PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a,b) = (z^4 + z^3 + z^2 + 1, z^3 + z^2 + z + 1)$

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Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a,b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a,b) in $\mathbb{F}_2[x] \setminus \mathbb{F}_2$. In this paper we prove the conjecture for $(a,b) = (z^4 + z^3 + z^2 + 1, z^3 + z^2 + z + 1)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \ldots)$ of elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{\ddots + \cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a,b)-Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \ldots) = (a, b, b, a, b, a, a, b, \ldots)$$

that we shall denote

$$\mathrm{CF}(a,b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When* $(a,b) = (z^4+z^3+z^2+1, z^3+z^2+z+1)$*, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4 + p_3(z)y^3 + p_2(z)y^2 + p_1(z)y + p_0(z) = 0$$

where, for $\mathrm{CF}(a,b)$*,*

$$\begin{aligned}
p_0(z) &= z^{17}+z^{15}+z^{14}+z^{13}+z^{11}+z^9+z^8+z^7+z^6+z^4+z^3+z^2+z+1,\\
p_1(z) &= z^{21}+z^{19}+z^{17}+z^{16}+z^{13}+z^{12}+z^8+z^5+z^2+z,\\
p_2(z) &= z^{24}+z^{20}+z^{18}+z^{12}+z^{10}+z^8+z^6+z^4+z^2+1,\\
p_3(z) &= z^{21}+z^{19}+z^{17}+z^{16}+z^{13}+z^{12}+z^8+z^5+z^2+z,\\
p_4(z) &= z^{18}+z^{17}+z^{15}+z^{14}+z^{13}+z^{12}+z^{11}+z^{10}+z^9+z^8+z^7+z^4+z^3\\
&\quad +z+1,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$*,*

$$\begin{aligned}
p_0(z) &= z^{17}+z^{15}+z^{14}+z^{13}+z^{12}+z^{11}+z^9+z^7+z^6+z^3+z^2+z+1,\\
p_1(z) &= z^{21}+z^{19}+z^{17}+z^{16}+z^{13}+z^{12}+z^8+z^5+z^2+z,\\
p_2(z) &= z^{24}+z^{20}+z^{18}+z^{12}+z^{10}+z^8+z^6+z^4+z^2+1,\\
p_3(z) &= z^{21}+z^{19}+z^{17}+z^{16}+z^{13}+z^{12}+z^8+z^5+z^2+z,\\
p_4(z) &= z^{18}+z^{17}+z^{15}+z^{14}+z^{13}+z^{11}+z^{10}+z^9+z^7+z^3+z+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z) \\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1 \\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} \in \mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x) &= x^{\deg(t_{2^n-1})}\begin{pmatrix} t_{2^n-1}(1/x) & 1 \\ 1 & 0\end{pmatrix} \times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1 \\ 1 & 0\end{pmatrix} \times \\
&\qquad \cdots \times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1 \\ 1 & 0\end{pmatrix},\\
W_n(x) &= x^{\deg(\bar t_{2^n-1})}\begin{pmatrix} \bar t_{2^n-1}(1/x) & 1 \\ 1 & 0\end{pmatrix} \times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix} \bar t_{2^n-2}(1/x) & 1 \\ 1 & 0\end{pmatrix} \times \\
&\qquad \cdots \times x^{\deg(\bar t_0)}\begin{pmatrix} \bar t_0(1/x) & 1 \\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n \geq 0$

$$\begin{aligned}
M_{n+1}(x) &= W_n(x)\cdot M_n(x),\\
W_{n+1}(x) &= M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(175, 175, 175, 175, 175)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z\left(\phi(W^o,0,0)(x,z),\ z^{12}\cdot\phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 630 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 630, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z\left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ to order 770 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 770, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. $\square$

Define

$$R^e = \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} \qquad\text{and}\qquad R^o = \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned}
M^e[:14u] &= M^e[:7u] + x^{2u}M^e[:5u] + x^{3u}M^e[:4u] + x^{4u}M^e[:3u] + x^{2u}M^e[:7u]\\
&\quad + x^{4u}M^e[:5u] + x^{5u}M^e[:4u] + x^{6u}M^e[:3u] + x^{3u}M^e[:7u]\\
&\quad + x^{5u}M^e[:5u] + x^{6u}M^e[:4u] + x^{7u}M^e[:3u] + x^{8u}M^e[:6u]\\
&\quad + (x^{7u} + x^{9u} + x^{10u})R^e,\\
W^e[:14u] &= W^e[:7u] + x^{2u}W^e[:5u] + x^{3u}W^e[:4u] + x^{4u}W^e[:3u] + x^{2u}W^e[:7u]\\
&\quad + x^{4u}W^e[:5u] + x^{5u}W^e[:4u] + x^{6u}W^e[:3u] + x^{3u}W^e[:7u]
\end{aligned}$$

$$+ x^{5u}W^e[:5u] + x^{6u}W^e[:4u] + x^{7u}W^e[:3u] + x^{8u}W^e[:6u] + (x^{7u} + x^{9u} + x^{10u})R^e.$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$\begin{aligned} M^o[:14u] &= M^o[:7u] + x^{2u}M^o[:5u] + x^{3u}M^o[:4u] + x^{4u}M^o[:3u] + x^{2u}M^o[:7u] \\ &+ x^{4u}M^o[:5u] + x^{5u}M^o[:4u] + x^{6u}M^o[:3u] + x^{3u}M^o[:7u] \\ &+ x^{5u}M^o[:5u] + x^{6u}M^o[:4u] + x^{7u}M^o[:3u] + x^{8u}M^o[:6u] \\ &+ (x^{7u} + x^{9u} + x^{10u})R^o, \\ W^o[:14u] &= W^o[:7u] + x^{2u}W^o[:5u] + x^{3u}W^o[:4u] + x^{4u}W^o[:3u] + x^{2u}W^o[:7u] \\ &+ x^{4u}W^o[:5u] + x^{5u}W^o[:4u] + x^{6u}W^o[:3u] + x^{3u}W^o[:7u] \\ &+ x^{5u}W^o[:5u] + x^{6u}W^o[:4u] + x^{7u}W^o[:3u] + x^{8u}W^o[:6u] \\ &+ (x^{7u} + x^{9u} + x^{10u})R^o. \end{aligned}$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$\begin{aligned} T[:14u] &= T[:7u] + x^{2u}T[:5u] + x^{3u}T[:4u] + x^{4u}T[:3u] + x^{2u}T[:7u] + x^{4u}T[:5u] \\ &+ x^{5u}T[:4u] + x^{6u}T[:3u] + x^{3u}T[:7u] + x^{5u}T[:5u] + x^{6u}T[:4u] \\ &+ x^{7u}T[:3u] + x^{8u}T[:6u] + (x^{7u} + x^{9u} + x^{10u}). \end{aligned}$$

The proof of the other 15 cases are similar. We break down the above identity into 7 parts:

$$\begin{aligned} x^7u &= x^{7u}T[:u] + x^{4u}T[3u:4u] + x^{3u}T[4u:5u] + x^{2u}T[5u:6u] \\ &+ T[7u:8u], \\ 0 &= x^{8u}T[:u] + x^{7u}T[u:2u] + x^{4u}T[4u:5u] + x^{3u}T[5u:6u] \\ &+ x^{2u}T[6u:7u] + T[8u:9u], \\ x^9u &= x^{8u}T[u:2u] + x^{7u}T[2u:3u] + x^{6u}T[3u:4u] + x^{5u}T[4u:5u] \\ &+ x^{3u}T[6u:7u] + T[9u:10u], \\ x^10u &= x^{8u}T[2u:3u] + T[10u:11u], \\ 0 &= x^{8u}T[3u:4u] + T[11u:12u], \\ 0 &= x^{8u}T[4u:5u] + T[12u:13u], \\ 0 &= x^{8u}T[5u:6u] + T[13u:14u], \end{aligned}$$

which can be rewritten as

$$0 = T[[w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[111w]_2], \tag{2.7}$$

$$\begin{aligned} 0 &= T[[w]_2] + T[[1w]_2] + T[[100w]_2] + T[[101w]_2] + T[[110w]_2] \\ &+ T[[1000w]_2], \end{aligned} \tag{2.8}$$

$$\begin{aligned} 0 &= T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[110w]_2] \\ &+ T[[1001w]_2], \end{aligned} \tag{2.9}$$

$$0 = T[[10w]_2] + T[[1010w]_2], \tag{2.10}$$
$$0 = T[[11w]_2] + T[[1011w]_2], \tag{2.11}$$
$$0 = T[[100w]_2] + T[[1100w]_2], \tag{2.12}$$
$$0 = T[[101w]_2] + T[[1101w]_2], \tag{2.13}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$1 = T[[w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[111w]_2], \tag{2.14}$$
$$\begin{aligned} 0 = {} & T[[w]_2] + T[[1w]_2] + T[[100w]_2] + T[[101w]_2] + T[[110w]_2] \\ & + T[[1000w]_2], \end{aligned} \tag{2.15}$$
$$\begin{aligned} 1 = {} & T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[110w]_2] \\ & + T[[1001w]_2], \end{aligned} \tag{2.16}$$
$$1 = T[[10w]_2] + T[[1010w]_2], \tag{2.17}$$
$$0 = T[[11w]_2] + T[[1011w]_2], \tag{2.18}$$
$$0 = T[[100w]_2] + T[[1100w]_2], \tag{2.19}$$
$$0 = T[[101w]_2] + T[[1101w]_2], \tag{2.20}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 1063 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.20) can be written as

$$\begin{aligned} 0 = {} & \tau(A(s, \epsilon)) + \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 101)) \\ & + \tau(A(s, 111)), \end{aligned} \tag{2.21}$$
$$\begin{aligned} 0 = {} & \tau(A(s, \epsilon)) + \tau(A(s, 1)) + \tau(A(s, 100)) + \tau(A(s, 101)) \\ & + \tau(A(s, 110)) + \tau(A(s, 1000)), \end{aligned} \tag{2.22}$$
$$\begin{aligned} 0 = {} & \tau(A(s, 1)) + \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 100)) \\ & + \tau(A(s, 110)) + \tau(A(s, 1001)), \end{aligned} \tag{2.23}$$
$$0 = \tau(A(s, 10)) + \tau(A(s, 1010)), \tag{2.24}$$
$$0 = \tau(A(s, 11)) + \tau(A(s, 1011)), \tag{2.25}$$
$$0 = \tau(A(s, 100)) + \tau(A(s, 1100)), \tag{2.26}$$
$$0 = \tau(A(s, 101)) + \tau(A(s, 1101)), \tag{2.27}$$

for all $s \in E_{2k}$, and

$$\begin{aligned} 1 = {} & \tau(A(s, \epsilon)) + \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 101)) \\ & + \tau(A(s, 111)), \end{aligned} \tag{2.28}$$
$$\begin{aligned} 0 = {} & \tau(A(s, \epsilon)) + \tau(A(s, 1)) + \tau(A(s, 100)) + \tau(A(s, 101)) \\ & + \tau(A(s, 110)) + \tau(A(s, 1000)), \end{aligned} \tag{2.29}$$
$$1 = \tau(A(s, 1)) + \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 100))$$

$$+\tau(A(s,110))+\tau(A(s,1001)), \tag{2.30}$$

$$1=\tau(A(s,10))+\tau(A(s,1010)), \tag{2.31}$$

$$0=\tau(A(s,11))+\tau(A(s,1011)), \tag{2.32}$$

$$0=\tau(A(s,100))+\tau(A(s,1100)), \tag{2.33}$$

$$0=\tau(A(s,101))+\tau(A(s,1101)), \tag{2.34}$$

for $s=A(i,0^{2k})$. We find that $(A(i,0^{28}),E_{28})=(A(i,0^{16}),E_{16})$, so that we only have to verify that identities (2.21) through (2.34) hold for $2\le k\le 14$, which turns out to be true. ☐

