PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a,b) = (z^2+1, z^5+z^4+z^3+z^2+z+1)$

GUO-NIU HAN* AND YINING HU*

Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a,b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a,b) in $\mathbb{F}_2[x]\setminus\mathbb{F}_2$. In this paper we prove the conjecture for $(a,b) = (z^2+1, z^5+z^4+z^3+z^2+z+1)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \ldots)$ of elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{\ddots + \cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a,b)-Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \ldots) = (a, b, b, a, b, a, a, b, \ldots)$$

that we shall denote

$$\mathrm{CF}(a,b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When* $(a,b) = (z^2+1, z^5+z^4+z^3+z^2+z+1)$, *the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$, *with minimal polynomials of the form*

$$p_4(z)y^4 + p_3(z)y^3 + p_2(z)y^2 + p_1(z)y + p_0(z) = 0$$

where, for $\mathrm{CF}(a,b)$,

$$\begin{aligned}
p_0(z) &= z^{17}+z^{15}+z^{14}+z^{11}+z^6+z^5+z^2+z+1,\\
p_1(z) &= z^{22}+z^{20}+z^{19}+z^{18}+z^{15}+z^{13}+z^{12}+z^{11}+z^{10}+z^7+z^5+z^3+z^2\\
&\quad +z,\\
p_2(z) &= z^{24}+z^{20}+z^{16}+z^{14}+z^8+z^4+z^2+1,\\
p_3(z) &= z^{22}+z^{20}+z^{19}+z^{18}+z^{15}+z^{13}+z^{12}+z^{11}+z^{10}+z^7+z^5+z^3+z^2\\
&\quad +z,\\
p_4(z) &= z^{20}+z^{17}+z^{15}+z^{11}+z^6+z^5+z+1,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$,

$$\begin{aligned}
p_0(z) &= z^{17}+z^{16}+z^{15}+z^{14}+z^{11}+z^8+z^6+z^5+z^4+z^2+z+1,\\
p_1(z) &= z^{22}+z^{20}+z^{19}+z^{18}+z^{15}+z^{13}+z^{12}+z^{11}+z^{10}+z^7+z^5+z^3+z^2\\
&\quad +z,\\
p_2(z) &= z^{24}+z^{20}+z^{16}+z^{14}+z^8+z^4+z^2+1,\\
p_3(z) &= z^{22}+z^{20}+z^{19}+z^{18}+z^{15}+z^{13}+z^{12}+z^{11}+z^{10}+z^7+z^5+z^3+z^2\\
&\quad +z,\\
p_4(z) &= z^{20}+z^{17}+z^{16}+z^{15}+z^{11}+z^8+z^6+z^5+z^4+z+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\ge 0}$ and $(Q_n(z))_{n\ge 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z) \\ P_{n-1}(z) & Q_{n-1}(z) \end{pmatrix} := \begin{pmatrix} t_n(z) & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t_{n-1}(z) & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} t_0(z) & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} \in \mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x) &= x^{\deg(t_{2^n-1})} \begin{pmatrix} t_{2^n-1}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times x^{\deg(t_{2^n-2})} \begin{pmatrix} t_{2^n-2}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times \\
&\qquad \cdots \times x^{\deg(t_0)} \begin{pmatrix} t_0(1/x) & 1 \\ 1 & 0 \end{pmatrix},\\
W_n(x) &= x^{\deg(\bar t_{2^n-1})} \begin{pmatrix} \bar t_{2^n-1}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times x^{\deg(\bar t_{2^n-2})} \begin{pmatrix} \bar t_{2^n-2}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times \\
&\qquad \cdots \times x^{\deg(\bar t_0)} \begin{pmatrix} \bar t_0(1/x) & 1 \\ 1 & 0 \end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n \geq 0$

$$\begin{aligned} M_{n+1}(x) &= W_n(x) \cdot M_n(x), \\ W_{n+1}(x) &= M_n(x) \cdot W_n(x). \end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$\begin{aligned} M_n(x)_{0,1} &= x^{d_n} \tilde{P}_{2^n-1}(x), \\ M_n(x)_{0,0} &= x^{d_n} \tilde{Q}_{2^n-1}(x), \end{aligned}$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \leq i,j \leq 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \leq i,j \leq 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \leq i,j \leq 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(175, 175, 175, 175, 175)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \geq 0$. If for some $n \geq 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \geq 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z\left(\phi(W^o,0,0)(x,z),\ z^{12}\cdot\phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 630 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 630, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z\left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ to order 770 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 770, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. $\square$

Define

$$R^e = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned} M^e[:14u] = {} & M^e[:7u] + x^u M^e[:6u] + x^{4u} M^e[:3u] + x^{5u} M^e[:2u] + x^{6u} M^e[:u] \\ & + x^u M^e[:7u] + x^{2u} M^e[:6u] + x^{5u} M^e[:3u] + x^{6u} M^e[:2u] \\ & + x^{7u} M^e[:u] + x^{2u} M^e[:7u] + x^{3u} M^e[:6u] + x^{6u} M^e[:3u] \\ & + x^{7u} M^e[:2u] + x^{8u} M^e[:u] + x^{3u} M^e[:7u] + x^{4u} M^e[:6u] \end{aligned}$$

$$
\begin{aligned}
&+ x^{7u}M^e[:3u] + x^{8u}M^e[:2u] + x^{9u}M^e[:u] + x^{6u}M^e[:7u]\\
&+ x^{7u}M^e[:6u] + x^{10u}M^e[:3u] + x^{11u}M^e[:2u] + x^{12u}M^e[:u]\\
&+ x^{7u}M^e[:7u] + x^{8u}M^e[:6u] + x^{11u}M^e[:3u] + x^{13u}M^e[:u]\\
&+ (x^{7u} + x^{8u} + x^{9u} + x^{10u} + x^{13u})R^e,\\
W^e[:14u] = {}& W^e[:7u] + x^uW^e[:6u] + x^{4u}W^e[:3u] + x^{5u}W^e[:2u] + x^{6u}W^e[:u]\\
&+ x^uW^e[:7u] + x^{2u}W^e[:6u] + x^{5u}W^e[:3u] + x^{6u}W^e[:2u]\\
&+ x^{7u}W^e[:u] + x^{2u}W^e[:7u] + x^{3u}W^e[:6u] + x^{6u}W^e[:3u]\\
&+ x^{7u}W^e[:2u] + x^{8u}W^e[:u] + x^{3u}W^e[:7u] + x^{4u}W^e[:6u]\\
&+ x^{7u}W^e[:3u] + x^{8u}W^e[:2u] + x^{9u}W^e[:u] + x^{6u}W^e[:7u]\\
&+ x^{7u}W^e[:6u] + x^{10u}W^e[:3u] + x^{11u}W^e[:2u] + x^{12u}W^e[:u]\\
&+ x^{7u}W^e[:7u] + x^{8u}W^e[:6u] + x^{11u}W^e[:3u] + x^{13u}W^e[:u]\\
&+ (x^{7u} + x^{8u} + x^{9u} + x^{10u} + x^{13u})R^e.
\end{aligned}
$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$
\begin{aligned}
M^o[:14u] = {}& M^o[:7u] + x^uM^o[:6u] + x^{4u}M^o[:3u] + x^{5u}M^o[:2u] + x^{6u}M^o[:u]\\
&+ x^uM^o[:7u] + x^{2u}M^o[:6u] + x^{5u}M^o[:3u] + x^{6u}M^o[:2u]\\
&+ x^{7u}M^o[:u] + x^{2u}M^o[:7u] + x^{3u}M^o[:6u] + x^{6u}M^o[:3u]\\
&+ x^{7u}M^o[:2u] + x^{8u}M^o[:u] + x^{3u}M^o[:7u] + x^{4u}M^o[:6u]\\
&+ x^{7u}M^o[:3u] + x^{8u}M^o[:2u] + x^{9u}M^o[:u] + x^{6u}M^o[:7u]\\
&+ x^{7u}M^o[:6u] + x^{10u}M^o[:3u] + x^{11u}M^o[:2u] + x^{12u}M^o[:u]\\
&+ x^{7u}M^o[:7u] + x^{8u}M^o[:6u] + x^{11u}M^o[:3u] + x^{13u}M^o[:u]\\
&+ (x^{7u} + x^{8u} + x^{9u} + x^{10u} + x^{13u})R^o,\\
W^o[:14u] = {}& W^o[:7u] + x^uW^o[:6u] + x^{4u}W^o[:3u] + x^{5u}W^o[:2u] + x^{6u}W^o[:u]\\
&+ x^uW^o[:7u] + x^{2u}W^o[:6u] + x^{5u}W^o[:3u] + x^{6u}W^o[:2u]\\
&+ x^{7u}W^o[:u] + x^{2u}W^o[:7u] + x^{3u}W^o[:6u] + x^{6u}W^o[:3u]\\
&+ x^{7u}W^o[:2u] + x^{8u}W^o[:u] + x^{3u}W^o[:7u] + x^{4u}W^o[:6u]\\
&+ x^{7u}W^o[:3u] + x^{8u}W^o[:2u] + x^{9u}W^o[:u] + x^{6u}W^o[:7u]\\
&+ x^{7u}W^o[:6u] + x^{10u}W^o[:3u] + x^{11u}W^o[:2u] + x^{12u}W^o[:u]\\
&+ x^{7u}W^o[:7u] + x^{8u}W^o[:6u] + x^{11u}W^o[:3u] + x^{13u}W^o[:u]\\
&+ (x^{7u} + x^{8u} + x^{9u} + x^{10u} + x^{13u})R^o.
\end{aligned}
$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$
\begin{aligned}
T[:14u] = {}& T[:7u] + x^uT[:6u] + x^{4u}T[:3u] + x^{5u}T[:2u] + x^{6u}T[:u] + x^uT[:7u]\\
&+ x^{2u}T[:6u] + x^{5u}T[:3u] + x^{6u}T[:2u] + x^{7u}T[:u] + x^{2u}T[:7u]\\
&+ x^{3u}T[:6u] + x^{6u}T[:3u] + x^{7u}T[:2u] + x^{8u}T[:u] + x^{3u}T[:7u]
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{4u}T[:6u] + x^{7u}T[:3u] + x^{8u}T[:2u] + x^{9u}T[:u] + x^{6u}T[:7u]\\
&+ x^{7u}T[:6u] + x^{10u}T[:3u] + x^{11u}T[:2u] + x^{12u}T[:u] + x^{7u}T[:7u]\\
&+ x^{8u}T[:6u] + x^{11u}T[:3u] + x^{13u}T[:u]\\
&+ (x^{7u} + x^{8u} + x^{9u} + x^{10u} + x^{13u}).
\end{aligned}
$$

