\\
&+ x^{106} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97}\\
&+ x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86}\\
&+ x^{85} + x^{84} + x^{82} + x^{81} + x^{80} + x^{77} + x^{74} + x^{73} + x^{67} + x^{63} + x^{62}\\
&+ x^{60} + x^{59} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{50} + x^{49} + x^{48} + x^{45}\\
&+ x^{42} + x^{41} + x^{35} + x^{31} + x^{30} + x^{28} + x^{27} + x^{24} + x^{23} + x^{22} + x^{21}\\
&+ x^{20} + x^{18} + x^{17} + x^{16} + x^{13} + x^{10} + x^{9},\\
p_{12}(x) &= x^{132} + x^{124} + x^{120} + x^{108} + x^{104} + x^{100} + x^{96} + x^{92} + x^{88} + x^{80}\\
&+ x^{64} + x^{60} + x^{56} + x^{52} + x^{32} + x^{28} + x^{24} + x^{20} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 1, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0].
For $\phi(W^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{129} + x^{127} + x^{124} + x^{121} + x^{117} + x^{116} + x^{114} + x^{112} + x^{110} + x^{104}\\
&+ x^{102} + x^{99} + x^{98} + x^{96} + x^{95} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{86}\\
&+ x^{85} + x^{82} + x^{81} + x^{79} + x^{77} + x^{74} + x^{73} + x^{70} + x^{69} + x^{66} + x^{61}\\
&+ x^{60} + x^{58} + x^{57} + x^{54} + x^{53} + x^{52} + x^{51} + x^{46} + x^{44} + x^{43} + x^{42},\\
p_3(x) &= x^{128} + x^{125} + x^{122} + x^{117} + x^{114} + x^{112} + x^{107} + x^{106} + x^{105} + x^{103}\\
&+ x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97} + x^{92} + x^{89} + x^{87} + x^{86} + x^{85}\\
&+ x^{84} + x^{83} + x^{80} + x^{78} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{70} + x^{69}\\
&+ x^{67} + x^{66} + x^{64} + x^{63} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{53}\\
&+ x^{52} + x^{47} + x^{44} + x^{42} + x^{40} + x^{39} + x^{38} + x^{37} + x^{34} + x^{30} + x^{28}
\end{aligned}
$$

$$\begin{aligned}
&+ x^{27} + x^{26},\\
p_6(x) &= x^{128} + x^{125} + x^{123} + x^{122} + x^{119} + x^{118} + x^{115} + x^{114} + x^{110} + x^{108}\\
&+ x^{107} + x^{106} + x^{105} + x^{104} + x^{101} + x^{100} + x^{98} + x^{95} + x^{94} + x^{93}\\
&+ x^{88} + x^{87} + x^{84} + x^{83} + x^{81} + x^{80} + x^{78} + x^{77} + x^{74} + x^{70} + x^{69}\\
&+ x^{68} + x^{67} + x^{66} + x^{63} + x^{61} + x^{59} + x^{58} + x^{57} + x^{51} + x^{49} + x^{48}\\
&+ x^{47} + x^{43} + x^{42} + x^{41} + x^{39} + x^{36} + x^{34} + x^{33} + x^{29} + x^{28} + x^{27}\\
&+ x^{26} + x^{23} + x^{21} + x^{20} + x^{17} + x^{15} + x^{14} + x^{12} + x^{11} + x^{9} + x^{7} + x^{5}\\
&+ x^{3} + x + 1,\\
p_9(x) &= x^{128} + x^{127} + x^{125} + x^{123} + x^{121} + x^{119} + x^{117} + x^{116} + x^{108} + x^{107}\\
&+ x^{105} + x^{103} + x^{101} + x^{100} + x^{92} + x^{91} + x^{89} + x^{87} + x^{85} + x^{84}\\
&+ x^{80} + x^{79} + x^{77} + x^{76} + x^{64} + x^{63} + x^{61} + x^{59} + x^{57} + x^{55} + x^{53}\\
&+ x^{52} + x^{48} + x^{47} + x^{45} + x^{44} + x^{32} + x^{31} + x^{29} + x^{27} + x^{25} + x^{23}\\
&+ x^{21} + x^{20} + x^{16} + x^{15} + x^{13} + x^{12},\\
p_{12}(x) &= x^{128} + x^{127} + x^{125} + x^{124} + x^{116} + x^{115} + x^{113} + x^{112} + x^{108} + x^{107}\\
&+ x^{105} + x^{103} + x^{101} + x^{100} + x^{92} + x^{91} + x^{89} + x^{87} + x^{85} + x^{84}\\
&+ x^{80} + x^{79} + x^{77} + x^{75} + x^{73} + x^{71} + x^{69} + x^{67} + x^{65} + x^{63} + x^{61}\\
&+ x^{60} + x^{52} + x^{51} + x^{49} + x^{47} + x^{45} + x^{43} + x^{41} + x^{39} + x^{37} + x^{35}\\
&+ x^{33} + x^{31} + x^{29} + x^{28} + x^{20} + x^{19} + x^{17} + x^{15} + x^{13} + x^{11} + x^{9}\\
&+ x^{7} + x^{5} + x^{3} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 0, 0, 1].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

| $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | [1, 2] | 196 | [316, 208] | 392 | [561, 249] | 588 | [730, 731] | 784 | [717, 402] |
| 1 | [3, 4] | 197 | [317, 318] | 393 | [562, 325] | 589 | [134, 732] | 785 | [871, 157] |
| 2 | [5, 6] | 198 | [111, 260] | 394 | [563, 564] | 590 | [379, 436] | 786 | [136, 757] |
| 3 | [7, 8] | 199 | [319, 243] | 395 | [565, 566] | 591 | [733, 649] | 787 | [171, 123] |
| 4 | [9, 10] | 200 | [320, 321] | 396 | [16, 555] | 592 | [361, 410] | 788 | [872, 55] |
