PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a,b) = (z^6+z^4+1, z+1)$

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Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a,b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a,b) in $\mathbb{F}_2[x] \setminus \mathbb{F}_2$. In this paper we prove the conjecture for $(a,b) = (z^6+z^4+1, z+1)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \ldots)$ of elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{\ddots + \cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a,b)-Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \ldots) = (a, b, b, a, b, a, a, b, \ldots)$$

that we shall denote

$$\mathrm{CF}(a,b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When $(a,b)=(z^6+z^4+1,z+1)$, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

where, for $\mathrm{CF}(a,b)$*,*

$$\begin{aligned}
p_0(z) &= z^{21}+z^{20}+z^{19}+z^{18}+z^{17}+z^{16}+z^{14}+z^{12}+z^{11}+z^8+z^7+z^6+z^5\\
&\quad +z^2+z+1,\\
p_1(z) &= z^{27}+z^{26}+z^{22}+z^{19}+z^{18}+z^{16}+z^{15}+z^{14}+z^{13}+z^{11}+z^9+z^4+z^2\\
&\quad +z,\\
p_2(z) &= z^{28}+z^{26}+z^{20}+z^{18}+z^{16}+z^8+z^2+1,\\
p_3(z) &= z^{27}+z^{26}+z^{22}+z^{19}+z^{18}+z^{16}+z^{15}+z^{14}+z^{13}+z^{11}+z^9+z^4+z^2\\
&\quad +z,\\
p_4(z) &= z^{26}+z^{21}+z^{20}+z^{19}+z^{17}+z^{14}+z^{12}+z^{11}+z^7+z^6+z^5+z+1,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$*,*

$$\begin{aligned}
p_0(z) &= z^{24}+z^{21}+z^{20}+z^{19}+z^{18}+z^{17}+z^{14}+z^{12}+z^{11}+z^8+z^7+z^6+z^5\\
&\quad +z^4+z^2+z+1,\\
p_1(z) &= z^{27}+z^{26}+z^{22}+z^{19}+z^{18}+z^{16}+z^{15}+z^{14}+z^{13}+z^{11}+z^9+z^4+z^2\\
&\quad +z,\\
p_2(z) &= z^{28}+z^{26}+z^{20}+z^{18}+z^{16}+z^8+z^2+1,\\
p_3(z) &= z^{27}+z^{26}+z^{22}+z^{19}+z^{18}+z^{16}+z^{15}+z^{14}+z^{13}+z^{11}+z^9+z^4+z^2\\
&\quad +z,\\
p_4(z) &= z^{26}+z^{24}+z^{21}+z^{20}+z^{19}+z^{17}+z^{16}+z^{14}+z^{12}+z^{11}+z^7+z^6+z^5\\
&\quad +z^4+z+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\ge 0}$ and $(Q_n(z))_{n\ge 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x) &= x^{\deg(t_{2^n-1})}\begin{pmatrix} t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x) &= x^{\deg(\bar t_{2^n-1})}\begin{pmatrix} \bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix} \bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times
\end{aligned}$$

$$\cdots \times x^{\deg(\bar{t}_0)} \begin{pmatrix} \bar{t}_0(1/x) & 1 \\ 1 & 0 \end{pmatrix},$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n \geq 0$

$$M_{n+1}(x) = W_n(x) \cdot M_n(x),$$
$$W_{n+1}(x) = M_n(x) \cdot W_n(x).$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n} \tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n} \tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \leq i,j \leq 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \leq i,j \leq 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \leq i,j \leq 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(175, 175, 175, 175, 175)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \geq 0$. If for some $n \geq 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \geq 0$, $q_1(0) = 1$, and $q_j(0) = 0$

for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0} M^o_{0,0}$ and $W^o_{0,1} M^o_{1,0}$. We know that

$$P(x, y) = \mathrm{Res}_z \left(\phi(W^o, 0, 0)(x, z),\ z^{12} \cdot \phi(M^o, 0, 0)(x, y/z) \right)$$

is an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0} M^o_{0,0}$, that will be called $\phi_0(x, y)$. To prove that $\phi_0(x, y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x, y)$ of multiplicity m and that $Q(x, y) := P(x, y)/\phi_0(x, y)^m$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0} M^o_{0,0}$ to order 630 and substitute it for y in $Q(x, y)$. We get a series of valuation less than 630, which proves that $Q(x, y)$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We find the minimal polynomial $\phi_1(x, y)$ of $W^o_{0,1} M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e, 0, 0)$ is the minimal polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. We know that

$$S(x, y) = \mathrm{Res}_z \left(\phi_0(x, z),\ \phi_1(x, y + z) \right)$$

is an annihilating of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. We verify that $\phi(M^e, 0, 0)$ is an irreducible factor of $S(x, y)$ of multiplicity μ, and that $Q(x, y) := S(x, y)/\phi(M^e, 0, 0)^\mu$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$ to order 770 and substitute it for y in $Q(x, y)$. We get a series of valuation less than 770, and therefore $Q(x, y)$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e, 0, 0)$, we know that the two series are one and the same. ☐

Define

$$R^e = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned} M^e[:14u] = {} & M^e[:7u] + x^u M^e[:6u] + x^{4u} M^e[:3u] + x^{5u} M^e[:2u] + x^{6u} M^e[:u] \\ & + x^u M^e[:7u] + x^{2u} M^e[:6u] + x^{5u} M^e[:3u] + x^{6u} M^e[:2u] \end{aligned}$$

$$
\begin{aligned}
&+ x^{7u}M^e[:u] + x^{2u}M^e[:7u] + x^{3u}M^e[:6u] + x^{6u}M^e[:3u]\\
&+ x^{7u}M^e[:2u] + x^{8u}M^e[:u] + x^{3u}M^e[:7u] + x^{4u}M^e[:6u]\\
&+ x^{7u}M^e[:3u] + x^{8u}M^e[:2u] + x^{9u}M^e[:u] + x^{6u}M^e[:7u]\\
&+ x^{7u}M^e[:6u] + x^{10u}M^e[:3u] + x^{11u}M^e[:2u] + x^{12u}M^e[:u]\\
&+ x^{7u}M^e[:7u] + x^{8u}M^e[:6u] + x^{11u}M^e[:3u] + x^{12u}M^e[:2u]\\
&+ x^{13u}M^e[:u] + (x^{7u} + x^{8u} + x^{9u} + x^{10u} + x^{13u})R^e,\\
W^e[:14u] = {} & W^e[:7u] + x^uW^e[:6u] + x^{4u}W^e[:3u] + x^{5u}W^e[:2u] + x^{6u}W^e[:u]\\
&+ x^uW^e[:7u] + x^{2u}W^e[:6u] + x^{5u}W^e[:3u] + x^{6u}W^e[:2u]\\
&+ x^{7u}W^e[:u] + x^{2u}W^e[:7u] + x^{3u}W^e[:6u] + x^{6u}W^e[:3u]\\
&+ x^{7u}W^e[:2u] + x^{8u}W^e[:u] + x^{3u}W^e[:7u] + x^{4u}W^e[:6u]\\
&+ x^{7u}W^e[:3u] + x^{8u}W^e[:2u] + x^{9u}W^e[:u] + x^{6u}W^e[:7u]\\
&+ x^{7u}W^e[:6u] + x^{10u}W^e[:3u] + x^{11u}W^e[:2u] + x^{12u}W^e[:u]\\
&+ x^{7u}W^e[:7u] + x^{8u}W^e[:6u] + x^{11u}W^e[:3u] + x^{12u}W^e[:2u]\\
&+ x^{13u}W^e[:u] + (x^{7u} + x^{8u} + x^{9u} + x^{10u} + x^{13u})R^e.
\end{aligned}
$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$
\begin{aligned}
M^o[:14u] = {} & M^o[:7u] + x^uM^o[:6u] + x^{4u}M^o[:3u] + x^{5u}M^o[:2u] + x^{6u}M^o[:u]\\
&+ x^uM^o[:7u] + x^{2u}M^o[:6u] + x^{5u}M^o[:3u] + x^{6u}M^o[:2u]\\
&+ x^{7u}M^o[:u] + x^{2u}M^o[:7u] + x^{3u}M^o[:6u] + x^{6u}M^o[:3u]\\
&+ x^{7u}M^o[:2u] + x^{8u}M^o[:u] + x^{3u}M^o[:7u] + x^{4u}M^o[:6u]\\
&+ x^{7u}M^o[:3u] + x^{8u}M^o[:2u] + x^{9u}M^o[:u] + x^{6u}M^o[:7u]\\
&+ x^{7u}M^o[:6u] + x^{10u}M^o[:3u] + x^{11u}M^o[:2u] + x^{12u}M^o[:u]\\
&+ x^{7u}M^o[:7u] + x^{8u}M^o[:6u] + x^{11u}M^o[:3u] + x^{12u}M^o[:2u]\\
&+ x^{13u}M^o[:u] + (x^{7u} + x^{8u} + x^{9u} + x^{10u} + x^{13u})R^o,\\
W^o[:14u] = {} & W^o[:7u] + x^uW^o[:6u] + x^{4u}W^o[:3u] + x^{5u}W^o[:2u] + x^{6u}W^o[:u]\\
&+ x^uW^o[:7u] + x^{2u}W^o[:6u] + x^{5u}W^o[:3u] + x^{6u}W^o[:2u]\\
&+ x^{7u}W^o[:u] + x^{2u}W^o[:7u] + x^{3u}W^o[:6u] + x^{6u}W^o[:3u]\\
&+ x^{7u}W^o[:2u] + x^{8u}W^o[:u] + x^{3u}W^o[:7u] + x^{4u}W^o[:6u]\\
&+ x^{7u}W^o[:3u] + x^{8u}W^o[:2u] + x^{9u}W^o[:u] + x^{6u}W^o[:7u]\\
&+ x^{7u}W^o[:6u] + x^{10u}W^o[:3u] + x^{11u}W^o[:2u] + x^{12u}W^o[:u]\\
&+ x^{7u}W^o[:7u] + x^{8u}W^o[:6u] + x^{11u}W^o[:3u] + x^{12u}W^o[:2u]\\
&+ x^{13u}W^o[:u] + (x^{7u} + x^{8u} + x^{9u} + x^{10u} + x^{13u})R^o.
\end{aligned}
$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$T[:14u] = T[:7u] + x^uT[:6u] + x^{4u}T[:3u] + x^{5u}T[:2u] + x^{6u}T[:u] + x^uT[:7u]$$

$$
\begin{aligned}
&+ x^{2u}T[:6u] + x^{5u}T[:3u] + x^{6u}T[:2u] + x^{7u}T[:u] + x^{2u}T[:7u]\\
&+ x^{3u}T[:6u] + x^{6u}T[:3u] + x^{7u}T[:2u] + x^{8u}T[:u] + x^{3u}T[:7u]\\
&+ x^{4u}T[:6u] + x^{7u}T[:3u] + x^{8u}T[:2u] + x^{9u}T[:u] + x^{6u}T[:7u]\\
&+ x^{7u}T[:6u] + x^{10u}T[:3u] + x^{11u}T[:2u] + x^{12u}T[:u] + x^{7u}T[:7u]\\
&+ x^{8u}T[:6u] + x^{11u}T[:3u] + x^{12u}T[:2u] + x^{13u}T[:u]\\
&+ (x^{7u} + x^{8u} + x^{9u} + x^{10u} + x^{13u}).
\end{aligned}
$$

The proof of the other 15 cases are similar. We break down the above identity into 7 parts:

$$
\begin{aligned}
x^7u &= x^{7u}T[:u] + x^{6u}T[u:2u] + x^{5u}T[2u:3u] + x^{4u}T[3u:4u]\\
&\quad + x^{u}T[6u:7u] + T[7u:8u],\\
x^8u &= x^{8u}T[:u] + x^{4u}T[4u:5u] + x^{2u}T[6u:7u] + T[8u:9u],\\
x^9u &= x^{9u}T[:u] + x^{7u}T[2u:3u] + x^{6u}T[3u:4u] + x^{4u}T[5u:6u]\\
&\quad + x^{3u}T[6u:7u] + T[9u:10u],\\
x^10u &= x^{10u}T[:u] + x^{8u}T[2u:3u] + x^{6u}T[4u:5u] + T[10u:11u],\\
0 &= x^{10u}T[u:2u] + x^{8u}T[3u:4u] + x^{6u}T[5u:6u] + T[11u:12u],\\
0 &= x^{10u}T[2u:3u] + x^{8u}T[4u:5u] + x^{6u}T[6u:7u] + T[12u:13u],\\
x^13u &= x^{13u}T[:u] + x^{12u}T[u:2u] + x^{11u}T[2u:3u] + x^{8u}T[5u:6u]\\
&\quad + x^{7u}T[6u:7u] + T[13u:14u],
\end{aligned}
$$

which can be rewritten as

$$
\begin{aligned}
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[110w]_2]\\
&\quad + T[[111w]_2], \qquad (2.7)\\
0 &= T[[w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1000w]_2], \qquad (2.8)\\
0 &= T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2]\\
&\quad + T[[1001w]_2], \qquad (2.9)\\
0 &= T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[1010w]_2], \qquad (2.10)\\
0 &= T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1011w]_2], \qquad (2.11)\\
0 &= T[[10w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1100w]_2], \qquad (2.12)\\
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[101w]_2] + T[[110w]_2]\\
&\quad + T[[1101w]_2], \qquad (2.13)
\end{aligned}
$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$
\begin{aligned}
1 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[110w]_2]\\
&\quad + T[[111w]_2], \qquad (2.14)\\
1 &= T[[w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1000w]_2], \qquad (2.15)\\
1 &= T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2]\\
&\quad + T[[1001w]_2], \qquad (2.16)\\
1 &= T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[1010w]_2], \qquad (2.17)
\end{aligned}
$$

$$0 = T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1011w]_2], \tag{2.18}$$

$$0 = T[[10w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1100w]_2], \tag{2.19}$$

$$\begin{aligned} 1 = {} & T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[101w]_2] + T[[110w]_2] \\ & + T[[1101w]_2], \end{aligned} \tag{2.20}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 903 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.20) can be written as

$$\begin{aligned} 0 = {} & \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,110)) \\ & + \tau(A(s,111)), \end{aligned} \tag{2.21}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,100)) + \tau(A(s,110)) + \tau(A(s,1000)), \tag{2.22}$$

$$\begin{aligned} 0 = {} & \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101)) \\ & + \tau(A(s,110)) + \tau(A(s,1001)), \end{aligned} \tag{2.23}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,1010)), \tag{2.24}$$

$$0 = \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,1011)), \tag{2.25}$$

$$0 = \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,110)) + \tau(A(s,1100)), \tag{2.26}$$

$$\begin{aligned} 0 = {} & \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,101)) \\ & + \tau(A(s,110)) + \tau(A(s,1101)), \end{aligned} \tag{2.27}$$

for all $s \in E_{2k}$, and

$$\begin{aligned} 1 = {} & \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,110)) \\ & + \tau(A(s,111)), \end{aligned} \tag{2.28}$$

$$1 = \tau(A(s,\epsilon)) + \tau(A(s,100)) + \tau(A(s,110)) + \tau(A(s,1000)), \tag{2.29}$$

$$\begin{aligned} 1 = {} & \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101)) \\ & + \tau(A(s,110)) + \tau(A(s,1001)), \end{aligned} \tag{2.30}$$

$$1 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,1010)), \tag{2.31}$$

$$0 = \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,1011)), \tag{2.32}$$

$$0 = \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,110)) + \tau(A(s,1100)), \tag{2.33}$$

$$\begin{aligned} 1 = {} & \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,101)) \\ & + \tau(A(s,110)) + \tau(A(s,1101)), \end{aligned} \tag{2.34}$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{20}), E_{20}) = (A(i, 0^{16}), E_{16})$, so that we only have to verify that identities (2.21) through (2.34) hold for $2 \leq k \leq 10$, which turns out to be true. $\square$