The proof of the other 15 cases are similar. We break down the above identity into 7 parts:

$$
\begin{aligned}
x^7u &= x^{7u}T[:u] + x^{6u}T[u:2u] + x^{5u}T[2u:3u] + x^{4u}T[3u:4u]\\
&\quad + x^{u}T[6u:7u] + T[7u:8u],\\
x^8u &= x^{8u}T[:u] + x^{4u}T[4u:5u] + x^{2u}T[6u:7u] + T[8u:9u],\\
x^9u &= x^{9u}T[:u] + x^{7u}T[2u:3u] + x^{6u}T[3u:4u] + x^{4u}T[5u:6u]\\
&\quad + x^{3u}T[6u:7u] + T[9u:10u],\\
x^10u &= x^{10u}T[:u] + x^{8u}T[2u:3u] + x^{6u}T[4u:5u] + T[10u:11u],\\
0 &= x^{10u}T[u:2u] + x^{8u}T[3u:4u] + x^{6u}T[5u:6u] + T[11u:12u],\\
0 &= x^{12u}T[:u] + x^{10u}T[2u:3u] + x^{8u}T[4u:5u] + x^{6u}T[6u:7u]\\
&\quad + T[12u:13u],\\
x^13u &= x^{13u}T[:u] + x^{11u}T[2u:3u] + x^{8u}T[5u:6u] + x^{7u}T[6u:7u]\\
&\quad + T[13u:14u],
\end{aligned}
$$

which can be rewritten as

$$
\begin{aligned}
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[110w]_2]\\
&\quad + T[[111w]_2], \qquad (2.7)\\
0 &= T[[w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1000w]_2], \qquad (2.8)\\
0 &= T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2]\\
&\quad + T[[1001w]_2], \qquad (2.9)\\
0 &= T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[1010w]_2], \qquad (2.10)\\
0 &= T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1011w]_2], \qquad (2.11)\\
0 &= T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1100w]_2], \qquad (2.12)\\
0 &= T[[w]_2] + T[[10w]_2] + T[[101w]_2] + T[[110w]_2] + T[[1101w]_2], \qquad (2.13)
\end{aligned}
$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$
\begin{aligned}
1 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[110w]_2]\\
&\quad + T[[111w]_2], \qquad (2.14)\\
1 &= T[[w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1000w]_2], \qquad (2.15)\\
1 &= T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2]\\
&\quad + T[[1001w]_2], \qquad (2.16)\\
1 &= T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[1010w]_2], \qquad (2.17)\\
0 &= T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1011w]_2], \qquad (2.18)\\
0 &= T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1100w]_2], \qquad (2.19)
\end{aligned}
$$

$$1 = T[[w]_2] + T[[10w]_2] + T[[101w]_2] + T[[110w]_2] + T[[1101w]_2], \tag{2.20}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 442 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.20) can be written as

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,110)) + \tau(A(s,111)), \tag{2.21}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,100)) + \tau(A(s,110)) + \tau(A(s,1000)), \tag{2.22}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,110)) + \tau(A(s,1001)), \tag{2.23}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,1010)), \tag{2.24}$$

$$0 = \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,1011)), \tag{2.25}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,110)) + \tau(A(s,1100)), \tag{2.26}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,101)) + \tau(A(s,110)) + \tau(A(s,1101)), \tag{2.27}$$

for all $s \in E_{2k}$, and

$$1 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,110)) + \tau(A(s,111)), \tag{2.28}$$

$$1 = \tau(A(s,\epsilon)) + \tau(A(s,100)) + \tau(A(s,110)) + \tau(A(s,1000)), \tag{2.29}$$

$$1 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,110)) + \tau(A(s,1001)), \tag{2.30}$$

$$1 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,1010)), \tag{2.31}$$

$$0 = \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,1011)), \tag{2.32}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,110)) + \tau(A(s,1100)), \tag{2.33}$$

$$1 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,101)) + \tau(A(s,110)) + \tau(A(s,1101)), \tag{2.34}$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{16}), E_{16}) = (A(i, 0^{14}), E_{14})$, so that we only have to verify that identities (2.21) through (2.34) hold for $2 \leq k \leq 8$, which turns out to be true. $\square$

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \geq 2$, and $u = 2^{2k-1}$,*

$$M_{2k} = M^e[:7u] + x^u M^e[:6u] + x^{4u} M^e[:3u] + x^{5u} M^e[:2u] + x^{6u} M^e[:u]$$

$$
\begin{aligned}
&\quad + x^{7u} R^e,\\
W_{2k} &= W^e[:7u] + x^u W^e[:6u] + x^{4u} W^e[:3u] + x^{5u} W^e[:2u] + x^{6u} W^e[:u]\\
&\quad + x^{7u} R^e.
\end{aligned}
$$

For all integer $k \geq 2$, and $u = 2^{2k}$,

$$
\begin{aligned}
M_{2k+1} &= M^o[:7u] + x^u M^o[:6u] + x^{4u} M^o[:3u] + x^{5u} M^o[:2u] + x^{6u} M^o[:u]\\
&\quad + x^{7u} R^o,\\
W_{2k+1} &= W^o[:7u] + x^u W^o[:6u] + x^{4u} W^o[:3u] + x^{5u} W^o[:2u] + x^{6u} W^o[:u]\\
&\quad + x^{7u} R^o.
\end{aligned}
$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \geq 2$, we claim that

$$
\begin{aligned}
M_{2k} \wedge W_{2k} &\Rightarrow M_{2k+1} \wedge W_{2k+1},\\
M_{2k+1} \wedge W_{2k+1} &\Rightarrow M_{2k+2} \wedge W_{2k+2}.
\end{aligned}
$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$
\begin{aligned}
W_{2k} M_{2k} &= \big(W^e[:7v] + x^v W^e[:6v] + x^{4v} W^e[:3v] + x^{5v} W^e[:2v] + x^{6v} W^e[:v]\\
&\quad + x^{7u} R^e\big) \times \big(M^e[:7v] + x^v M^e[:6v] + x^{4v} M^e[:3v] + x^{5v} M^e[:2v]\\
&\quad + x^{6v} M^e[:v] + x^{7u} R^e\big); \qquad (2.35)
\end{aligned}
$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{14v} R^o$. Therefore we only need to prove that their difference is $O(x^{14v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{14v}, to

$$
\begin{aligned}
&W^e[:14v] M^e[:14v] + x^{2v} W^e[:12v] M^e[:12v] + x^{8v} W^e[:6v] M^e[:6v]\\
&\qquad\qquad + x^{10v} W^e[:4v] M^e[:4v] + x^{12v} W^e[:2v] M^e[:2v].
\end{aligned}
$$

For all $n \leq 14$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$
\begin{aligned}
X &:= x^v,\\
a_n &:= W^e[n \cdot v : (n+1) \cdot v] / X^n,\\
b_n &:= M^e[n \cdot v : (n+1) \cdot v] / X^n,\\
c &:= R^e.
\end{aligned}
$$

Using the notation introduced above, we can represent the expressions lhs (2.35) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{14}, b_1, ..., b_{14}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{14})$, which completes the proof. $\square$

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a,b)$; for $\mathrm{CF}(b,a)$, the proof is similar. By Lemma 2.3, we have For all $0 \le j,k \le 1$,

$$\lim_{n\to\infty} M_{2n,j,k} = M^e_{j,k},$$
$$\lim_{n\to\infty} M_{2n+1,j,k} = M^o_{j,k},$$
$$\lim_{n\to\infty} W_{2n,j,k} = W^e_{j,k},$$
$$\lim_{n\to\infty} W_{2n+1,j,k} = W^o_{j,k}.$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a,b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e,0,1)$ and $\phi(M^e,0,0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x,y) = \mathrm{Res}_t\left(\phi(M^e,0,1)(x,t), y^{12}\phi(M^e,0,1)(x,t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x,y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned}
q_0(x) &= x^{24}+x^{23}+x^{22}+x^{19}+x^{18}+x^{13}+x^{10}+x^9+x^7,\\
q_1(x) &= x^{23}+x^{22}+x^{21}+x^{19}+x^{17}+x^{14}+x^{13}+x^{12}+x^{11}+x^9+x^6+x^5\\
&\quad +x^4+x^2,\\
q_2(x) &= x^{24}+x^{22}+x^{20}+x^{16}+x^{10}+x^8+x^4+1,\\
q_3(x) &= x^{23}+x^{22}+x^{21}+x^{19}+x^{17}+x^{14}+x^{13}+x^{12}+x^{11}+x^9+x^6+x^5\\
&\quad +x^4+x^2,\\
q_4(x) &= x^{24}+x^{23}+x^{19}+x^{18}+x^{13}+x^9+x^7+x^4.
\end{aligned}$$

The polynomial $Q(x,y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and $R(x,y) := P(x,y)/Q(x,y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 216, and substitute it for y in $R(z,y)$, we get a series with valuation smaller than 216, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b) = f(1/z)$. □

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e,0,0)$:

$$p_0(x) = x^{138}+x^{128}+x^{118}+x^{116}+x^{114}+x^{112}+x^{110}+x^{108}+x^{104}+x^{102}$$

$$
\begin{aligned}
&+ x^{100} + x^{96} + x^{92} + x^{86} + x^{76} + x^{66} + x^{62} + x^{58} + x^{52} + x^{50} + x^{48}\\
&+ x^{44} + x^{30} + x^{26} + x^{24},\\
p_3(x) &= x^{136} + x^{134} + x^{130} + x^{128} + x^{126} + x^{118} + x^{112} + x^{104} + x^{102} + x^{94}\\
&+ x^{86} + x^{82} + x^{74} + x^{72} + x^{66} + x^{64} + x^{48} + x^{46} + x^{42} + x^{26} + x^{22}\\
&+ x^{16},\\
p_6(x) &= x^{136} + x^{132} + x^{128} + x^{126} + x^{120} + x^{118} + x^{116} + x^{110} + x^{108} + x^{106}\\
&+ x^{102} + x^{100} + x^{94} + x^{92} + x^{84} + x^{78} + x^{76} + x^{70} + x^{68} + x^{64} + x^{62}\\
&+ x^{58} + x^{56} + x^{54} + x^{52} + x^{50} + x^{48} + x^{46} + x^{42} + x^{40} + x^{36} + x^{32}\\
&+ x^{30} + x^{24} + x^{20} + x^{18} + x^{16} + x^{8} + x^{6} + x^{4} + x^{2} + 1,\\
p_9(x) &= x^{136} + x^{134} + x^{132} + x^{130} + x^{128} + x^{126} + x^{124} + x^{122} + x^{120} + x^{116}\\
&+ x^{114} + x^{106} + x^{104} + x^{102} + x^{100} + x^{92} + x^{90} + x^{86} + x^{80} + x^{78}\\
&+ x^{72} + x^{66} + x^{62} + x^{56} + x^{54} + x^{52} + x^{46} + x^{42} + x^{40} + x^{38} + x^{34}\\
&+ x^{30} + x^{26} + x^{22} + x^{12} + x^{10} + x^{6} + x^{4},\\
p_{12}(x) &= x^{138} + x^{136} + x^{130} + x^{128} + x^{106} + x^{104} + x^{98} + x^{96} + x^{58} + x^{56}\\
&+ x^{50} + x^{48} + x^{26} + x^{24} + x^{18} + x^{16} + x^{10} + x^{8} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0].