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k\ge 2$, and $u=2^{2k-1}$,*

$$M_{2k}=M^e[:7u]+x^{2u}M^e[:5u]+x^{3u}M^e[:4u]+x^{4u}M^e[:3u]+x^{7u}R^e,$$
$$W_{2k}=W^e[:7u]+x^{2u}W^e[:5u]+x^{3u}W^e[:4u]+x^{4u}W^e[:3u]+x^{7u}R^e.$$

For all integer $k\ge 2$, and $u=2^{2k}$,

$$M_{2k+1}=M^o[:7u]+x^{2u}M^o[:5u]+x^{3u}M^o[:4u]+x^{4u}M^o[:3u]+x^{7u}R^o,$$
$$W_{2k+1}=W^o[:7u]+x^{2u}W^o[:5u]+x^{3u}W^o[:4u]+x^{4u}W^o[:3u]+x^{7u}R^o.$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n=2$, the identities can be verified directly. For $n\ge 2$, we claim that

$$M_{2k}\wedge W_{2k}\Rightarrow M_{2k+1}\wedge W_{2k+1},$$
$$M_{2k+1}\wedge W_{2k+1}\Rightarrow M_{2k+2}\wedge W_{2k+2}.$$

We give the proof of

$$M_{2k}\wedge W_{2k}\Rightarrow M_{2k+1},$$

the other ones being similar. Set $u=2^{2k}$ and $v=2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$\begin{aligned}W_{2k}M_{2k}=\big(&W^e[:7v]+x^{2v}W^e[:5v]+x^{3v}W^e[:4v]+x^{4v}W^e[:3v]+x^{7u}R^e\\ &\big)\times\big(M^e[:7v]+x^{2v}M^e[:5v]+x^{3v}M^e[:4v]+x^{4v}M^e[:3v]+x^{7u}R^e\\ &\big);\end{aligned}\tag{2.35}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{14v}R^o$. Therefore we only need to prove that their difference is $O(x^{14v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{14v}, to

$$W^e[:14v]M^e[:14v]+x^{4v}W^e[:10v]M^e[:10v]+x^{6v}W^e[:8v]M^e[:8v]\\ +x^{8v}W^e[:6v]M^e[:6v].$$

For all $n\le 14$, replace the occurrences of $W^e[:n\cdot v]$ and $M^e[:n\cdot v]$ in the above expression by the reduction modulo $x^{n\cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$X:=x^v,$$
$$a_n:=W^e[n\cdot v:(n+1)\cdot v]/X^n,$$

$$\begin{aligned} b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.35) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{14}, b_1, ..., b_{14}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{14})$, which completes the proof. $\square$

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \le j, k \le 1$,

$$\begin{aligned} \lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k}, \\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k}, \\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k}, \\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}. \end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a, b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x, y) = \mathrm{Res}_t \left(\phi(M^e, 0, 1)(x, t), y^{12}\phi(M^e, 0, 1)(x, t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x, y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned} q_0(x) &= x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{18} + x^{17} + x^{16} + x^{15} + x^{13} + x^{11} + x^{10} \\ &\quad + x^9 + x^7, \\ q_1(x) &= x^{23} + x^{22} + x^{19} + x^{16} + x^{12} + x^{11} + x^8 + x^7 + x^5 + x^3, \\ q_2(x) &= x^{24} + x^{22} + x^{20} + x^{18} + x^{16} + x^{14} + x^{12} + x^6 + x^4 + 1, \\ q_3(x) &= x^{23} + x^{22} + x^{19} + x^{16} + x^{12} + x^{11} + x^8 + x^7 + x^5 + x^3, \\ q_4(x) &= x^{24} + x^{23} + x^{21} + x^{20} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} \\ &\quad + x^9 + x^7 + x^6. \end{aligned}$$

The polynomial $Q(x, y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x, y)$ of multiplicity m and $R(x, y) := P(x, y)/Q(x, y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 216, and substitute it for y in $R(z, y)$, we get a series with valuation smaller than 216, which proves that $R(z, y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z, y)$ is the minimal polynomial of $\mathrm{CF}(a, b) = f(1/z)$. $\square$

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{118} + x^{116} + x^{114} + x^{112} + x^{108} + x^{104} + x^{102} + x^{96} + x^{92} \\
&\quad + x^{90} + x^{84} + x^{82} + x^{78} + x^{74} + x^{70} + x^{66} + x^{54} + x^{52} + x^{46} + x^{44} \\
&\quad + x^{38} + x^{36}, \\
p_3(x) &= x^{124} + x^{122} + x^{120} + x^{118} + x^{116} + x^{106} + x^{100} + x^{94} + x^{92} + x^{90} \\
&\quad + x^{86} + x^{80} + x^{78} + x^{74} + x^{72} + x^{68} + x^{66} + x^{64} + x^{58} + x^{56} + x^{52} \\
&\quad + x^{46} + x^{44} + x^{42} + x^{40} + x^{36} + x^{34} + x^{30} + x^{28} + x^{20}, \\
p_6(x) &= x^{124} + x^{118} + x^{116} + x^{114} + x^{110} + x^{108} + x^{106} + x^{102} + x^{100} + x^{98} \\
&\quad + x^{96} + x^{86} + x^{80} + x^{76} + x^{74} + x^{72} + x^{64} + x^{58} + x^{56} + x^{50} + x^{46} \\
&\quad + x^{44} + x^{40} + x^{38} + x^{32} + x^{28} + x^{26} + x^{20} + x^{18} + x^{16} + x^{12} + x^{8} \\
&\quad + x^{4} + 1, \\
p_9(x) &= x^{124} + x^{122} + x^{118} + x^{116} + x^{112} + x^{108} + x^{102} + x^{96} + x^{94} + x^{92} \\
&\quad + x^{90} + x^{78} + x^{74} + x^{72} + x^{64} + x^{62} + x^{54} + x^{52} + x^{50} + x^{46} + x^{44} \\
&\quad + x^{40} + x^{34} + x^{26} + x^{16} + x^{6}, \\
p_{12}(x) &= x^{126} + x^{124} + x^{120} + x^{118} + x^{116} + x^{112} + x^{78} + x^{76} + x^{72} + x^{70} \\
&\quad + x^{68} + x^{64} + x^{46} + x^{44} + x^{40} + x^{38} + x^{36} + x^{32} + x^{14} + x^{12} + x^{8} \\
&\quad + x^{6} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0].

For $\phi(M^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{129} + x^{128} + x^{127} + x^{126} + x^{122} + x^{115} + x^{113} + x^{107} + x^{106} + x^{104} \\
&\quad + x^{102} + x^{100} + x^{99} + x^{98} + x^{95} + x^{92} + x^{90} + x^{89} + x^{88} + x^{82} + x^{81} \\
&\quad + x^{79} + x^{78} + x^{77} + x^{74} + x^{72} + x^{70} + x^{65} + x^{63} + x^{58} + x^{56} + x^{54} \\
&\quad + x^{52} + x^{51} + x^{49} + x^{46} + x^{44} + x^{43} + x^{42} + x^{40} + x^{39}, \\
p_3(x) &= x^{120} + x^{117} + x^{106} + x^{104} + x^{103} + x^{102} + x^{101} + x^{99} + x^{98} + x^{96} \\
&\quad + x^{95} + x^{94} + x^{93} + x^{91} + x^{90} + x^{88} + x^{87} + x^{86} + x^{85} + x^{83} + x^{82} \\
&\quad + x^{79} + x^{78} + x^{75} + x^{74} + x^{71} + x^{70} + x^{67} + x^{64} + x^{61} + x^{58} + x^{55} \\
&\quad + x^{54} + x^{51} + x^{42} + x^{39} + x^{38} + x^{35} + x^{34} + x^{32} + x^{31} + x^{30} + x^{29} \\
&\quad + x^{27}, \\
p_6(x) &= x^{123} + x^{122} + x^{120} + x^{119} + x^{118} + x^{116} + x^{113} + x^{112} + x^{110} + x^{109} \\
&\quad + x^{108} + x^{106} + x^{105} + x^{104} + x^{103} + x^{101} + x^{99} + x^{97} + x^{95} + x^{92} \\
&\quad + x^{87} + x^{86} + x^{85} + x^{82} + x^{79} + x^{78} + x^{76} + x^{69} + x^{68} + x^{67} + x^{64}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{63} + x^{62} + x^{61} + x^{59} + x^{57} + x^{55} + x^{52} + x^{45} + x^{44} + x^{43} + x^{40} \\
&+ x^{39} + x^{38} + x^{37} + x^{34} + x^{33} + x^{32} + x^{31} + x^{28} + x^{27} + x^{26} + x^{25} \\
&+ x^{23} + x^{21} + x^{18}, \\
p_9(x) = {}& x^{126} + x^{124} + x^{123} + x^{121} + x^{116} + x^{113} + x^{110} + x^{107} + x^{106} + x^{104} \\
&+ x^{103} + x^{102} + x^{101} + x^{99} + x^{98} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} \\
&+ x^{89} + x^{88} + x^{85} + x^{82} + x^{80} + x^{79} + x^{78} + x^{77} + x^{75} + x^{72} + x^{69} \\
&+ x^{66} + x^{63} + x^{60} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{51} + x^{46} \\
&+ x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{37} + x^{34} + x^{32} + x^{31} + x^{30} \\
&+ x^{29} + x^{27} + x^{22} + x^{19} + x^{18} + x^{15} + x^{14} + x^{12} + x^{11} + x^{9}, \\
p_{12}(x) = {}& x^{129} + x^{128} + x^{127} + x^{125} + x^{123} + x^{120} + x^{117} + x^{116} + x^{115} + x^{112} \\
&+ x^{81} + x^{80} + x^{79} + x^{77} + x^{75} + x^{72} + x^{69} + x^{68} + x^{67} + x^{64} + x^{49} \\
&+ x^{48} + x^{47} + x^{45} + x^{43} + x^{40} + x^{37} + x^{36} + x^{35} + x^{32} + x^{17} + x^{16} \\
&+ x^{15} + x^{13} + x^{11} + x^{8} + x^{5} + x^{4} + x^{3} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 1, 1, 1, 1, 0, 0].