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \geq 2$, and $u = 2^{2k-1}$,*

$$\begin{aligned} M_{2k} &= M^e[:7u] + x^u M^e[:6u] + x^{4u} M^e[:3u] + x^{5u} M^e[:2u] + x^{6u} M^e[:u] \\ &\quad + x^{7u} R^e, \\ W_{2k} &= W^e[:7u] + x^u W^e[:6u] + x^{4u} W^e[:3u] + x^{5u} W^e[:2u] + x^{6u} W^e[:u] \\ &\quad + x^{7u} R^e. \end{aligned}$$

For all integer $k \geq 2$, and $u = 2^{2k}$,

$$\begin{aligned} M_{2k+1} &= M^o[:7u] + x^u M^o[:6u] + x^{4u} M^o[:3u] + x^{5u} M^o[:2u] + x^{6u} M^o[:u] \\ &\quad + x^{7u} R^o, \\ W_{2k+1} &= W^o[:7u] + x^u W^o[:6u] + x^{4u} W^o[:3u] + x^{5u} W^o[:2u] + x^{6u} W^o[:u] \\ &\quad + x^{7u} R^o. \end{aligned}$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \geq 2$, we claim that

$$\begin{aligned} M_{2k} \wedge W_{2k} &\Rightarrow M_{2k+1} \wedge W_{2k+1}, \\ M_{2k+1} \wedge W_{2k+1} &\Rightarrow M_{2k+2} \wedge W_{2k+2}. \end{aligned}$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$\begin{aligned} W_{2k} M_{2k} &= \big(W^e[:7v] + x^v W^e[:6v] + x^{4v} W^e[:3v] + x^{5v} W^e[:2v] + x^{6v} W^e[:v] \\ &\quad + x^{7u} R^e\big) \times \big(M^e[:7v] + x^v M^e[:6v] + x^{4v} M^e[:3v] + x^{5v} M^e[:2v] \\ &\quad + x^{6v} M^e[:v] + x^{7u} R^e\big); \end{aligned} \tag{2.35}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{14v} R^o$. Therefore we only need to prove that their difference is $O(x^{14v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{14v}, to

$$\begin{aligned} &W^e[:14v] M^e[:14v] + x^{2v} W^e[:12v] M^e[:12v] + x^{8v} W^e[:6v] M^e[:6v] \\ &\qquad + x^{10v} W^e[:4v] M^e[:4v] + x^{12v} W^e[:2v] M^e[:2v]. \end{aligned}$$

For all $n \leq 14$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.35) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{14}, b_1, ..., b_{14}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{14})$, which completes the proof. □

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a,b)$; for $\mathrm{CF}(b,a)$, the proof is similar. By Lemma 2.3, we have For all $0 \le j,k \le 1$,

$$\begin{aligned}\lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k},\\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k},\\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k},\\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}.\end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a,b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e,0,1)$ and $\phi(M^e,0,0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x,y) = \mathrm{Res}_t\left(\phi(M^e,0,1)(x,t), y^{12}\phi(M^e,0,1)(x,t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x,y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned}q_0(x) &= x^{28}+x^{27}+x^{26}+x^{23}+x^{22}+x^{21}+x^{20}+x^{17}+x^{16}+x^{14}+x^{12}+x^{11}\\ &\quad +x^{10}+x^9+x^8+x^7,\\ q_1(x) &= x^{27}+x^{26}+x^{24}+x^{19}+x^{17}+x^{15}+x^{14}+x^{13}+x^{12}+x^{10}+x^9+x^6\\ &\quad +x^2+x,\\ q_2(x) &= x^{28}+x^{26}+x^{20}+x^{12}+x^{10}+x^8+x^2+1,\\ q_3(x) &= x^{27}+x^{26}+x^{24}+x^{19}+x^{17}+x^{15}+x^{14}+x^{13}+x^{12}+x^{10}+x^9+x^6\\ &\quad +x^2+x,\\ q_4(x) &= x^{28}+x^{27}+x^{23}+x^{22}+x^{21}+x^{17}+x^{16}+x^{14}+x^{11}+x^9+x^8+x^7\\ &\quad +x^2.\end{aligned}$$

The polynomial $Q(x,y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and $R(x,y) := P(x,y)/Q(x,y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 216, and substitute it for y in $R(z,y)$, we get a series with valuation smaller than 216, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b) = f(1/z)$. □

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{132} + x^{130} + x^{128} + x^{126} + x^{124} + x^{120} + x^{114} + x^{110} + x^{106} + x^{104} \\
&\quad + x^{102} + x^{96} + x^{94} + x^{90} + x^{88} + x^{84} + x^{80} + x^{78} + x^{76} + x^{74} + x^{70} \\
&\quad + x^{64} + x^{62} + x^{54} + x^{48} + x^{42} + x^{22} + x^{16} + x^{12}, \\
p_3(x) &= x^{130} + x^{126} + x^{124} + x^{122} + x^{120} + x^{110} + x^{106} + x^{104} + x^{100} + x^{98} \\
&\quad + x^{96} + x^{92} + x^{90} + x^{86} + x^{84} + x^{78} + x^{68} + x^{66} + x^{64} + x^{60} + x^{58} \\
&\quad + x^{56} + x^{54} + x^{52} + x^{50} + x^{48} + x^{44} + x^{38} + x^{36} + x^{34} + x^{22} + x^{18} \\
&\quad + x^{10} + x^{8}, \\
p_6(x) &= x^{130} + x^{128} + x^{120} + x^{118} + x^{116} + x^{114} + x^{112} + x^{100} + x^{98} + x^{96} \\
&\quad + x^{94} + x^{92} + x^{90} + x^{88} + x^{80} + x^{76} + x^{68} + x^{66} + x^{64} + x^{62} + x^{60} \\
&\quad + x^{58} + x^{56} + x^{54} + x^{52} + x^{44} + x^{42} + x^{40} + x^{36} + x^{28} + x^{26} + x^{22} \\
&\quad + x^{20} + x^{18} + x^{16} + x^{12} + x^{10} + x^{4} + x^{2} + 1, \\
p_9(x) &= x^{130} + x^{124} + x^{118} + x^{116} + x^{112} + x^{108} + x^{106} + x^{104} + x^{100} + x^{96} \\
&\quad + x^{92} + x^{78} + x^{68} + x^{64} + x^{58} + x^{48} + x^{40} + x^{32} + x^{26} + x^{22} + x^{20} \\
&\quad + x^{16} + x^{12} + x^{10} + x^{6} + x^{2}, \\
p_{12}(x) &= x^{132} + x^{120} + x^{116} + x^{112} + x^{52} + x^{40} + x^{36} + x^{32} + x^{20} + x^{8} + x^{4} + 1 \\
&\quad ,
\end{aligned}$$

and the initial terms are [1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0].

For $\phi(M^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{147} + x^{146} + x^{144} + x^{143} + x^{140} + x^{139} + x^{134} + x^{132} + x^{127} + x^{124} \\
&\quad + x^{122} + x^{120} + x^{114} + x^{110} + x^{109} + x^{108} + x^{106} + x^{105} + x^{104} \\
&\quad + x^{103} + x^{102} + x^{100} + x^{99} + x^{97} + x^{95} + x^{94} + x^{90} + x^{89} + x^{88} + x^{87} \\
&\quad + x^{85} + x^{83} + x^{81} + x^{77} + x^{76} + x^{74} + x^{72} + x^{70} + x^{60} + x^{59} + x^{58} \\
&\quad + x^{56} + x^{47} + x^{46} + x^{45} + x^{44} + x^{42} + x^{40} + x^{38} + x^{37} + x^{36} + x^{33} \\
&\quad + x^{32} + x^{31} + x^{27}, \\
p_3(x) &= x^{138} + x^{136} + x^{135} + x^{131} + x^{130} + x^{127} + x^{125} + x^{122} + x^{120} + x^{119} \\
&\quad + x^{118} + x^{114} + x^{113} + x^{111} + x^{110} + x^{109} + x^{107} + x^{106} + x^{105} \\
&\quad + x^{103} + x^{101} + x^{98} + x^{96} + x^{95} + x^{94} + x^{89} + x^{88} + x^{86} + x^{85} + x^{82} \\
&\quad + x^{81} + x^{80} + x^{79} + x^{75} + x^{72} + x^{69} + x^{67} + x^{64} + x^{62} + x^{61} + x^{58} \\
&\quad + x^{57} + x^{55} + x^{53} + x^{48} + x^{46} + x^{45} + x^{42} + x^{41} + x^{40} + x^{39} + x^{35} \\
&\quad + x^{34} + x^{31} + x^{29} + x^{26} + x^{24} + x^{23} + x^{22} + x^{17} + x^{16} + x^{14} + x^{13}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{9},\\
p_6(x) &= x^{141} + x^{140} + x^{139} + x^{138} + x^{137} + x^{136} + x^{135} + x^{134} + x^{133} + x^{132}\\
&+ x^{131} + x^{130} + x^{127} + x^{126} + x^{123} + x^{122} + x^{119} + x^{118} + x^{115}\\
&+ x^{114} + x^{109} + x^{108} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100}\\
&+ x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85}\\
&+ x^{84} + x^{83} + x^{82} + x^{79} + x^{78} + x^{73} + x^{72} + x^{71} + x^{70} + x^{65} + x^{64}\\
&+ x^{63} + x^{62} + x^{59} + x^{58} + x^{51} + x^{50} + x^{47} + x^{46} + x^{43} + x^{42} + x^{41}\\
&+ x^{40} + x^{39} + x^{38} + x^{35} + x^{34} + x^{29} + x^{28} + x^{25} + x^{24} + x^{21} + x^{20}\\
&+ x^{17} + x^{16} + x^{13} + x^{12} + x^{11} + x^{10} + x^{7} + x^{6},\\
p_9(x) &= x^{144} + x^{141} + x^{140} + x^{139} + x^{138} + x^{137} + x^{134} + x^{133} + x^{131} + x^{130}\\
&+ x^{129} + x^{127} + x^{126} + x^{125} + x^{124} + x^{123} + x^{122} + x^{120} + x^{116}\\
&+ x^{115} + x^{114} + x^{113} + x^{109} + x^{108} + x^{106} + x^{105} + x^{104} + x^{102}\\
&+ x^{100} + x^{98} + x^{96} + x^{91} + x^{84} + x^{82} + x^{81} + x^{77} + x^{74} + x^{72} + x^{71}\\
&+ x^{70} + x^{68} + x^{63} + x^{62} + x^{60} + x^{59} + x^{58} + x^{56} + x^{54} + x^{52} + x^{48}\\
&+ x^{47} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{37} + x^{36} + x^{33} + x^{31} + x^{30}\\
&+ x^{27} + x^{26} + x^{25} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18} + x^{16} + x^{12}\\
&+ x^{11} + x^{9} + x^{8} + x^{7} + x^{5} + x^{3},\\
p_{12}(x) &= x^{147} + x^{146} + x^{141} + x^{140} + x^{139} + x^{138} + x^{137} + x^{136} + x^{131} + x^{130}\\
&+ x^{125} + x^{124} + x^{121} + x^{120} + x^{117} + x^{116} + x^{115} + x^{114} + x^{113}\\
&+ x^{112} + x^{67} + x^{66} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{51} + x^{50}\\
&+ x^{45} + x^{44} + x^{41} + x^{40} + x^{37} + x^{36} + x^{33} + x^{32} + x^{29} + x^{28} + x^{27}\\
&+ x^{26} + x^{25} + x^{24} + x^{19} + x^{18} + x^{13} + x^{12} + x^{9} + x^{8} + x^{5} + x^{4} + x^{3}\\
&+ x^{2} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 1, 1, 1].