| 5 | [11, 12] | 201 | [220, 322] | 397 | [100, 567] | 593 | [362, 660] | 789 | [873, 346] |
| 6 | [13, 14] | 202 | [323, 324] | 398 | [568, 569] | 594 | [734, 445] | 790 | [460, 14] |
| 7 | [15, 16] | 203 | [239, 325] | 399 | [570, 571] | 595 | [141, 662] | 791 | [124, 178] |
| 8 | [17, 18] | 204 | [198, 153] | 400 | [487, 572] | 596 | [120, 168] | 792 | [413, 874] |
| 9 | [19, 20] | 205 | [326, 327] | 401 | [20, 573] | 597 | [44, 53] | 793 | [875, 136] |
| 10 | [21, 22] | 206 | [328, 161] | 402 | [574, 575] | 598 | [397, 448] | 794 | [184, 184] |
| 11 | [23, 24] | 207 | [329, 330] | 403 | [576, 518] | 599 | [735, 675] | 795 | [876, 647] |
| 12 | [25, 26] | 208 | [331, 332] | 404 | [577, 578] | 600 | [412, 138] | 796 | [389, 466] |
| 13 | [27, 28] | 209 | [333, 334] | 405 | [579, 580] | 601 | [736, 181] | 797 | [877, 377] |
| 14 | [29, 30] | 210 | [335, 336] | 406 | [581, 544] | 602 | [451, 646] | 798 | [696, 666] |

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|---|---|---|---|---|---|---|---|---|---|
| 15 | [31, 32] | 211 | [337, 182] | 407 | [277, 80] | 603 | [404, 459] | 799 | [456, 371] |
| 16 | [33, 34] | 212 | [338, 339] | 408 | [582, 583] | 604 | [662, 401] | 800 | [878, 752] |
| 17 | [35, 36] | 213 | [340, 341] | 409 | [99, 584] | 605 | [467, 461] | 801 | [663, 429] |
| 18 | [18, 18] | 214 | [342, 343] | 410 | [312, 113] | 606 | [737, 738] | 802 | [733, 385] |
| 19 | [37, 38] | 215 | [344, 184] | 411 | [544, 296] | 607 | [739, 174] | 803 | [701, 671] |
| 20 | [39, 40] | 216 | [145, 119] | 412 | [585, 586] | 608 | [8, 396] | 804 | [682, 428] |
| 21 | [41, 42] | 217 | [345, 346] | 413 | [517, 587] | 609 | [740, 139] | 805 | [668, 432] |
| 22 | [43, 44] | 218 | [347, 348] | 414 | [588, 589] | 610 | [741, 742] | 806 | [706, 136] |
| 23 | [31, 45] | 219 | [349, 350] | 415 | [555, 234] | 611 | [655, 743] | 807 | [879, 738] |
| 24 | [46, 47] | 220 | [351, 352] | 416 | [550, 557] | 612 | [744, 745] | 808 | [196, 343] |
| 25 | [48, 49] | 221 | [353, 38] | 417 | [590, 591] | 613 | [57, 53] | 809 | [880, 881] |
| 26 | [50, 51] | 222 | [354, 355] | 418 | [65, 99] | 614 | [746, 684] | 810 | [882, 662] |
| 27 | [52, 53] | 223 | [117, 41] | 419 | [227, 592] | 615 | [355, 654] | 811 | [883, 46] |
| 28 | [54, 55] | 224 | [356, 357] | 420 | [593, 73] | 616 | [747, 429] | 812 | [884, 885] |
| 29 | [56, 57] | 225 | [358, 359] | 421 | [594, 595] | 617 | [748, 152] | 813 | [763, 881] |
| 30 | [58, 59] | 226 | [360, 361] | 422 | [596, 597] | 618 | [749, 149] | 814 | [744, 886] |
| 31 | [60, 61] | 227 | [121, 169] | 423 | [598, 599] | 619 | [164, 41] | 815 | [710, 385] |
| 32 | [62, 63] | 228 | [362, 363] | 424 | [600, 601] | 620 | [48, 707] | 816 | [887, 888] |
| 33 | [64, 65] | 229 | [364, 365] | 425 | [602, 513] | 621 | [9, 662] | 817 | [139, 396] |
| 34 | [66, 67] | 230 | [366, 367] | 426 | [603, 511] | 622 | [171, 133] | 818 | [461, 440] |
| 35 | [68, 69] | 231 | [368, 369] | 427 | [604, 605] | 623 | [750, 40] | 819 | [889, 890] |
| 36 | [70, 71] | 232 | [159, 58] | 428 | [606, 607] | 624 | [751, 752] | 820 | [331, 418] |
| 37 | [72, 73] | 233 | [370, 371] | 429 | [584, 497] | 625 | [430, 732] | 821 | [891, 373] |
| 38 | [74, 75] | 234 | [372, 373] | 430 | [608, 609] | 626 | [422, 402] | 822 | [892, 45] |
| 39 | [76, 77] | 235 | [374, 375] | 431 | [314, 75] | 627 | [753, 331] | 823 | [871, 159] |
| 40 | [78, 79] | 236 | [376, 377] | 432 | [75, 595] | 628 | [652, 117] | 824 | [893, 365] |
| 41 | [80, 81] | 237 | [330, 8] | 433 | [278, 610] | 629 | [725, 754] | 825 | [894, 380] |
| 42 | [82, 83] | 238 | [18, 378] | 434 | [611, 314] | 630 | [755, 756] | 826 | [642, 718] |
| 43 | [84, 85] | 239 | [379, 380] | 435 | [612, 613] | 631 | [746, 757] | 827 | [373, 371] |
| 44 | [86, 87] | 240 | [381, 382] | 436 | [614, 592] | 632 | [709, 392] | 828 | [706, 895] |
| 45 | [88, 89] | 241 | [383, 174] | 437 | [615, 616] | 633 | [758, 715] | 829 | [887, 896] |
| 46 | [90, 91] | 242 | [384, 385] | 438 | [617, 618] | 634 | [683, 757] | 830 | [8, 140] |
| 47 | [92, 93] | 243 | [386, 387] | 439 | [292, 29] | 635 | [759, 760] | 831 | [694, 722] |