For $\phi(M^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) = {}& x^{129} + x^{128} + x^{127} + x^{126} + x^{122} + x^{115} + x^{113} + x^{107} + x^{106} + x^{104} \\
&+ x^{102} + x^{100} + x^{99} + x^{98} + x^{95} + x^{92} + x^{90} + x^{89} + x^{88} + x^{82} + x^{81} \\
&+ x^{79} + x^{78} + x^{77} + x^{74} + x^{72} + x^{70} + x^{65} + x^{63} + x^{58} + x^{56} + x^{54} \\
&+ x^{52} + x^{51} + x^{49} + x^{46} + x^{44} + x^{43} + x^{42} + x^{40} + x^{39}, \\
p_3(x) = {}& x^{120} + x^{117} + x^{106} + x^{104} + x^{103} + x^{102} + x^{101} + x^{99} + x^{98} + x^{96} \\
&+ x^{95} + x^{94} + x^{93} + x^{91} + x^{90} + x^{88} + x^{87} + x^{86} + x^{85} + x^{83} + x^{82} \\
&+ x^{79} + x^{78} + x^{75} + x^{74} + x^{71} + x^{70} + x^{67} + x^{64} + x^{61} + x^{58} + x^{55} \\
&+ x^{54} + x^{51} + x^{42} + x^{39} + x^{38} + x^{35} + x^{34} + x^{32} + x^{31} + x^{30} + x^{29} \\
&+ x^{27}, \\
p_6(x) = {}& x^{123} + x^{122} + x^{120} + x^{119} + x^{118} + x^{116} + x^{113} + x^{112} + x^{110} + x^{109} \\
&+ x^{108} + x^{106} + x^{105} + x^{104} + x^{103} + x^{101} + x^{99} + x^{97} + x^{95} + x^{92} \\
&+ x^{87} + x^{86} + x^{85} + x^{82} + x^{79} + x^{78} + x^{76} + x^{69} + x^{68} + x^{67} + x^{64} \\
&+ x^{63} + x^{62} + x^{61} + x^{59} + x^{57} + x^{55} + x^{52} + x^{45} + x^{44} + x^{43} + x^{40} \\
&+ x^{39} + x^{38} + x^{37} + x^{34} + x^{33} + x^{32} + x^{31} + x^{28} + x^{27} + x^{26} + x^{25} \\
&+ x^{23} + x^{21} + x^{18}, \\
p_9(x) = {}& x^{126} + x^{124} + x^{123} + x^{121} + x^{116} + x^{113} + x^{110} + x^{107} + x^{106} + x^{104} \\
&+ x^{103} + x^{102} + x^{101} + x^{99} + x^{98} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} \\
&+ x^{89} + x^{88} + x^{85} + x^{82} + x^{80} + x^{79} + x^{78} + x^{77} + x^{75} + x^{72} + x^{69} \\
&+ x^{66} + x^{63} + x^{60} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{51} + x^{46} \\
&+ x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{37} + x^{34} + x^{32} + x^{31} + x^{30}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{29}+x^{27}+x^{22}+x^{19}+x^{18}+x^{15}+x^{14}+x^{12}+x^{11}+x^{9},\\
p_{12}(x) &= x^{129}+x^{128}+x^{127}+x^{125}+x^{123}+x^{120}+x^{117}+x^{116}+x^{115}+x^{112}\\
&+x^{81}+x^{80}+x^{79}+x^{77}+x^{75}+x^{72}+x^{69}+x^{68}+x^{67}+x^{64}+x^{49}\\
&+x^{48}+x^{47}+x^{45}+x^{43}+x^{40}+x^{37}+x^{36}+x^{35}+x^{32}+x^{17}+x^{16}\\
&+x^{15}+x^{13}+x^{11}+x^{8}+x^{5}+x^{4}+x^{3}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 1, 1, 1, 1, 0, 0].

For $\phi(M^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{132}+x^{130}+x^{128}+x^{126}+x^{124}+x^{120}+x^{118}+x^{110}+x^{108}+x^{106}\\
&+x^{102}+x^{94}+x^{92}+x^{86}+x^{80}+x^{76}+x^{72}+x^{68}+x^{66}+x^{64}+x^{60}\\
&+x^{56}+x^{54}+x^{52}+x^{50}+x^{48}+x^{42},\\
p_3(x) &= x^{130}+x^{126}+x^{116}+x^{114}+x^{112}+x^{108}+x^{106}+x^{100}+x^{96}+x^{90}\\
&+x^{88}+x^{84}+x^{78}+x^{76}+x^{74}+x^{70}+x^{66}+x^{58}+x^{52}+x^{44}+x^{40}\\
&+x^{30}+x^{20}+x^{18},\\
p_6(x) &= x^{130}+x^{126}+x^{122}+x^{120}+x^{118}+x^{114}+x^{112}+x^{110}+x^{108}+x^{102}\\
&+x^{96}+x^{92}+x^{86}+x^{78}+x^{72}+x^{68}+x^{64}+x^{62}+x^{56}+x^{48}+x^{42}\\
&+x^{34}+x^{22}+x^{20}+x^{18}+x^{16}+x^{14}+x^{12}+x^{8}+x^{6}+x^{4}+1,\\
p_9(x) &= x^{130}+x^{126}+x^{120}+x^{114}+x^{112}+x^{110}+x^{108}+x^{106}+x^{102}+x^{98}\\
&+x^{90}+x^{88}+x^{86}+x^{84}+x^{82}+x^{80}+x^{76}+x^{66}+x^{64}+x^{58}+x^{54}\\
&+x^{46}+x^{44}+x^{40}+x^{38}+x^{36}+x^{30}+x^{28}+x^{26}+x^{22}+x^{20}+x^{18}\\
&+x^{12}+x^{6},\\
p_{12}(x) &= x^{132}+x^{128}+x^{108}+x^{104}+x^{100}+x^{96}+x^{92}+x^{88}+x^{76}+x^{72}+x^{68}\\
&+x^{64}+x^{52}+x^{48}+x^{28}+x^{24}+x^{12}+x^{8}+x^{4}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0].

For $\phi(W^e, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{132}+x^{130}+x^{126}+x^{124}+x^{122}+x^{120}+x^{118}+x^{112}+x^{108}+x^{96}\\
&+x^{94}+x^{90}+x^{84}+x^{82}+x^{74}+x^{72}+x^{64}+x^{62}+x^{56}+x^{52}+x^{50}\\
&+x^{48}+x^{42}+x^{38}+x^{36},\\
p_3(x) &= x^{130}+x^{124}+x^{122}+x^{120}+x^{114}+x^{112}+x^{110}+x^{106}+x^{104}+x^{96}\\
&+x^{94}+x^{92}+x^{90}+x^{74}+x^{72}+x^{64}+x^{62}+x^{58}+x^{56}+x^{50}+x^{46}\\
&+x^{38}+x^{34}+x^{32}+x^{30}+x^{28}+x^{26}+x^{20},\\
p_6(x) &= x^{130}+x^{128}+x^{124}+x^{120}+x^{118}+x^{116}+x^{114}+x^{112}+x^{108}+x^{104}\\
&+x^{102}+x^{96}+x^{94}+x^{92}+x^{88}+x^{86}+x^{72}+x^{70}+x^{66}+x^{64}+x^{60}\\
&+x^{56}+x^{52}+x^{50}+x^{46}+x^{44}+x^{42}+x^{38}+x^{36}+x^{32}+x^{26}+x^{22}\\
&+x^{20}+x^{14}+x^{8}+x^{6}+x^{4}+1,\\
p_9(x) &= x^{130}+x^{126}+x^{120}+x^{114}+x^{112}+x^{110}+x^{108}+x^{106}+x^{102}+x^{98}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{90}+x^{88}+x^{86}+x^{84}+x^{82}+x^{80}+x^{76}+x^{66}+x^{64}+x^{58}+x^{54}\\
&+x^{46}+x^{44}+x^{40}+x^{38}+x^{36}+x^{30}+x^{28}+x^{26}+x^{22}+x^{20}+x^{18}\\
&+x^{12}+x^{6},\\
p_{12}(x)&=x^{132}+x^{128}+x^{108}+x^{104}+x^{100}+x^{96}+x^{92}+x^{88}+x^{76}+x^{72}+x^{68}\\
&+x^{64}+x^{52}+x^{48}+x^{28}+x^{24}+x^{12}+x^{8}+x^{4}+1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0].

For $\phi(W^e, 0, 1)$:

$$
\begin{aligned}
p_0(x)&=x^{129}+x^{128}+x^{127}+x^{126}+x^{123}+x^{120}+x^{112}+x^{110}+x^{107}+x^{106}\\
&+x^{105}+x^{104}+x^{102}+x^{100}+x^{99}+x^{98}+x^{97}+x^{95}+x^{94}+x^{90}\\
&+x^{88}+x^{86}+x^{85}+x^{84}+x^{81}+x^{78}+x^{74}+x^{73}+x^{72}+x^{70}+x^{68}\\
&+x^{67}+x^{65}+x^{64}+x^{62}+x^{60}+x^{59}+x^{57}+x^{55}+x^{54}+x^{53}+x^{52}\\
&+x^{51}+x^{48}+x^{47}+x^{43}+x^{42}+x^{40}+x^{39},\\
p_3(x)&=x^{120}+x^{117}+x^{106}+x^{104}+x^{103}+x^{102}+x^{101}+x^{99}+x^{98}+x^{96}\\
&+x^{95}+x^{94}+x^{93}+x^{91}+x^{90}+x^{88}+x^{87}+x^{86}+x^{85}+x^{83}+x^{82}\\
&+x^{79}+x^{78}+x^{75}+x^{74}+x^{71}+x^{70}+x^{67}+x^{64}+x^{61}+x^{58}+x^{55}\\
&+x^{54}+x^{51}+x^{42}+x^{39}+x^{38}+x^{35}+x^{34}+x^{32}+x^{31}+x^{30}+x^{29}\\
&+x^{27},\\
p_6(x)&=x^{123}+x^{122}+x^{120}+x^{119}+x^{118}+x^{116}+x^{113}+x^{112}+x^{110}+x^{109}\\
&+x^{108}+x^{106}+x^{105}+x^{104}+x^{103}+x^{101}+x^{99}+x^{97}+x^{95}+x^{92}\\
&+x^{87}+x^{86}+x^{85}+x^{82}+x^{79}+x^{78}+x^{76}+x^{69}+x^{68}+x^{67}+x^{64}\\
&+x^{63}+x^{62}+x^{61}+x^{59}+x^{57}+x^{55}+x^{52}+x^{45}+x^{44}+x^{43}+x^{40}\\
&+x^{39}+x^{38}+x^{37}+x^{34}+x^{33}+x^{32}+x^{31}+x^{28}+x^{27}+x^{26}+x^{25}\\
&+x^{23}+x^{21}+x^{18},\\
p_9(x)&=x^{126}+x^{124}+x^{123}+x^{121}+x^{116}+x^{113}+x^{110}+x^{107}+x^{106}+x^{104}\\
&+x^{103}+x^{102}+x^{101}+x^{99}+x^{98}+x^{96}+x^{95}+x^{94}+x^{93}+x^{92}+x^{91}\\
&+x^{89}+x^{88}+x^{85}+x^{82}+x^{80}+x^{79}+x^{78}+x^{77}+x^{75}+x^{72}+x^{69}\\
&+x^{66}+x^{63}+x^{60}+x^{58}+x^{57}+x^{56}+x^{55}+x^{54}+x^{53}+x^{51}+x^{46}\\
&+x^{44}+x^{43}+x^{42}+x^{41}+x^{40}+x^{39}+x^{37}+x^{34}+x^{32}+x^{31}+x^{30}\\
&+x^{29}+x^{27}+x^{22}+x^{19}+x^{18}+x^{15}+x^{14}+x^{12}+x^{11}+x^{9},\\
p_{12}(x)&=x^{129}+x^{128}+x^{127}+x^{125}+x^{123}+x^{120}+x^{117}+x^{116}+x^{115}+x^{112}\\
&+x^{81}+x^{80}+x^{79}+x^{77}+x^{75}+x^{72}+x^{69}+x^{68}+x^{67}+x^{64}+x^{49}\\
&+x^{48}+x^{47}+x^{45}+x^{43}+x^{40}+x^{37}+x^{36}+x^{35}+x^{32}+x^{17}+x^{16}\\
&+x^{15}+x^{13}+x^{11}+x^{8}+x^{5}+x^{4}+x^{3}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 1, 0, 0, 1, 1, 0, 1, 1, 1, 1, 1, 0].