For $\phi(M^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{147} + x^{146} + x^{144} + x^{143} + x^{140} + x^{139} + x^{134} + x^{132} + x^{127} + x^{124}\\
&+ x^{122} + x^{120} + x^{114} + x^{110} + x^{109} + x^{108} + x^{106} + x^{105} + x^{104}\\
&+ x^{103} + x^{102} + x^{100} + x^{99} + x^{97} + x^{95} + x^{94} + x^{90} + x^{89} + x^{88} + x^{87}\\
&+ x^{85} + x^{83} + x^{81} + x^{77} + x^{76} + x^{74} + x^{72} + x^{70} + x^{60} + x^{59} + x^{58}\\
&+ x^{56} + x^{47} + x^{46} + x^{45} + x^{44} + x^{42} + x^{40} + x^{38} + x^{37} + x^{36} + x^{33}\\
&+ x^{32} + x^{31} + x^{27},\\
p_3(x) &= x^{138} + x^{136} + x^{135} + x^{131} + x^{130} + x^{127} + x^{125} + x^{122} + x^{120} + x^{119}\\
&+ x^{118} + x^{114} + x^{113} + x^{111} + x^{110} + x^{109} + x^{107} + x^{106} + x^{105}\\
&+ x^{103} + x^{101} + x^{98} + x^{96} + x^{95} + x^{94} + x^{89} + x^{88} + x^{86} + x^{85} + x^{82}\\
&+ x^{81} + x^{80} + x^{79} + x^{75} + x^{72} + x^{69} + x^{67} + x^{64} + x^{62} + x^{61} + x^{58}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{57} + x^{55} + x^{53} + x^{48} + x^{46} + x^{45} + x^{42} + x^{41} + x^{40} + x^{39} + x^{35}\\
&+ x^{34} + x^{31} + x^{29} + x^{26} + x^{24} + x^{23} + x^{22} + x^{17} + x^{16} + x^{14} + x^{13}\\
&+ x^{9},\\
p_6(x) &= x^{141} + x^{140} + x^{139} + x^{138} + x^{137} + x^{136} + x^{135} + x^{134} + x^{133} + x^{132}\\
&+ x^{131} + x^{130} + x^{127} + x^{126} + x^{123} + x^{122} + x^{119} + x^{118} + x^{115}\\
&+ x^{114} + x^{109} + x^{108} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100}\\
&+ x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85}\\
&+ x^{84} + x^{83} + x^{82} + x^{79} + x^{78} + x^{73} + x^{72} + x^{71} + x^{70} + x^{65} + x^{64}\\
&+ x^{63} + x^{62} + x^{59} + x^{58} + x^{51} + x^{50} + x^{47} + x^{46} + x^{43} + x^{42} + x^{41}\\
&+ x^{40} + x^{39} + x^{38} + x^{35} + x^{34} + x^{29} + x^{28} + x^{25} + x^{24} + x^{21} + x^{20}\\
&+ x^{17} + x^{16} + x^{13} + x^{12} + x^{11} + x^{10} + x^{7} + x^{6},\\
p_9(x) &= x^{144} + x^{141} + x^{140} + x^{139} + x^{138} + x^{137} + x^{134} + x^{133} + x^{131} + x^{130}\\
&+ x^{129} + x^{127} + x^{126} + x^{125} + x^{124} + x^{123} + x^{122} + x^{120} + x^{116}\\
&+ x^{115} + x^{114} + x^{113} + x^{109} + x^{108} + x^{106} + x^{105} + x^{104} + x^{102}\\
&+ x^{100} + x^{98} + x^{96} + x^{91} + x^{84} + x^{82} + x^{81} + x^{77} + x^{74} + x^{72} + x^{71}\\
&+ x^{70} + x^{68} + x^{63} + x^{62} + x^{60} + x^{59} + x^{58} + x^{56} + x^{54} + x^{52} + x^{48}\\
&+ x^{47} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{37} + x^{36} + x^{33} + x^{31} + x^{30}\\
&+ x^{27} + x^{26} + x^{25} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18} + x^{16} + x^{12}\\
&+ x^{11} + x^{9} + x^{8} + x^{7} + x^{5} + x^{3},\\
p_{12}(x) &= x^{147} + x^{146} + x^{141} + x^{140} + x^{139} + x^{138} + x^{137} + x^{136} + x^{131} + x^{130}\\
&+ x^{125} + x^{124} + x^{121} + x^{120} + x^{117} + x^{116} + x^{115} + x^{114} + x^{113}\\
&+ x^{112} + x^{67} + x^{66} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{51} + x^{50}\\
&+ x^{45} + x^{44} + x^{41} + x^{40} + x^{37} + x^{36} + x^{33} + x^{32} + x^{29} + x^{28} + x^{27}\\
&+ x^{26} + x^{25} + x^{24} + x^{19} + x^{18} + x^{13} + x^{12} + x^{9} + x^{8} + x^{5} + x^{4} + x^{3}\\
&+ x^{2} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 1, 1, 1].

For $\phi(M^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{162} + x^{158} + x^{156} + x^{150} + x^{148} + x^{140} + x^{138} + x^{122} + x^{120} + x^{116}\\
&+ x^{110} + x^{106} + x^{104} + x^{102} + x^{100} + x^{94} + x^{90} + x^{88} + x^{86} + x^{82}\\
&+ x^{80} + x^{78} + x^{76} + x^{74} + x^{70} + x^{66} + x^{58} + x^{56} + x^{54} + x^{46} + x^{42},\\
p_3(x) &= x^{160} + x^{158} + x^{150} + x^{148} + x^{138} + x^{136} + x^{130} + x^{128} + x^{126} + x^{124}\\
&+ x^{122} + x^{116} + x^{114} + x^{112} + x^{104} + x^{100} + x^{96} + x^{92} + x^{90} + x^{88}\\
&+ x^{82} + x^{80} + x^{66} + x^{62} + x^{56} + x^{52} + x^{40} + x^{36} + x^{30} + x^{28} + x^{26}\\
&+ x^{20} + x^{8} + x^{6},\\
p_6(x) &= x^{160} + x^{158} + x^{156} + x^{144} + x^{142} + x^{136} + x^{130} + x^{128} + x^{126} + x^{118}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{114} + x^{112} + x^{106} + x^{104} + x^{100} + x^{98} + x^{90} + x^{86} + x^{82} + x^{78}\\
&+ x^{74} + x^{72} + x^{70} + x^{68} + x^{66} + x^{62} + x^{60} + x^{56} + x^{50} + x^{48} + x^{46}\\
&+ x^{40} + x^{38} + x^{36} + x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{20} + x^{18} + x^{12}\\
&+ x^{6} + 1,\\
p_9(x) &= x^{160} + x^{158} + x^{154} + x^{152} + x^{146} + x^{142} + x^{140} + x^{138} + x^{136} + x^{130}\\
&+ x^{128} + x^{126} + x^{124} + x^{116} + x^{104} + x^{100} + x^{94} + x^{86} + x^{84} + x^{78}\\
&+ x^{76} + x^{66} + x^{64} + x^{62} + x^{56} + x^{54} + x^{50} + x^{48} + x^{46} + x^{44} + x^{42}\\
&+ x^{38} + x^{32} + x^{26} + x^{20} + x^{18} + x^{14} + x^{12} + x^{4} + x^{2},\\
p_{12}(x) &= x^{162} + x^{160} + x^{130} + x^{128} + x^{114} + x^{112} + x^{82} + x^{80} + x^{34} + x^{32}\\
&+ x^{18} + x^{16} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0].
For $\phi(W^e, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{162} + x^{154} + x^{150} + x^{146} + x^{140} + x^{132} + x^{130} + x^{128} + x^{124} + x^{122}\\
&+ x^{114} + x^{110} + x^{108} + x^{106} + x^{104} + x^{98} + x^{94} + x^{86} + x^{80} + x^{78}\\
&+ x^{76} + x^{72} + x^{70} + x^{68} + x^{64} + x^{62} + x^{60} + x^{58} + x^{48} + x^{38} + x^{36}\\
&+ x^{32} + x^{30} + x^{28} + x^{22} + x^{18} + x^{16} + x^{14} + x^{12},\\
p_3(x) &= x^{160} + x^{158} + x^{156} + x^{152} + x^{144} + x^{138} + x^{136} + x^{134} + x^{132} + x^{130}\\
&+ x^{128} + x^{118} + x^{116} + x^{112} + x^{104} + x^{96} + x^{94} + x^{92} + x^{88} + x^{82}\\
&+ x^{80} + x^{66} + x^{64} + x^{62} + x^{60} + x^{58} + x^{56} + x^{50} + x^{46} + x^{44} + x^{42}\\
&+ x^{38} + x^{36} + x^{26} + x^{22} + x^{20} + x^{18} + x^{16} + x^{10} + x^{8},\\
p_6(x) &= x^{160} + x^{156} + x^{154} + x^{150} + x^{148} + x^{146} + x^{144} + x^{142} + x^{136} + x^{134}\\
&+ x^{132} + x^{130} + x^{128} + x^{124} + x^{120} + x^{118} + x^{116} + x^{114} + x^{106}\\
&+ x^{104} + x^{102} + x^{100} + x^{98} + x^{92} + x^{90} + x^{88} + x^{86} + x^{82} + x^{78} + x^{74}\\
&+ x^{72} + x^{70} + x^{68} + x^{60} + x^{54} + x^{50} + x^{44} + x^{42} + x^{38} + x^{34} + x^{32}\\
&+ x^{26} + x^{16} + x^{12} + x^{8} + x^{6} + x^{4} + 1,\\
p_9(x) &= x^{160} + x^{158} + x^{154} + x^{152} + x^{146} + x^{142} + x^{140} + x^{138} + x^{136} + x^{130}\\
&+ x^{128} + x^{126} + x^{124} + x^{116} + x^{104} + x^{100} + x^{94} + x^{86} + x^{84} + x^{78}\\
&+ x^{76} + x^{66} + x^{64} + x^{62} + x^{56} + x^{54} + x^{50} + x^{48} + x^{46} + x^{44} + x^{42}\\
&+ x^{38} + x^{32} + x^{26} + x^{20} + x^{18} + x^{14} + x^{12} + x^{4} + x^{2},\\
p_{12}(x) &= x^{162} + x^{160} + x^{130} + x^{128} + x^{114} + x^{112} + x^{82} + x^{80} + x^{34} + x^{32}\\
&+ x^{18} + x^{16} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0].
For $\phi(W^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{147} + x^{146} + x^{144} + x^{143} + x^{141} + x^{138} + x^{134} + x^{131} + x^{129} + x^{128}\\
&+ x^{125} + x^{123} + x^{121} + x^{120} + x^{119} + x^{117} + x^{113} + x^{112} + x^{111}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{110} + x^{109} + x^{108} + x^{107} + x^{106} + x^{105} + x^{104} + x^{103} + x^{102}\\
&+ x^{101} + x^{100} + x^{99} + x^{98} + x^{96} + x^{90} + x^{88} + x^{87} + x^{85} + x^{84} + x^{83}\\
&+ x^{81} + x^{79} + x^{78} + x^{77} + x^{75} + x^{70} + x^{69} + x^{68} + x^{64} + x^{63} + x^{62}\\
&+ x^{60} + x^{58} + x^{56} + x^{54} + x^{52} + x^{51} + x^{50} + x^{48} + x^{47} + x^{46} + x^{45}\\
&+ x^{42} + x^{39} + x^{38} + x^{33} + x^{32} + x^{31} + x^{28} + x^{27} + x^{22} + x^{21} + x^{19}\\
&+ x^{18},\\
p_3(x) &= x^{138} + x^{136} + x^{135} + x^{131} + x^{130} + x^{127} + x^{125} + x^{122} + x^{120} + x^{119}\\
&+ x^{118} + x^{114} + x^{113} + x^{111} + x^{110} + x^{109} + x^{107} + x^{106} + x^{105}\\
&+ x^{103} + x^{101} + x^{98} + x^{96} + x^{95} + x^{94} + x^{89} + x^{88} + x^{86} + x^{85} + x^{82}\\
&+ x^{81} + x^{80} + x^{79} + x^{75} + x^{72} + x^{69} + x^{67} + x^{64} + x^{62} + x^{61} + x^{58}\\
&+ x^{57} + x^{55} + x^{53} + x^{48} + x^{46} + x^{45} + x^{42} + x^{41} + x^{40} + x^{39} + x^{35}\\
&+ x^{34} + x^{31} + x^{29} + x^{26} + x^{24} + x^{23} + x^{22} + x^{17} + x^{16} + x^{14} + x^{13}\\
&+ x^{9},\\
p_6(x) &= x^{141} + x^{140} + x^{139} + x^{138} + x^{137} + x^{136} + x^{135} + x^{134} + x^{133} + x^{132}\\
&+ x^{131} + x^{130} + x^{127} + x^{126} + x^{123} + x^{122} + x^{119} + x^{118} + x^{115}\\
&+ x^{114} + x^{109} + x^{108} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100}\\
&+ x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85}\\
&+ x^{84} + x^{83} + x^{82} + x^{79} + x^{78} + x^{73} + x^{72} + x^{71} + x^{70} + x^{65} + x^{64}\\
&+ x^{63} + x^{62} + x^{59} + x^{58} + x^{51} + x^{50} + x^{47} + x^{46} + x^{43} + x^{42} + x^{41}\\
&+ x^{40} + x^{39} + x^{38} + x^{35} + x^{34} + x^{29} + x^{28} + x^{25} + x^{24} + x^{21} + x^{20}\\
&+ x^{17} + x^{16} + x^{13} + x^{12} + x^{11} + x^{10} + x^{7} + x^{6},\\
p_9(x) &= x^{144} + x^{141} + x^{140} + x^{139} + x^{138} + x^{137} + x^{134} + x^{133} + x^{131} + x^{130}\\
&+ x^{129} + x^{127} + x^{126} + x^{125} + x^{124} + x^{123} + x^{122} + x^{120} + x^{116}\\
&+ x^{115} + x^{114} + x^{113} + x^{109} + x^{108} + x^{106} + x^{105} + x^{104} + x^{102}\\
&+ x^{100} + x^{98} + x^{96} + x^{91} + x^{84} + x^{82} + x^{81} + x^{77} + x^{74} + x^{72} + x^{71}\\
&+ x^{70} + x^{68} + x^{63} + x^{62} + x^{60} + x^{59} + x^{58} + x^{56} + x^{54} + x^{52} + x^{48}\\
&+ x^{47} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{37} + x^{36} + x^{33} + x^{31} + x^{30}\\
&+ x^{27} + x^{26} + x^{25} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18} + x^{16} + x^{12}\\
&+ x^{11} + x^{9} + x^{8} + x^{7} + x^{5} + x^{3},\\
p_{12}(x) &= x^{147} + x^{146} + x^{141} + x^{140} + x^{139} + x^{138} + x^{137} + x^{136} + x^{131} + x^{130}\\
&+ x^{125} + x^{124} + x^{121} + x^{120} + x^{117} + x^{116} + x^{115} + x^{114} + x^{113}\\
&+ x^{112} + x^{67} + x^{66} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{51} + x^{50}\\
&+ x^{45} + x^{44} + x^{41} + x^{40} + x^{37} + x^{36} + x^{33} + x^{32} + x^{29} + x^{28} + x^{27}\\
&+ x^{26} + x^{25} + x^{24} + x^{19} + x^{18} + x^{13} + x^{12} + x^{9} + x^{8} + x^{5} + x^{4} + x^{3}\\
&+ x^{2} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 1, 1, 1, 0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 1].