For $\phi(M^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{129} + x^{128} + x^{127} + x^{125} + x^{124} + x^{123} + x^{122} + x^{119} + x^{117} + x^{115}\\
&+ x^{114} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} + x^{106} + x^{105} + x^{104}\\
&+ x^{102} + x^{101} + x^{99} + x^{97} + x^{92} + x^{91} + x^{89} + x^{88} + x^{87} + x^{85} + x^{84}\\
&+ x^{82} + x^{80} + x^{79} + x^{75} + x^{74} + x^{72} + x^{69} + x^{68} + x^{66} + x^{65} + x^{64}\\
&+ x^{63} + x^{61} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{49} + x^{47} + x^{45} + x^{44}\\
&+ x^{42} + x^{41} + x^{39} + x^{34} + x^{33},\\
p_3(x) &= x^{120} + x^{118} + x^{117} + x^{116} + x^{114} + x^{111} + x^{110} + x^{109} + x^{107} + x^{104}\\
&+ x^{103} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{91} + x^{90} + x^{72}\\
&+ x^{70} + x^{69} + x^{68} + x^{66} + x^{63} + x^{62} + x^{61} + x^{59} + x^{56} + x^{55} + x^{51}\\
&+ x^{50} + x^{49} + x^{47} + x^{44} + x^{43} + x^{39} + x^{38} + x^{37} + x^{35} + x^{32} + x^{31}\\
&+ x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{19} + x^{18},\\
p_6(x) &= x^{123} + x^{122} + x^{119} + x^{118} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108}\\
&+ x^{107} + x^{106} + x^{105} + x^{104} + x^{91} + x^{90} + x^{81} + x^{80} + x^{73} + x^{72}\\
&+ x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{53} + x^{52} + x^{51} + x^{50} + x^{45}\\
&+ x^{44} + x^{41} + x^{40} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30}\\
&+ x^{29} + x^{28} + x^{27} + x^{26} + x^{23} + x^{22} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15}\\
&+ x^{14} + x^{13} + x^{12},\\
p_9(x) &= x^{126} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{117} + x^{116} + x^{108} + x^{106}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{105} + x^{104} + x^{100} + x^{98} + x^{97} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88}\\
&+ x^{87} + x^{86} + x^{85} + x^{83} + x^{82} + x^{80} + x^{78} + x^{77} + x^{76} + x^{74} + x^{72}\\
&+ x^{71} + x^{70} + x^{68} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{57}\\
&+ x^{56} + x^{54} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43}\\
&+ x^{41} + x^{38} + x^{37} + x^{33} + x^{32} + x^{31} + x^{29} + x^{25} + x^{24} + x^{23} + x^{21}\\
&+ x^{18} + x^{17} + x^{14} + x^{13} + x^{9} + x^{7} + x^{6},\\
p_{12}(x) = {}& x^{129} + x^{128} + x^{127} + x^{126} + x^{123} + x^{122} + x^{121} + x^{120} + x^{113} + x^{112}\\
&+ x^{111} + x^{110} + x^{107} + x^{106} + x^{103} + x^{102} + x^{99} + x^{98} + x^{95} + x^{94}\\
&+ x^{91} + x^{90} + x^{87} + x^{86} + x^{83} + x^{82} + x^{81} + x^{80} + x^{73} + x^{72} + x^{71}\\
&+ x^{70} + x^{67} + x^{66} + x^{63} + x^{62} + x^{59} + x^{58} + x^{57} + x^{56} + x^{41} + x^{40}\\
&+ x^{39} + x^{38} + x^{35} + x^{34} + x^{31} + x^{30} + x^{27} + x^{26} + x^{25} + x^{24} + x^{9}\\
&+ x^{8} + x^{7} + x^{6} + x^{3} + x^{2} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 1, 0, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1].

For $\phi(M^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) = {}& x^{129} + x^{128} + x^{127} + x^{125} + x^{124} + x^{123} + x^{122} + x^{119} + x^{117} + x^{115}\\
&+ x^{114} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} + x^{106} + x^{105} + x^{104}\\
&+ x^{102} + x^{101} + x^{99} + x^{97} + x^{92} + x^{91} + x^{89} + x^{88} + x^{87} + x^{85} + x^{84}\\
&+ x^{82} + x^{80} + x^{79} + x^{75} + x^{74} + x^{72} + x^{69} + x^{68} + x^{66} + x^{65} + x^{64}\\
&+ x^{63} + x^{61} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{49} + x^{47} + x^{45} + x^{44}\\
&+ x^{42} + x^{41} + x^{39} + x^{34} + x^{33},\\
p_3(x) = {}& x^{120} + x^{118} + x^{117} + x^{116} + x^{114} + x^{111} + x^{110} + x^{109} + x^{107} + x^{104}\\
&+ x^{103} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{91} + x^{90} + x^{72}\\
&+ x^{70} + x^{69} + x^{68} + x^{66} + x^{63} + x^{62} + x^{61} + x^{59} + x^{56} + x^{55} + x^{51}\\
&+ x^{50} + x^{49} + x^{47} + x^{44} + x^{43} + x^{39} + x^{38} + x^{37} + x^{35} + x^{32} + x^{31}\\
&+ x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{19} + x^{18},\\
p_6(x) = {}& x^{123} + x^{122} + x^{119} + x^{118} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108}\\
&+ x^{107} + x^{106} + x^{105} + x^{104} + x^{91} + x^{90} + x^{81} + x^{80} + x^{73} + x^{72}\\
&+ x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{53} + x^{52} + x^{51} + x^{50} + x^{45}\\
&+ x^{44} + x^{41} + x^{40} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30}\\
&+ x^{29} + x^{28} + x^{27} + x^{26} + x^{23} + x^{22} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15}\\
&+ x^{14} + x^{13} + x^{12},\\
p_9(x) = {}& x^{126} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{117} + x^{116} + x^{108} + x^{106}\\
&+ x^{105} + x^{104} + x^{100} + x^{98} + x^{97} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88}\\
&+ x^{87} + x^{86} + x^{85} + x^{83} + x^{82} + x^{80} + x^{78} + x^{77} + x^{76} + x^{74} + x^{72}\\
&+ x^{71} + x^{70} + x^{68} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{57}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{56} + x^{54} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43}\\
&+ x^{41} + x^{38} + x^{37} + x^{33} + x^{32} + x^{31} + x^{29} + x^{25} + x^{24} + x^{23} + x^{21}\\
&+ x^{18} + x^{17} + x^{14} + x^{13} + x^{9} + x^{7} + x^{6},\\
p_{12}(x) &= x^{129} + x^{128} + x^{127} + x^{126} + x^{123} + x^{122} + x^{121} + x^{120} + x^{113} + x^{112}\\
&+ x^{111} + x^{110} + x^{107} + x^{106} + x^{103} + x^{102} + x^{99} + x^{98} + x^{95} + x^{94}\\
&+ x^{91} + x^{90} + x^{87} + x^{86} + x^{83} + x^{82} + x^{81} + x^{80} + x^{73} + x^{72} + x^{71}\\
&+ x^{70} + x^{67} + x^{66} + x^{63} + x^{62} + x^{59} + x^{58} + x^{57} + x^{56} + x^{41} + x^{40}\\
&+ x^{39} + x^{38} + x^{35} + x^{34} + x^{31} + x^{30} + x^{27} + x^{26} + x^{25} + x^{24} + x^{9}\\
&+ x^{8} + x^{7} + x^{6} + x^{3} + x^{2} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 1, 0, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1].

For $\phi(M^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{120} + x^{118} + x^{116} + x^{112} + x^{102} + x^{94} + x^{90} + x^{88} + x^{84} + x^{80} + x^{76}\\
&+ x^{74} + x^{68} + x^{62} + x^{54} + x^{52} + x^{50} + x^{48} + x^{42},\\
p_3(x) &= x^{118} + x^{110} + x^{104} + x^{96} + x^{92} + x^{88} + x^{86} + x^{76} + x^{72} + x^{70} + x^{64}\\
&+ x^{60} + x^{54} + x^{48} + x^{44} + x^{36} + x^{32} + x^{30} + x^{16} + x^{12},\\
p_6(x) &= x^{118} + x^{112} + x^{104} + x^{102} + x^{100} + x^{92} + x^{90} + x^{88} + x^{86} + x^{78} + x^{76}\\
&+ x^{70} + x^{68} + x^{62} + x^{58} + x^{56} + x^{54} + x^{50} + x^{48} + x^{46} + x^{44} + x^{32}\\
&+ x^{30} + x^{28} + x^{26} + x^{24} + x^{22} + x^{20} + x^{18} + x^{16} + x^{12} + x^{10} + x^{8} + 1,\\
p_9(x) &= x^{118} + x^{106} + x^{102} + x^{100} + x^{98} + x^{94} + x^{90} + x^{86} + x^{84} + x^{80} + x^{78}\\
&+ x^{76} + x^{74} + x^{70} + x^{64} + x^{60} + x^{56} + x^{50} + x^{48} + x^{46} + x^{44} + x^{42}\\
&+ x^{36} + x^{28} + x^{26} + x^{16} + x^{14} + x^{10} + x^{8} + x^{4},\\
p_{12}(x) &= x^{120} + x^{116} + x^{112} + x^{72} + x^{68} + x^{64} + x^{56} + x^{52} + x^{48} + x^{40} + x^{36}\\
&+ x^{32} + x^{24} + x^{20} + x^{16} + x^{8} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{120} + x^{118} + x^{112} + x^{110} + x^{106} + x^{104} + x^{100} + x^{94} + x^{88} + x^{84}\\
&+ x^{76} + x^{70} + x^{68} + x^{66} + x^{64} + x^{60} + x^{58} + x^{54} + x^{48} + x^{46} + x^{44}\\
&+ x^{42} + x^{30} + x^{28} + x^{24},\\
p_3(x) &= x^{118} + x^{114} + x^{112} + x^{110} + x^{108} + x^{106} + x^{102} + x^{98} + x^{96} + x^{88}\\
&+ x^{86} + x^{82} + x^{80} + x^{72} + x^{68} + x^{62} + x^{58} + x^{54} + x^{52} + x^{50} + x^{38}\\
&+ x^{34} + x^{32} + x^{24} + x^{22} + x^{20} + x^{18} + x^{16},\\
p_6(x) &= x^{118} + x^{116} + x^{114} + x^{106} + x^{98} + x^{96} + x^{92} + x^{90} + x^{86} + x^{84} + x^{82}\\
&+ x^{80} + x^{78} + x^{60} + x^{56} + x^{54} + x^{52} + x^{48} + x^{46} + x^{38} + x^{34} + x^{26}\\
&+ x^{16} + x^{12} + x^{10} + 1,\\
p_9(x) &= x^{118} + x^{106} + x^{102} + x^{100} + x^{98} + x^{94} + x^{90} + x^{86} + x^{84} + x^{80} + x^{78}
\end{aligned}
$$

$$+ x^{76} + x^{74} + x^{70} + x^{64} + x^{60} + x^{56} + x^{50} + x^{48} + x^{46} + x^{44} + x^{42} + x^{36} + x^{28} + x^{26} + x^{16} + x^{14} + x^{10} + x^{8} + x^{4},$$

$$p_{12}(x) = x^{120} + x^{116} + x^{112} + x^{72} + x^{68} + x^{64} + x^{56} + x^{52} + x^{48} + x^{40} + x^{36} + x^{32} + x^{24} + x^{20} + x^{16} + x^{8} + x^{4} + 1,$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 1)$:

$$p_0(x) = x^{129} + x^{128} + x^{127} + x^{125} + x^{124} + x^{119} + x^{117} + x^{114} + x^{113} + x^{111} + x^{109} + x^{106} + x^{105} + x^{104} + x^{102} + x^{98} + x^{95} + x^{94} + x^{93} + x^{91} + x^{90} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{79} + x^{76} + x^{75} + x^{74} + x^{71} + x^{67} + x^{62} + x^{60} + x^{59} + x^{56} + x^{55} + x^{53} + x^{50} + x^{49} + x^{47} + x^{45} + x^{44} + x^{41} + x^{39} + x^{38} + x^{36} + x^{33},$$