For $\phi(W^e, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{129} + x^{128} + x^{127} + x^{126} + x^{123} + x^{120} + x^{112} + x^{110} + x^{107} + x^{106} \\
&\quad + x^{105} + x^{104} + x^{102} + x^{100} + x^{99} + x^{98} + x^{97} + x^{95} + x^{94} + x^{90} \\
&\quad + x^{88} + x^{86} + x^{85} + x^{84} + x^{81} + x^{78} + x^{74} + x^{73} + x^{72} + x^{70} + x^{68} \\
&\quad + x^{67} + x^{65} + x^{64} + x^{62} + x^{60} + x^{59} + x^{57} + x^{55} + x^{54} + x^{53} + x^{52} \\
&\quad + x^{51} + x^{48} + x^{47} + x^{43} + x^{42} + x^{40} + x^{39}, \\
p_3(x) &= x^{120} + x^{117} + x^{106} + x^{104} + x^{103} + x^{102} + x^{101} + x^{99} + x^{98} + x^{96} \\
&\quad + x^{95} + x^{94} + x^{93} + x^{91} + x^{90} + x^{88} + x^{87} + x^{86} + x^{85} + x^{83} + x^{82} \\
&\quad + x^{79} + x^{78} + x^{75} + x^{74} + x^{71} + x^{70} + x^{67} + x^{64} + x^{61} + x^{58} + x^{55} \\
&\quad + x^{54} + x^{51} + x^{42} + x^{39} + x^{38} + x^{35} + x^{34} + x^{32} + x^{31} + x^{30} + x^{29} \\
&\quad + x^{27}, \\
p_6(x) &= x^{123} + x^{122} + x^{120} + x^{119} + x^{118} + x^{116} + x^{113} + x^{112} + x^{110} + x^{109} \\
&\quad + x^{108} + x^{106} + x^{105} + x^{104} + x^{103} + x^{101} + x^{99} + x^{97} + x^{95} + x^{92} \\
&\quad + x^{87} + x^{86} + x^{85} + x^{82} + x^{79} + x^{78} + x^{76} + x^{69} + x^{68} + x^{67} + x^{64} \\
&\quad + x^{63} + x^{62} + x^{61} + x^{59} + x^{57} + x^{55} + x^{52} + x^{45} + x^{44} + x^{43} + x^{40} \\
&\quad + x^{39} + x^{38} + x^{37} + x^{34} + x^{33} + x^{32} + x^{31} + x^{28} + x^{27} + x^{26} + x^{25} \\
&\quad + x^{23} + x^{21} + x^{18}, \\
p_9(x) &= x^{126} + x^{124} + x^{123} + x^{121} + x^{116} + x^{113} + x^{110} + x^{107} + x^{106} + x^{104} \\
&\quad + x^{103} + x^{102} + x^{101} + x^{99} + x^{98} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} \\
&\quad + x^{89} + x^{88} + x^{85} + x^{82} + x^{80} + x^{79} + x^{78} + x^{77} + x^{75} + x^{72} + x^{69} \\
&\quad + x^{66} + x^{63} + x^{60} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{51} + x^{46} \\
&\quad + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{37} + x^{34} + x^{32} + x^{31} + x^{30} \\
&\quad + x^{29} + x^{27} + x^{22} + x^{19} + x^{18} + x^{15} + x^{14} + x^{12} + x^{11} + x^{9}, \\
p_{12}(x) &= x^{129} + x^{128} + x^{127} + x^{125} + x^{123} + x^{120} + x^{117} + x^{116} + x^{115} + x^{112} \\
&\quad + x^{81} + x^{80} + x^{79} + x^{77} + x^{75} + x^{72} + x^{69} + x^{68} + x^{67} + x^{64} + x^{49} \\
&\quad + x^{48} + x^{47} + x^{45} + x^{43} + x^{40} + x^{37} + x^{36} + x^{35} + x^{32} + x^{17} + x^{16} \\
&\quad + x^{15} + x^{13} + x^{11} + x^{8} + x^{5} + x^{4} + x^{3} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 1, 1, 0, 0, 1, 1, 0, 1, 1, 1, 1, 1, 0].

For $\phi(W^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{122} + x^{120} + x^{112} + x^{110} + x^{106} + x^{104} + x^{102} + x^{100} + x^{96} \\
&\quad + x^{94} + x^{92} + x^{90} + x^{86} + x^{84} + x^{82} + x^{80} + x^{78} + x^{72} + x^{68} + x^{64} \\
&\quad + x^{58} + x^{52} + x^{50} + x^{42}, \\
p_3(x) &= x^{124} + x^{122} + x^{118} + x^{114} + x^{110} + x^{106} + x^{90} + x^{88} + x^{82} + x^{80} \\
&\quad + x^{78} + x^{74} + x^{68} + x^{64} + x^{58} + x^{56} + x^{52} + x^{48} + x^{46} + x^{40} + x^{38} \\
&\quad + x^{36} + x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{24} + x^{20} + x^{18},
\end{aligned}$$

$$\begin{aligned}
p_6(x) &= x^{124} + x^{122} + x^{114} + x^{110} + x^{108} + x^{106} + x^{104} + x^{100} + x^{98} + x^{94} \\
&\quad + x^{92} + x^{90} + x^{86} + x^{80} + x^{72} + x^{68} + x^{60} + x^{58} + x^{56} + x^{40} + x^{38} \\
&\quad + x^{34} + x^{24} + x^{20} + x^{18} + x^{8} + x^{4} + 1, \\
p_9(x) &= x^{124} + x^{122} + x^{118} + x^{116} + x^{112} + x^{108} + x^{102} + x^{96} + x^{94} + x^{92} \\
&\quad + x^{90} + x^{78} + x^{74} + x^{72} + x^{64} + x^{62} + x^{54} + x^{52} + x^{50} + x^{46} + x^{44} \\
&\quad + x^{40} + x^{34} + x^{26} + x^{16} + x^{6}, \\
p_{12}(x) &= x^{126} + x^{124} + x^{120} + x^{118} + x^{116} + x^{112} + x^{78} + x^{76} + x^{72} + x^{70} \\
&\quad + x^{68} + x^{64} + x^{46} + x^{44} + x^{40} + x^{38} + x^{36} + x^{32} + x^{14} + x^{12} + x^{8} \\
&\quad + x^{6} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0].

For $\phi(M^o, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{129} + x^{128} + x^{126} + x^{124} + x^{123} + x^{121} + x^{118} + x^{117} + x^{114} + x^{112} \\
&\quad + x^{111} + x^{108} + x^{107} + x^{104} + x^{103} + x^{100} + x^{97} + x^{95} + x^{94} + x^{93} \\
&\quad + x^{91} + x^{90} + x^{88} + x^{85} + x^{84} + x^{83} + x^{82} + x^{80} + x^{79} + x^{77} + x^{76} \\
&\quad + x^{74} + x^{73} + x^{71} + x^{70} + x^{68} + x^{67} + x^{66} + x^{65} + x^{63} + x^{62} + x^{61} \\
&\quad + x^{60} + x^{59} + x^{57} + x^{56} + x^{55} + x^{53} + x^{52} + x^{51} + x^{48} + x^{46} + x^{45} \\
&\quad + x^{43} + x^{39} + x^{38} + x^{36}, \\
p_3(x) &= x^{125} + x^{124} + x^{122} + x^{120} + x^{119} + x^{117} + x^{114} + x^{113} + x^{112} + x^{111} \\
&\quad + x^{109} + x^{107} + x^{106} + x^{105} + x^{103} + x^{98} + x^{93} + x^{89} + x^{88} + x^{86} \\
&\quad + x^{85} + x^{83} + x^{82} + x^{79} + x^{77} + x^{76} + x^{75} + x^{74} + x^{71} + x^{69} + x^{68} \\
&\quad + x^{66} + x^{64} + x^{62} + x^{59} + x^{57} + x^{56} + x^{53} + x^{51} + x^{49} + x^{46} + x^{42} \\
&\quad + x^{41} + x^{37} + x^{35} + x^{34} + x^{33} + x^{30} + x^{28} + x^{27} + x^{26} + x^{24}, \\
p_6(x) &= x^{127} + x^{126} + x^{125} + x^{124} + x^{121} + x^{120} + x^{118} + x^{113} + x^{112} + x^{111} \\
&\quad + x^{110} + x^{108} + x^{107} + x^{106} + x^{104} + x^{103} + x^{102} + x^{100} + x^{97} + x^{96} \\
&\quad + x^{94} + x^{93} + x^{90} + x^{89} + x^{87} + x^{86} + x^{85} + x^{84} + x^{82} + x^{81} + x^{80} \\
&\quad + x^{79} + x^{78} + x^{74} + x^{73} + x^{71} + x^{70} + x^{68} + x^{67} + x^{61} + x^{59} + x^{56} \\
&\quad + x^{55} + x^{54} + x^{53} + x^{52} + x^{50} + x^{49} + x^{48} + x^{47} + x^{39} + x^{37} + x^{36} \\
&\quad + x^{35} + x^{32} + x^{27} + x^{25} + x^{21} + x^{20} + x^{18} + x^{17} + x^{14} + x^{13} + x^{12} \\
&\quad + x^{11} + x^{8} + x^{5} + x^{4} + x^{3} + 1, \\
p_9(x) &= x^{125} + x^{124} + x^{123} + x^{120} + x^{113} + x^{112} + x^{111} + x^{109} + x^{107} + x^{104} \\
&\quad + x^{97} + x^{96} + x^{95} + x^{93} + x^{91} + x^{88} + x^{81} + x^{80} + x^{79} + x^{76} + x^{45} \\
&\quad + x^{44} + x^{43} + x^{40} + x^{33} + x^{32} + x^{31} + x^{29} + x^{27} + x^{24} + x^{17} + x^{16} \\
&\quad + x^{15} + x^{12}, \\
p_{12}(x) &= x^{129} + x^{128} + x^{127} + x^{124} + x^{117} + x^{116} + x^{115} + x^{113} + x^{111} + x^{109} \\
&\quad + x^{107} + x^{104} + x^{97} + x^{96} + x^{95} + x^{93} + x^{91} + x^{88} + x^{77} + x^{76} + x^{75}
\end{aligned}$$

$$+ x^{72} + x^{69} + x^{68} + x^{67} + x^{64} + x^{49} + x^{48} + x^{47} + x^{44} + x^{37} + x^{36} + x^{35} + x^{33} + x^{31} + x^{29} + x^{27} + x^{24} + x^{13} + x^{12} + x^{11} + x^{8} + x^{5} + x^{4} + x^{3} + 1,$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 0, 0, 1, 1].