| 48 | [94, 95] | 244 | [388, 46] | 440 | [619, 291] | 636 | [697, 168] | 832 | [659, 874] |
| 49 | [96, 6] | 245 | [187, 361] | 441 | [320, 564] | 637 | [693, 384] | 833 | [892, 32] |
| 50 | [97, 98] | 246 | [357, 44] | 442 | [620, 535] | 638 | [761, 378] | 834 | [897, 898] |
| 51 | [24, 99] | 247 | [389, 333] | 443 | [621, 322] | 639 | [762, 188] | 835 | [899, 383] |
| 52 | [100, 101] | 248 | [390, 391] | 444 | [622, 575] | 640 | [763, 764] | 836 | [41, 654] |
| 53 | [102, 103] | 249 | [392, 393] | 445 | [623, 289] | 641 | [765, 622] | 837 | [645, 452] |
| 54 | [104, 105] | 250 | [137, 202] | 446 | [515, 257] | 642 | [325, 766] | 838 | [356, 43] |
| 55 | [106, 107] | 251 | [394, 142] | 447 | [204, 210] | 643 | [767, 508] | 839 | [900, 461] |
| 56 | [108, 109] | 252 | [395, 396] | 448 | [624, 625] | 644 | [230, 638] | 840 | [901, 179] |
| 57 | [110, 111] | 253 | [397, 398] | 449 | [626, 627] | 645 | [78, 285] | 841 | [734, 440] |
| 58 | [112, 113] | 254 | [399, 400] | 450 | [628, 528] | 646 | [768, 769] | 842 | [902, 745] |

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|---|---|---|---|---|---|---|---|---|---|
| 59 | [114, 115] | 255 | [10, 401] | 451 | [285, 629] | 647 | [770, 281] | 843 | [40, 123] |
| 60 | [116, 117] | 256 | [402, 128] | 452 | [630, 631] | 648 | [481, 771] | 844 | [903, 57] |
| 61 | [118, 119] | 257 | [403, 198] | 453 | [68, 61] | 649 | [591, 538] | 845 | [904, 905] |
| 62 | [120, 121] | 258 | [404, 405] | 454 | [508, 560] | 650 | [772, 773] | 846 | [906, 907] |
| 63 | [122, 123] | 259 | [406, 396] | 455 | [632, 95] | 651 | [774, 775] | 847 | [128, 14] |
| 64 | [124, 125] | 260 | [192, 192] | 456 | [504, 562] | 652 | [484, 567] | 848 | [908, 896] |
| 65 | [126, 51] | 261 | [151, 154] | 457 | [80, 211] | 653 | [776, 777] | 849 | [326, 194] |
| 66 | [127, 128] | 262 | [407, 398] | 458 | [633, 634] | 654 | [263, 532] | 850 | [909, 890] |
| 67 | [129, 130] | 263 | [408, 186] | 459 | [635, 517] | 655 | [545, 226] | 851 | [462, 651] |
| 68 | [131, 132] | 264 | [409, 410] | 460 | [615, 20] | 656 | [778, 618] | 852 | [377, 348] |
| 69 | [40, 133] | 265 | [360, 188] | 461 | [546, 227] | 657 | [561, 779] | 853 | [910, 383] |
| 70 | [134, 135] | 266 | [411, 199] | 462 | [636, 90] | 658 | [94, 561] | 854 | [435, 380] |
| 71 | [119, 136] | 267 | [412, 202] | 463 | [577, 597] | 659 | [780, 86] | 855 | [911, 647] |
| 72 | [137, 138] | 268 | [388, 376] | 464 | [616, 573] | 660 | [781, 295] | 856 | [392, 384] |
| 73 | [139, 140] | 269 | [413, 414] | 465 | [637, 638] | 661 | [779, 551] | 857 | [394, 719] |
| 74 | [45, 141] | 270 | [415, 136] | 466 | [547, 546] | 662 | [89, 619] | 858 | [375, 673] |
| 75 | [142, 143] | 271 | [194, 167] | 467 | [522, 619] | 663 | [782, 783] | 859 | [912, 885] |
| 76 | [144, 145] | 272 | [416, 363] | 468 | [628, 639] | 664 | [521, 318] | 860 | [913, 914] |
| 77 | [146, 147] | 273 | [417, 418] | 469 | [640, 479] | 665 | [784, 255] | 861 | [233, 847] |
| 78 | [148, 149] | 274 | [13, 419] | 470 | [411, 330] | 666 | [475, 265] | 862 | [915, 235] |
| 79 | [150, 151] | 275 | [130, 128] | 471 | [641, 467] | 667 | [785, 82] | 863 | [916, 917] |
| 80 | [152, 149] | 276 | [144, 339] | 472 | [642, 643] | 668 | [73, 279] | 864 | [576, 587] |
| 81 | [153, 154] | 277 | [420, 421] | 473 | [644, 418] | 669 | [27, 786] | 865 | [918, 305] |
| 82 | [155, 156] | 278 | [422, 130] | 474 | [645, 646] | 670 | [787, 788] | 866 | [919, 777] |
| 83 | [157, 58] | 279 | [138, 203] | 475 | [647, 147] | 671 | [789, 790] | 867 | [315, 920] |
| 84 | [158, 159] | 280 | [193, 327] | 476 | [648, 382] | 672 | [791, 16] | 868 | [921, 589] |
| 85 | [160, 161] | 281 | [423, 369] | 477 | [463, 36] | 673 | [26, 510] | 869 | [598, 922] |
| 86 | [162, 163] | 282 | [342, 197] | 478 | [649, 392] | 674 | [792, 85] | 870 | [923, 559] |
| 87 | [154, 150] | 283 | [424, 187] | 479 | [650, 352] | 675 | [793, 83] | 871 | [519, 112] |
| 88 | [116, 164] | 284 | [425, 341] | 480 | [651, 46] | 676 | [207, 308] | 872 | [924, 475] |
| 89 | [165, 166] | 285 | [426, 427] | 481 | [652, 164] | 677 | [794, 485] | 873 | [543, 529] |