For $\phi(W^e, 1, 0)$:

$$\begin{aligned}
p_0(x) = {} & x^{147} + x^{146} + x^{144} + x^{143} + x^{141} + x^{138} + x^{134} + x^{131} + x^{129} + x^{128} \\
& + x^{125} + x^{123} + x^{121} + x^{120} + x^{119} + x^{117} + x^{113} + x^{112} + x^{111} \\
& + x^{110} + x^{109} + x^{108} + x^{107} + x^{106} + x^{105} + x^{104} + x^{103} + x^{102} \\
& + x^{101} + x^{100} + x^{99} + x^{98} + x^{96} + x^{90} + x^{88} + x^{87} + x^{85} + x^{84} + x^{83} \\
& + x^{81} + x^{79} + x^{78} + x^{77} + x^{75} + x^{70} + x^{69} + x^{68} + x^{64} + x^{63} + x^{62} \\
& + x^{60} + x^{58} + x^{56} + x^{54} + x^{52} + x^{51} + x^{50} + x^{48} + x^{47} + x^{46} + x^{45} \\
& + x^{42} + x^{39} + x^{38} + x^{33} + x^{32} + x^{31} + x^{28} + x^{27} + x^{22} + x^{21} + x^{19} \\
& + x^{18}, \\
p_3(x) = {} & x^{138} + x^{136} + x^{135} + x^{131} + x^{130} + x^{127} + x^{125} + x^{122} + x^{120} + x^{119} \\
& + x^{118} + x^{114} + x^{113} + x^{111} + x^{110} + x^{109} + x^{107} + x^{106} + x^{105} \\
& + x^{103} + x^{101} + x^{98} + x^{96} + x^{95} + x^{94} + x^{89} + x^{88} + x^{86} + x^{85} + x^{82} \\
& + x^{81} + x^{80} + x^{79} + x^{75} + x^{72} + x^{69} + x^{67} + x^{64} + x^{62} + x^{61} + x^{58} \\
& + x^{57} + x^{55} + x^{53} + x^{48} + x^{46} + x^{45} + x^{42} + x^{41} + x^{40} + x^{39} + x^{35} \\
& + x^{34} + x^{31} + x^{29} + x^{26} + x^{24} + x^{23} + x^{22} + x^{17} + x^{16} + x^{14} + x^{13} \\
& + x^{9}, \\
p_6(x) = {} & x^{141} + x^{140} + x^{139} + x^{138} + x^{137} + x^{136} + x^{135} + x^{134} + x^{133} + x^{132} \\
& + x^{131} + x^{130} + x^{127} + x^{126} + x^{123} + x^{122} + x^{119} + x^{118} + x^{115} \\
& + x^{114} + x^{109} + x^{108} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} \\
& + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} \\
& + x^{84} + x^{83} + x^{82} + x^{79} + x^{78} + x^{73} + x^{72} + x^{71} + x^{70} + x^{65} + x^{64} \\
& + x^{63} + x^{62} + x^{59} + x^{58} + x^{51} + x^{50} + x^{47} + x^{46} + x^{43} + x^{42} + x^{41} \\
& + x^{40} + x^{39} + x^{38} + x^{35} + x^{34} + x^{29} + x^{28} + x^{25} + x^{24} + x^{21} + x^{20} \\
& + x^{17} + x^{16} + x^{13} + x^{12} + x^{11} + x^{10} + x^{7} + x^{6}, \\
p_9(x) = {} & x^{144} + x^{141} + x^{140} + x^{139} + x^{138} + x^{137} + x^{134} + x^{133} + x^{131} + x^{130} \\
& + x^{129} + x^{127} + x^{126} + x^{125} + x^{124} + x^{123} + x^{122} + x^{120} + x^{116} \\
& + x^{115} + x^{114} + x^{113} + x^{109} + x^{108} + x^{106} + x^{105} + x^{104} + x^{102} \\
& + x^{100} + x^{98} + x^{96} + x^{91} + x^{84} + x^{82} + x^{81} + x^{77} + x^{74} + x^{72} + x^{71} \\
& + x^{70} + x^{68} + x^{63} + x^{62} + x^{60} + x^{59} + x^{58} + x^{56} + x^{54} + x^{52} + x^{48} \\
& + x^{47} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{37} + x^{36} + x^{33} + x^{31} + x^{30} \\
& + x^{27} + x^{26} + x^{25} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18} + x^{16} + x^{12} \\
& + x^{11} + x^{9} + x^{8} + x^{7} + x^{5} + x^{3}, \\
p_{12}(x) = {} & x^{147} + x^{146} + x^{141} + x^{140} + x^{139} + x^{138} + x^{137} + x^{136} + x^{131} + x^{130} \\
& + x^{125} + x^{124} + x^{121} + x^{120} + x^{117} + x^{116} + x^{115} + x^{114} + x^{113}
\end{aligned}$$

$$+ x^{112} + x^{67} + x^{66} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{51} + x^{50} + x^{45} + x^{44} + x^{41} + x^{40} + x^{37} + x^{36} + x^{33} + x^{32} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{19} + x^{18} + x^{13} + x^{12} + x^{9} + x^{8} + x^{5} + x^{4} + x^{3} + x^{2} + x + 1,$$

and the initial terms are [0, 1, 1, 1, 1, 0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 1].

For $\phi(W^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{132} + x^{130} + x^{126} + x^{122} + x^{116} + x^{110} + x^{100} + x^{98} + x^{96} + x^{94} \\
&\quad + x^{92} + x^{90} + x^{84} + x^{80} + x^{78} + x^{72} + x^{68} + x^{64} + x^{62} + x^{60} + x^{50} \\
&\quad + x^{46} + x^{42} + x^{32} + x^{30} + x^{28} + x^{24}, \\
p_3(x) &= x^{130} + x^{122} + x^{110} + x^{108} + x^{106} + x^{104} + x^{96} + x^{94} + x^{86} + x^{82} \\
&\quad + x^{80} + x^{76} + x^{64} + x^{62} + x^{56} + x^{52} + x^{48} + x^{46} + x^{38} + x^{36} + x^{28} \\
&\quad + x^{24} + x^{22} + x^{20} + x^{18} + x^{14} + x^{12} + x^{6}, \\
p_6(x) &= x^{130} + x^{126} + x^{118} + x^{116} + x^{114} + x^{112} + x^{108} + x^{100} + x^{96} + x^{82} \\
&\quad + x^{78} + x^{76} + x^{68} + x^{64} + x^{56} + x^{52} + x^{50} + x^{46} + x^{44} + x^{42} + x^{40} \\
&\quad + x^{34} + x^{32} + x^{26} + x^{24} + x^{22} + x^{18} + x^{14} + x^{8} + x^{6} + x^{2} + 1, \\
p_9(x) &= x^{130} + x^{124} + x^{118} + x^{116} + x^{112} + x^{108} + x^{106} + x^{104} + x^{100} + x^{96} \\
&\quad + x^{92} + x^{78} + x^{68} + x^{64} + x^{58} + x^{48} + x^{40} + x^{32} + x^{26} + x^{22} + x^{20} \\
&\quad + x^{16} + x^{12} + x^{10} + x^{6} + x^{2}, \\
p_{12}(x) &= x^{132} + x^{120} + x^{116} + x^{112} + x^{52} + x^{40} + x^{36} + x^{32} + x^{20} + x^{8} + x^{4} + 1 \\
&\quad ,
\end{aligned}$$

and the initial terms are [0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0].

For $\phi(M^o, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{147} + x^{146} + x^{145} + x^{144} + x^{142} + x^{141} + x^{139} + x^{138} + x^{137} + x^{136} \\
&\quad + x^{135} + x^{134} + x^{133} + x^{132} + x^{130} + x^{129} + x^{126} + x^{125} + x^{124} \\
&\quad + x^{121} + x^{120} + x^{117} + x^{112} + x^{108} + x^{107} + x^{106} + x^{105} + x^{102} \\
&\quad + x^{100} + x^{99} + x^{97} + x^{96} + x^{95} + x^{93} + x^{92} + x^{91} + x^{87} + x^{85} + x^{84} \\
&\quad + x^{81} + x^{80} + x^{79} + x^{77} + x^{76} + x^{72} + x^{71} + x^{68} + x^{67} + x^{64} + x^{63} \\
&\quad + x^{61} + x^{58} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} \\
&\quad + x^{46} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{36} + x^{35} + x^{34} + x^{32} \\
&\quad + x^{31} + x^{28} + x^{27} + x^{23} + x^{22} + x^{21} + x^{18} + x^{17} + x^{16} + x^{13} + x^{12}, \\
p_3(x) &= x^{143} + x^{142} + x^{141} + x^{140} + x^{138} + x^{137} + x^{135} + x^{134} + x^{133} + x^{132} \\
&\quad + x^{131} + x^{122} + x^{121} + x^{119} + x^{115} + x^{113} + x^{112} + x^{111} + x^{109} \\
&\quad + x^{108} + x^{107} + x^{101} + x^{100} + x^{98} + x^{97} + x^{95} + x^{93} + x^{92} + x^{89} \\
&\quad + x^{87} + x^{83} + x^{80} + x^{79} + x^{78} + x^{72} + x^{69} + x^{68} + x^{67} + x^{66} + x^{64} \\
&\quad + x^{62} + x^{59} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{49} + x^{47} + x^{45}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{44} + x^{43} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{31} + x^{30} + x^{29}\\
&+ x^{28} + x^{27} + x^{25} + x^{23} + x^{21} + x^{20} + x^{17} + x^{15} + x^{9} + x^{8},\\
p_6(x) &= x^{145} + x^{144} + x^{142} + x^{140} + x^{134} + x^{133} + x^{132} + x^{131} + x^{129} + x^{127}\\
&+ x^{125} + x^{123} + x^{122} + x^{116} + x^{114} + x^{111} + x^{110} + x^{109} + x^{106}\\
&+ x^{105} + x^{103} + x^{101} + x^{99} + x^{97} + x^{95} + x^{94} + x^{91} + x^{90} + x^{89} + x^{88}\\
&+ x^{87} + x^{86} + x^{84} + x^{82} + x^{79} + x^{78} + x^{74} + x^{73} + x^{70} + x^{68} + x^{65}\\
&+ x^{63} + x^{62} + x^{61} + x^{56} + x^{54} + x^{52} + x^{49} + x^{48} + x^{47} + x^{46} + x^{44}\\
&+ x^{41} + x^{36} + x^{35} + x^{34} + x^{33} + x^{29} + x^{28} + x^{25} + x^{24} + x^{23} + x^{21}\\
&+ x^{18} + x^{17} + x^{16} + x^{12} + x^{11} + x^{9} + x^{8} + x^{3} + x^{2} + x + 1,\\
p_9(x) &= x^{143} + x^{142} + x^{139} + x^{138} + x^{137} + x^{136} + x^{131} + x^{130} + x^{127} + x^{126}\\
&+ x^{125} + x^{124} + x^{123} + x^{122} + x^{119} + x^{118} + x^{117} + x^{116} + x^{63} + x^{62}\\
&+ x^{59} + x^{58} + x^{57} + x^{56} + x^{51} + x^{50} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43}\\
&+ x^{42} + x^{39} + x^{38} + x^{37} + x^{36} + x^{31} + x^{30} + x^{27} + x^{26} + x^{25} + x^{24}\\
&+ x^{19} + x^{18} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^{7} + x^{6} + x^{5} + x^{4},\\
p_{12}(x) &= x^{147} + x^{146} + x^{143} + x^{142} + x^{141} + x^{140} + x^{127} + x^{126} + x^{123} + x^{122}\\
&+ x^{121} + x^{120} + x^{119} + x^{118} + x^{115} + x^{114} + x^{113} + x^{112} + x^{67} + x^{66}\\
&+ x^{63} + x^{62} + x^{61} + x^{60} + x^{47} + x^{46} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39}\\
&+ x^{38} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{15} + x^{14} + x^{11} + x^{10}\\
&+ x^{9} + x^{8} + x^{7} + x^{6} + x^{3} + x^{2} + x + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 1, 0, 1, 0, 1, 1, 1, 1, 1, 0, 1, 1, 0, 0].
For $\phi(M^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{164} + x^{161} + x^{159} + x^{156} + x^{155} + x^{153} + x^{149} + x^{147} + x^{143} + x^{142}\\
&+ x^{139} + x^{137} + x^{136} + x^{133} + x^{132} + x^{131} + x^{128} + x^{127} + x^{126}\\
&+ x^{125} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117} + x^{115} + x^{114}\\
&+ x^{110} + x^{106} + x^{105} + x^{104} + x^{102} + x^{100} + x^{99} + x^{97} + x^{95} + x^{94}\\
&+ x^{93} + x^{92} + x^{91} + x^{88} + x^{85} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77}\\
&+ x^{76} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{63} + x^{61}\\
&+ x^{60} + x^{58} + x^{57} + x^{56} + x^{53} + x^{51} + x^{48} + x^{46} + x^{45} + x^{44} + x^{43}\\
&+ x^{38} + x^{36} + x^{35} + x^{34} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27},\\
p_3(x) &= x^{155} + x^{154} + x^{153} + x^{150} + x^{146} + x^{144} + x^{142} + x^{136} + x^{132} + x^{131}\\
&+ x^{129} + x^{128} + x^{126} + x^{123} + x^{122} + x^{121} + x^{120} + x^{118} + x^{116}\\
&+ x^{110} + x^{107} + x^{105} + x^{104} + x^{98} + x^{94} + x^{92} + x^{91} + x^{89} + x^{86}\\
&+ x^{84} + x^{83} + x^{81} + x^{78} + x^{76} + x^{74} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64}\\
&+ x^{60} + x^{58} + x^{56} + x^{51} + x^{49} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{36}\\
&+ x^{34} + x^{32} + x^{27} + x^{25} + x^{20} + x^{16} + x^{14} + x^{12} + x^{11} + x^{10} + x^{9},
\end{aligned}
$$