For $\phi(M^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{132} + x^{129} + x^{126} + x^{125} + x^{124} + x^{122} + x^{121} + x^{118} + x^{113} + x^{109} \\
&\quad + x^{108} + x^{107} + x^{106} + x^{105} + x^{104} + x^{103} + x^{102} + x^{100} + x^{97} + x^{95} \\
&\quad + x^{92} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{83} + x^{80} + x^{79} + x^{77} + x^{76} \\
&\quad + x^{75} + x^{74} + x^{72} + x^{71} + x^{70} + x^{69} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} \\
&\quad + x^{57} + x^{55} + x^{54} + x^{52} + x^{49} + x^{48} + x^{47} + x^{40} + x^{39}, \\
p_3(x) &= x^{123} + x^{122} + x^{121} + x^{120} + x^{118} + x^{115} + x^{114} + x^{113} + x^{111} + x^{108} \\
&\quad + x^{107} + x^{106} + x^{104} + x^{101} + x^{100} + x^{98} + x^{96} + x^{93} + x^{92} + x^{90} \\
&\quad + x^{88} + x^{85} + x^{84} + x^{83} + x^{81} + x^{78} + x^{77} + x^{76} + x^{74} + x^{71} + x^{70} \\
&\quad + x^{69} + x^{67} + x^{64} + x^{63} + x^{62} + x^{60} + x^{59} + x^{58} + x^{55} + x^{54} + x^{53} \\
&\quad + x^{50} + x^{49} + x^{47} + x^{44} + x^{41} + x^{38} + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} \\
&\quad + x^{31} + x^{29} + x^{27}, \\
p_6(x) &= x^{126} + x^{124} + x^{116} + x^{112} + x^{110} + x^{106} + x^{104} + x^{100} + x^{96} + x^{90} \\
&\quad + x^{88} + x^{84} + x^{82} + x^{72} + x^{70} + x^{68} + x^{66} + x^{58} + x^{54} + x^{52} + x^{50} \\
&\quad + x^{48} + x^{46} + x^{38} + x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{18}, \\
p_9(x) &= x^{129} + x^{128} + x^{124} + x^{122} + x^{121} + x^{119} + x^{118} + x^{116} + x^{115} + x^{113} \\
&\quad + x^{112} + x^{110} + x^{108} + x^{107} + x^{106} + x^{103} + x^{102} + x^{100} + x^{99} + x^{97} \\
&\quad + x^{96} + x^{94} + x^{92} + x^{91} + x^{90} + x^{87} + x^{86} + x^{84} + x^{83} + x^{78} + x^{75} \\
&\quad + x^{73} + x^{49} + x^{48} + x^{44} + x^{42} + x^{41} + x^{39} + x^{38} + x^{36} + x^{35} + x^{33} \\
&\quad + x^{32} + x^{30} + x^{28} + x^{27} + x^{26} + x^{23} + x^{22} + x^{20} + x^{19} + x^{14} + x^{11} \\
&\quad + x^{9}, \\
p_{12}(x) &= x^{132} + x^{104} + x^{100} + x^{96} + x^{88} + x^{80} + x^{72} + x^{68} + x^{64} + x^{52} + x^{24} \\
&\quad + x^{16} + x^{8} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 1, 0, 1, 1, 0, 1].

For $\phi(M^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{132} + x^{129} + x^{125} + x^{120} + x^{118} + x^{117} + x^{114} + x^{113} + x^{111} + x^{110} \\
&\quad + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{94} + x^{91} + x^{89} \\
&\quad + x^{88} + x^{84} + x^{82} + x^{81} + x^{78} + x^{76} + x^{74} + x^{70} + x^{69} + x^{65} + x^{64} \\
&\quad + x^{63} + x^{60} + x^{58} + x^{57} + x^{55} + x^{51} + x^{48} + x^{47} + x^{46} + x^{43} + x^{40} \\
&\quad + x^{39}, \\
p_3(x) &= x^{123} + x^{122} + x^{121} + x^{120} + x^{118} + x^{112} + x^{109} + x^{107} + x^{83} + x^{82}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{81} + x^{80} + x^{78} + x^{75} + x^{74} + x^{73} + x^{70} + x^{69} + x^{66} + x^{65} + x^{62} \\
&+ x^{61} + x^{58} + x^{57} + x^{54} + x^{53} + x^{51} + x^{48} + x^{45} + x^{42} + x^{41} + x^{40} \\
&+ x^{38} + x^{32} + x^{29} + x^{27}, \\
p_6(x) = {}& x^{126} + x^{124} + x^{118} + x^{114} + x^{106} + x^{104} + x^{96} + x^{92} + x^{90} + x^{80} \\
&+ x^{78} + x^{76} + x^{72} + x^{66} + x^{64} + x^{62} + x^{56} + x^{52} + x^{48} + x^{42} + x^{40} \\
&+ x^{36} + x^{34} + x^{30} + x^{28} + x^{26} + x^{22} + x^{18}, \\
p_9(x) = {}& x^{129} + x^{128} + x^{124} + x^{122} + x^{120} + x^{119} + x^{118} + x^{117} + x^{116} + x^{115} \\
&+ x^{114} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} + x^{107} + x^{106} \\
&+ x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} \\
&+ x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} \\
&+ x^{84} + x^{83} + x^{82} + x^{79} + x^{78} + x^{77} + x^{75} + x^{73} + x^{49} + x^{48} + x^{44} \\
&+ x^{42} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} \\
&+ x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} \\
&+ x^{19} + x^{18} + x^{15} + x^{14} + x^{13} + x^{11} + x^{9}, \\
p_{12}(x) = {}& x^{132} + x^{124} + x^{120} + x^{116} + x^{108} + x^{104} + x^{100} + x^{96} + x^{92} + x^{88} \\
&+ x^{80} + x^{64} + x^{52} + x^{44} + x^{40} + x^{36} + x^{28} + x^{24} + x^{16} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 1, 0, 0, 1, 1, 1, 1, 1, 1, 0, 0, 1].

For $\phi(M^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) = {}& x^{129} + x^{128} + x^{126} + x^{125} + x^{123} + x^{122} + x^{121} + x^{118} + x^{117} + x^{116} \\
&+ x^{112} + x^{110} + x^{107} + x^{104} + x^{101} + x^{100} + x^{98} + x^{97} + x^{95} + x^{94} \\
&+ x^{93} + x^{91} + x^{90} + x^{88} + x^{85} + x^{83} + x^{82} + x^{81} + x^{80} + x^{78} + x^{76} \\
&+ x^{74} + x^{72} + x^{69} + x^{67} + x^{66} + x^{63} + x^{60} + x^{57} + x^{56} + x^{54} + x^{53} \\
&+ x^{51} + x^{50} + x^{48} + x^{45} + x^{42}, \\
p_3(x) = {}& x^{125} + x^{124} + x^{122} + x^{121} + x^{119} + x^{118} + x^{116} + x^{113} + x^{112} + x^{111} \\
&+ x^{109} + x^{103} + x^{102} + x^{101} + x^{100} + x^{97} + x^{96} + x^{94} + x^{92} + x^{90} \\
&+ x^{84} + x^{83} + x^{79} + x^{77} + x^{76} + x^{75} + x^{74} + x^{71} + x^{69} + x^{68} + x^{67} \\
&+ x^{65} + x^{64} + x^{61} + x^{60} + x^{57} + x^{53} + x^{50} + x^{49} + x^{48} + x^{46} + x^{43} \\
&+ x^{41} + x^{40} + x^{37} + x^{35} + x^{34} + x^{32} + x^{29} + x^{26}, \\
p_6(x) = {}& x^{127} + x^{126} + x^{122} + x^{113} + x^{112} + x^{109} + x^{108} + x^{107} + x^{104} + x^{103} \\
&+ x^{102} + x^{100} + x^{97} + x^{96} + x^{94} + x^{92} + x^{91} + x^{90} + x^{87} + x^{85} + x^{84} \\
&+ x^{83} + x^{81} + x^{76} + x^{74} + x^{73} + x^{71} + x^{70} + x^{69} + x^{64} + x^{63} + x^{62} \\
&+ x^{55} + x^{54} + x^{51} + x^{50} + x^{49} + x^{47} + x^{45} + x^{44} + x^{43} + x^{40} + x^{38} \\
&+ x^{37} + x^{34} + x^{26} + x^{23} + x^{20} + x^{16} + x^{15} + x^{14} + x^{13} + x^{11} + x^{8} \\
&+ x^{5} + x^{4} + x^{3} + 1, \\
p_9(x) = {}& x^{125} + x^{124} + x^{123} + x^{120} + x^{113} + x^{112} + x^{111} + x^{109} + x^{107} + x^{104}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{97} + x^{96} + x^{95} + x^{93} + x^{91} + x^{88} + x^{81} + x^{80} + x^{79} + x^{76} + x^{45} \\
&+ x^{44} + x^{43} + x^{40} + x^{33} + x^{32} + x^{31} + x^{29} + x^{27} + x^{24} + x^{17} + x^{16} \\
&+ x^{15} + x^{12}, \\
p_{12}(x) &= x^{129} + x^{128} + x^{127} + x^{124} + x^{117} + x^{116} + x^{115} + x^{113} + x^{111} + x^{109} \\
&+ x^{107} + x^{104} + x^{97} + x^{96} + x^{95} + x^{93} + x^{91} + x^{88} + x^{77} + x^{76} + x^{75} \\
&+ x^{72} + x^{69} + x^{68} + x^{67} + x^{64} + x^{49} + x^{48} + x^{47} + x^{44} + x^{37} + x^{36} \\
&+ x^{35} + x^{33} + x^{31} + x^{29} + x^{27} + x^{24} + x^{13} + x^{12} + x^{11} + x^{8} + x^{5} + x^{4} \\
&+ x^{3} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 0, 0, 0, 1, 1].

For $\phi(W^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{129} + x^{128} + x^{126} + x^{124} + x^{123} + x^{121} + x^{118} + x^{117} + x^{114} + x^{112} \\
&+ x^{111} + x^{108} + x^{107} + x^{104} + x^{103} + x^{100} + x^{97} + x^{95} + x^{94} + x^{93} \\
&+ x^{91} + x^{90} + x^{88} + x^{85} + x^{84} + x^{83} + x^{82} + x^{80} + x^{79} + x^{77} + x^{76} \\
&+ x^{74} + x^{73} + x^{71} + x^{70} + x^{68} + x^{67} + x^{66} + x^{65} + x^{63} + x^{62} + x^{61} \\
&+ x^{60} + x^{59} + x^{57} + x^{56} + x^{55} + x^{53} + x^{52} + x^{51} + x^{48} + x^{46} + x^{45} \\
&+ x^{43} + x^{39} + x^{38} + x^{36}, \\
p_3(x) &= x^{125} + x^{124} + x^{122} + x^{120} + x^{119} + x^{117} + x^{114} + x^{113} + x^{112} + x^{111} \\
&+ x^{109} + x^{107} + x^{106} + x^{105} + x^{103} + x^{98} + x^{93} + x^{89} + x^{88} + x^{86} \\
&+ x^{85} + x^{83} + x^{82} + x^{79} + x^{77} + x^{76} + x^{75} + x^{74} + x^{71} + x^{69} + x^{68} \\
&+ x^{66} + x^{64} + x^{62} + x^{59} + x^{57} + x^{56} + x^{53} + x^{51} + x^{49} + x^{46} + x^{42} \\
&+ x^{41} + x^{37} + x^{35} + x^{34} + x^{33} + x^{30} + x^{28} + x^{27} + x^{26} + x^{24}, \\
p_6(x) &= x^{127} + x^{126} + x^{125} + x^{124} + x^{121} + x^{120} + x^{118} + x^{113} + x^{112} + x^{111} \\
&+ x^{110} + x^{108} + x^{107} + x^{106} + x^{104} + x^{103} + x^{102} + x^{100} + x^{97} + x^{96} \\
&+ x^{94} + x^{93} + x^{90} + x^{89} + x^{87} + x^{86} + x^{85} + x^{84} + x^{82} + x^{81} + x^{80} \\
&+ x^{79} + x^{78} + x^{74} + x^{73} + x^{71} + x^{70} + x^{68} + x^{67} + x^{61} + x^{59} + x^{56} \\
&+ x^{55} + x^{54} + x^{53} + x^{52} + x^{50} + x^{49} + x^{48} + x^{47} + x^{39} + x^{37} + x^{36} \\
&+ x^{35} + x^{32} + x^{27} + x^{25} + x^{21} + x^{20} + x^{18} + x^{17} + x^{14} + x^{13} + x^{12} \\
&+ x^{11} + x^{8} + x^{5} + x^{4} + x^{3} + 1, \\
p_9(x) &= x^{125} + x^{124} + x^{123} + x^{120} + x^{113} + x^{112} + x^{111} + x^{109} + x^{107} + x^{104} \\
&+ x^{97} + x^{96} + x^{95} + x^{93} + x^{91} + x^{88} + x^{81} + x^{80} + x^{79} + x^{76} + x^{45} \\
&+ x^{44} + x^{43} + x^{40} + x^{33} + x^{32} + x^{31} + x^{29} + x^{27} + x^{24} + x^{17} + x^{16} \\
&+ x^{15} + x^{12}, \\
p_{12}(x) &= x^{129} + x^{128} + x^{127} + x^{124} + x^{117} + x^{116} + x^{115} + x^{113} + x^{111} + x^{109} \\
&+ x^{107} + x^{104} + x^{97} + x^{96} + x^{95} + x^{93} + x^{91} + x^{88} + x^{77} + x^{76} + x^{75} \\
&+ x^{72} + x^{69} + x^{68} + x^{67} + x^{64} + x^{49} + x^{48} + x^{47} + x^{44} + x^{37} + x^{36}
\end{aligned}
$$

$$+ x^{35} + x^{33} + x^{31} + x^{29} + x^{27} + x^{24} + x^{13} + x^{12} + x^{11} + x^{8} + x^{5} + x^{4} + x^{3} + 1,$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 0, 0, 1, 1].