| 90 | [37, 167] | 286 | [336, 150] | 482 | [653, 654] | 678 | [770, 492] | 874 | [925, 926] |
| 91 | [168, 169] | 287 | [131, 428] | 483 | [353, 167] | 679 | [795, 625] | 875 | [784, 609] |
| 92 | [52, 166] | 288 | [410, 429] | 484 | [335, 154] | 680 | [796, 797] | 876 | [108, 270] |
| 93 | [170, 171] | 289 | [188, 410] | 485 | [446, 198] | 681 | [798, 623] | 877 | [103, 502] |
| 94 | [172, 173] | 290 | [430, 135] | 486 | [655, 656] | 682 | [258, 488] | 878 | [73, 610] |
| 95 | [172, 174] | 291 | [399, 59] | 487 | [143, 332] | 683 | [799, 584] | 879 | [219, 621] |
| 96 | [175, 176] | 292 | [165, 53] | 488 | [129, 402] | 684 | [572, 303] | 880 | [477, 766] |
| 97 | [177, 178] | 293 | [431, 432] | 489 | [657, 643] | 685 | [514, 282] | 881 | [927, 928] |
| 98 | [179, 180] | 294 | [433, 434] | 490 | [658, 201] | 686 | [800, 801] | 882 | [83, 28] |
| 99 | [45, 181] | 295 | [337, 149] | 491 | [659, 414] | 687 | [802, 566] | 883 | [636, 543] |
| 100 | [152, 182] | 296 | [435, 436] | 492 | [167, 431] | 688 | [803, 248] | 884 | [21, 929] |
| 101 | [183, 184] | 297 | [437, 438] | 493 | [327, 167] | 689 | [804, 482] | 885 | [930, 517] |
| 102 | [185, 186] | 298 | [439, 440] | 494 | [416, 660] | 690 | [88, 522] | 886 | [931, 932] |

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|---|---|---|---|---|---|---|---|---|---|
| 103 | [187, 188] | 299 | [441, 49] | 495 | [661, 371] | 691 | [805, 806] | 887 | [831, 90] |
| 104 | [158, 157] | 300 | [442, 443] | 496 | [47, 348] | 692 | [585, 534] | 888 | [933, 934] |
| 105 | [189, 190] | 301 | [334, 444] | 497 | [181, 662] | 693 | [325, 478] | 889 | [19, 616] |
| 106 | [191, 163] | 302 | [439, 445] | 498 | [192, 184] | 694 | [593, 278] | 890 | [935, 225] |
| 107 | [184, 192] | 303 | [46, 377] | 499 | [663, 664] | 695 | [807, 320] | 891 | [936, 917] |
| 108 | [193, 194] | 304 | [446, 150] | 500 | [665, 152] | 696 | [780, 257] | 892 | [596, 578] |
| 109 | [195, 156] | 305 | [447, 336] | 501 | [358, 666] | 697 | [111, 282] | 893 | [772, 937] |
| 110 | [196, 197] | 306 | [407, 448] | 502 | [667, 355] | 698 | [79, 629] | 894 | [938, 622] |
| 111 | [198, 151] | 307 | [449, 410] | 503 | [167, 668] | 699 | [808, 485] | 895 | [490, 521] |
| 112 | [32, 141] | 308 | [450, 46] | 504 | [35, 464] | 700 | [603, 471] | 896 | [939, 277] |
| 113 | [199, 8] | 309 | [451, 452] | 505 | [669, 438] | 701 | [533, 586] | 897 | [494, 601] |
| 114 | [200, 201] | 310 | [38, 431] | 506 | [670, 401] | 702 | [809, 324] | 898 | [940, 504] |
| 115 | [202, 203] | 311 | [453, 405] | 507 | [349, 671] | 703 | [810, 480] | 899 | [522, 290] |
| 116 | [204, 80] | 312 | [370, 454] | 508 | [672, 387] | 704 | [232, 29] | 900 | [941, 788] |
| 117 | [205, 206] | 313 | [455, 142] | 509 | [376, 47] | 705 | [305, 285] | 901 | [15, 554] |
| 118 | [207, 208] | 314 | [456, 454] | 510 | [382, 130] | 706 | [811, 572] | 902 | [311, 112] |
| 119 | [209, 115] | 315 | [457, 458] | 511 | [203, 673] | 707 | [812, 526] | 903 | [942, 599] |
| 120 | [210, 211] | 316 | [453, 459] | 512 | [424, 360] | 708 | [813, 243] | 904 | [306, 307] |
| 121 | [212, 213] | 317 | [460, 419] | 513 | [674, 675] | 709 | [238, 562] | 905 | [943, 769] |
| 122 | [214, 215] | 318 | [334, 461] | 514 | [420, 427] | 710 | [632, 561] | 906 | [791, 554] |
| 123 | [77, 216] | 319 | [462, 388] | 515 | [447, 154] | 711 | [814, 775] | 907 | [944, 298] |
| 124 | [17, 17] | 320 | [463, 464] | 516 | [676, 187] | 712 | [815, 591] | 908 | [527, 639] |
| 125 | [217, 218] | 321 | [465, 384] | 517 | [426, 421] | 713 | [816, 490] | 909 | [83, 786] |
| 126 | [219, 220] | 322 | [466, 467] | 518 | [677, 153] | 714 | [817, 818] | 910 | [483, 929] |
| 127 | [221, 222] | 323 | [468, 388] | 519 | [329, 199] | 715 | [819, 230] | 911 | [942, 922] |
| 128 | [69, 223] | 324 | [469, 391] | 520 | [466, 334] | 716 | [820, 821] | 912 | [89, 290] |
| 129 | [224, 225] | 325 | [378, 18] | 521 | [643, 661] | 717 | [60, 69] | 913 | [897, 945] |
| 130 | [226, 227] | 326 | [470, 471] | 522 | [159, 399] | 718 | [822, 773] | 914 | [55, 666] |
| 131 | [228, 209] | 327 | [472, 473] | 523 | [678, 679] | 719 | [23, 65] | 915 | [143, 418] |