$$p_3(x) = x^{120} + x^{118} + x^{117} + x^{116} + x^{114} + x^{111} + x^{110} + x^{109} + x^{107} + x^{104} + x^{103} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{91} + x^{90} + x^{72} + x^{70} + x^{69} + x^{68} + x^{66} + x^{63} + x^{62} + x^{61} + x^{59} + x^{56} + x^{55} + x^{51} + x^{50} + x^{49} + x^{47} + x^{44} + x^{43} + x^{39} + x^{38} + x^{37} + x^{35} + x^{32} + x^{31} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{19} + x^{18},$$

$$p_6(x) = x^{123} + x^{122} + x^{119} + x^{118} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} + x^{107} + x^{106} + x^{105} + x^{104} + x^{91} + x^{90} + x^{81} + x^{80} + x^{73} + x^{72} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{53} + x^{52} + x^{51} + x^{50} + x^{45} + x^{44} + x^{41} + x^{40} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{23} + x^{22} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12},$$

$$p_9(x) = x^{126} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{117} + x^{116} + x^{108} + x^{106} + x^{105} + x^{104} + x^{100} + x^{98} + x^{97} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{83} + x^{82} + x^{80} + x^{78} + x^{77} + x^{76} + x^{74} + x^{72} + x^{71} + x^{70} + x^{68} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{57} + x^{56} + x^{54} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{41} + x^{38} + x^{37} + x^{33} + x^{32} + x^{31} + x^{29} + x^{25} + x^{24} + x^{23} + x^{21} + x^{18} + x^{17} + x^{14} + x^{13} + x^{9} + x^{7} + x^{6},$$

$$p_{12}(x) = x^{129} + x^{128} + x^{127} + x^{126} + x^{123} + x^{122} + x^{121} + x^{120} + x^{113} + x^{112} + x^{111} + x^{110} + x^{107} + x^{106} + x^{103} + x^{102} + x^{99} + x^{98} + x^{95} + x^{94} + x^{91} + x^{90} + x^{87} + x^{86} + x^{83} + x^{82} + x^{81} + x^{80} + x^{73} + x^{72} + x^{71} + x^{70} + x^{67} + x^{66} + x^{63} + x^{62} + x^{59} + x^{58} + x^{57} + x^{56} + x^{41} + x^{40} + x^{39} + x^{38} + x^{35} + x^{34} + x^{31} + x^{30} + x^{27} + x^{26} + x^{25} + x^{24} + x^{9} + x^{8} + x^{7} + x^{6} + x^{3} + x^{2} + x + 1,$$

and the initial terms are [0, 0, 0, 0, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1].

For $\phi(W^e, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{129} + x^{128} + x^{127} + x^{125} + x^{124} + x^{119} + x^{117} + x^{114} + x^{113} + x^{111}\\
&\quad + x^{109} + x^{106} + x^{105} + x^{104} + x^{102} + x^{98} + x^{95} + x^{94} + x^{93} + x^{91}\\
&\quad + x^{90} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{79} + x^{76}\\
&\quad + x^{75} + x^{74} + x^{71} + x^{67} + x^{62} + x^{60} + x^{59} + x^{56} + x^{55} + x^{53} + x^{50}\\
&\quad + x^{49} + x^{47} + x^{45} + x^{44} + x^{41} + x^{39} + x^{38} + x^{36} + x^{33},\\
p_3(x) &= x^{120} + x^{118} + x^{117} + x^{116} + x^{114} + x^{111} + x^{110} + x^{109} + x^{107} + x^{104}\\
&\quad + x^{103} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{91} + x^{90} + x^{72}\\
&\quad + x^{70} + x^{69} + x^{68} + x^{66} + x^{63} + x^{62} + x^{61} + x^{59} + x^{56} + x^{55} + x^{51}\\
&\quad + x^{50} + x^{49} + x^{47} + x^{44} + x^{43} + x^{39} + x^{38} + x^{37} + x^{35} + x^{32} + x^{31}\\
&\quad + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{19} + x^{18},\\
p_6(x) &= x^{123} + x^{122} + x^{119} + x^{118} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108}\\
&\quad + x^{107} + x^{106} + x^{105} + x^{104} + x^{91} + x^{90} + x^{81} + x^{80} + x^{73} + x^{72}\\
&\quad + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{53} + x^{52} + x^{51} + x^{50} + x^{45}\\
&\quad + x^{44} + x^{41} + x^{40} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30}\\
&\quad + x^{29} + x^{28} + x^{27} + x^{26} + x^{23} + x^{22} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15}\\
&\quad + x^{14} + x^{13} + x^{12},\\
p_9(x) &= x^{126} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{117} + x^{116} + x^{108} + x^{106}\\
&\quad + x^{105} + x^{104} + x^{100} + x^{98} + x^{97} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88}\\
&\quad + x^{87} + x^{86} + x^{85} + x^{83} + x^{82} + x^{80} + x^{78} + x^{77} + x^{76} + x^{74} + x^{72}\\
&\quad + x^{71} + x^{70} + x^{68} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{57}\\
&\quad + x^{56} + x^{54} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43}\\
&\quad + x^{41} + x^{38} + x^{37} + x^{33} + x^{32} + x^{31} + x^{29} + x^{25} + x^{24} + x^{23} + x^{21}\\
&\quad + x^{18} + x^{17} + x^{14} + x^{13} + x^{9} + x^{7} + x^{6},\\
p_{12}(x) &= x^{129} + x^{128} + x^{127} + x^{126} + x^{123} + x^{122} + x^{121} + x^{120} + x^{113} + x^{112}\\
&\quad + x^{111} + x^{110} + x^{107} + x^{106} + x^{103} + x^{102} + x^{99} + x^{98} + x^{95} + x^{94}\\
&\quad + x^{91} + x^{90} + x^{87} + x^{86} + x^{83} + x^{82} + x^{81} + x^{80} + x^{73} + x^{72} + x^{71}\\
&\quad + x^{70} + x^{67} + x^{66} + x^{63} + x^{62} + x^{59} + x^{58} + x^{57} + x^{56} + x^{41} + x^{40}\\
&\quad + x^{39} + x^{38} + x^{35} + x^{34} + x^{31} + x^{30} + x^{27} + x^{26} + x^{25} + x^{24} + x^{9}\\
&\quad + x^{8} + x^{7} + x^{6} + x^{3} + x^{2} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1].

For $\phi(W^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{138} + x^{134} + x^{132} + x^{130} + x^{128} + x^{124} + x^{120} + x^{116} + x^{110} + x^{106}\\
&\quad + x^{98} + x^{86} + x^{84} + x^{82} + x^{80} + x^{78} + x^{76} + x^{74} + x^{72} + x^{70} + x^{68}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{62} + x^{58} + x^{56} + x^{50} + x^{46} + x^{42}, \\
p_3(x) &= x^{136} + x^{134} + x^{132} + x^{130} + x^{128} + x^{122} + x^{120} + x^{118} + x^{108} + x^{106} \\
&+ x^{104} + x^{102} + x^{98} + x^{92} + x^{86} + x^{84} + x^{82} + x^{80} + x^{78} + x^{74} + x^{72} \\
&+ x^{70} + x^{60} + x^{56} + x^{54} + x^{52} + x^{46} + x^{40} + x^{38} + x^{36} + x^{34} + x^{30} \\
&+ x^{28} + x^{26} + x^{22} + x^{20} + x^{14} + x^{12}, \\
p_6(x) &= x^{136} + x^{134} + x^{132} + x^{130} + x^{120} + x^{114} + x^{110} + x^{106} + x^{104} + x^{102} \\
&+ x^{100} + x^{98} + x^{94} + x^{92} + x^{90} + x^{88} + x^{86} + x^{84} + x^{80} + x^{78} + x^{68} \\
&+ x^{64} + x^{62} + x^{60} + x^{58} + x^{50} + x^{44} + x^{42} + x^{40} + x^{38} + x^{30} + x^{26} \\
&+ x^{22} + x^{20} + x^{18} + x^{14} + x^{12} + x^{10} + x^{6} + x^{4} + x^{2} + 1, \\
p_9(x) &= x^{136} + x^{134} + x^{132} + x^{130} + x^{128} + x^{126} + x^{124} + x^{122} + x^{120} + x^{116} \\
&+ x^{114} + x^{106} + x^{104} + x^{102} + x^{100} + x^{92} + x^{90} + x^{86} + x^{80} + x^{78} \\
&+ x^{72} + x^{66} + x^{62} + x^{56} + x^{54} + x^{52} + x^{46} + x^{42} + x^{40} + x^{38} + x^{34} \\
&+ x^{30} + x^{26} + x^{22} + x^{12} + x^{10} + x^{6} + x^{4}, \\
p_{12}(x) &= x^{138} + x^{136} + x^{130} + x^{128} + x^{106} + x^{104} + x^{98} + x^{96} + x^{58} + x^{56} \\
&+ x^{50} + x^{48} + x^{26} + x^{24} + x^{18} + x^{16} + x^{10} + x^{8} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0].

For $\phi(M^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{129} + x^{128} + x^{125} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{117} + x^{115} \\
&+ x^{114} + x^{109} + x^{107} + x^{106} + x^{105} + x^{103} + x^{101} + x^{99} + x^{98} + x^{97} \\
&+ x^{96} + x^{90} + x^{88} + x^{86} + x^{85} + x^{81} + x^{79} + x^{78} + x^{75} + x^{74} + x^{72} \\
&+ x^{71} + x^{70} + x^{69} + x^{66} + x^{65} + x^{63} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} \\
&+ x^{55} + x^{52} + x^{50} + x^{47} + x^{45} + x^{42} + x^{41} + x^{40} + x^{38} + x^{36} + x^{34} \\
&+ x^{33} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24}, \\
p_3(x) &= x^{125} + x^{124} + x^{121} + x^{119} + x^{118} + x^{115} + x^{112} + x^{111} + x^{107} + x^{103} \\
&+ x^{100} + x^{98} + x^{97} + x^{94} + x^{93} + x^{92} + x^{89} + x^{88} + x^{85} + x^{84} + x^{81} \\
&+ x^{80} + x^{79} + x^{78} + x^{75} + x^{74} + x^{73} + x^{69} + x^{68} + x^{67} + x^{64} + x^{62} \\
&+ x^{59} + x^{57} + x^{56} + x^{55} + x^{52} + x^{51} + x^{49} + x^{48} + x^{46} + x^{45} + x^{44} \\
&+ x^{42} + x^{40} + x^{39} + x^{37} + x^{36} + x^{34} + x^{33} + x^{32} + x^{30} + x^{28} + x^{27} \\
&+ x^{24} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18} + x^{17} + x^{16}, \\
p_6(x) &= x^{127} + x^{126} + x^{125} + x^{119} + x^{117} + x^{116} + x^{114} + x^{111} + x^{110} + x^{109} \\
&+ x^{108} + x^{105} + x^{103} + x^{100} + x^{98} + x^{97} + x^{95} + x^{93} + x^{92} + x^{91} \\
&+ x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{80} + x^{78} + x^{77} + x^{73} + x^{72} + x^{71} \\
&+ x^{70} + x^{69} + x^{67} + x^{64} + x^{61} + x^{59} + x^{57} + x^{55} + x^{54} + x^{53} + x^{49} \\
&+ x^{47} + x^{46} + x^{42} + x^{40} + x^{36} + x^{35} + x^{33} + x^{31} + x^{28} + x^{23} + x^{21} \\
&+ x^{19} + x^{16} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^{7} + x^{6} + x^{3} + x^{2} + x + 1,
\end{aligned}
$$