For $\phi(W^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{132} + x^{129} + x^{125} + x^{120} + x^{118} + x^{117} + x^{114} + x^{113} + x^{111} + x^{110} \\
&\quad + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{94} + x^{91} + x^{89} \\
&\quad + x^{88} + x^{84} + x^{82} + x^{81} + x^{78} + x^{76} + x^{74} + x^{70} + x^{69} + x^{65} + x^{64} \\
&\quad + x^{63} + x^{60} + x^{58} + x^{57} + x^{55} + x^{51} + x^{48} + x^{47} + x^{46} + x^{43} + x^{40} \\
&\quad + x^{39}, \\
p_3(x) &= x^{123} + x^{122} + x^{121} + x^{120} + x^{118} + x^{112} + x^{109} + x^{107} + x^{83} + x^{82} \\
&\quad + x^{81} + x^{80} + x^{78} + x^{75} + x^{74} + x^{73} + x^{70} + x^{69} + x^{66} + x^{65} + x^{62} \\
&\quad + x^{61} + x^{58} + x^{57} + x^{54} + x^{53} + x^{51} + x^{48} + x^{45} + x^{42} + x^{41} + x^{40} \\
&\quad + x^{38} + x^{32} + x^{29} + x^{27}, \\
p_6(x) &= x^{126} + x^{124} + x^{118} + x^{114} + x^{106} + x^{104} + x^{96} + x^{92} + x^{90} + x^{80} \\
&\quad + x^{78} + x^{76} + x^{72} + x^{66} + x^{64} + x^{62} + x^{56} + x^{52} + x^{48} + x^{42} + x^{40} \\
&\quad + x^{36} + x^{34} + x^{30} + x^{28} + x^{26} + x^{22} + x^{18}, \\
p_9(x) &= x^{129} + x^{128} + x^{124} + x^{122} + x^{120} + x^{119} + x^{118} + x^{117} + x^{116} + x^{115} \\
&\quad + x^{114} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} + x^{107} + x^{106} \\
&\quad + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} \\
&\quad + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} \\
&\quad + x^{84} + x^{83} + x^{82} + x^{79} + x^{78} + x^{77} + x^{75} + x^{73} + x^{49} + x^{48} + x^{44} \\
&\quad + x^{42} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} \\
&\quad + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} \\
&\quad + x^{19} + x^{18} + x^{15} + x^{14} + x^{13} + x^{11} + x^{9}, \\
p_{12}(x) &= x^{132} + x^{124} + x^{120} + x^{116} + x^{108} + x^{104} + x^{100} + x^{96} + x^{92} + x^{88} \\
&\quad + x^{80} + x^{64} + x^{52} + x^{44} + x^{40} + x^{36} + x^{28} + x^{24} + x^{16} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 1, 1, 0, 0, 1, 1, 1, 1, 1, 1, 0, 0, 1].

For $\phi(W^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{132} + x^{129} + x^{126} + x^{125} + x^{124} + x^{122} + x^{121} + x^{118} + x^{113} + x^{109} \\
&\quad + x^{108} + x^{107} + x^{106} + x^{105} + x^{104} + x^{103} + x^{102} + x^{100} + x^{97} + x^{95} \\
&\quad + x^{92} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{83} + x^{80} + x^{79} + x^{77} + x^{76} \\
&\quad + x^{75} + x^{74} + x^{72} + x^{71} + x^{70} + x^{69} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} \\
&\quad + x^{57} + x^{55} + x^{54} + x^{52} + x^{49} + x^{48} + x^{47} + x^{40} + x^{39}, \\
p_3(x) &= x^{123} + x^{122} + x^{121} + x^{120} + x^{118} + x^{115} + x^{114} + x^{113} + x^{111} + x^{108} \\
&\quad + x^{107} + x^{106} + x^{104} + x^{101} + x^{100} + x^{98} + x^{96} + x^{93} + x^{92} + x^{90}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{88} + x^{85} + x^{84} + x^{83} + x^{81} + x^{78} + x^{77} + x^{76} + x^{74} + x^{71} + x^{70}\\
&+ x^{69} + x^{67} + x^{64} + x^{63} + x^{62} + x^{60} + x^{59} + x^{58} + x^{55} + x^{54} + x^{53}\\
&+ x^{50} + x^{49} + x^{47} + x^{44} + x^{41} + x^{38} + x^{37} + x^{36} + x^{35} + x^{33} + x^{32}\\
&+ x^{31} + x^{29} + x^{27},\\
p_6(x) &= x^{126} + x^{124} + x^{116} + x^{112} + x^{110} + x^{106} + x^{104} + x^{100} + x^{96} + x^{90}\\
&+ x^{88} + x^{84} + x^{82} + x^{72} + x^{70} + x^{68} + x^{66} + x^{58} + x^{54} + x^{52} + x^{50}\\
&+ x^{48} + x^{46} + x^{38} + x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{18},\\
p_9(x) &= x^{129} + x^{128} + x^{124} + x^{122} + x^{121} + x^{119} + x^{118} + x^{116} + x^{115} + x^{113}\\
&+ x^{112} + x^{110} + x^{108} + x^{107} + x^{106} + x^{103} + x^{102} + x^{100} + x^{99} + x^{97}\\
&+ x^{96} + x^{94} + x^{92} + x^{91} + x^{90} + x^{87} + x^{86} + x^{84} + x^{83} + x^{78} + x^{75}\\
&+ x^{73} + x^{49} + x^{48} + x^{44} + x^{42} + x^{41} + x^{39} + x^{38} + x^{36} + x^{35} + x^{33}\\
&+ x^{32} + x^{30} + x^{28} + x^{27} + x^{26} + x^{23} + x^{22} + x^{20} + x^{19} + x^{14} + x^{11}\\
&+ x^{9},\\
p_{12}(x) &= x^{132} + x^{104} + x^{100} + x^{96} + x^{88} + x^{80} + x^{72} + x^{68} + x^{64} + x^{52} + x^{24}\\
&+ x^{16} + x^{8} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 1, 0, 1, 1, 0, 1].

For $\phi(W^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{129} + x^{128} + x^{126} + x^{125} + x^{123} + x^{122} + x^{121} + x^{118} + x^{117} + x^{116}\\
&+ x^{112} + x^{110} + x^{107} + x^{104} + x^{101} + x^{100} + x^{98} + x^{97} + x^{95} + x^{94}\\
&+ x^{93} + x^{91} + x^{90} + x^{88} + x^{85} + x^{83} + x^{82} + x^{81} + x^{80} + x^{78} + x^{76}\\
&+ x^{74} + x^{72} + x^{69} + x^{67} + x^{66} + x^{63} + x^{60} + x^{57} + x^{56} + x^{54} + x^{53}\\
&+ x^{51} + x^{50} + x^{48} + x^{45} + x^{42},\\
p_3(x) &= x^{125} + x^{124} + x^{122} + x^{121} + x^{119} + x^{118} + x^{116} + x^{113} + x^{112} + x^{111}\\
&+ x^{109} + x^{103} + x^{102} + x^{101} + x^{100} + x^{97} + x^{96} + x^{94} + x^{92} + x^{90}\\
&+ x^{84} + x^{83} + x^{79} + x^{77} + x^{76} + x^{75} + x^{74} + x^{71} + x^{69} + x^{68} + x^{67}\\
&+ x^{65} + x^{64} + x^{61} + x^{60} + x^{57} + x^{53} + x^{50} + x^{49} + x^{48} + x^{46} + x^{43}\\
&+ x^{41} + x^{40} + x^{37} + x^{35} + x^{34} + x^{32} + x^{29} + x^{26},\\
p_6(x) &= x^{127} + x^{126} + x^{122} + x^{113} + x^{112} + x^{109} + x^{108} + x^{107} + x^{104} + x^{103}\\
&+ x^{102} + x^{100} + x^{97} + x^{96} + x^{94} + x^{92} + x^{91} + x^{90} + x^{87} + x^{85} + x^{84}\\
&+ x^{83} + x^{81} + x^{76} + x^{74} + x^{73} + x^{71} + x^{70} + x^{69} + x^{64} + x^{63} + x^{62}\\
&+ x^{55} + x^{54} + x^{51} + x^{50} + x^{49} + x^{47} + x^{45} + x^{44} + x^{43} + x^{40} + x^{38}\\
&+ x^{37} + x^{34} + x^{26} + x^{23} + x^{20} + x^{16} + x^{15} + x^{14} + x^{13} + x^{11} + x^{8}\\
&+ x^{5} + x^{4} + x^{3} + 1,\\
p_9(x) &= x^{125} + x^{124} + x^{123} + x^{120} + x^{113} + x^{112} + x^{111} + x^{109} + x^{107} + x^{104}\\
&+ x^{97} + x^{96} + x^{95} + x^{93} + x^{91} + x^{88} + x^{81} + x^{80} + x^{79} + x^{76} + x^{45}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{44} + x^{43} + x^{40} + x^{33} + x^{32} + x^{31} + x^{29} + x^{27} + x^{24} + x^{17} + x^{16} \\
&+ x^{15} + x^{12}, \\
p_{12}(x) = {}& x^{129} + x^{128} + x^{127} + x^{124} + x^{117} + x^{116} + x^{115} + x^{113} + x^{111} + x^{109} \\
&+ x^{107} + x^{104} + x^{97} + x^{96} + x^{95} + x^{93} + x^{91} + x^{88} + x^{77} + x^{76} + x^{75} \\
&+ x^{72} + x^{69} + x^{68} + x^{67} + x^{64} + x^{49} + x^{48} + x^{47} + x^{44} + x^{37} + x^{36} \\
&+ x^{35} + x^{33} + x^{31} + x^{29} + x^{27} + x^{24} + x^{13} + x^{12} + x^{11} + x^{8} + x^{5} + x^{4} \\
&+ x^{3} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 0, 0, 0, 1, 1].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ ($\Lambda(n) := [\delta(n, 0), \delta(n, 1)]$):