| 132 | [229, 230] | 328 | [474, 475] | 524 | [467, 444] | 720 | [600, 495] | 916 | [880, 764] |
| 133 | [231, 232] | 329 | [18, 17] | 525 | [680, 179] | 721 | [823, 231] | 917 | [12, 703] |
| 134 | [233, 234] | 330 | [476, 300] | 526 | [681, 401] | 722 | [824, 825] | 918 | [191, 754] |
| 135 | [235, 236] | 331 | [477, 478] | 527 | [682, 132] | 723 | [826, 827] | 919 | [403, 150] |
| 136 | [208, 237] | 332 | [479, 480] | 528 | [169, 666] | 724 | [253, 828] | 920 | [361, 747] |
| 137 | [238, 239] | 333 | [481, 482] | 529 | [683, 684] | 725 | [829, 830] | 921 | [946, 674] |
| 138 | [240, 26] | 334 | [483, 22] | 530 | [685, 686] | 726 | [831, 543] | 922 | [947, 668] |
| 139 | [241, 242] | 335 | [484, 101] | 531 | [58, 400] | 727 | [832, 231] | 923 | [648, 129] |
| 140 | [243, 244] | 336 | [485, 107] | 532 | [339, 119] | 728 | [833, 98] | 924 | [676, 360] |
| 141 | [245, 246] | 337 | [486, 487] | 533 | [383, 173] | 729 | [834, 549] | 925 | [948, 756] |
| 142 | [247, 248] | 338 | [266, 488] | 534 | [687, 378] | 730 | [608, 255] | 926 | [949, 432] |
| 143 | [95, 249] | 339 | [489, 490] | 535 | [688, 34] | 731 | [835, 581] | 927 | [950, 176] |
| 144 | [250, 209] | 340 | [491, 82] | 536 | [381, 129] | 732 | [571, 275] | 928 | [741, 348] |
| 145 | [251, 252] | 341 | [492, 493] | 537 | [689, 380] | 733 | [563, 321] | 929 | [716, 951] |
| 146 | [253, 213] | 342 | [494, 495] | 538 | [385, 392] | 734 | [836, 222] | 930 | [952, 434] |

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|---|---|---|---|---|---|---|---|---|---|
| 147 | [254, 103] | 343 | [496, 497] | 539 | [690, 466] | 735 | [837, 102] | 931 | [953, 954] |
| 148 | [228, 114] | 344 | [498, 215] | 540 | [366, 691] | 736 | [838, 21] | 932 | [860, 174] |
| 149 | [255, 256] | 345 | [92, 499] | 541 | [393, 385] | 737 | [97, 839] | 933 | [955, 905] |
| 150 | [210, 81] | 346 | [500, 277] | 542 | [157, 399] | 738 | [840, 841] | 934 | [956, 163] |
| 151 | [257, 87] | 347 | [501, 502] | 543 | [355, 42] | 739 | [842, 21] | 935 | [902, 886] |
| 152 | [258, 259] | 348 | [91, 503] | 544 | [692, 693] | 740 | [796, 553] | 936 | [468, 651] |
| 153 | [107, 111] | 349 | [504, 239] | 545 | [694, 695] | 741 | [843, 844] | 937 | [141, 10] |
| 154 | [260, 261] | 350 | [505, 506] | 546 | [696, 359] | 742 | [844, 806] | 938 | [957, 945] |
| 155 | [262, 263] | 351 | [507, 508] | 547 | [697, 121] | 743 | [845, 808] | 939 | [958, 868] |
| 156 | [264, 265] | 352 | [509, 510] | 548 | [698, 170] | 744 | [272, 268] | 940 | [959, 954] |
| 157 | [266, 259] | 353 | [470, 511] | 549 | [699, 654] | 745 | [846, 581] | 941 | [720, 706] |
| 158 | [267, 268] | 354 | [512, 513] | 550 | [700, 695] | 746 | [817, 627] | 942 | [338, 145] |
| 159 | [250, 114] | 355 | [514, 260] | 551 | [693, 393] | 747 | [830, 571] | 943 | [960, 674] |
| 160 | [269, 105] | 356 | [515, 86] | 552 | [701, 350] | 748 | [555, 847] | 944 | [961, 33] |
| 161 | [270, 271] | 357 | [516, 102] | 553 | [702, 703] | 749 | [848, 827] | 945 | [962, 854] |
| 162 | [272, 273] | 358 | [517, 518] | 554 | [704, 128] | 750 | [849, 583] | 946 | [276, 531] |
| 163 | [274, 275] | 359 | [281, 493] | 555 | [32, 181] | 751 | [314, 594] | 947 | [322, 963] |
| 164 | [276, 263] | 360 | [519, 312] | 556 | [705, 170] | 752 | [850, 825] | 948 | [312, 964] |
| 165 | [277, 210] | 361 | [520, 521] | 557 | [145, 706] | 753 | [851, 479] | 949 | [965, 963] |
| 166 | [109, 271] | 362 | [287, 522] | 558 | [441, 707] | 754 | [852, 236] | 950 | [836, 966] |
| 167 | [278, 279] | 363 | [523, 214] | 559 | [708, 180] | 755 | [830, 274] | 951 | [963, 605] |
| 168 | [280, 281] | 364 | [507, 524] | 560 | [333, 467] | 756 | [853, 854] | 952 | [74, 594] |
| 169 | [282, 261] | 365 | [525, 526] | 561 | [709, 693] | 757 | [609, 256] | 953 | [803, 967] |
| 170 | [283, 284] | 366 | [527, 528] | 562 | [710, 649] | 758 | [799, 496] | 954 | [968, 506] |
| 171 | [285, 286] | 367 | [529, 503] | 563 | [378, 378] | 759 | [530, 270] | 955 | [582, 969] |
| 172 | [287, 89] | 368 | [530, 109] | 564 | [711, 367] | 760 | [855, 801] | 956 | [970, 858] |
| 173 | [288, 289] | 369 | [531, 532] | 565 | [689, 436] | 761 | [856, 544] | 957 | [316, 308] |