$$\begin{aligned}
p_9(x) = {} & x^{125} + x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{117} + x^{116} + x^{115} + x^{114} \\
& + x^{111} + x^{110} + x^{109} + x^{108} + x^{101} + x^{100} + x^{99} + x^{98} + x^{95} + x^{94} \\
& + x^{93} + x^{92} + x^{85} + x^{84} + x^{83} + x^{82} + x^{79} + x^{78} + x^{75} + x^{74} + x^{73} \\
& + x^{72} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{53} + x^{52} + x^{51} + x^{50} \\
& + x^{47} + x^{46} + x^{43} + x^{42} + x^{41} + x^{40} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} \\
& + x^{24} + x^{21} + x^{20} + x^{19} + x^{18} + x^{15} + x^{14} + x^{11} + x^{10} + x^{9} + x^{8}, \\
p_{12}(x) = {} & x^{129} + x^{128} + x^{127} + x^{126} + x^{125} + x^{124} + x^{117} + x^{116} + x^{115} + x^{114} \\
& + x^{113} + x^{112} + x^{109} + x^{108} + x^{107} + x^{106} + x^{103} + x^{102} + x^{101} \\
& + x^{100} + x^{93} + x^{92} + x^{91} + x^{90} + x^{87} + x^{86} + x^{85} + x^{84} + x^{81} + x^{80} \\
& + x^{79} + x^{78} + x^{75} + x^{74} + x^{71} + x^{70} + x^{67} + x^{66} + x^{63} + x^{62} + x^{61} \\
& + x^{60} + x^{53} + x^{52} + x^{51} + x^{50} + x^{47} + x^{46} + x^{43} + x^{42} + x^{39} + x^{38} \\
& + x^{35} + x^{34} + x^{31} + x^{30} + x^{29} + x^{28} + x^{21} + x^{20} + x^{19} + x^{18} + x^{15} \\
& + x^{14} + x^{11} + x^{10} + x^{7} + x^{6} + x^{3} + x^{2} + x + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0].
For $\phi(M^o, 0, 1)$:

$$\begin{aligned}
p_0(x) = {} & x^{128} + x^{125} + x^{124} + x^{123} + x^{121} + x^{119} + x^{118} + x^{117} + x^{116} + x^{114} \\
& + x^{112} + x^{110} + x^{107} + x^{105} + x^{103} + x^{101} + x^{100} + x^{99} + x^{94} + x^{92} \\
& + x^{91} + x^{89} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{79} + x^{78} \\
& + x^{77} + x^{76} + x^{75} + x^{73} + x^{72} + x^{65} + x^{63} + x^{62} + x^{61} + x^{60} + x^{55} \\
& + x^{54} + x^{53} + x^{51} + x^{48} + x^{44} + x^{35} + x^{33}, \\
p_3(x) = {} & x^{119} + x^{118} + x^{116} + x^{113} + x^{112} + x^{111} + x^{108} + x^{107} + x^{106} + x^{105} \\
& + x^{102} + x^{101} + x^{100} + x^{99} + x^{96} + x^{93} + x^{92} + x^{90} + x^{71} + x^{70} + x^{68} \\
& + x^{65} + x^{64} + x^{63} + x^{60} + x^{59} + x^{58} + x^{57} + x^{54} + x^{53} + x^{52} + x^{51} \\
& + x^{48} + x^{47} + x^{46} + x^{45} + x^{42} + x^{41} + x^{40} + x^{39} + x^{36} + x^{35} + x^{34} \\
& + x^{33} + x^{30} + x^{29} + x^{28} + x^{27} + x^{24} + x^{21} + x^{20} + x^{18}, \\
p_6(x) = {} & x^{122} + x^{120} + x^{116} + x^{110} + x^{108} + x^{104} + x^{90} + x^{88} + x^{86} + x^{82} \\
& + x^{78} + x^{74} + x^{70} + x^{64} + x^{62} + x^{60} + x^{58} + x^{54} + x^{48} + x^{34} + x^{32} \\
& + x^{28} + x^{26} + x^{24} + x^{20} + x^{18} + x^{16} + x^{12}, \\
p_9(x) = {} & x^{125} + x^{124} + x^{123} + x^{120} + x^{115} + x^{114} + x^{113} + x^{110} + x^{107} + x^{106} \\
& + x^{105} + x^{102} + x^{99} + x^{98} + x^{97} + x^{94} + x^{91} + x^{90} + x^{89} + x^{86} + x^{83} \\
& + x^{82} + x^{81} + x^{78} + x^{77} + x^{76} + x^{74} + x^{73} + x^{72} + x^{70} + x^{61} + x^{60} \\
& + x^{59} + x^{56} + x^{51} + x^{50} + x^{49} + x^{46} + x^{45} + x^{44} + x^{42} + x^{41} + x^{40} \\
& + x^{38} + x^{29} + x^{28} + x^{27} + x^{24} + x^{19} + x^{18} + x^{17} + x^{14} + x^{13} + x^{12} \\
& + x^{10} + x^{9} + x^{8} + x^{6}, \\
p_{12}(x) = {} & x^{128} + x^{124} + x^{116} + x^{112} + x^{108} + x^{100} + x^{92} + x^{84} + x^{80} + x^{60}
\end{aligned}$$

$$+ x^{52} + x^{28} + x^{20} + 1,$$

and the initial terms are [0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0].

For $\phi(M^o, 1, 0)$:

$$\begin{aligned}
p_0(x) = {} & x^{138} + x^{136} + x^{135} + x^{134} + x^{130} + x^{126} + x^{125} + x^{123} + x^{120} + x^{119} \\
& + x^{112} + x^{111} + x^{109} + x^{108} + x^{107} + x^{105} + x^{103} + x^{101} + x^{100} \\
& + x^{95} + x^{94} + x^{92} + x^{87} + x^{84} + x^{82} + x^{80} + x^{79} + x^{78} + x^{73} + x^{72} \\
& + x^{71} + x^{70} + x^{69} + x^{67} + x^{65} + x^{62} + x^{61} + x^{59} + x^{58} + x^{55} + x^{51} \\
& + x^{49} + x^{48} + x^{47} + x^{46} + x^{43} + x^{42} + x^{37} + x^{35} + x^{34} + x^{33}, \\
p_3(x) = {} & x^{129} + x^{128} + x^{127} + x^{124} + x^{123} + x^{122} + x^{120} + x^{119} + x^{118} + x^{117} \\
& + x^{114} + x^{105} + x^{104} + x^{103} + x^{100} + x^{99} + x^{98} + x^{96} + x^{95} + x^{94} \\
& + x^{93} + x^{90} + x^{81} + x^{80} + x^{79} + x^{76} + x^{75} + x^{74} + x^{72} + x^{71} + x^{70} \\
& + x^{69} + x^{66} + x^{33} + x^{32} + x^{31} + x^{28} + x^{27} + x^{26} + x^{24} + x^{23} + x^{22} \\
& + x^{21} + x^{18}, \\
p_6(x) = {} & x^{132} + x^{128} + x^{126} + x^{124} + x^{120} + x^{114} + x^{110} + x^{102} + x^{98} + x^{96} \\
& + x^{84} + x^{78} + x^{76} + x^{74} + x^{70} + x^{62} + x^{58} + x^{50} + x^{46} + x^{44} + x^{28} \\
& + x^{24} + x^{22} + x^{16} + x^{14} + x^{12}, \\
p_9(x) = {} & x^{135} + x^{134} + x^{132} + x^{131} + x^{130} + x^{128} + x^{125} + x^{124} + x^{122} + x^{121} \\
& + x^{120} + x^{119} + x^{117} + x^{115} + x^{113} + x^{111} + x^{108} + x^{107} + x^{106} \\
& + x^{104} + x^{103} + x^{102} + x^{101} + x^{99} + x^{97} + x^{95} + x^{92} + x^{91} + x^{90} \\
& + x^{88} + x^{85} + x^{84} + x^{82} + x^{81} + x^{80} + x^{79} + x^{77} + x^{75} + x^{73} + x^{71} \\
& + x^{68} + x^{67} + x^{66} + x^{64} + x^{61} + x^{60} + x^{58} + x^{57} + x^{56} + x^{54} + x^{53} \\
& + x^{52} + x^{50} + x^{49} + x^{48} + x^{47} + x^{45} + x^{43} + x^{41} + x^{39} + x^{36} + x^{35} \\
& + x^{34} + x^{32} + x^{29} + x^{28} + x^{26} + x^{25} + x^{24} + x^{22} + x^{21} + x^{20} + x^{18} \\
& + x^{17} + x^{16} + x^{15} + x^{13} + x^{11} + x^{9} + x^{6}, \\
p_{12}(x) = {} & x^{138} + x^{136} + x^{134} + x^{132} + x^{126} + x^{124} + x^{114} + x^{112} + x^{110} + x^{108} \\
& + x^{94} + x^{92} + x^{90} + x^{88} + x^{86} + x^{84} + x^{74} + x^{72} + x^{70} + x^{68} + x^{66} \\
& + x^{64} + x^{62} + x^{60} + x^{58} + x^{56} + x^{54} + x^{52} + x^{50} + x^{48} + x^{42} + x^{40} \\
& + x^{38} + x^{36} + x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{24} + x^{22} + x^{20} + x^{18} \\
& + x^{16} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 1, 1, 0, 0, 1, 1, 1, 1, 1, 1, 0].