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	266	[150, 165]	532	[285, 742]	798	[732, 930]
1	[3, 4]	267	[440, 167]	533	[310, 743]	799	[20, 913]
2	[5, 6]	268	[132, 131]	534	[266, 744]	800	[368, 305]
3	[7, 8]	269	[44, 441]	535	[745, 746]	801	[556, 738]
4	[9, 10]	270	[168, 442]	536	[705, 669]	802	[931, 932]
5	[11, 12]	271	[443, 444]	537	[596, 747]	803	[933, 68]
6	[13, 14]	272	[445, 446]	538	[748, 715]	804	[691, 934]
7	[15, 16]	273	[447, 402]	539	[728, 749]	805	[553, 365]
8	[17, 18]	274	[448, 449]	540	[723, 628]	806	[688, 935]
9	[19, 20]	275	[450, 451]	541	[571, 635]	807	[252, 936]
10	[21, 22]	276	[452, 453]	542	[750, 751]	808	[752, 83]
11	[23, 24]	277	[454, 205]	543	[642, 699]	809	[20, 299]
12	[25, 26]	278	[174, 455]	544	[752, 753]	810	[734, 937]
13	[27, 28]	279	[139, 456]	545	[705, 558]	811	[757, 609]
14	[29, 30]	280	[457, 172]	546	[754, 689]	812	[321, 748]
15	[31, 32]	281	[458, 168]	547	[658, 755]	813	[919, 654]
16	[33, 34]	282	[459, 408]	548	[272, 756]	814	[938, 939]
17	[35, 36]	283	[460, 461]	549	[757, 732]	815	[940, 243]
18	[37, 38]	284	[129, 156]	550	[758, 759]	816	[294, 262]
19	[39, 40]	285	[462, 463]	551	[760, 481]	817	[941, 117]
20	[41, 42]	286	[464, 465]	552	[761, 762]	818	[942, 359]
21	[43, 44]	287	[145, 133]	553	[196, 8]	819	[943, 235]
22	[45, 46]	288	[466, 437]	554	[143, 763]	820	[944, 232]
23	[47, 48]	289	[41, 423]	555	[450, 764]	821	[945, 946]
24	[49, 50]	290	[467, 468]	556	[478, 765]	822	[947, 635]
25	[51, 52]	291	[469, 470]	557	[440, 403]	823	[933, 110]
26	[53, 54]	292	[471, 472]	558	[162, 476]	824	[644, 649]
27	[55, 56]	293	[441, 166]	559	[513, 128]	825	[222, 948]
28	[57, 58]	294	[164, 43]	560	[766, 547]	826	[949, 950]
29	[57, 59]	295	[149, 43]	561	[767, 400]	827	[331, 559]

30	[60, 61]	296	[44, 206]	562	[507, 768]	828	[327, 951]
31	[62, 63]	297	[165, 441]	563	[769, 475]	829	[952, 924]
32	[64, 65]	298	[384, 392]	564	[770, 771]	830	[953, 954]
33	[66, 67]	299	[163, 473]	565	[772, 773]	831	[690, 944]
34	[68, 69]	300	[474, 475]	566	[774, 464]	832	[245, 93]
35	[70, 71]	301	[476, 477]	567	[775, 505]	833	[955, 956]
36	[72, 73]	302	[478, 479]	568	[514, 776]	834	[957, 562]
37	[74, 75]	303	[165, 206]	569	[415, 777]	835	[958, 272]
38	[76, 77]	304	[480, 376]	570	[778, 543]	836	[725, 742]
39	[23, 78]	305	[481, 482]	571	[779, 780]	837	[959, 743]
40	[79, 77]	306	[483, 484]	572	[781, 782]	838	[591, 934]
41	[80, 81]	307	[214, 485]	573	[38, 783]	839	[960, 961]
42	[82, 83]	308	[486, 487]	574	[193, 784]	840	[954, 663]
43	[84, 85]	309	[438, 468]	575	[785, 786]	841	[598, 115]
44	[86, 87]	310	[488, 158]	576	[182, 787]	842	[962, 747]
45	[88, 89]	311	[489, 490]	577	[191, 498]	843	[606, 347]
46	[90, 91]	312	[491, 481]	578	[142, 788]	844	[566, 646]
47	[92, 93]	313	[471, 492]	579	[789, 422]	845	[720, 297]
48	[94, 95]	314	[493, 494]	580	[790, 791]	846	[963, 602]
49	[96, 97]	315	[183, 495]	581	[144, 792]	847	[131, 637]
50	[98, 99]	316	[197, 496]	582	[793, 794]	848	[719, 245]
51	[100, 101]	317	[497, 498]	583	[775, 57]	849	[964, 626]
52	[102, 103]	318	[186, 465]	584	[795, 796]	850	[965, 75]
53	[104, 105]	319	[499, 496]	585	[788, 797]	851	[669, 966]
54	[106, 107]	320	[189, 500]	586	[470, 548]	852	[244, 962]
55	[108, 109]	321	[496, 182]	587	[370, 798]	853	[623, 703]
56	[110, 111]	322	[389, 501]	588	[527, 799]	854	[967, 968]
57	[112, 113]	323	[502, 389]	589	[49, 800]	855	[969, 227]
58	[114, 115]	324	[469, 32]	590	[476, 545]	856	[965, 970]
59	[92, 116]	325	[141, 503]	591	[801, 802]	857	[656, 296]
60	[117, 118]	326	[504, 505]	592	[176, 803]	858	[605, 596]
61	[119, 120]	327	[457, 506]	593	[35, 8]	859	[971, 972]
62	[121, 122]	328	[507, 508]	594	[195, 482]	860	[973, 974]
63	[123, 124]	329	[499, 181]	595	[535, 804]	861	[701, 975]
64	[125, 126]	330	[509, 510]	596	[42, 770]	862	[976, 355]
65	[127, 128]	331	[511, 189]	597	[178, 805]	863	[977, 686]
66	[129, 130]	332	[198, 512]	598	[51, 806]	864	[978, 86]
67	[131, 132]	333	[513, 418]	599	[441, 440]	865	[630, 979]
68	[133, 134]	334	[514, 124]	600	[807, 808]	866	[630, 980]
69	[135, 136]	335	[515, 516]	601	[60, 168]	867	[70, 974]
70	[137, 138]	336	[32, 517]	602	[175, 809]	868	[960, 357]
71	[139, 140]	337	[36, 122]	603	[421, 810]	869	[324, 981]
72	[141, 142]	338	[518, 519]	604	[200, 782]	870	[73, 608]
73	[143, 144]	339	[520, 374]	605	[146, 212]	871	[982, 983]

74	[145, 146]	340	[521, 522]	606	[811, 812]	872	[4, 982]
75	[147, 148]	341	[412, 487]	607	[813, 786]	873	[938, 613]
76	[149, 150]	342	[386, 523]	608	[142, 515]	874	[709, 984]
77	[151, 152]	343	[394, 524]	609	[814, 370]	875	[567, 647]
78	[153, 154]	344	[525, 526]	610	[815, 816]	876	[552, 646]
79	[155, 156]	345	[9, 527]	611	[33, 817]	877	[985, 986]
80	[157, 158]	346	[528, 529]	612	[174, 818]	878	[987, 17]
81	[159, 160]	347	[530, 531]	613	[819, 378]	879	[336, 741]
82	[161, 162]	348	[496, 509]	614	[820, 484]	880	[988, 989]
83	[163, 54]	349	[380, 529]	615	[821, 382]	881	[16, 73]
84	[164, 150]	350	[399, 532]	616	[61, 169]	882	[990, 991]
85	[43, 165]	351	[533, 534]	617	[793, 822]	883	[602, 949]
86	[132, 132]	352	[535, 536]	618	[173, 823]	884	[992, 360]
87	[166, 167]	353	[61, 442]	619	[824, 765]	885	[607, 670]
88	[168, 169]	354	[537, 430]	620	[825, 420]	886	[926, 759]
89	[170, 140]	355	[538, 539]	621	[406, 826]	887	[235, 673]
90	[171, 172]	356	[161, 490]	622	[213, 771]	888	[967, 668]
91	[173, 174]	357	[467, 439]	623	[217, 827]	889	[993, 994]
92	[175, 176]	358	[540, 387]	624	[378, 415]	890	[639, 912]
93	[177, 178]	359	[48, 541]	625	[419, 828]	891	[352, 693]
94	[179, 180]	360	[542, 543]	626	[170, 456]	892	[995, 996]
95	[146, 134]	361	[544, 545]	627	[537, 452]	893	[997, 582]
96	[181, 182]	362	[430, 453]	628	[829, 764]	894	[956, 998]
97	[183, 184]	363	[181, 509]	629	[830, 408]	895	[999, 555]
98	[185, 186]	364	[546, 547]	630	[778, 831]	896	[954, 277]
99	[138, 187]	365	[32, 548]	631	[447, 832]	897	[27, 712]
100	[188, 126]	366	[8, 122]	632	[833, 834]	898	[947, 572]
101	[189, 190]	367	[194, 481]	633	[835, 836]	899	[333, 330]
102	[191, 192]	368	[549, 550]	634	[425, 373]	900	[342, 564]
103	[8, 193]	369	[551, 552]	635	[162, 544]	901	[1000, 1001]
104	[194, 195]	370	[553, 554]	636	[837, 838]	902	[220, 18]
105	[196, 36]	371	[80, 555]	637	[434, 46]	903	[567, 1002]
106	[197, 181]	372	[556, 557]	638	[424, 470]	904	[717, 350]
107	[198, 199]	373	[95, 558]	639	[549, 370]	905	[1003, 415]
108	[200, 201]	374	[559, 560]	640	[827, 50]	906	[801, 1004]
109	[153, 202]	375	[561, 562]	641	[839, 840]	907	[135, 841]
110	[177, 203]	376	[563, 564]	642	[789, 810]	908	[171, 506]
111	[204, 205]	377	[565, 279]	643	[210, 841]	909	[401, 832]
112	[206, 166]	378	[566, 567]	644	[823, 455]	910	[1005, 501]
113	[207, 208]	379	[568, 569]	645	[814, 550]	911	[1006, 492]
114	[209, 210]	380	[347, 570]	646	[481, 379]	912	[1007, 763]
115	[158, 211]	381	[571, 572]	647	[842, 843]	913	[40, 1008]
116	[133, 212]	382	[254, 554]	648	[812, 844]	914	[1009, 1010]
117	[42, 213]	383	[573, 574]	649	[845, 794]	915	[125, 1011]