$$\begin{aligned}
p_6(x) &= x^{158} + x^{156} + x^{152} + x^{146} + x^{144} + x^{132} + x^{130} + x^{128} + x^{120} + x^{118} \\
&\quad + x^{114} + x^{112} + x^{108} + x^{106} + x^{104} + x^{102} + x^{94} + x^{92} + x^{88} + x^{82} \\
&\quad + x^{80} + x^{68} + x^{62} + x^{58} + x^{56} + x^{54} + x^{52} + x^{48} + x^{44} + x^{42} + x^{40} \\
&\quad + x^{38} + x^{26} + x^{24} + x^{22} + x^{16} + x^{8} + x^{6}, \\
p_9(x) &= x^{161} + x^{160} + x^{158} + x^{155} + x^{154} + x^{153} + x^{152} + x^{151} + x^{149} + x^{148} \\
&\quad + x^{147} + x^{145} + x^{142} + x^{140} + x^{138} + x^{135} + x^{134} + x^{133} + x^{132} \\
&\quad + x^{130} + x^{129} + x^{127} + x^{125} + x^{120} + x^{116} + x^{115} + x^{81} + x^{80} + x^{78} \\
&\quad + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{62} + x^{60} \\
&\quad + x^{58} + x^{55} + x^{54} + x^{53} + x^{52} + x^{50} + x^{48} + x^{47} + x^{46} + x^{45} + x^{43} \\
&\quad + x^{42} + x^{41} + x^{39} + x^{37} + x^{33} + x^{30} + x^{28} + x^{26} + x^{23} + x^{22} + x^{21} \\
&\quad + x^{20} + x^{18} + x^{17} + x^{15} + x^{13} + x^{8} + x^{4} + x^{3}, \\
p_{12}(x) &= x^{164} + x^{148} + x^{144} + x^{140} + x^{136} + x^{128} + x^{112} + x^{84} + x^{68} + x^{64} \\
&\quad + x^{60} + x^{56} + x^{52} + x^{48} + x^{36} + x^{28} + x^{24} + x^{16} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 1].

For $\phi(M^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{144} + x^{141} + x^{140} + x^{139} + x^{138} + x^{137} + x^{136} + x^{133} + x^{131} + x^{130} \\
&\quad + x^{129} + x^{128} + x^{126} + x^{125} + x^{116} + x^{110} + x^{109} + x^{108} + x^{107} \\
&\quad + x^{105} + x^{103} + x^{102} + x^{98} + x^{96} + x^{94} + x^{92} + x^{90} + x^{89} + x^{88} + x^{86} \\
&\quad + x^{85} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{76} + x^{75} + x^{74} + x^{72} + x^{71} \\
&\quad + x^{70} + x^{69} + x^{68} + x^{67} + x^{62} + x^{61} + x^{59} + x^{57} + x^{56} + x^{53} + x^{51} \\
&\quad + x^{49} + x^{48} + x^{44} + x^{40} + x^{38} + x^{37} + x^{35} + x^{34} + x^{33} + x^{32} + x^{29} \\
&\quad + x^{28} + x^{26} + x^{24} + x^{22} + x^{21} + x^{18}, \\
p_3(x) &= x^{135} + x^{134} + x^{133} + x^{131} + x^{129} + x^{124} + x^{120} + x^{119} + x^{117} + x^{116} \\
&\quad + x^{115} + x^{113} + x^{110} + x^{108} + x^{106} + x^{100} + x^{96} + x^{95} + x^{93} + x^{92} \\
&\quad + x^{91} + x^{89} + x^{87} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{79} + x^{78} + x^{77} \\
&\quad + x^{75} + x^{73} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{61} \\
&\quad + x^{59} + x^{57} + x^{56} + x^{50} + x^{47} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39} \\
&\quad + x^{38} + x^{37} + x^{35} + x^{33} + x^{28} + x^{24} + x^{23} + x^{21} + x^{20} + x^{19} + x^{17} \\
&\quad + x^{15} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9}, \\
p_6(x) &= x^{138} + x^{136} + x^{134} + x^{128} + x^{126} + x^{124} + x^{118} + x^{114} + x^{112} + x^{110} \\
&\quad + x^{108} + x^{106} + x^{90} + x^{88} + x^{84} + x^{82} + x^{80} + x^{78} + x^{70} + x^{68} + x^{64} \\
&\quad + x^{52} + x^{46} + x^{42} + x^{40} + x^{28} + x^{26} + x^{24} + x^{18} + x^{14} + x^{12} + x^{10} \\
&\quad + x^{8} + x^{6}, \\
p_9(x) &= x^{141} + x^{140} + x^{138} + x^{137} + x^{136} + x^{135} + x^{133} + x^{132} + x^{130} + x^{125} \\
&\quad + x^{123} + x^{119} + x^{116} + x^{115} + x^{61} + x^{60} + x^{58} + x^{57} + x^{56} + x^{55}
\end{aligned}$$

$$\begin{aligned}
&+ x^{53} + x^{52} + x^{50} + x^{45} + x^{43} + x^{39} + x^{36} + x^{35} + x^{29} + x^{28} + x^{26}\\
&+ x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{18} + x^{13} + x^{11} + x^{7} + x^{4} + x^{3},\\
p_{12}(x) &= x^{144} + x^{140} + x^{124} + x^{120} + x^{116} + x^{112} + x^{64} + x^{60} + x^{44} + x^{40}\\
&+ x^{36} + x^{28} + x^{12} + x^{8} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [0, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 1, 1, 0].

For $\phi(M^o, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{147} + x^{146} + x^{145} + x^{144} + x^{143} + x^{140} + x^{139} + x^{138} + x^{135} + x^{133}\\
&+ x^{130} + x^{129} + x^{127} + x^{126} + x^{125} + x^{124} + x^{122} + x^{121} + x^{119}\\
&+ x^{117} + x^{116} + x^{112} + x^{110} + x^{108} + x^{107} + x^{105} + x^{98} + x^{95} + x^{93}\\
&+ x^{92} + x^{88} + x^{87} + x^{84} + x^{83} + x^{76} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64}\\
&+ x^{63} + x^{62} + x^{61} + x^{59} + x^{58} + x^{56} + x^{53} + x^{51} + x^{48} + x^{43} + x^{42}\\
&+ x^{40} + x^{38} + x^{36} + x^{33},\\
p_3(x) &= x^{143} + x^{142} + x^{141} + x^{140} + x^{139} + x^{136} + x^{133} + x^{132} + x^{131} + x^{127}\\
&+ x^{126} + x^{125} + x^{124} + x^{123} + x^{120} + x^{119} + x^{114} + x^{111} + x^{107}\\
&+ x^{105} + x^{104} + x^{103} + x^{102} + x^{99} + x^{96} + x^{95} + x^{93} + x^{92} + x^{89}\\
&+ x^{86} + x^{85} + x^{84} + x^{82} + x^{80} + x^{75} + x^{74} + x^{73} + x^{65} + x^{63} + x^{61}\\
&+ x^{60} + x^{58} + x^{48} + x^{46} + x^{42} + x^{40} + x^{37} + x^{36} + x^{35} + x^{26} + x^{24}\\
&+ x^{23} + x^{21} + x^{20} + x^{17} + x^{14} + x^{13} + x^{12},\\
p_6(x) &= x^{145} + x^{144} + x^{143} + x^{141} + x^{139} + x^{138} + x^{134} + x^{133} + x^{132} + x^{131}\\
&+ x^{129} + x^{127} + x^{124} + x^{123} + x^{122} + x^{119} + x^{118} + x^{117} + x^{115}\\
&+ x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{107} + x^{105} + x^{102} + x^{100}\\
&+ x^{98} + x^{96} + x^{95} + x^{94} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{83} + x^{81}\\
&+ x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{72} + x^{70} + x^{69} + x^{64} + x^{61}\\
&+ x^{59} + x^{58} + x^{56} + x^{54} + x^{52} + x^{49} + x^{48} + x^{45} + x^{41} + x^{39} + x^{38}\\
&+ x^{36} + x^{33} + x^{31} + x^{30} + x^{27} + x^{26} + x^{25} + x^{24} + x^{22} + x^{20} + x^{18}\\
&+ x^{17} + x^{16} + x^{12} + x^{10} + x^{5} + x^{4} + x^{3} + x^{2} + x + 1,\\
p_9(x) &= x^{143} + x^{142} + x^{139} + x^{138} + x^{137} + x^{136} + x^{131} + x^{130} + x^{127} + x^{126}\\
&+ x^{125} + x^{124} + x^{123} + x^{122} + x^{119} + x^{118} + x^{117} + x^{116} + x^{63} + x^{62}\\
&+ x^{59} + x^{58} + x^{57} + x^{56} + x^{51} + x^{50} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43}\\
&+ x^{42} + x^{39} + x^{38} + x^{37} + x^{36} + x^{31} + x^{30} + x^{27} + x^{26} + x^{25} + x^{24}\\
&+ x^{19} + x^{18} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^{7} + x^{6} + x^{5} + x^{4},\\
p_{12}(x) &= x^{147} + x^{146} + x^{143} + x^{142} + x^{141} + x^{140} + x^{127} + x^{126} + x^{123} + x^{122}\\
&+ x^{121} + x^{120} + x^{119} + x^{118} + x^{115} + x^{114} + x^{113} + x^{112} + x^{67} + x^{66}\\
&+ x^{63} + x^{62} + x^{61} + x^{60} + x^{47} + x^{46} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39}\\
&+ x^{38} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{15} + x^{14} + x^{11} + x^{10}
\end{aligned}$$

$$+ x^9 + x^8 + x^7 + x^6 + x^3 + x^2 + x + 1,$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 1, 0, 0].

For $\phi(W^o, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{147} + x^{146} + x^{145} + x^{144} + x^{142} + x^{141} + x^{139} + x^{138} + x^{137} + x^{136}\\
&\quad + x^{135} + x^{134} + x^{133} + x^{132} + x^{130} + x^{129} + x^{126} + x^{125} + x^{124}\\
&\quad + x^{121} + x^{120} + x^{117} + x^{112} + x^{108} + x^{107} + x^{106} + x^{105} + x^{102}\\
&\quad + x^{100} + x^{99} + x^{97} + x^{96} + x^{95} + x^{93} + x^{92} + x^{91} + x^{87} + x^{85} + x^{84}\\
&\quad + x^{81} + x^{80} + x^{79} + x^{77} + x^{76} + x^{72} + x^{71} + x^{68} + x^{67} + x^{64} + x^{63}\\
&\quad + x^{61} + x^{58} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47}\\
&\quad + x^{46} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{36} + x^{35} + x^{34} + x^{32}\\
&\quad + x^{31} + x^{28} + x^{27} + x^{23} + x^{22} + x^{21} + x^{18} + x^{17} + x^{16} + x^{13} + x^{12},\\
p_3(x) &= x^{143} + x^{142} + x^{141} + x^{140} + x^{138} + x^{137} + x^{135} + x^{134} + x^{133} + x^{132}\\
&\quad + x^{131} + x^{122} + x^{121} + x^{119} + x^{115} + x^{113} + x^{112} + x^{111} + x^{109}\\
&\quad + x^{108} + x^{107} + x^{101} + x^{100} + x^{98} + x^{97} + x^{95} + x^{93} + x^{92} + x^{89}\\
&\quad + x^{87} + x^{83} + x^{80} + x^{79} + x^{78} + x^{72} + x^{69} + x^{68} + x^{67} + x^{66} + x^{64}\\
&\quad + x^{62} + x^{59} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{49} + x^{47} + x^{45}\\
&\quad + x^{44} + x^{43} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{31} + x^{30} + x^{29}\\
&\quad + x^{28} + x^{27} + x^{25} + x^{23} + x^{21} + x^{20} + x^{17} + x^{15} + x^{9} + x^{8},\\
p_6(x) &= x^{145} + x^{144} + x^{142} + x^{140} + x^{134} + x^{133} + x^{132} + x^{131} + x^{129} + x^{127}\\
&\quad + x^{125} + x^{123} + x^{122} + x^{116} + x^{114} + x^{111} + x^{110} + x^{109} + x^{106}\\
&\quad + x^{105} + x^{103} + x^{101} + x^{99} + x^{97} + x^{95} + x^{94} + x^{91} + x^{90} + x^{89} + x^{88}\\
&\quad + x^{87} + x^{86} + x^{84} + x^{82} + x^{79} + x^{78} + x^{74} + x^{73} + x^{70} + x^{68} + x^{65}\\
&\quad + x^{63} + x^{62} + x^{61} + x^{56} + x^{54} + x^{52} + x^{49} + x^{48} + x^{47} + x^{46} + x^{44}\\
&\quad + x^{41} + x^{36} + x^{35} + x^{34} + x^{33} + x^{29} + x^{28} + x^{25} + x^{24} + x^{23} + x^{21}\\
&\quad + x^{18} + x^{17} + x^{16} + x^{12} + x^{11} + x^{9} + x^{8} + x^{3} + x^{2} + x + 1,\\
p_9(x) &= x^{143} + x^{142} + x^{139} + x^{138} + x^{137} + x^{136} + x^{131} + x^{130} + x^{127} + x^{126}\\
&\quad + x^{125} + x^{124} + x^{123} + x^{122} + x^{119} + x^{118} + x^{117} + x^{116} + x^{63} + x^{62}\\
&\quad + x^{59} + x^{58} + x^{57} + x^{56} + x^{51} + x^{50} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43}\\
&\quad + x^{42} + x^{39} + x^{38} + x^{37} + x^{36} + x^{31} + x^{30} + x^{27} + x^{26} + x^{25} + x^{24}\\
&\quad + x^{19} + x^{18} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^{7} + x^{6} + x^{5} + x^{4},\\
p_{12}(x) &= x^{147} + x^{146} + x^{143} + x^{142} + x^{141} + x^{140} + x^{127} + x^{126} + x^{123} + x^{122}\\
&\quad + x^{121} + x^{120} + x^{119} + x^{118} + x^{115} + x^{114} + x^{113} + x^{112} + x^{67} + x^{66}\\
&\quad + x^{63} + x^{62} + x^{61} + x^{60} + x^{47} + x^{46} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39}\\
&\quad + x^{38} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{15} + x^{14} + x^{11} + x^{10}\\
&\quad + x^{9} + x^{8} + x^{7} + x^{6} + x^{3} + x^{2} + x + 1,
\end{aligned}$$

and the initial terms are [1, 0, 1, 0, 1, 0, 1, 1, 1, 1, 1, 0, 1, 1, 0, 0].