For $\phi(M^o, 1, 1)$:

$$\begin{aligned}
p_0(x) = {} & x^{129} + x^{128} + x^{124} + x^{123} + x^{122} + x^{119} + x^{113} + x^{112} + x^{111} + x^{108} \\
& + x^{106} + x^{105} + x^{103} + x^{101} + x^{100} + x^{99} + x^{98} + x^{96} + x^{95} + x^{94} \\
& + x^{93} + x^{90} + x^{84} + x^{82} + x^{81} + x^{80} + x^{78} + x^{77} + x^{75} + x^{74} + x^{72} \\
& + x^{71} + x^{70} + x^{69} + x^{68} + x^{64} + x^{63} + x^{61} + x^{57} + x^{55} + x^{54} + x^{51}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{50} + x^{49} + x^{45} + x^{44} + x^{42},\\
p_3(x) &= x^{125} + x^{124} + x^{120} + x^{119} + x^{118} + x^{115} + x^{113} + x^{110} + x^{109} + x^{108}\\
&+ x^{107} + x^{105} + x^{104} + x^{102} + x^{100} + x^{99} + x^{97} + x^{94} + x^{91} + x^{90}\\
&+ x^{85} + x^{84} + x^{81} + x^{80} + x^{79} + x^{78} + x^{75} + x^{74} + x^{72} + x^{69} + x^{68}\\
&+ x^{67} + x^{65} + x^{63} + x^{61} + x^{60} + x^{59} + x^{54} + x^{52} + x^{50} + x^{46} + x^{43}\\
&+ x^{40} + x^{38} + x^{34} + x^{31} + x^{28} + x^{26} + x^{24} + x^{22},\\
p_6(x) &= x^{127} + x^{126} + x^{124} + x^{119} + x^{117} + x^{116} + x^{115} + x^{113} + x^{112} + x^{111}\\
&+ x^{110} + x^{104} + x^{102} + x^{101} + x^{98} + x^{97} + x^{95} + x^{91} + x^{89} + x^{88}\\
&+ x^{87} + x^{86} + x^{82} + x^{81} + x^{79} + x^{77} + x^{75} + x^{74} + x^{68} + x^{66} + x^{65}\\
&+ x^{63} + x^{62} + x^{61} + x^{59} + x^{56} + x^{53} + x^{51} + x^{50} + x^{48} + x^{42} + x^{41}\\
&+ x^{36} + x^{35} + x^{32} + x^{31} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{20}\\
&+ x^{18} + x^{16} + x^{14} + x^{9} + x^{8} + x^{7} + x^{6} + x^{3} + x^{2} + x + 1,\\
p_9(x) &= x^{125} + x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{117} + x^{116} + x^{115} + x^{114}\\
&+ x^{111} + x^{110} + x^{109} + x^{108} + x^{101} + x^{100} + x^{99} + x^{98} + x^{95} + x^{94}\\
&+ x^{93} + x^{92} + x^{85} + x^{84} + x^{83} + x^{82} + x^{79} + x^{78} + x^{75} + x^{74} + x^{73}\\
&+ x^{72} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{53} + x^{52} + x^{51} + x^{50}\\
&+ x^{47} + x^{46} + x^{43} + x^{42} + x^{41} + x^{40} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25}\\
&+ x^{24} + x^{21} + x^{20} + x^{19} + x^{18} + x^{15} + x^{14} + x^{11} + x^{10} + x^{9} + x^{8},\\
p_{12}(x) &= x^{129} + x^{128} + x^{127} + x^{126} + x^{125} + x^{124} + x^{117} + x^{116} + x^{115} + x^{114}\\
&+ x^{113} + x^{112} + x^{109} + x^{108} + x^{107} + x^{106} + x^{103} + x^{102} + x^{101}\\
&+ x^{100} + x^{93} + x^{92} + x^{91} + x^{90} + x^{87} + x^{86} + x^{85} + x^{84} + x^{81} + x^{80}\\
&+ x^{79} + x^{78} + x^{75} + x^{74} + x^{71} + x^{70} + x^{67} + x^{66} + x^{63} + x^{62} + x^{61}\\
&+ x^{60} + x^{53} + x^{52} + x^{51} + x^{50} + x^{47} + x^{46} + x^{43} + x^{42} + x^{39} + x^{38}\\
&+ x^{35} + x^{34} + x^{31} + x^{30} + x^{29} + x^{28} + x^{21} + x^{20} + x^{19} + x^{18} + x^{15}\\
&+ x^{14} + x^{11} + x^{10} + x^{7} + x^{6} + x^{3} + x^{2} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 1, 0, 0].

For $\phi(W^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{129} + x^{128} + x^{125} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{117} + x^{115}\\
&+ x^{114} + x^{109} + x^{107} + x^{106} + x^{105} + x^{103} + x^{101} + x^{99} + x^{98} + x^{97}\\
&+ x^{96} + x^{90} + x^{88} + x^{86} + x^{85} + x^{81} + x^{79} + x^{78} + x^{75} + x^{74} + x^{72}\\
&+ x^{71} + x^{70} + x^{69} + x^{66} + x^{65} + x^{63} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57}\\
&+ x^{55} + x^{52} + x^{50} + x^{47} + x^{45} + x^{42} + x^{41} + x^{40} + x^{38} + x^{36} + x^{34}\\
&+ x^{33} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24},\\
p_3(x) &= x^{125} + x^{124} + x^{121} + x^{119} + x^{118} + x^{115} + x^{112} + x^{111} + x^{107} + x^{103}\\
&+ x^{100} + x^{98} + x^{97} + x^{94} + x^{93} + x^{92} + x^{89} + x^{88} + x^{85} + x^{84} + x^{81}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{80} + x^{79} + x^{78} + x^{75} + x^{74} + x^{73} + x^{69} + x^{68} + x^{67} + x^{64} + x^{62} \\
&+ x^{59} + x^{57} + x^{56} + x^{55} + x^{52} + x^{51} + x^{49} + x^{48} + x^{46} + x^{45} + x^{44} \\
&+ x^{42} + x^{40} + x^{39} + x^{37} + x^{36} + x^{34} + x^{33} + x^{32} + x^{30} + x^{28} + x^{27} \\
&+ x^{24} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18} + x^{17} + x^{16}, \\
p_6(x) &= x^{127} + x^{126} + x^{125} + x^{119} + x^{117} + x^{116} + x^{114} + x^{111} + x^{110} + x^{109} \\
&+ x^{108} + x^{105} + x^{103} + x^{100} + x^{98} + x^{97} + x^{95} + x^{93} + x^{92} + x^{91} \\
&+ x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{80} + x^{78} + x^{77} + x^{73} + x^{72} + x^{71} \\
&+ x^{70} + x^{69} + x^{67} + x^{64} + x^{61} + x^{59} + x^{57} + x^{55} + x^{54} + x^{53} + x^{49} \\
&+ x^{47} + x^{46} + x^{42} + x^{40} + x^{36} + x^{35} + x^{33} + x^{31} + x^{28} + x^{23} + x^{21} \\
&+ x^{19} + x^{16} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^{7} + x^{6} + x^{3} + x^{2} + x + 1, \\
p_9(x) &= x^{125} + x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{117} + x^{116} + x^{115} + x^{114} \\
&+ x^{111} + x^{110} + x^{109} + x^{108} + x^{101} + x^{100} + x^{99} + x^{98} + x^{95} + x^{94} \\
&+ x^{93} + x^{92} + x^{85} + x^{84} + x^{83} + x^{82} + x^{79} + x^{78} + x^{75} + x^{74} + x^{73} \\
&+ x^{72} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{53} + x^{52} + x^{51} + x^{50} \\
&+ x^{47} + x^{46} + x^{43} + x^{42} + x^{41} + x^{40} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} \\
&+ x^{24} + x^{21} + x^{20} + x^{19} + x^{18} + x^{15} + x^{14} + x^{11} + x^{10} + x^{9} + x^{8}, \\
p_{12}(x) &= x^{129} + x^{128} + x^{127} + x^{126} + x^{125} + x^{124} + x^{117} + x^{116} + x^{115} + x^{114} \\
&+ x^{113} + x^{112} + x^{109} + x^{108} + x^{107} + x^{106} + x^{103} + x^{102} + x^{101} \\
&+ x^{100} + x^{93} + x^{92} + x^{91} + x^{90} + x^{87} + x^{86} + x^{85} + x^{84} + x^{81} + x^{80} \\
&+ x^{79} + x^{78} + x^{75} + x^{74} + x^{71} + x^{70} + x^{67} + x^{66} + x^{63} + x^{62} + x^{61} \\
&+ x^{60} + x^{53} + x^{52} + x^{51} + x^{50} + x^{47} + x^{46} + x^{43} + x^{42} + x^{39} + x^{38} \\
&+ x^{35} + x^{34} + x^{31} + x^{30} + x^{29} + x^{28} + x^{21} + x^{20} + x^{19} + x^{18} + x^{15} \\
&+ x^{14} + x^{11} + x^{10} + x^{7} + x^{6} + x^{3} + x^{2} + x + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0].

For $\phi(W^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{138} + x^{136} + x^{135} + x^{134} + x^{130} + x^{126} + x^{125} + x^{123} + x^{120} + x^{119} \\
&+ x^{112} + x^{111} + x^{109} + x^{108} + x^{107} + x^{105} + x^{103} + x^{101} + x^{100} \\
&+ x^{95} + x^{94} + x^{92} + x^{87} + x^{84} + x^{82} + x^{80} + x^{79} + x^{78} + x^{73} + x^{72} \\
&+ x^{71} + x^{70} + x^{69} + x^{67} + x^{65} + x^{62} + x^{61} + x^{59} + x^{58} + x^{55} + x^{51} \\
&+ x^{49} + x^{48} + x^{47} + x^{46} + x^{43} + x^{42} + x^{37} + x^{35} + x^{34} + x^{33}, \\
p_3(x) &= x^{129} + x^{128} + x^{127} + x^{124} + x^{123} + x^{122} + x^{120} + x^{119} + x^{118} + x^{117} \\
&+ x^{114} + x^{105} + x^{104} + x^{103} + x^{100} + x^{99} + x^{98} + x^{96} + x^{95} + x^{94} \\
&+ x^{93} + x^{90} + x^{81} + x^{80} + x^{79} + x^{76} + x^{75} + x^{74} + x^{72} + x^{71} + x^{70} \\
&+ x^{69} + x^{66} + x^{33} + x^{32} + x^{31} + x^{28} + x^{27} + x^{26} + x^{24} + x^{23} + x^{22} \\
&+ x^{21} + x^{18},
\end{aligned}
$$

$$
\begin{aligned}
p_6(x) &= x^{132}+x^{128}+x^{126}+x^{124}+x^{120}+x^{114}+x^{110}+x^{102}+x^{98}+x^{96}\\
&\quad +x^{84}+x^{78}+x^{76}+x^{74}+x^{70}+x^{62}+x^{58}+x^{50}+x^{46}+x^{44}+x^{28}\\
&\quad +x^{24}+x^{22}+x^{16}+x^{14}+x^{12},\\
p_9(x) &= x^{135}+x^{134}+x^{132}+x^{131}+x^{130}+x^{128}+x^{125}+x^{124}+x^{122}+x^{121}\\
&\quad +x^{120}+x^{119}+x^{117}+x^{115}+x^{113}+x^{111}+x^{108}+x^{107}+x^{106}\\
&\quad +x^{104}+x^{103}+x^{102}+x^{101}+x^{99}+x^{97}+x^{95}+x^{92}+x^{91}+x^{90}\\
&\quad +x^{88}+x^{85}+x^{84}+x^{82}+x^{81}+x^{80}+x^{79}+x^{77}+x^{75}+x^{73}+x^{71}\\
&\quad +x^{68}+x^{67}+x^{66}+x^{64}+x^{61}+x^{60}+x^{58}+x^{57}+x^{56}+x^{54}+x^{53}\\
&\quad +x^{52}+x^{50}+x^{49}+x^{48}+x^{47}+x^{45}+x^{43}+x^{41}+x^{39}+x^{36}+x^{35}\\
&\quad +x^{34}+x^{32}+x^{29}+x^{28}+x^{26}+x^{25}+x^{24}+x^{22}+x^{21}+x^{20}+x^{18}\\
&\quad +x^{17}+x^{16}+x^{15}+x^{13}+x^{11}+x^{9}+x^{6},\\
p_{12}(x) &= x^{138}+x^{136}+x^{134}+x^{132}+x^{126}+x^{124}+x^{114}+x^{112}+x^{110}+x^{108}\\
&\quad +x^{94}+x^{92}+x^{90}+x^{88}+x^{86}+x^{84}+x^{74}+x^{72}+x^{70}+x^{68}+x^{66}\\
&\quad +x^{64}+x^{62}+x^{60}+x^{58}+x^{56}+x^{54}+x^{52}+x^{50}+x^{48}+x^{42}+x^{40}\\
&\quad +x^{38}+x^{36}+x^{34}+x^{32}+x^{30}+x^{28}+x^{26}+x^{24}+x^{22}+x^{20}+x^{18}\\
&\quad +x^{16}+x^{2}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 1, 1, 0, 0, 1, 1, 1, 1, 1, 1, 0].