| 174 | [290, 291] | 370 | [533, 534] | 566 | [712, 384] | 762 | [857, 858] | 958 | [262, 531] |
| 175 | [292, 293] | 371 | [535, 67] | 567 | [713, 343] | 763 | [95, 779] | 959 | [64, 24] |
| 176 | [294, 295] | 372 | [536, 535] | 568 | [714, 715] | 764 | [859, 841] | 960 | [205, 920] |
| 177 | [241, 296] | 373 | [537, 538] | 569 | [716, 432] | 765 | [705, 721] | 961 | [767, 524] |
| 178 | [297, 298] | 374 | [539, 220] | 570 | [717, 130] | 766 | [860, 173] | 962 | [971, 907] |
| 179 | [299, 300] | 375 | [540, 541] | 571 | [718, 661] | 767 | [690, 333] | 963 | [7, 139] |
| 180 | [301, 244] | 376 | [542, 93] | 572 | [719, 143] | 768 | [861, 731] | 964 | [681, 972] |
| 181 | [61, 223] | 377 | [543, 91] | 573 | [38, 668] | 769 | [862, 429] | 965 | [658, 673] |
| 182 | [302, 303] | 378 | [544, 242] | 574 | [720, 119] | 770 | [685, 458] | 966 | [59, 732] |
| 183 | [304, 305] | 379 | [545, 546] | 575 | [721, 171] | 771 | [327, 38] | 967 | [402, 973] |
| 184 | [305, 79] | 380 | [293, 30] | 576 | [150, 153] | 772 | [177, 125] | 968 | [974, 33] |
| 185 | [306, 77] | 381 | [547, 226] | 577 | [700, 722] | 773 | [863, 443] | 969 | [975, 59] |
| 186 | [307, 216] | 382 | [548, 549] | 578 | [723, 58] | 774 | [864, 357] | 970 | [976, 656] |
| 187 | [267, 273] | 383 | [550, 63] | 579 | [724, 679] | 775 | [865, 123] | 971 | [552, 797] |
| 188 | [308, 237] | 384 | [551, 541] | 580 | [725, 163] | 776 | [162, 754] | 972 | [223, 71] |
| 189 | [309, 284] | 385 | [249, 551] | 581 | [726, 367] | 777 | [866, 153] | 973 | [225, 246] |
| 190 | [310, 232] | 386 | [552, 553] | 582 | [457, 686] | 778 | [867, 868] | 974 | [822, 937] |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 191 | [311, 312] | 387 | [554, 555] | 583 | [727, 190] | 779 | [726, 691] | 975 | [977, 235] |
| 192 | [79, 286] | 388 | [556, 92] | 584 | [728, 375] | 780 | [148, 182] | 976 | [804, 771] |
| 193 | [72, 278] | 389 | [62, 557] | 585 | [692, 392] | 781 | [869, 760] | 977 | [883, 376] |
| 194 | [313, 314] | 390 | [558, 559] | 586 | [729, 464] | 782 | [200, 673] | | |
| 195 | [315, 206] | 391 | [524, 560] | 587 | [434, 421] | 783 | [870, 331] | | |

Output function $n \mapsto \tau(n)$:

| $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 140 | 1 | 280 | 0 | 420 | 0 | 560 | 1 | 700 | 1 | 840 | 0 |
| 1 | 0 | 141 | 1 | 281 | 1 | 421 | 1 | 561 | 0 | 701 | 1 | 841 | 1 |
| 2 | 0 | 142 | 1 | 282 | 1 | 422 | 1 | 562 | 0 | 702 | 1 | 842 | 0 |
| 3 | 0 | 143 | 0 | 283 | 1 | 423 | 1 | 563 | 1 | 703 | 1 | 843 | 0 |
| 4 | 0 | 144 | 0 | 284 | 1 | 424 | 1 | 564 | 1 | 704 | 0 | 844 | 1 |
| 5 | 0 | 145 | 1 | 285 | 1 | 425 | 1 | 565 | 1 | 705 | 0 | 845 | 0 |
| 6 | 1 | 146 | 1 | 286 | 0 | 426 | 1 | 566 | 0 | 706 | 0 | 846 | 0 |
| 7 | 0 | 147 | 0 | 287 | 0 | 427 | 0 | 567 | 0 | 707 | 1 | 847 | 0 |
| 8 | 0 | 148 | 0 | 288 | 1 | 428 | 1 | 568 | 1 | 708 | 0 | 848 | 1 |
| 9 | 0 | 149 | 1 | 289 | 1 | 429 | 1 | 569 | 1 | 709 | 0 | 849 | 1 |
| 10 | 1 | 150 | 1 | 290 | 0 | 430 | 0 | 570 | 0 | 710 | 0 | 850 | 1 |
| 11 | 0 | 151 | 1 | 291 | 0 | 431 | 0 | 571 | 1 | 711 | 1 | 851 | 0 |
| 12 | 0 | 152 | 1 | 292 | 0 | 432 | 1 | 572 | 0 | 712 | 0 | 852 | 0 |
| 13 | 1 | 153 | 0 | 293 | 0 | 433 | 1 | 573 | 0 | 713 | 0 | 853 | 1 |
| 14 | 0 | 154 | 1 | 294 | 1 | 434 | 0 | 574 | 1 | 714 | 1 | 854 | 1 |
| 15 | 0 | 155 | 0 | 295 | 0 | 435 | 1 | 575 | 0 | 715 | 0 | 855 | 1 |
| 16 | 0 | 156 | 0 | 296 | 1 | 436 | 1 | 576 | 1 | 716 | 1 | 856 | 0 |
| 17 | 0 | 157 | 1 | 297 | 0 | 437 | 0 | 577 | 1 | 717 | 0 | 857 | 1 |
| 18 | 0 | 158 | 1 | 298 | 0 | 438 | 1 | 578 | 1 | 718 | 1 | 858 | 1 |
| 19 | 0 | 159 | 0 | 299 | 1 | 439 | 0 | 579 | 1 | 719 | 0 | 859 | 0 |
| 20 | 0 | 160 | 1 | 300 | 0 | 440 | 0 | 580 | 0 | 720 | 1 | 860 | 1 |
| 21 | 1 | 161 | 1 | 301 | 1 | 441 | 1 | 581 | 0 | 721 | 0 | 861 | 1 |
| 22 | 1 | 162 | 1 | 302 | 0 | 442 | 0 | 582 | 1 | 722 | 1 | 862 | 0 |
| 23 | 0 | 163 | 1 | 303 | 0 | 443 | 0 | 583 | 0 | 723 | 1 | 863 | 1 |
| 24 | 0 | 164 | 0 | 304 | 0 | 444 | 1 | 584 | 1 | 724 | 1 | 864 | 1 |
| 25 | 0 | 165 | 0 | 305 | 0 | 445 | 1 | 585 | 1 | 725 | 0 | 865 | 0 |
| 26 | 1 | 166 | 1 | 306 | 0 | 446 | 0 | 586 | 1 | 726 | 0 | 866 | 1 |