For $\phi(W^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{144} + x^{141} + x^{140} + x^{139} + x^{138} + x^{137} + x^{136} + x^{133} + x^{131} + x^{130} \\
&\quad + x^{129} + x^{128} + x^{126} + x^{125} + x^{116} + x^{110} + x^{109} + x^{108} + x^{107} \\
&\quad + x^{105} + x^{103} + x^{102} + x^{98} + x^{96} + x^{94} + x^{92} + x^{90} + x^{89} + x^{88} + x^{86} \\
&\quad + x^{85} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{76} + x^{75} + x^{74} + x^{72} + x^{71} \\
&\quad + x^{70} + x^{69} + x^{68} + x^{67} + x^{62} + x^{61} + x^{59} + x^{57} + x^{56} + x^{53} + x^{51} \\
&\quad + x^{49} + x^{48} + x^{44} + x^{40} + x^{38} + x^{37} + x^{35} + x^{34} + x^{33} + x^{32} + x^{29} \\
&\quad + x^{28} + x^{26} + x^{24} + x^{22} + x^{21} + x^{18}, \\
p_3(x) &= x^{135} + x^{134} + x^{133} + x^{131} + x^{129} + x^{124} + x^{120} + x^{119} + x^{117} + x^{116} \\
&\quad + x^{115} + x^{113} + x^{110} + x^{108} + x^{106} + x^{100} + x^{96} + x^{95} + x^{93} + x^{92} \\
&\quad + x^{91} + x^{89} + x^{87} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{79} + x^{78} + x^{77} \\
&\quad + x^{75} + x^{73} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{61} \\
&\quad + x^{59} + x^{57} + x^{56} + x^{50} + x^{47} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39} \\
&\quad + x^{38} + x^{37} + x^{35} + x^{33} + x^{28} + x^{24} + x^{23} + x^{21} + x^{20} + x^{19} + x^{17} \\
&\quad + x^{15} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9}, \\
p_6(x) &= x^{138} + x^{136} + x^{134} + x^{128} + x^{126} + x^{124} + x^{118} + x^{114} + x^{112} + x^{110} \\
&\quad + x^{108} + x^{106} + x^{90} + x^{88} + x^{84} + x^{82} + x^{80} + x^{78} + x^{70} + x^{68} + x^{64} \\
&\quad + x^{52} + x^{46} + x^{42} + x^{40} + x^{28} + x^{26} + x^{24} + x^{18} + x^{14} + x^{12} + x^{10} \\
&\quad + x^{8} + x^{6}, \\
p_9(x) &= x^{141} + x^{140} + x^{138} + x^{137} + x^{136} + x^{135} + x^{133} + x^{132} + x^{130} + x^{125} \\
&\quad + x^{123} + x^{119} + x^{116} + x^{115} + x^{61} + x^{60} + x^{58} + x^{57} + x^{56} + x^{55} \\
&\quad + x^{53} + x^{52} + x^{50} + x^{45} + x^{43} + x^{39} + x^{36} + x^{35} + x^{29} + x^{28} + x^{26} \\
&\quad + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{18} + x^{13} + x^{11} + x^{7} + x^{4} + x^{3}, \\
p_{12}(x) &= x^{144} + x^{140} + x^{124} + x^{120} + x^{116} + x^{112} + x^{64} + x^{60} + x^{44} + x^{40} \\
&\quad + x^{36} + x^{28} + x^{12} + x^{8} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [0, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 1, 1, 0].

For $\phi(W^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{164} + x^{161} + x^{159} + x^{156} + x^{155} + x^{153} + x^{149} + x^{147} + x^{143} + x^{142} \\
&\quad + x^{139} + x^{137} + x^{136} + x^{133} + x^{132} + x^{131} + x^{128} + x^{127} + x^{126} \\
&\quad + x^{125} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117} + x^{115} + x^{114} \\
&\quad + x^{110} + x^{106} + x^{105} + x^{104} + x^{102} + x^{100} + x^{99} + x^{97} + x^{95} + x^{94} \\
&\quad + x^{93} + x^{92} + x^{91} + x^{88} + x^{85} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} \\
&\quad + x^{76} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{63} + x^{61} \\
&\quad + x^{60} + x^{58} + x^{57} + x^{56} + x^{53} + x^{51} + x^{48} + x^{46} + x^{45} + x^{44} + x^{43}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{38} + x^{36} + x^{35} + x^{34} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27},\\
p_3(x) &= x^{155} + x^{154} + x^{153} + x^{150} + x^{146} + x^{144} + x^{142} + x^{136} + x^{132} + x^{131}\\
&+ x^{129} + x^{128} + x^{126} + x^{123} + x^{122} + x^{121} + x^{120} + x^{118} + x^{116}\\
&+ x^{110} + x^{107} + x^{105} + x^{104} + x^{98} + x^{94} + x^{92} + x^{91} + x^{89} + x^{86}\\
&+ x^{84} + x^{83} + x^{81} + x^{78} + x^{76} + x^{74} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64}\\
&+ x^{60} + x^{58} + x^{56} + x^{51} + x^{49} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{36}\\
&+ x^{34} + x^{32} + x^{27} + x^{25} + x^{20} + x^{16} + x^{14} + x^{12} + x^{11} + x^{10} + x^{9},\\
p_6(x) &= x^{158} + x^{156} + x^{152} + x^{146} + x^{144} + x^{132} + x^{130} + x^{128} + x^{120} + x^{118}\\
&+ x^{114} + x^{112} + x^{108} + x^{106} + x^{104} + x^{102} + x^{94} + x^{92} + x^{88} + x^{82}\\
&+ x^{80} + x^{68} + x^{62} + x^{58} + x^{56} + x^{54} + x^{52} + x^{48} + x^{44} + x^{42} + x^{40}\\
&+ x^{38} + x^{26} + x^{24} + x^{22} + x^{16} + x^{8} + x^{6},\\
p_9(x) &= x^{161} + x^{160} + x^{158} + x^{155} + x^{154} + x^{153} + x^{152} + x^{151} + x^{149} + x^{148}\\
&+ x^{147} + x^{145} + x^{142} + x^{140} + x^{138} + x^{135} + x^{134} + x^{133} + x^{132}\\
&+ x^{130} + x^{129} + x^{127} + x^{125} + x^{120} + x^{116} + x^{115} + x^{81} + x^{80} + x^{78}\\
&+ x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{62} + x^{60}\\
&+ x^{58} + x^{55} + x^{54} + x^{53} + x^{52} + x^{50} + x^{48} + x^{47} + x^{46} + x^{45} + x^{43}\\
&+ x^{42} + x^{41} + x^{39} + x^{37} + x^{33} + x^{30} + x^{28} + x^{26} + x^{23} + x^{22} + x^{21}\\
&+ x^{20} + x^{18} + x^{17} + x^{15} + x^{13} + x^{8} + x^{4} + x^{3},\\
p_{12}(x) &= x^{164} + x^{148} + x^{144} + x^{140} + x^{136} + x^{128} + x^{112} + x^{84} + x^{68} + x^{64}\\
&+ x^{60} + x^{56} + x^{52} + x^{48} + x^{36} + x^{28} + x^{24} + x^{16} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 1].

For $\phi(W^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{147} + x^{146} + x^{145} + x^{144} + x^{143} + x^{140} + x^{139} + x^{138} + x^{135} + x^{133}\\
&+ x^{130} + x^{129} + x^{127} + x^{126} + x^{125} + x^{124} + x^{122} + x^{121} + x^{119}\\
&+ x^{117} + x^{116} + x^{112} + x^{110} + x^{108} + x^{107} + x^{105} + x^{98} + x^{95} + x^{93}\\
&+ x^{92} + x^{88} + x^{87} + x^{84} + x^{83} + x^{76} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64}\\
&+ x^{63} + x^{62} + x^{61} + x^{59} + x^{58} + x^{56} + x^{53} + x^{51} + x^{48} + x^{43} + x^{42}\\
&+ x^{40} + x^{38} + x^{36} + x^{33},\\
p_3(x) &= x^{143} + x^{142} + x^{141} + x^{140} + x^{139} + x^{136} + x^{133} + x^{132} + x^{131} + x^{127}\\
&+ x^{126} + x^{125} + x^{124} + x^{123} + x^{120} + x^{119} + x^{114} + x^{111} + x^{107}\\
&+ x^{105} + x^{104} + x^{103} + x^{102} + x^{99} + x^{96} + x^{95} + x^{93} + x^{92} + x^{89}\\
&+ x^{86} + x^{85} + x^{84} + x^{82} + x^{80} + x^{75} + x^{74} + x^{73} + x^{65} + x^{63} + x^{61}\\
&+ x^{60} + x^{58} + x^{48} + x^{46} + x^{42} + x^{40} + x^{37} + x^{36} + x^{35} + x^{26} + x^{24}\\
&+ x^{23} + x^{21} + x^{20} + x^{17} + x^{14} + x^{13} + x^{12},\\
p_6(x) &= x^{145} + x^{144} + x^{143} + x^{141} + x^{139} + x^{138} + x^{134} + x^{133} + x^{132} + x^{131}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{129} + x^{127} + x^{124} + x^{123} + x^{122} + x^{119} + x^{118} + x^{117} + x^{115}\\
&+ x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{107} + x^{105} + x^{102} + x^{100}\\
&+ x^{98} + x^{96} + x^{95} + x^{94} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{83} + x^{81}\\
&+ x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{72} + x^{70} + x^{69} + x^{64} + x^{61}\\
&+ x^{59} + x^{58} + x^{56} + x^{54} + x^{52} + x^{49} + x^{48} + x^{45} + x^{41} + x^{39} + x^{38}\\
&+ x^{36} + x^{33} + x^{31} + x^{30} + x^{27} + x^{26} + x^{25} + x^{24} + x^{22} + x^{20} + x^{18}\\
&+ x^{17} + x^{16} + x^{12} + x^{10} + x^{5} + x^{4} + x^{3} + x^{2} + x + 1,\\
p_9(x) = {}& x^{143} + x^{142} + x^{139} + x^{138} + x^{137} + x^{136} + x^{131} + x^{130} + x^{127} + x^{126}\\
&+ x^{125} + x^{124} + x^{123} + x^{122} + x^{119} + x^{118} + x^{117} + x^{116} + x^{63} + x^{62}\\
&+ x^{59} + x^{58} + x^{57} + x^{56} + x^{51} + x^{50} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43}\\
&+ x^{42} + x^{39} + x^{38} + x^{37} + x^{36} + x^{31} + x^{30} + x^{27} + x^{26} + x^{25} + x^{24}\\
&+ x^{19} + x^{18} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^{7} + x^{6} + x^{5} + x^{4},\\
p_{12}(x) = {}& x^{147} + x^{146} + x^{143} + x^{142} + x^{141} + x^{140} + x^{127} + x^{126} + x^{123} + x^{122}\\
&+ x^{121} + x^{120} + x^{119} + x^{118} + x^{115} + x^{114} + x^{113} + x^{112} + x^{67} + x^{66}\\
&+ x^{63} + x^{62} + x^{61} + x^{60} + x^{47} + x^{46} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39}\\
&+ x^{38} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{15} + x^{14} + x^{11} + x^{10}\\
&+ x^{9} + x^{8} + x^{7} + x^{6} + x^{3} + x^{2} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 1, 0, 0].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	181	[318, 319]	362	[182, 500]	543	[737, 462]	724	[508, 186]
1	[3, 4]	182	[320, 321]	363	[38, 37]	544	[53, 737]	725	[212, 192]
2	[5, 6]	183	[322, 323]	364	[55, 430]	545	[740, 448]	726	[389, 10]
3	[7, 8]	184	[324, 325]	365	[143, 142]	546	[167, 493]	727	[751, 202]
4	[9, 10]	185	[326, 327]	366	[501, 167]	547	[473, 394]	728	[780, 781]
5	[11, 12]	186	[328, 329]	367	[502, 503]	548	[469, 47]	729	[782, 783]
6	[13, 14]	187	[330, 331]	368	[447, 8]	549	[188, 728]	730	[784, 785]
7	[15, 16]	188	[332, 333]	369	[504, 485]	550	[132, 390]	731	[786, 787]
8	[17, 18]	189	[334, 335]	370	[215, 407]	551	[43, 398]	732	[788, 789]
9	[19, 20]	190	[336, 337]	371	[411, 467]	552	[741, 394]	733	[790, 69]
10	[21, 22]	191	[338, 339]	372	[505, 506]	553	[175, 196]	734	[791, 346]
11	[23, 24]	192	[340, 341]	373	[54, 442]	554	[742, 392]	735	[792, 638]
12	[25, 26]	193	[342, 343]	374	[153, 500]	555	[427, 512]	736	[793, 542]
13	[27, 28]	194	[344, 345]	375	[131, 150]	556	[740, 203]	737	[794, 795]
14	[29, 30]	195	[346, 347]	376	[149, 38]	557	[418, 441]	738	[796, 797]
15	[31, 32]	196	[348, 349]	377	[507, 482]	558	[743, 149]	739	[798, 799]
16	[33, 34]	197	[350, 351]	378	[145, 435]	559	[506, 181]	740	[120, 524]