For $\phi(W^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{128}+x^{125}+x^{124}+x^{123}+x^{121}+x^{119}+x^{118}+x^{117}+x^{116}+x^{114}\\
&\quad +x^{112}+x^{110}+x^{107}+x^{105}+x^{103}+x^{101}+x^{100}+x^{99}+x^{94}+x^{92}\\
&\quad +x^{91}+x^{89}+x^{87}+x^{86}+x^{85}+x^{84}+x^{83}+x^{82}+x^{81}+x^{79}+x^{78}\\
&\quad +x^{77}+x^{76}+x^{75}+x^{73}+x^{72}+x^{65}+x^{63}+x^{62}+x^{61}+x^{60}+x^{55}\\
&\quad +x^{54}+x^{53}+x^{51}+x^{48}+x^{44}+x^{35}+x^{33},\\
p_3(x) &= x^{119}+x^{118}+x^{116}+x^{113}+x^{112}+x^{111}+x^{108}+x^{107}+x^{106}+x^{105}\\
&\quad +x^{102}+x^{101}+x^{100}+x^{99}+x^{96}+x^{93}+x^{92}+x^{90}+x^{71}+x^{70}+x^{68}\\
&\quad +x^{65}+x^{64}+x^{63}+x^{60}+x^{59}+x^{58}+x^{57}+x^{54}+x^{53}+x^{52}+x^{51}\\
&\quad +x^{48}+x^{47}+x^{46}+x^{45}+x^{42}+x^{41}+x^{40}+x^{39}+x^{36}+x^{35}+x^{34}\\
&\quad +x^{33}+x^{30}+x^{29}+x^{28}+x^{27}+x^{24}+x^{21}+x^{20}+x^{18},\\
p_6(x) &= x^{122}+x^{120}+x^{116}+x^{110}+x^{108}+x^{104}+x^{90}+x^{88}+x^{86}+x^{82}\\
&\quad +x^{78}+x^{74}+x^{70}+x^{64}+x^{62}+x^{60}+x^{58}+x^{54}+x^{48}+x^{34}+x^{32}\\
&\quad +x^{28}+x^{26}+x^{24}+x^{20}+x^{18}+x^{16}+x^{12},\\
p_9(x) &= x^{125}+x^{124}+x^{123}+x^{120}+x^{115}+x^{114}+x^{113}+x^{110}+x^{107}+x^{106}\\
&\quad +x^{105}+x^{102}+x^{99}+x^{98}+x^{97}+x^{94}+x^{91}+x^{90}+x^{89}+x^{86}+x^{83}\\
&\quad +x^{82}+x^{81}+x^{78}+x^{77}+x^{76}+x^{74}+x^{73}+x^{72}+x^{70}+x^{61}+x^{60}\\
&\quad +x^{59}+x^{56}+x^{51}+x^{50}+x^{49}+x^{46}+x^{45}+x^{44}+x^{42}+x^{41}+x^{40}
\end{aligned}
$$

$$\begin{aligned}
&+ x^{38} + x^{29} + x^{28} + x^{27} + x^{24} + x^{19} + x^{18} + x^{17} + x^{14} + x^{13} + x^{12}\\
&+ x^{10} + x^{9} + x^{8} + x^{6},\\
p_{12}(x) &= x^{128} + x^{124} + x^{116} + x^{112} + x^{108} + x^{100} + x^{92} + x^{84} + x^{80} + x^{60}\\
&+ x^{52} + x^{28} + x^{20} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0].

For $\phi(W^o, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{129} + x^{128} + x^{124} + x^{123} + x^{122} + x^{119} + x^{113} + x^{112} + x^{111} + x^{108}\\
&+ x^{106} + x^{105} + x^{103} + x^{101} + x^{100} + x^{99} + x^{98} + x^{96} + x^{95} + x^{94}\\
&+ x^{93} + x^{90} + x^{84} + x^{82} + x^{81} + x^{80} + x^{78} + x^{77} + x^{75} + x^{74} + x^{72}\\
&+ x^{71} + x^{70} + x^{69} + x^{68} + x^{64} + x^{63} + x^{61} + x^{57} + x^{55} + x^{54} + x^{51}\\
&+ x^{50} + x^{49} + x^{45} + x^{44} + x^{42},\\
p_3(x) &= x^{125} + x^{124} + x^{120} + x^{119} + x^{118} + x^{115} + x^{113} + x^{110} + x^{109} + x^{108}\\
&+ x^{107} + x^{105} + x^{104} + x^{102} + x^{100} + x^{99} + x^{97} + x^{94} + x^{91} + x^{90}\\
&+ x^{85} + x^{84} + x^{81} + x^{80} + x^{79} + x^{78} + x^{75} + x^{74} + x^{72} + x^{69} + x^{68}\\
&+ x^{67} + x^{65} + x^{63} + x^{61} + x^{60} + x^{59} + x^{54} + x^{52} + x^{50} + x^{46} + x^{43}\\
&+ x^{40} + x^{38} + x^{34} + x^{31} + x^{28} + x^{26} + x^{24} + x^{22},\\
p_6(x) &= x^{127} + x^{126} + x^{124} + x^{119} + x^{117} + x^{116} + x^{115} + x^{113} + x^{112} + x^{111}\\
&+ x^{110} + x^{104} + x^{102} + x^{101} + x^{98} + x^{97} + x^{95} + x^{91} + x^{89} + x^{88}\\
&+ x^{87} + x^{86} + x^{82} + x^{81} + x^{79} + x^{77} + x^{75} + x^{74} + x^{68} + x^{66} + x^{65}\\
&+ x^{63} + x^{62} + x^{61} + x^{59} + x^{56} + x^{53} + x^{51} + x^{50} + x^{48} + x^{42} + x^{41}\\
&+ x^{36} + x^{35} + x^{32} + x^{31} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{20}\\
&+ x^{18} + x^{16} + x^{14} + x^{9} + x^{8} + x^{7} + x^{6} + x^{3} + x^{2} + x + 1,\\
p_9(x) &= x^{125} + x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{117} + x^{116} + x^{115} + x^{114}\\
&+ x^{111} + x^{110} + x^{109} + x^{108} + x^{101} + x^{100} + x^{99} + x^{98} + x^{95} + x^{94}\\
&+ x^{93} + x^{92} + x^{85} + x^{84} + x^{83} + x^{82} + x^{79} + x^{78} + x^{75} + x^{74} + x^{73}\\
&+ x^{72} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{53} + x^{52} + x^{51} + x^{50}\\
&+ x^{47} + x^{46} + x^{43} + x^{42} + x^{41} + x^{40} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25}\\
&+ x^{24} + x^{21} + x^{20} + x^{19} + x^{18} + x^{15} + x^{14} + x^{11} + x^{10} + x^{9} + x^{8},\\
p_{12}(x) &= x^{129} + x^{128} + x^{127} + x^{126} + x^{125} + x^{124} + x^{117} + x^{116} + x^{115} + x^{114}\\
&+ x^{113} + x^{112} + x^{109} + x^{108} + x^{107} + x^{106} + x^{103} + x^{102} + x^{101}\\
&+ x^{100} + x^{93} + x^{92} + x^{91} + x^{90} + x^{87} + x^{86} + x^{85} + x^{84} + x^{81} + x^{80}\\
&+ x^{79} + x^{78} + x^{75} + x^{74} + x^{71} + x^{70} + x^{67} + x^{66} + x^{63} + x^{62} + x^{61}\\
&+ x^{60} + x^{53} + x^{52} + x^{51} + x^{50} + x^{47} + x^{46} + x^{43} + x^{42} + x^{39} + x^{38}\\
&+ x^{35} + x^{34} + x^{31} + x^{30} + x^{29} + x^{28} + x^{21} + x^{20} + x^{19} + x^{18} + x^{15}\\
&+ x^{14} + x^{11} + x^{10} + x^{7} + x^{6} + x^{3} + x^{2} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 1, 0, 0].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	89	[157, 158]	178	[289, 290]	267	[57, 334]	356	[43, 326]
1	[3, 4]	90	[37, 159]	179	[291, 292]	268	[116, 116]	357	[33, 304]
2	[5, 6]	91	[143, 139]	180	[293, 294]	269	[301, 307]	358	[168, 116]
3	[7, 8]	92	[138, 160]	181	[295, 167]	270	[337, 14]	359	[120, 307]
4	[9, 10]	93	[141, 161]	182	[118, 168]	271	[11, 157]	360	[427, 428]
5	[11, 12]	94	[47, 137]	183	[154, 117]	272	[335, 326]	361	[176, 332]
6	[13, 14]	95	[49, 162]	184	[154, 118]	273	[303, 34]	362	[152, 58]
7	[15, 16]	96	[163, 163]	185	[167, 117]	274	[117, 32]	363	[47, 334]
8	[17, 18]	97	[164, 165]	186	[32, 118]	275	[116, 118]	364	[161, 329]
9	[19, 20]	98	[47, 144]	187	[119, 168]	276	[311, 325]	365	[299, 127]
10	[21, 22]	99	[8, 166]	188	[126, 55]	277	[335, 123]	366	[163, 149]
11	[23, 24]	100	[167, 168]	189	[296, 297]	278	[32, 167]	367	[429, 330]
12	[25, 26]	101	[168, 154]	190	[46, 298]	279	[138, 328]	368	[57, 324]
13	[27, 28]	102	[169, 170]	191	[299, 55]	280	[338, 339]	369	[317, 327]
14	[29, 30]	103	[171, 158]	192	[174, 174]	281	[149, 340]	370	[118, 32]
15	[31, 32]	104	[172, 173]	193	[150, 300]	282	[341, 151]	371	[54, 127]
16	[33, 34]	105	[49, 160]	194	[128, 137]	283	[333, 324]	372	[129, 140]
17	[35, 36]	106	[174, 175]	195	[301, 302]	284	[167, 32]	373	[141, 338]
18	[37, 38]	107	[176, 177]	196	[179, 38]	285	[301, 158]	374	[131, 127]
19	[39, 36]	108	[128, 48]	197	[303, 304]	286	[40, 321]	375	[145, 338]
20	[40, 41]	109	[39, 178]	198	[167, 154]	287	[41, 157]	376	[333, 58]
21	[42, 12]	110	[179, 159]	199	[132, 148]	288	[308, 123]	377	[341, 300]
22	[43, 44]	111	[45, 129]	200	[305, 165]	289	[128, 144]	378	[146, 180]
23	[45, 46]	112	[142, 162]	201	[136, 48]	290	[53, 342]	379	[154, 32]
24	[47, 48]	113	[52, 161]	202	[143, 129]	291	[142, 328]	380	[426, 310]
25	[49, 50]	114	[136, 180]	203	[54, 306]	292	[175, 133]	381	[152, 334]
26	[46, 51]	115	[181, 182]	204	[157, 307]	293	[319, 139]	382	[338, 342]
27	[52, 53]	116	[183, 61]	205	[308, 309]	294	[152, 324]	383	[120, 302]
28	[54, 55]	117	[184, 183]	206	[8, 310]	295	[343, 184]	384	[322, 314]
29	[45, 56]	118	[182, 185]	207	[117, 118]	296	[344, 345]	385	[305, 336]
30	[57, 58]	119	[186, 187]	208	[175, 148]	297	[346, 347]	386	[323, 58]
31	[59, 60]	120	[188, 189]	209	[117, 167]	298	[88, 348]	387	[303, 156]
32	[61, 62]	121	[24, 190]	210	[116, 32]	299	[349, 350]	388	[319, 56]
33	[63, 64]	122	[191, 192]	211	[119, 154]	300	[351, 352]	389	[147, 328]
34	[65, 66]	123	[193, 194]	212	[311, 178]	301	[28, 353]	390	[143, 56]
35	[67, 68]	124	[195, 196]	213	[312, 313]	302	[30, 354]	391	[323, 334]
36	[69, 70]	125	[197, 198]	214	[41, 120]	303	[355, 356]	392	[133, 340]
37	[71, 72]	126	[92, 199]	215	[37, 314]	304	[357, 358]	393	[145, 296]
38	[73, 74]	127	[200, 201]	216	[315, 161]	305	[359, 360]	394	[430, 139]