| 27 | 1 | 167 | 1 | 307 | 1 | 447 | 0 | 587 | 0 | 727 | 0 | 867 | 1 |
| 28 | 1 | 168 | 0 | 308 | 1 | 448 | 0 | 588 | 0 | 728 | 1 | 868 | 0 |
| 29 | 0 | 169 | 1 | 309 | 1 | 449 | 1 | 589 | 1 | 729 | 1 | 869 | 1 |
| 30 | 1 | 170 | 1 | 310 | 0 | 450 | 1 | 590 | 0 | 730 | 0 | 870 | 1 |
| 31 | 0 | 171 | 1 | 311 | 0 | 451 | 1 | 591 | 1 | 731 | 0 | 871 | 0 |
| 32 | 1 | 172 | 0 | 312 | 1 | 452 | 0 | 592 | 0 | 732 | 1 | 872 | 0 |
| 33 | 0 | 173 | 1 | 313 | 0 | 453 | 0 | 593 | 0 | 733 | 1 | 873 | 0 |
| 34 | 1 | 174 | 0 | 314 | 0 | 454 | 0 | 594 | 1 | 734 | 1 | 874 | 1 |
| 35 | 0 | 175 | 0 | 315 | 1 | 455 | 0 | 595 | 1 | 735 | 0 | 875 | 0 |

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 36 | 1 | 176 | 1 | 316 | 0 | 456 | 0 | 596 | 1 | 736 | 0 | 876 | 0 |
| 37 | 0 | 177 | 1 | 317 | 0 | 457 | 1 | 597 | 1 | 737 | 1 | 877 | 1 |
| 38 | 0 | 178 | 0 | 318 | 1 | 458 | 0 | 598 | 1 | 738 | 0 | 878 | 1 |
| 39 | 0 | 179 | 1 | 319 | 0 | 459 | 0 | 599 | 0 | 739 | 0 | 879 | 0 |
| 40 | 0 | 180 | 1 | 320 | 1 | 460 | 0 | 600 | 1 | 740 | 1 | 880 | 1 |
| 41 | 1 | 181 | 0 | 321 | 1 | 461 | 0 | 601 | 0 | 741 | 0 | 881 | 1 |
| 42 | 0 | 182 | 0 | 322 | 0 | 462 | 0 | 602 | 1 | 742 | 1 | 882 | 1 |
| 43 | 1 | 183 | 0 | 323 | 1 | 463 | 1 | 603 | 1 | 743 | 0 | 883 | 0 |
| 44 | 1 | 184 | 0 | 324 | 0 | 464 | 0 | 604 | 0 | 744 | 1 | 884 | 1 |
| 45 | 0 | 185 | 0 | 325 | 1 | 465 | 1 | 605 | 0 | 745 | 0 | 885 | 0 |
| 46 | 0 | 186 | 1 | 326 | 1 | 466 | 0 | 606 | 1 | 746 | 1 | 886 | 1 |
| 47 | 1 | 187 | 1 | 327 | 1 | 467 | 0 | 607 | 0 | 747 | 0 | 887 | 0 |
| 48 | 0 | 188 | 1 | 328 | 0 | 468 | 1 | 608 | 0 | 748 | 1 | 888 | 1 |
| 49 | 0 | 189 | 1 | 329 | 0 | 469 | 0 | 609 | 1 | 749 | 1 | 889 | 0 |
| 50 | 1 | 190 | 0 | 330 | 1 | 470 | 1 | 610 | 0 | 750 | 1 | 890 | 0 |
| 51 | 0 | 191 | 0 | 331 | 1 | 471 | 0 | 611 | 0 | 751 | 0 | 891 | 1 |
| 52 | 1 | 192 | 1 | 332 | 1 | 472 | 1 | 612 | 1 | 752 | 1 | 892 | 1 |
| 53 | 0 | 193 | 0 | 333 | 1 | 473 | 1 | 613 | 0 | 753 | 0 | 893 | 1 |
| 54 | 1 | 194 | 0 | 334 | 1 | 474 | 0 | 614 | 1 | 754 | 0 | 894 | 0 |
| 55 | 0 | 195 | 1 | 335 | 1 | 475 | 0 | 615 | 0 | 755 | 0 | 895 | 0 |
| 56 | 0 | 196 | 0 | 336 | 0 | 476 | 1 | 616 | 0 | 756 | 1 | 896 | 0 |
| 57 | 0 | 197 | 0 | 337 | 0 | 477 | 1 | 617 | 1 | 757 | 1 | 897 | 1 |
| 58 | 1 | 198 | 0 | 338 | 1 | 478 | 1 | 618 | 1 | 758 | 0 | 898 | 0 |
| 59 | 1 | 199 | 0 | 339 | 0 | 479 | 1 | 619 | 0 | 759 | 0 | 899 | 0 |
| 60 | 0 | 200 | 1 | 340 | 0 | 480 | 1 | 620 | 0 | 760 | 1 | 900 | 1 |
| 61 | 0 | 201 | 0 | 341 | 1 | 481 | 1 | 621 | 0 | 761 | 0 | 901 | 0 |
| 62 | 1 | 202 | 1 | 342 | 1 | 482 | 1 | 622 | 1 | 762 | 1 | 902 | 0 |
| 63 | 1 | 203 | 0 | 343 | 1 | 483 | 1 | 623 | 1 | 763 | 0 | 903 | 1 |
| 64 | 0 | 204 | 0 | 344 | 1 | 484 | 1 | 624 | 0 | 764 | 0 | 904 | 0 |
| 65 | 0 | 205 | 1 | 345 | 1 | 485 | 0 | 625 | 0 | 765 | 0 | 905 | 1 |
| 66 | 1 | 206 | 0 | 346 | 0 | 486 | 0 | 626 | 1 | 766 | 1 | 906 | 0 |
| 67 | 0 | 207 | 0 | 347 | 1 | 487 | 0 | 627 | 0 | 767 | 0 | 907 | 0 |
| 68 | 0 | 208 | 1 | 348 | 0 | 488 | 0 | 628 | 1 | 768 | 1 | 908 | 1 |
| 69 | 0 | 209 | 1 | 349 | 0 | 489 | 0 | 629 | 0 | 769 | 0 | 909 | 1 |
| 70 | 1 | 210 | 1 | 350 | 1 | 490 | 0 | 630 | 0 | 770 | 0 | 910 | 1 |
| 71 | 1 | 211 | 0 | 351 | 0 | 491 | 0 | 631 | 1 | 771 | 1 | 911 | 1 |
| 72 | 0 | 212 | 1 | 352 | 1 | 492 | 1 | 632 | 0 | 772 | 1 | 912 | 0 |
| 73 | 1 | 213 | 0 | 353 | 1 | 493 | 1 | 633 | 0 | 773 | 1 | 913 | 1 |
| 74 | 0 | 214 | 1 | 354 | 1 | 494 | 1 | 634 | 0 | 774 | 1 | 914 | 0 |
| 75 | 1 | 215 | 1 | 355 | 0 | 495 | 0 | 635 | 0 | 775 | 0 | 915 | 0 |
| 76 | 0 | 216 | 1 | 356 | 0 | 496 | 1 | 636 | 0 | 776 | 1 | 916 | 1 |
| 77 | 1 | 217 | 1 | 357 | 0 | 497 | 0 | 637 | 1 | 777 | 1 | 917 | 0 |
| 78 | 0 | 218 | 1 | 358 | 1 | 498 | 1 | 638 | 0 | 778 | 1 | 918 | 0 |
| 79 | 1 | 219 | 0 | 359 | 1 | 499 | 1 | 639 | 1 | 779 | 0 | 919 | 1 |

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 80 | 1 | 220 | 0 | 360 | 0 | 500 | 0 | 640 | 0 | 780 | 0 | 920 | 0 |
| 81 | 0 | 221 | 1 | 361 | 0 | 501 | 1 | 641 | 0 | 781 | 1 | 921 | 0 |
| 82 | 0 | 222 | 1 | 362 | 0 | 502 | 0 | 642 | 1 | 782 | 1 | 922 | 0 |
| 83 | 1 | 223 | 1 | 363 | 0 | 503 | 1 | 643 | 0 | 783 | 1 | 923 | 1 |