17	[35, 36]	198	[352, 353]	379	[508, 33]	560	[744, 745]	741	[800, 801]
18	[37, 38]	199	[354, 355]	380	[424, 509]	561	[501, 194]	742	[802, 574]
19	[39, 40]	200	[356, 357]	381	[510, 180]	562	[136, 142]	743	[803, 804]
20	[41, 42]	201	[358, 359]	382	[468, 421]	563	[481, 46]	744	[805, 806]
21	[43, 44]	202	[360, 361]	383	[480, 168]	564	[47, 140]	745	[807, 808]
22	[45, 46]	203	[362, 363]	384	[474, 489]	565	[746, 432]	746	[809, 810]
23	[47, 48]	204	[364, 365]	385	[511, 512]	566	[433, 43]	747	[319, 811]
24	[49, 50]	205	[366, 367]	386	[490, 209]	567	[502, 400]	748	[812, 813]
25	[51, 52]	206	[368, 369]	387	[37, 155]	568	[747, 489]	749	[814, 815]
26	[53, 54]	207	[370, 371]	388	[163, 211]	569	[208, 413]	750	[816, 817]
27	[55, 56]	208	[372, 373]	389	[513, 514]	570	[143, 748]	751	[818, 819]
28	[57, 58]	209	[374, 375]	390	[515, 516]	571	[749, 503]	752	[820, 821]
29	[59, 60]	210	[226, 376]	391	[517, 518]	572	[139, 750]	753	[822, 823]
30	[61, 62]	211	[377, 378]	392	[519, 65]	573	[457, 482]	754	[824, 825]
31	[63, 64]	212	[379, 297]	393	[520, 521]	574	[208, 32]	755	[826, 827]
32	[65, 66]	213	[380, 381]	394	[522, 523]	575	[463, 178]	756	[828, 829]
33	[67, 68]	214	[382, 383]	395	[268, 77]	576	[474, 496]	757	[830, 831]
34	[69, 70]	215	[274, 384]	396	[524, 525]	577	[751, 484]	758	[832, 780]
35	[71, 72]	216	[385, 386]	397	[526, 527]	578	[457, 752]	759	[833, 834]
36	[73, 74]	217	[387, 388]	398	[528, 529]	579	[192, 449]	760	[835, 836]
37	[75, 76]	218	[163, 166]	399	[530, 531]	580	[753, 497]	761	[837, 838]
38	[77, 78]	219	[131, 151]	400	[532, 533]	581	[754, 142]	762	[839, 840]
39	[79, 80]	220	[389, 390]	401	[534, 535]	582	[428, 131]	763	[841, 271]
40	[81, 82]	221	[391, 392]	402	[536, 537]	583	[389, 42]	764	[842, 843]
41	[83, 84]	222	[136, 43]	403	[538, 539]	584	[755, 178]	765	[844, 845]
42	[85, 86]	223	[393, 394]	404	[540, 541]	585	[509, 443]	766	[846, 847]
43	[87, 88]	224	[37, 148]	405	[542, 543]	586	[212, 445]	767	[848, 370]
44	[89, 90]	225	[395, 149]	406	[544, 545]	587	[418, 417]	768	[383, 849]
45	[91, 92]	226	[148, 152]	407	[546, 547]	588	[756, 129]	769	[850, 851]
46	[93, 94]	227	[8, 396]	408	[305, 548]	589	[753, 206]	770	[852, 853]
47	[95, 96]	228	[397, 390]	409	[549, 550]	590	[482, 58]	771	[854, 855]
48	[97, 98]	229	[142, 398]	410	[551, 552]	591	[127, 757]	772	[856, 857]
49	[99, 100]	230	[399, 400]	411	[553, 554]	592	[758, 209]	773	[858, 859]
50	[101, 102]	231	[401, 402]	412	[555, 556]	593	[33, 173]	774	[64, 526]
51	[103, 104]	232	[403, 188]	413	[557, 558]	594	[759, 512]	775	[860, 861]
52	[105, 106]	233	[148, 37]	414	[373, 559]	595	[35, 205]	776	[862, 863]
53	[107, 108]	234	[404, 164]	415	[94, 238]	596	[153, 205]	777	[864, 865]
54	[109, 110]	235	[212, 169]	416	[560, 561]	597	[447, 39]	778	[866, 867]
55	[111, 112]	236	[405, 203]	417	[562, 563]	598	[491, 135]	779	[386, 868]
56	[113, 114]	237	[406, 217]	418	[564, 565]	599	[455, 745]	780	[402, 464]
57	[115, 116]	238	[389, 407]	419	[566, 567]	600	[739, 452]	781	[157, 443]
58	[117, 118]	239	[62, 408]	420	[533, 568]	601	[457, 760]	782	[413, 869]
59	[119, 120]	240	[409, 410]	421	[569, 242]	602	[137, 460]	783	[9, 42]
60	[121, 122]	241	[411, 200]	422	[570, 571]	603	[182, 154]	784	[127, 181]

61	[123, 124]	242	[412, 202]	423	[572, 573]	604	[210, 189]	785	[870, 159]
62	[125, 126]	243	[413, 414]	424	[574, 575]	605	[214, 194]	786	[731, 440]
63	[7, 127]	244	[415, 185]	425	[576, 577]	606	[761, 394]	787	[466, 200]
64	[128, 34]	245	[416, 417]	426	[578, 579]	607	[141, 471]	788	[141, 436]
65	[35, 129]	246	[418, 419]	427	[580, 581]	608	[476, 460]	789	[213, 449]
66	[130, 131]	247	[420, 421]	428	[256, 582]	609	[32, 414]	790	[415, 137]
67	[132, 133]	248	[144, 398]	429	[583, 584]	610	[436, 762]	791	[411, 871]
68	[134, 135]	249	[418, 422]	430	[381, 585]	611	[421, 139]	792	[415, 765]
69	[136, 49]	250	[423, 211]	431	[586, 279]	612	[505, 39]	793	[508, 737]
70	[137, 138]	251	[8, 171]	432	[587, 588]	613	[412, 484]	794	[175, 489]
71	[139, 140]	252	[424, 188]	433	[589, 590]	614	[33, 156]	795	[736, 202]
72	[141, 142]	253	[425, 426]	434	[591, 592]	615	[465, 757]	796	[134, 872]
73	[143, 144]	254	[427, 199]	435	[593, 594]	616	[447, 506]	797	[413, 774]
74	[145, 146]	255	[130, 428]	436	[595, 596]	617	[512, 200]	798	[468, 492]
75	[130, 147]	256	[150, 150]	437	[597, 598]	618	[438, 129]	799	[142, 444]
76	[148, 149]	257	[38, 148]	438	[599, 600]	619	[743, 152]	800	[413, 873]
77	[150, 151]	258	[428, 189]	439	[601, 602]	620	[474, 410]	801	[870, 127]
78	[152, 38]	259	[37, 395]	440	[603, 604]	621	[62, 442]	802	[499, 463]
79	[153, 154]	260	[151, 150]	441	[605, 606]	622	[216, 763]	803	[210, 428]
80	[155, 149]	261	[210, 147]	442	[607, 608]	623	[483, 158]	804	[149, 37]
81	[128, 156]	262	[147, 189]	443	[575, 609]	624	[764, 765]	805	[215, 42]
82	[157, 158]	263	[147, 131]	444	[610, 611]	625	[748, 44]	806	[128, 462]
83	[159, 160]	264	[429, 430]	445	[612, 613]	626	[438, 184]	807	[51, 874]
84	[132, 10]	265	[431, 432]	446	[614, 615]	627	[756, 36]	808	[188, 774]
85	[57, 161]	266	[433, 142]	447	[616, 617]	628	[413, 446]	809	[753, 875]
86	[162, 146]	267	[434, 435]	448	[618, 619]	629	[758, 32]	810	[144, 44]
87	[163, 164]	268	[395, 152]	449	[620, 621]	630	[54, 426]	811	[729, 876]
88	[165, 166]	269	[143, 436]	450	[622, 623]	631	[733, 425]	812	[204, 419]
89	[167, 168]	270	[437, 139]	451	[624, 625]	632	[397, 198]	813	[416, 164]
90	[169, 170]	271	[438, 36]	452	[626, 627]	633	[766, 442]	814	[188, 873]
91	[39, 171]	272	[189, 151]	453	[628, 629]	634	[483, 443]	815	[731, 188]
92	[172, 127]	273	[411, 439]	454	[630, 631]	635	[767, 206]	816	[59, 877]
93	[128, 173]	274	[440, 158]	455	[632, 633]	636	[402, 467]	817	[509, 158]
94	[174, 62]	275	[416, 441]	456	[327, 634]	637	[174, 766]	818	[477, 425]
95	[175, 176]	276	[130, 189]	457	[635, 117]	638	[167, 768]	819	[403, 413]
96	[177, 178]	277	[425, 442]	458	[636, 637]	639	[765, 138]	820	[163, 419]
97	[179, 180]	278	[440, 443]	459	[638, 639]	640	[132, 769]	821	[772, 36]
98	[8, 181]	279	[43, 444]	460	[640, 641]	641	[33, 426]	822	[401, 177]
99	[182, 183]	280	[445, 170]	461	[642, 643]	642	[401, 463]	823	[878, 390]
100	[182, 184]	281	[188, 446]	462	[644, 645]	643	[478, 448]	824	[767, 875]
101	[43, 50]	282	[447, 448]	463	[646, 647]	644	[507, 432]	825	[480, 768]
102	[185, 138]	283	[449, 450]	464	[648, 649]	645	[497, 207]	826	[195, 489]
103	[53, 186]	284	[451, 452]	465	[122, 650]	646	[215, 390]	827	[134, 217]
104	[187, 188]	285	[141, 144]	466	[651, 652]	647	[59, 197]	828	[55, 738]

105	[150, 189]	286	[453, 454]	467	[653, 654]	648	[501, 752]	829	[501, 482]
106	[190, 164]	287	[455, 456]	468	[655, 95]	649	[469, 139]	830	[199, 439]
107	[191, 192]	288	[431, 57]	469	[656, 657]	650	[210, 131]	831	[506, 757]
108	[193, 194]	289	[457, 167]	470	[278, 658]	651	[132, 770]	832	[424, 157]
109	[195, 196]	290	[393, 458]	471	[659, 660]	652	[51, 485]	833	[191, 213]
110	[51, 197]	291	[438, 205]	472	[661, 662]	653	[136, 452]	834	[776, 43]
111	[132, 198]	292	[423, 166]	473	[663, 664]	654	[192, 170]	835	[416, 422]
112	[199, 200]	293	[432, 459]	474	[82, 665]	655	[499, 177]	836	[153, 184]
113	[201, 202]	294	[185, 460]	475	[666, 667]	656	[486, 448]	837	[39, 734]
114	[203, 160]	295	[405, 448]	476	[668, 97]	657	[510, 392]	838	[779, 32]
115	[204, 164]	296	[461, 197]	477	[669, 670]	658	[397, 10]	839	[471, 879]
116	[182, 205]	297	[195, 176]	478	[671, 672]	659	[153, 771]	840	[765, 460]
117	[142, 50]	298	[33, 462]	479	[673, 674]	660	[772, 129]	841	[151, 189]
118	[206, 207]	299	[463, 464]	480	[675, 676]	661	[432, 773]	842	[53, 33]
119	[174, 54]	300	[465, 181]	481	[677, 678]	662	[445, 449]	843	[880, 390]
120	[208, 209]	301	[405, 159]	482	[679, 680]	663	[440, 774]	844	[490, 188]
121	[130, 210]	302	[466, 467]	483	[681, 682]	664	[505, 203]	845	[504, 197]
122	[204, 211]	303	[213, 170]	484	[683, 17]	665	[41, 407]	846	[397, 770]
123	[212, 213]	304	[468, 469]	485	[684, 218]	666	[57, 775]	847	[742, 180]
124	[214, 57]	305	[433, 452]	486	[685, 277]	667	[399, 503]	848	[511, 411]
125	[215, 10]	306	[39, 396]	487	[686, 687]	668	[505, 8]	849	[421, 47]
126	[216, 217]	307	[470, 176]	488	[688, 689]	669	[764, 476]	850	[432, 775]
127	[218, 219]	308	[471, 472]	489	[690, 691]	670	[776, 142]	851	[741, 458]
128	[220, 221]	309	[473, 458]	490	[692, 693]	671	[731, 157]	852	[471, 881]
129	[222, 223]	310	[474, 475]	491	[694, 330]	672	[186, 426]	853	[778, 454]
130	[224, 225]	311	[391, 180]	492	[695, 696]	673	[744, 456]	854	[507, 194]
131	[225, 226]	312	[214, 432]	493	[697, 698]	674	[143, 43]	855	[749, 400]
132	[227, 228]	313	[476, 138]	494	[16, 699]	675	[182, 771]	856	[449, 487]
133	[229, 230]	314	[477, 54]	495	[700, 701]	676	[416, 419]	857	[746, 57]
134	[231, 232]	315	[478, 203]	496	[702, 703]	677	[440, 414]	858	[57, 773]
135	[233, 234]	316	[150, 131]	497	[704, 705]	678	[424, 413]	859	[492, 139]
136	[235, 89]	317	[479, 205]	498	[706, 707]	679	[153, 183]	860	[436, 879]
137	[236, 237]	318	[402, 178]	499	[708, 312]	680	[165, 211]	861	[875, 498]
138	[238, 239]	319	[127, 160]	500	[709, 710]	681	[416, 777]	862	[191, 445]
139	[240, 241]	320	[170, 450]	501	[711, 712]	682	[37, 37]	863	[194, 493]
140	[242, 243]	321	[457, 194]	502	[713, 714]	683	[395, 37]	864	[501, 480]
141	[244, 101]	322	[457, 480]	503	[715, 716]	684	[151, 131]	865	[434, 146]
142	[245, 246]	323	[481, 454]	504	[717, 301]	685	[424, 440]	866	[159, 757]
143	[247, 248]	324	[214, 482]	505	[359, 318]	686	[179, 392]	867	[447, 203]
144	[249, 250]	325	[453, 46]	506	[718, 719]	687	[188, 414]	868	[512, 467]
145	[251, 252]	326	[424, 483]	507	[720, 721]	688	[8, 40]	869	[882, 883]
146	[253, 254]	327	[201, 484]	508	[722, 644]	689	[474, 196]	870	[884, 299]
147	[255, 256]	328	[409, 475]	509	[388, 723]	690	[480, 493]	871	[885, 886]
148	[257, 258]	329	[59, 485]	510	[724, 692]	691	[730, 400]	872	[887, 888]