39	[75, 76]	128	[202, 203]	217	[299, 306]	306	[361, 362]	395	[333, 153]
40	[77, 78]	129	[204, 205]	218	[141, 296]	307	[363, 364]	396	[152, 48]
41	[79, 80]	130	[206, 207]	219	[57, 153]	308	[365, 366]	397	[139, 130]
42	[81, 82]	131	[203, 208]	220	[138, 162]	309	[367, 368]	398	[429, 135]
43	[83, 84]	132	[209, 210]	221	[133, 132]	310	[369, 370]	399	[47, 180]
44	[85, 86]	133	[211, 87]	222	[39, 316]	311	[371, 372]	400	[317, 125]
45	[87, 88]	134	[212, 213]	223	[43, 309]	312	[373, 374]	401	[154, 154]
46	[89, 90]	135	[214, 215]	224	[155, 170]	313	[375, 376]	402	[331, 177]
47	[91, 92]	136	[216, 217]	225	[119, 117]	314	[377, 378]	403	[146, 144]
48	[93, 94]	137	[218, 219]	226	[118, 154]	315	[379, 380]	404	[169, 156]
49	[95, 96]	138	[220, 221]	227	[317, 310]	316	[381, 382]	405	[118, 118]
50	[97, 98]	139	[222, 223]	228	[126, 318]	317	[383, 384]	406	[14, 301]
51	[99, 100]	140	[224, 225]	229	[148, 163]	318	[385, 386]	407	[179, 314]
52	[101, 102]	141	[226, 227]	230	[32, 116]	319	[241, 387]	408	[120, 158]
53	[103, 104]	142	[228, 229]	231	[124, 166]	320	[388, 389]	409	[431, 313]
54	[105, 106]	143	[230, 231]	232	[319, 129]	321	[390, 391]	410	[14, 171]
55	[107, 108]	144	[232, 233]	233	[128, 180]	322	[374, 392]	411	[43, 123]
56	[109, 110]	145	[234, 235]	234	[32, 32]	323	[393, 228]	412	[145, 161]
57	[111, 112]	146	[236, 237]	235	[33, 170]	324	[394, 395]	413	[430, 129]
58	[113, 114]	147	[238, 239]	236	[315, 296]	325	[396, 397]	414	[146, 137]
59	[115, 116]	148	[240, 241]	237	[147, 160]	326	[398, 399]	415	[340, 174]
60	[32, 117]	149	[210, 242]	238	[54, 318]	327	[400, 401]	416	[321, 171]
61	[118, 119]	150	[243, 244]	239	[149, 132]	328	[402, 403]	417	[122, 326]
62	[117, 119]	151	[245, 246]	240	[119, 167]	329	[404, 405]	418	[52, 338]
63	[120, 121]	152	[247, 248]	241	[118, 167]	330	[406, 407]	419	[12, 178]
64	[122, 123]	153	[249, 250]	242	[167, 116]	331	[408, 409]	420	[322, 173]
65	[124, 125]	154	[251, 186]	243	[35, 316]	332	[410, 411]	421	[154, 168]
66	[119, 116]	155	[252, 253]	244	[320, 14]	333	[412, 95]	422	[117, 154]
67	[126, 127]	156	[254, 255]	245	[321, 301]	334	[413, 414]	423	[311, 316]
68	[56, 51]	157	[256, 257]	246	[322, 38]	335	[389, 415]	424	[13, 428]
69	[128, 58]	158	[258, 259]	247	[45, 139]	336	[416, 417]	425	[32, 119]
70	[129, 130]	159	[260, 261]	248	[131, 318]	337	[418, 279]	426	[432, 433]
71	[131, 55]	160	[262, 263]	249	[52, 296]	338	[419, 420]	427	[434, 191]
72	[132, 133]	161	[264, 265]	250	[323, 324]	339	[102, 421]	428	[435, 381]
73	[134, 135]	162	[266, 267]	251	[168, 119]	340	[422, 211]	429	[436, 437]
74	[136, 137]	163	[268, 209]	252	[35, 325]	341	[423, 424]	430	[438, 99]
75	[138, 50]	164	[269, 270]	253	[308, 326]	342	[380, 425]	431	[439, 67]
76	[139, 140]	165	[271, 272]	254	[124, 327]	343	[295, 154]	432	[311, 36]
77	[141, 53]	166	[273, 274]	255	[116, 167]	344	[12, 316]	433	[172, 314]
78	[142, 50]	167	[60, 275]	256	[49, 328]	345	[122, 44]	434	[319, 46]
79	[143, 46]	168	[168, 168]	257	[161, 297]	346	[426, 166]	435	[315, 338]
80	[136, 144]	169	[276, 277]	258	[57, 144]	347	[168, 117]	436	[35, 178]
81	[145, 53]	170	[16, 278]	259	[296, 329]	348	[116, 119]	437	[440, 321]
82	[146, 48]	171	[279, 280]	260	[134, 330]	349	[126, 306]	438	[168, 167]

83	[147, 50]	172	[78, 281]	261	[323, 153]	350	[340, 175]	439	[430, 56]
84	[148, 149]	173	[282, 283]	262	[331, 332]	351	[11, 120]	440	[441, 25]
85	[150, 151]	174	[284, 101]	263	[333, 334]	352	[172, 38]	441	[430, 46]
86	[152, 153]	175	[242, 234]	264	[171, 307]	353	[53, 339]		
87	[154, 116]	176	[285, 286]	265	[335, 44]	354	[56, 298]		
88	[155, 156]	177	[287, 288]	266	[164, 336]	355	[301, 121]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	64	0	128	1	192	1	256	1	320	1	384	1
1	0	65	1	129	1	193	0	257	0	321	1	385	0
2	0	66	1	130	0	194	1	258	0	322	1	386	1
3	0	67	0	131	1	195	1	259	1	323	1	387	1
4	0	68	0	132	0	196	0	260	1	324	0	388	1
5	0	69	1	133	1	197	1	261	1	325	0	389	1
6	0	70	1	134	1	198	1	262	0	326	0	390	1
7	0	71	1	135	1	199	0	263	1	327	0	391	1
8	0	72	0	136	0	200	0	264	0	328	0	392	1
9	0	73	1	137	1	201	0	265	1	329	1	393	1
10	1	74	0	138	0	202	1	266	1	330	0	394	0
11	0	75	0	139	0	203	1	267	0	331	0	395	1
12	1	76	0	140	0	204	1	268	0	332	0	396	0
13	0	77	1	141	1	205	0	269	1	333	1	397	0
14	0	78	0	142	0	206	0	270	0	334	0	398	0
15	0	79	1	143	1	207	0	271	0	335	1	399	1
16	0	80	0	144	1	208	1	272	1	336	1	400	0
17	0	81	1	145	1	209	0	273	1	337	0	401	0
18	1	82	0	146	0	210	0	274	0	338	1	402	0
19	0	83	1	147	1	211	1	275	0	339	1	403	0
20	1	84	1	148	1	212	1	276	1	340	0	404	1
21	1	85	0	149	0	213	1	277	1	341	1	405	1
22	1	86	0	150	0	214	1	278	1	342	1	406	0
23	0	87	0	151	1	215	1	279	0	343	0	407	0
24	1	88	0	152	0	216	0	280	1	344	1	408	0
25	1	89	1	153	0	217	0	281	0	345	0	409	0
26	1	90	1	154	0	218	1	282	1	346	1	410	0
27	0	91	1	155	0	219	0	283	1	347	0	411	1
28	1	92	0	156	1	220	0	284	1	348	0	412	1
29	0	93	1	157	1	221	1	285	1	349	0	413	0
30	0	94	1	158	0	222	0	286	1	350	0	414	0
31	0	95	1	159	1	223	1	287	1	351	0	415	0
32	1	96	0	160	0	224	0	288	0	352	0	416	1
33	0	97	1	161	0	225	1	289	1	353	0	417	0
34	1	98	1	162	1	226	1	290	0	354	0	418	0

35	0	99	0	163	0	227	0	291	0	355	1	419	1
36	1	100	1	164	1	228	0	292	1	356	1	420	1
37	1	101	0	165	0	229	1	293	1	357	0	421	0
38	1	102	1	166	1	230	1	294	0	358	0	422	0
39	0	103	0	167	1	231	1	295	0	359	0	423	1
40	1	104	0	168	0	232	1	296	1	360	1	424	0
41	1	105	1	169	1	233	1	297	1	361	1	425	1
42	1	106	1	170	0	234	1	298	0	362	0	426	1
43	1	107	1	171	0	235	0	299	0	363	1	427	1
44	0	108	1	172	0	236	0	300	0	364	0	428	0
45	0	109	0	173	1	237	1	301	1	365	0	429	0
46	1	110	0	174	1	238	1	302	0	366	0	430	0
47	1	111	0	175	1	239	0	303	1	367	0	431	0
48	1	112	0	176	1	240	1	304	0	368	0	432	1
49	1	113	0	177	1	241	1	305	0	369	0	433	0
50	1	114	0	178	1	242	1	306	1	370	1	434	1
51	0	115	0	179	0	243	0	307	1	371	1	435	0
52	0	116	0	180	1	244	1	308	0	372	1	436	0
53	0	117	0	181	0	245	1	309	0	373	1	437	0
54	1	118	1	182	1	246	1	310	0	374	1	438	0
55	1	119	1	183	0	247	0	311	1	375	1	439	0
56	0	120	0	184	0	248	1	312	1	376	1	440	0
57	0	121	1	185	1	249	0	313	1	377	1	441	0
58	0	122	0	186	1	250	1	314	1	378	0		
59	0	123	0	187	1	251	0	315	0	379	0		
60	1	124	1	188	0	252	0	316	0	380	1		
61	1	125	1	189	1	253	0	317	0	381	0		
62	0	126	0	190	1	254	1	318	0	382	1		
63	0	127	0	191	0	255	0	319	1	383	0		

Université de Strasbourg, CNRS, IRMA UMR 7501, F-67000 Strasbourg, France
Email address: guoniu.han@unistra.fr

School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan, PR China
Email address: huyining@hust.edu.cn