| 84 | 1 | 224 | 0 | 364 | 0 | 504 | 0 | 644 | 1 | 784 | 0 | 924 | 0 |
| 85 | 1 | 225 | 1 | 365 | 1 | 505 | 1 | 645 | 0 | 785 | 0 | 925 | 1 |
| 86 | 1 | 226 | 0 | 366 | 1 | 506 | 1 | 646 | 1 | 786 | 1 | 926 | 0 |
| 87 | 1 | 227 | 1 | 367 | 0 | 507 | 0 | 647 | 0 | 787 | 1 | 927 | 1 |
| 88 | 0 | 228 | 0 | 368 | 0 | 508 | 0 | 648 | 1 | 788 | 0 | 928 | 0 |
| 89 | 0 | 229 | 0 | 369 | 1 | 509 | 1 | 649 | 1 | 789 | 0 | 929 | 1 |
| 90 | 0 | 230 | 1 | 370 | 1 | 510 | 1 | 650 | 1 | 790 | 0 | 930 | 0 |
| 91 | 0 | 231 | 0 | 371 | 1 | 511 | 0 | 651 | 1 | 791 | 0 | 931 | 1 |
| 92 | 1 | 232 | 0 | 372 | 0 | 512 | 1 | 652 | 1 | 792 | 1 | 932 | 1 |
| 93 | 1 | 233 | 1 | 373 | 1 | 513 | 1 | 653 | 1 | 793 | 0 | 933 | 1 |
| 94 | 0 | 234 | 0 | 374 | 0 | 514 | 0 | 654 | 1 | 794 | 0 | 934 | 1 |
| 95 | 0 | 235 | 0 | 375 | 1 | 515 | 0 | 655 | 0 | 795 | 0 | 935 | 0 |
| 96 | 0 | 236 | 1 | 376 | 1 | 516 | 0 | 656 | 1 | 796 | 1 | 936 | 1 |
| 97 | 1 | 237 | 1 | 377 | 0 | 517 | 1 | 657 | 0 | 797 | 1 | 937 | 1 |
| 98 | 1 | 238 | 0 | 378 | 1 | 518 | 0 | 658 | 0 | 798 | 0 | 938 | 0 |
| 99 | 0 | 239 | 0 | 379 | 0 | 519 | 0 | 659 | 0 | 799 | 0 | 939 | 0 |
| 100 | 1 | 240 | 0 | 380 | 0 | 520 | 0 | 660 | 1 | 800 | 1 | 940 | 0 |
| 101 | 0 | 241 | 1 | 381 | 0 | 521 | 0 | 661 | 0 | 801 | 1 | 941 | 1 |
| 102 | 0 | 242 | 1 | 382 | 1 | 522 | 0 | 662 | 0 | 802 | 1 | 942 | 1 |
| 103 | 1 | 243 | 1 | 383 | 1 | 523 | 0 | 663 | 1 | 803 | 1 | 943 | 1 |
| 104 | 1 | 244 | 0 | 384 | 1 | 524 | 0 | 664 | 0 | 804 | 1 | 944 | 0 |
| 105 | 1 | 245 | 1 | 385 | 0 | 525 | 1 | 665 | 0 | 805 | 1 | 945 | 1 |
| 106 | 0 | 246 | 0 | 386 | 1 | 526 | 0 | 666 | 0 | 806 | 0 | 946 | 0 |
| 107 | 0 | 247 | 1 | 387 | 0 | 527 | 1 | 667 | 0 | 807 | 0 | 947 | 0 |
| 108 | 0 | 248 | 1 | 388 | 0 | 528 | 1 | 668 | 1 | 808 | 0 | 948 | 1 |
| 109 | 1 | 249 | 0 | 389 | 1 | 529 | 0 | 669 | 1 | 809 | 1 | 949 | 0 |
| 110 | 0 | 250 | 0 | 390 | 1 | 530 | 0 | 670 | 1 | 810 | 1 | 950 | 1 |
| 111 | 0 | 251 | 1 | 391 | 0 | 531 | 1 | 671 | 0 | 811 | 0 | 951 | 0 |
| 112 | 1 | 252 | 1 | 392 | 0 | 532 | 0 | 672 | 0 | 812 | 1 | 952 | 0 |
| 113 | 0 | 253 | 1 | 393 | 0 | 533 | 1 | 673 | 1 | 813 | 0 | 953 | 1 |
| 114 | 1 | 254 | 0 | 394 | 1 | 534 | 1 | 674 | 1 | 814 | 1 | 954 | 1 |
| 115 | 1 | 255 | 1 | 395 | 1 | 535 | 1 | 675 | 0 | 815 | 0 | 955 | 1 |
| 116 | 0 | 256 | 1 | 396 | 0 | 536 | 0 | 676 | 0 | 816 | 0 | 956 | 1 |
| 117 | 1 | 257 | 1 | 397 | 1 | 537 | 1 | 677 | 0 | 817 | 1 | 957 | 0 |
| 118 | 0 | 258 | 1 | 398 | 1 | 538 | 0 | 678 | 0 | 818 | 0 | 958 | 0 |
| 119 | 1 | 259 | 0 | 399 | 0 | 539 | 0 | 679 | 0 | 819 | 0 | 959 | 0 |
| 120 | 1 | 260 | 1 | 400 | 0 | 540 | 1 | 680 | 1 | 820 | 1 | 960 | 1 |
| 121 | 1 | 261 | 1 | 401 | 0 | 541 | 0 | 681 | 0 | 821 | 1 | 961 | 0 |
| 122 | 1 | 262 | 0 | 402 | 1 | 542 | 1 | 682 | 1 | 822 | 1 | 962 | 1 |
| 123 | 1 | 263 | 1 | 403 | 1 | 543 | 0 | 683 | 0 | 823 | 0 | 963 | 0 |

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 124 | 0 | 264 | 0 | 404 | 1 | 544 | 1 | 684 | 0 | 824 | 1 | 964 | 0 |
| 125 | 1 | 265 | 0 | 405 | 1 | 545 | 0 | 685 | 0 | 825 | 0 | 965 | 0 |
| 126 | 0 | 266 | 1 | 406 | 0 | 546 | 0 | 686 | 1 | 826 | 1 | 966 | 1 |
| 127 | 1 | 267 | 1 | 407 | 0 | 547 | 0 | 687 | 1 | 827 | 1 | 967 | 1 |
| 128 | 0 | 268 | 0 | 408 | 1 | 548 | 1 | 688 | 1 | 828 | 0 | 968 | 1 |
| 129 | 0 | 269 | 1 | 409 | 0 | 549 | 0 | 689 | 1 | 829 | 0 | 969 | 0 |
| 130 | 0 | 270 | 1 | 410 | 1 | 550 | 1 | 690 | 0 | 830 | 0 | 970 | 1 |
| 131 | 0 | 271 | 0 | 411 | 1 | 551 | 1 | 691 | 1 | 831 | 0 | 971 | 1 |
| 132 | 0 | 272 | 1 | 412 | 1 | 552 | 1 | 692 | 1 | 832 | 0 | 972 | 1 |
| 133 | 0 | 273 | 0 | 413 | 1 | 553 | 1 | 693 | 1 | 833 | 1 | 973 | 1 |
| 134 | 1 | 274 | 1 | 414 | 0 | 554 | 0 | 694 | 0 | 834 | 1 | 974 | 1 |
| 135 | 0 | 275 | 0 | 415 | 1 | 555 | 1 | 695 | 0 | 835 | 0 | 975 | 0 |
| 136 | 1 | 276 | 0 | 416 | 1 | 556 | 0 | 696 | 0 | 836 | 1 | 976 | 1 |
| 137 | 0 | 277 | 0 | 417 | 0 | 557 | 1 | 697 | 0 | 837 | 0 | 977 | 0 |
| 138 | 0 | 278 | 1 | 418 | 0 | 558 | 1 | 698 | 1 | 838 | 0 | | |
| 139 | 1 | 279 | 0 | 419 | 1 | 559 | 0 | 699 | 0 | 839 | 1 | | |

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