149	[259, 260]	330	[486, 159]	511	[725, 653]	692	[405, 127]	873	[331, 83]
150	[261, 262]	331	[425, 408]	512	[726, 727]	693	[186, 462]	874	[889, 618]
151	[151, 151]	332	[204, 166]	513	[413, 728]	694	[174, 425]	875	[890, 113]
152	[76, 263]	333	[38, 38]	514	[729, 176]	695	[731, 509]	876	[891, 892]
153	[264, 265]	334	[38, 395]	515	[436, 472]	696	[461, 485]	877	[893, 894]
154	[266, 267]	335	[152, 37]	516	[730, 503]	697	[471, 762]	878	[895, 896]
155	[260, 268]	336	[170, 487]	517	[427, 466]	698	[169, 449]	879	[799, 897]
156	[269, 270]	337	[141, 43]	518	[187, 413]	699	[32, 774]	880	[781, 898]
157	[271, 272]	338	[477, 62]	519	[130, 130]	700	[157, 414]	881	[825, 899]
158	[273, 274]	339	[488, 489]	520	[159, 181]	701	[172, 159]	882	[199, 871]
159	[275, 276]	340	[490, 413]	521	[731, 413]	702	[167, 58]	883	[209, 774]
160	[277, 278]	341	[491, 217]	522	[199, 732]	703	[778, 46]	884	[505, 465]
161	[279, 280]	342	[420, 492]	523	[733, 54]	704	[731, 483]	885	[507, 57]
162	[281, 282]	343	[482, 493]	524	[54, 408]	705	[406, 135]	886	[437, 47]
163	[283, 284]	344	[35, 184]	525	[209, 414]	706	[397, 769]	887	[210, 210]
164	[285, 286]	345	[418, 164]	526	[127, 494]	707	[177, 464]	888	[900, 184]
165	[287, 288]	346	[159, 494]	527	[9, 407]	708	[415, 476]	889	[37, 743]
166	[289, 290]	347	[409, 489]	528	[57, 459]	709	[141, 748]	890	[486, 203]
167	[291, 292]	348	[432, 161]	529	[206, 498]	710	[761, 458]	891	[142, 44]
168	[293, 294]	349	[45, 454]	530	[8, 734]	711	[191, 169]	892	[162, 435]
169	[295, 296]	350	[210, 130]	531	[505, 448]	712	[748, 398]	893	[148, 38]
170	[297, 298]	351	[404, 419]	532	[411, 732]	713	[157, 774]	894	[900, 205]
171	[299, 300]	352	[480, 58]	533	[427, 411]	714	[779, 209]	895	[188, 869]
172	[301, 302]	353	[495, 435]	534	[420, 469]	715	[463, 467]	896	[409, 196]
173	[124, 303]	354	[409, 496]	535	[735, 194]	716	[759, 466]	897	[492, 47]
174	[304, 305]	355	[216, 135]	536	[397, 133]	717	[511, 466]	898	[470, 876]
175	[306, 307]	356	[143, 471]	537	[736, 484]	718	[418, 777]	899	[875, 207]
176	[308, 309]	357	[497, 498]	538	[447, 465]	719	[189, 150]	900	[901, 902]
177	[310, 311]	358	[499, 402]	539	[737, 426]	720	[764, 185]	901	[47, 750]
178	[312, 313]	359	[486, 127]	540	[429, 738]	721	[194, 58]	902	[451, 49]
179	[314, 315]	360	[38, 155]	541	[739, 49]	722	[767, 497]		
180	[316, 317]	361	[479, 184]	542	[490, 32]	723	[155, 152]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	130	0	260	0	390	0	520	0	650	1	780	1
1	0	131	0	261	1	391	1	521	1	651	0	781	1
2	1	132	0	262	0	392	0	522	0	652	1	782	1
3	0	133	0	263	0	393	0	523	0	653	0	783	1
4	1	134	1	264	1	394	0	524	0	654	1	784	1
5	1	135	1	265	0	395	0	525	1	655	0	785	0
6	0	136	0	266	1	396	0	526	1	656	0	786	1
7	0	137	1	267	1	397	1	527	1	657	0	787	0
8	0	138	1	268	0	398	0	528	0	658	1	788	0

9	1	139	0	269	1	399	0	529	1	659	1	789	0
10	1	140	1	270	1	400	1	530	0	660	1	790	0
11	1	141	0	271	1	401	1	531	0	661	1	791	1
12	1	142	0	272	1	402	1	532	1	662	0	792	0
13	0	143	1	273	1	403	1	533	1	663	0	793	0
14	0	144	1	274	0	404	1	534	1	664	0	794	1
15	0	145	0	275	0	405	1	535	0	665	0	795	0
16	0	146	1	276	0	406	1	536	1	666	0	796	1
17	0	147	0	277	1	407	1	537	0	667	0	797	1
18	0	148	1	278	0	408	1	538	1	668	0	798	0
19	1	149	0	279	1	409	0	539	1	669	1	799	0
20	0	150	1	280	0	410	1	540	1	670	1	800	1
21	1	151	0	281	0	411	1	541	0	671	1	801	0
22	1	152	1	282	1	412	1	542	1	672	0	802	0
23	1	153	1	283	1	413	1	543	1	673	0	803	1
24	0	154	1	284	1	414	0	544	1	674	1	804	0
25	1	155	0	285	0	415	0	545	0	675	0	805	0
26	1	156	1	286	1	416	0	546	1	676	0	806	1
27	0	157	1	287	1	417	0	547	0	677	0	807	1
28	0	158	1	288	0	418	1	548	0	678	0	808	0
29	0	159	0	289	0	419	1	549	0	679	1	809	1
30	0	160	1	290	0	420	1	550	0	680	1	810	1
31	0	161	1	291	1	421	0	551	1	681	0	811	1
32	0	162	0	292	0	422	1	552	1	682	0	812	0
33	0	163	1	293	1	423	0	553	1	683	0	813	0
34	0	164	0	294	0	424	0	554	0	684	0	814	0
35	0	165	1	295	1	425	1	555	1	685	0	815	1
36	1	166	0	296	1	426	0	556	0	686	1	816	0
37	0	167	1	297	0	427	1	557	1	687	0	817	1
38	1	168	1	298	0	428	1	558	1	688	0	818	1
39	1	169	1	299	0	429	1	559	1	689	1	819	1
40	1	170	0	300	0	430	0	560	0	690	0	820	1
41	0	171	0	301	1	431	0	561	1	691	1	821	1
42	0	172	1	302	0	432	1	562	0	692	1	822	1
43	1	173	0	303	0	433	1	563	0	693	0	823	0
44	1	174	0	304	0	434	1	564	1	694	0	824	0
45	1	175	1	305	1	435	0	565	1	695	1	825	0
46	1	176	1	306	1	436	0	566	1	696	1	826	0
47	1	177	1	307	0	437	1	567	1	697	1	827	1
48	1	178	0	308	1	438	1	568	1	698	1	828	0
49	0	179	1	309	0	439	0	569	0	699	0	829	1
50	1	180	1	310	1	440	0	570	1	700	1	830	0
51	1	181	1	311	1	441	0	571	0	701	1	831	1
52	1	182	0	312	0	442	0	572	0	702	1	832	0

53	1	183	0	313	0	443	0	573	0	703	0	833	1
54	0	184	0	314	1	444	0	574	0	704	1	834	1
55	0	185	0	315	1	445	0	575	0	705	1	835	0
56	0	186	0	316	1	446	0	576	1	706	1	836	1
57	0	187	0	317	0	447	1	577	1	707	1	837	1
58	0	188	0	318	1	448	1	578	0	708	0	838	1
59	0	189	1	319	1	449	1	579	1	709	0	839	1
60	0	190	0	320	0	450	0	580	1	710	1	840	1
61	0	191	1	321	0	451	1	581	0	711	1	841	0
62	0	192	1	322	0	452	1	582	1	712	0	842	1
63	0	193	1	323	0	453	1	583	1	713	1	843	1
64	1	194	0	324	0	454	0	584	0	714	1	844	1
65	0	195	0	325	1	455	1	585	1	715	0	845	0
66	0	196	1	326	0	456	0	586	0	716	1	846	1
67	0	197	1	327	0	457	0	587	1	717	0	847	0
68	1	198	0	328	0	458	1	588	0	718	1	848	0
69	0	199	0	329	0	459	1	589	1	719	1	849	0
70	1	200	1	330	0	460	0	590	1	720	1	850	1
71	0	201	0	331	1	461	1	591	1	721	0	851	1
72	0	202	1	332	0	462	1	592	0	722	0	852	1
73	1	203	0	333	1	463	0	593	0	723	0	853	0
74	0	204	0	334	1	464	1	594	1	724	0	854	1
75	0	205	1	335	1	465	0	595	0	725	0	855	0
76	1	206	1	336	0	466	0	596	1	726	1	856	1
77	1	207	0	337	0	467	0	597	1	727	1	857	1
78	1	208	0	338	1	468	0	598	0	728	1	858	0
79	1	209	1	339	0	469	0	599	1	729	1	859	1
80	0	210	1	340	1	470	0	600	0	730	1	860	0
81	1	211	1	341	0	471	1	601	0	731	1	861	0
82	1	212	0	342	1	472	1	602	1	732	0	862	1
83	0	213	0	343	1	473	0	603	0	733	0	863	0
84	0	214	0	344	0	474	1	604	1	734	1	864	1
85	0	215	0	345	1	475	0	605	0	735	0	865	1
86	0	216	0	346	0	476	0	606	1	736	0	866	0
87	1	217	0	347	0	477	1	607	0	737	1	867	1
88	1	218	1	348	1	478	1	608	0	738	1	868	1
89	1	219	0	349	1	479	0	609	0	739	0	869	0
90	1	220	1	350	1	480	0	610	0	740	0	870	0
91	1	221	1	351	1	481	0	611	0	741	1	871	1
92	1	222	0	352	0	482	1	612	0	742	0	872	1
93	1	223	0	353	1	483	0	613	1	743	1	873	1
94	0	224	0	354	0	484	0	614	0	744	0	874	0
95	1	225	0	355	0	485	0	615	0	745	1	875	0
96	1	226	1	356	1	486	0	616	1	746	1	876	0

97	1	227	0	357	1	487	1	617	1	747	1	877	1
98	0	228	1	358	0	488	0	618	1	748	0	878	0
99	0	229	0	359	0	489	0	619	1	749	0	879	0
100	0	230	0	360	1	490	1	620	1	750	0	880	1
101	1	231	1	361	0	491	0	621	0	751	1	881	0
102	0	232	1	362	0	492	1	622	0	752	1	882	0
103	1	233	1	363	1	493	1	623	0	753	1	883	1
104	0	234	1	364	0	494	0	624	1	754	0	884	0
105	1	235	0	365	1	495	1	625	0	755	0	885	1
106	0	236	1	366	1	496	1	626	1	756	0	886	1
107	1	237	1	367	1	497	1	627	0	757	0	887	1
108	1	238	1	368	1	498	1	628	1	758	0	888	1
109	0	239	0	369	0	499	0	629	0	759	1	889	0
110	1	240	0	370	0	500	0	630	0	760	0	890	0
111	0	241	1	371	1	501	1	631	0	761	1	891	0
112	0	242	1	372	0	502	1	632	1	762	1	892	0
113	0	243	1	373	0	503	0	633	1	763	0	893	1
114	0	244	0	374	1	504	0	634	0	764	1	894	1
115	0	245	0	375	0	505	0	635	0	765	1	895	0
116	0	246	1	376	0	506	1	636	1	766	1	896	0
117	0	247	1	377	1	507	1	637	0	767	0	897	1
118	1	248	1	378	0	508	0	638	1	768	0	898	0
119	0	249	1	379	0	509	1	639	1	769	1	899	0
120	0	250	0	380	0	510	0	640	0	770	1	900	1
121	0	251	0	381	0	511	0	641	0	771	1	901	1
122	0	252	0	382	0	512	1	642	1	772	1	902	1
123	0	253	1	383	0	513	1	643	1	773	0		
124	0	254	1	384	1	514	1	644	1	774	1		
125	0	255	0	385	0	515	0	645	1	775	0		
126	0	256	1	386	1	516	1	646	0	776	1		
127	1	257	1	387	0	517	1	647	0	777	1		
128	1	258	1	388	1	518	0	648	1	778	0		
129	0	259	0	389	1	519	0	649	0	779	1		

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