118	[214, 215]	384	[575, 576]	650	[846, 38]	916	[147, 47]
119	[179, 216]	385	[577, 578]	651	[847, 783]	917	[462, 889]
120	[217, 49]	386	[230, 243]	652	[788, 516]	918	[483, 1012]
121	[218, 219]	387	[579, 580]	653	[39, 436]	919	[833, 449]
122	[220, 221]	388	[246, 581]	654	[486, 413]	920	[511, 494]
123	[222, 223]	389	[582, 583]	655	[848, 135]	921	[416, 1010]
124	[224, 225]	390	[584, 585]	656	[155, 130]	922	[209, 135]
125	[226, 227]	391	[332, 329]	657	[849, 208]	923	[436, 900]
126	[228, 229]	392	[586, 587]	658	[850, 826]	924	[1013, 539]
127	[230, 231]	393	[226, 289]	659	[544, 477]	925	[388, 1005]
128	[232, 233]	394	[578, 588]	660	[851, 852]	926	[1014, 817]
129	[234, 235]	395	[552, 567]	661	[827, 800]	927	[374, 1015]
130	[236, 237]	396	[231, 244]	662	[770, 853]	928	[176, 1016]
131	[131, 131]	397	[589, 590]	663	[210, 136]	929	[158, 1017]
132	[76, 238]	398	[591, 592]	664	[830, 839]	930	[138, 1018]
133	[239, 240]	399	[234, 593]	665	[854, 176]	931	[850, 407]
134	[241, 242]	400	[594, 595]	666	[53, 473]	932	[1019, 541]
135	[243, 244]	401	[596, 597]	667	[448, 834]	933	[1020, 52]
136	[245, 116]	402	[598, 265]	668	[855, 148]	934	[835, 791]
137	[246, 247]	403	[599, 266]	669	[781, 201]	935	[186, 1021]
138	[65, 248]	404	[561, 600]	670	[856, 512]	936	[812, 1022]
139	[249, 250]	405	[63, 601]	671	[445, 519]	937	[1019, 1023]
140	[251, 252]	406	[340, 602]	672	[515, 797]	938	[772, 374]
141	[253, 254]	407	[603, 604]	673	[427, 816]	939	[864, 1024]
142	[255, 256]	408	[231, 605]	674	[857, 858]	940	[884, 146]
143	[257, 258]	409	[606, 252]	675	[785, 859]	941	[1025, 802]
144	[250, 259]	410	[19, 298]	676	[494, 190]	942	[474, 1026]
145	[260, 261]	411	[253, 553]	677	[491, 195]	943	[904, 863]
146	[66, 262]	412	[607, 363]	678	[190, 499]	944	[530, 1027]
147	[263, 264]	413	[336, 608]	679	[860, 861]	945	[378, 1028]
148	[114, 265]	414	[609, 610]	680	[862, 184]	946	[795, 1029]
149	[266, 267]	415	[611, 224]	681	[137, 863]	947	[1025, 1004]
150	[268, 269]	416	[612, 613]	682	[864, 444]	948	[443, 1024]
151	[270, 271]	417	[586, 614]	683	[500, 197]	949	[452, 866]
152	[272, 273]	418	[358, 615]	684	[766, 865]	950	[790, 836]
153	[274, 275]	419	[616, 617]	685	[500, 499]	951	[807, 1030]
154	[276, 277]	420	[350, 618]	686	[862, 495]	952	[157, 1031]
155	[278, 279]	421	[619, 620]	687	[774, 186]	953	[848, 210]
156	[280, 281]	422	[621, 622]	688	[470, 517]	954	[869, 49]
157	[282, 283]	423	[258, 623]	689	[391, 385]	955	[761, 858]
158	[284, 249]	424	[624, 272]	690	[430, 866]	956	[1007, 144]
159	[107, 96]	425	[625, 268]	691	[867, 214]	957	[534, 1029]
160	[285, 286]	426	[312, 626]	692	[868, 397]	958	[525, 852]
161	[287, 288]	427	[627, 628]	693	[767, 532]	959	[382, 1032]

162	[289, 290]	428	[86, 240]	694	[458, 61]	960	[899, 216]
163	[291, 292]	429	[629, 630]	695	[869, 827]	961	[1031, 211]
164	[293, 86]	430	[262, 625]	696	[122, 870]	962	[1033, 1027]
165	[294, 67]	431	[568, 631]	697	[871, 872]	963	[429, 452]
166	[295, 296]	432	[632, 20]	698	[873, 874]	964	[12, 1034]
167	[250, 297]	433	[116, 633]	699	[454, 875]	965	[1005, 390]
168	[298, 299]	434	[304, 634]	700	[494, 500]	966	[432, 898]
169	[300, 301]	435	[635, 636]	701	[497, 192]	967	[435, 40]
170	[302, 303]	436	[637, 637]	702	[370, 876]	968	[1013, 1035]
171	[304, 105]	437	[594, 337]	703	[423, 770]	969	[488, 1031]
172	[305, 306]	438	[638, 639]	704	[489, 162]	970	[538, 1035]
173	[282, 307]	439	[319, 560]	705	[411, 203]	971	[540, 523]
174	[308, 309]	440	[67, 21]	706	[877, 507]	972	[1036, 902]
175	[310, 308]	441	[85, 250]	707	[878, 879]	973	[868, 1037]
176	[311, 120]	442	[640, 641]	708	[815, 428]	974	[1006, 472]
177	[312, 313]	443	[620, 267]	709	[493, 189]	975	[820, 1012]
178	[314, 315]	444	[642, 643]	710	[190, 197]	976	[887, 901]
179	[64, 316]	445	[644, 645]	711	[880, 881]	977	[188, 1011]
180	[317, 318]	446	[646, 647]	712	[518, 446]	978	[824, 479]
181	[319, 320]	447	[629, 648]	713	[882, 768]	979	[502, 1005]
182	[106, 321]	448	[649, 650]	714	[509, 787]	980	[408, 840]
183	[322, 245]	449	[651, 652]	715	[182, 510]	981	[377, 414]
184	[241, 323]	450	[653, 654]	716	[38, 883]	982	[1038, 536]
185	[324, 271]	451	[655, 110]	717	[503, 788]	983	[878, 861]
186	[325, 326]	452	[656, 657]	718	[504, 57]	984	[546, 865]
187	[327, 328]	453	[658, 659]	719	[884, 133]	985	[185, 464]
188	[274, 264]	454	[660, 247]	720	[825, 828]	986	[846, 847]
189	[329, 330]	455	[661, 662]	721	[885, 463]	987	[823, 818]
190	[331, 319]	456	[276, 663]	722	[821, 886]	988	[414, 1028]
191	[332, 333]	457	[88, 292]	723	[887, 522]	989	[527, 1039]
192	[334, 335]	458	[664, 640]	724	[466, 160]	990	[1014, 34]
193	[16, 336]	459	[665, 666]	725	[888, 795]	991	[863, 187]
194	[337, 338]	460	[667, 668]	726	[122, 784]	992	[490, 544]
195	[248, 107]	461	[95, 669]	727	[819, 414]	993	[123, 776]
196	[339, 220]	462	[670, 364]	728	[885, 889]	994	[882, 508]
197	[316, 106]	463	[671, 672]	729	[890, 507]	995	[371, 527]
198	[340, 341]	464	[593, 673]	730	[534, 796]	996	[1040, 896]
199	[342, 343]	465	[674, 99]	731	[193, 870]	997	[480, 1034]
200	[344, 236]	466	[675, 676]	732	[845, 822]	998	[1040, 843]
201	[345, 346]	467	[677, 223]	733	[891, 892]	999	[886, 1032]
202	[251, 347]	468	[678, 332]	734	[542, 831]	1000	[847, 883]
203	[348, 333]	469	[578, 679]	735	[867, 780]	1001	[1028, 777]
204	[349, 350]	470	[333, 680]	736	[204, 875]	1002	[464, 1021]
205	[351, 352]	471	[681, 682]	737	[206, 440]	1003	[1041, 351]

206	[112, 303]	472	[368, 583]	738	[893, 152]	1004	[570, 1042]
207	[353, 254]	473	[683, 684]	739	[871, 894]	1005	[674, 708]
208	[338, 30]	474	[685, 686]	740	[895, 433]	1006	[1043, 649]
209	[354, 355]	475	[687, 688]	741	[528, 381]	1007	[287, 1044]
210	[356, 357]	476	[689, 690]	742	[842, 896]	1008	[78, 1045]
211	[358, 359]	477	[691, 592]	743	[854, 809]	1009	[1046, 609]
212	[360, 361]	478	[692, 682]	744	[893, 405]	1010	[577, 346]
213	[119, 362]	479	[693, 694]	745	[897, 894]	1011	[109, 1047]
214	[363, 364]	480	[341, 695]	746	[393, 154]	1012	[623, 944]
215	[17, 221]	481	[100, 684]	747	[895, 898]	1013	[1048, 319]
216	[365, 366]	482	[338, 696]	748	[856, 199]	1014	[239, 87]
217	[367, 368]	483	[697, 283]	749	[36, 193]	1015	[90, 652]
218	[369, 370]	484	[698, 699]	750	[886, 383]	1016	[308, 929]
219	[371, 10]	485	[72, 336]	751	[855, 47]	1017	[232, 662]
220	[372, 373]	486	[670, 700]	752	[899, 180]	1018	[350, 585]
221	[374, 375]	487	[701, 702]	753	[40, 900]	1019	[227, 1049]
222	[12, 376]	488	[703, 232]	754	[521, 901]	1020	[248, 321]
223	[377, 378]	489	[257, 689]	755	[880, 902]	1021	[582, 305]
224	[195, 379]	490	[704, 705]	756	[415, 903]	1022	[609, 930]
225	[380, 381]	491	[706, 707]	757	[904, 138]	1023	[93, 928]
226	[382, 383]	492	[98, 708]	758	[426, 503]	1024	[690, 703]
227	[384, 385]	493	[709, 676]	759	[37, 847]	1025	[625, 22]
228	[386, 387]	494	[710, 316]	760	[905, 568]	1026	[645, 1050]
229	[388, 389]	495	[360, 711]	761	[262, 21]	1027	[219, 975]
230	[389, 390]	496	[678, 348]	762	[906, 907]	1028	[235, 1051]
231	[391, 392]	497	[575, 315]	763	[599, 620]	1029	[655, 68]
232	[393, 202]	498	[712, 713]	764	[743, 309]	1030	[689, 623]
233	[394, 395]	499	[685, 714]	765	[908, 909]	1031	[22, 1052]
234	[396, 397]	500	[96, 715]	766	[910, 288]	1032	[946, 1053]
235	[372, 398]	501	[716, 328]	767	[278, 911]	1033	[1054, 1055]
236	[399, 400]	502	[717, 584]	768	[912, 587]	1034	[24, 1045]
237	[401, 402]	503	[105, 718]	769	[316, 248]	1035	[646, 1002]
238	[166, 403]	504	[719, 92]	770	[298, 913]	1036	[1056, 1057]
239	[45, 404]	505	[85, 720]	771	[698, 643]	1037	[1058, 1059]
240	[151, 405]	506	[721, 349]	772	[366, 914]	1038	[1060, 1061]
241	[406, 407]	507	[634, 594]	773	[915, 332]	1039	[311, 362]
242	[408, 409]	508	[345, 578]	774	[367, 582]	1040	[1062, 630]
243	[410, 42]	509	[683, 101]	775	[354, 916]	1041	[7, 36]
244	[411, 178]	510	[348, 329]	776	[917, 918]	1042	[213, 853]
245	[412, 413]	511	[321, 96]	777	[351, 749]	1043	[520, 773]
246	[414, 415]	512	[78, 722]	778	[919, 665]	1044	[829, 451]
247	[416, 417]	513	[723, 724]	779	[339, 17]	1045	[48, 1023]
248	[127, 418]	514	[616, 639]	780	[920, 670]	1046	[396, 1037]
249	[419, 420]	515	[254, 365]	781	[741, 921]	1047	[1036, 881]

250	[150, 44]	516	[725, 286]	782	[912, 614]	1048	[813, 859]
251	[421, 422]	517	[63, 726]	783	[922, 923]	1049	[1031, 1017]
252	[423, 213]	518	[692, 727]	784	[219, 702]	1050	[773, 375]
253	[424, 32]	519	[728, 352]	785	[750, 924]	1051	[863, 1018]
254	[425, 398]	520	[729, 561]	786	[746, 925]	1052	[849, 431]
255	[426, 142]	521	[611, 326]	787	[314, 576]	1053	[1028, 903]
256	[427, 428]	522	[730, 731]	788	[926, 927]	1054	[851, 526]
257	[429, 430]	523	[362, 108]	789	[720, 259]	1055	[860, 879]
258	[410, 423]	524	[732, 610]	790	[346, 679]	1056	[503, 515]
259	[207, 431]	525	[733, 734]	791	[718, 595]	1057	[505, 59]
260	[148, 48]	526	[735, 736]	792	[82, 753]	1058	[769, 1026]
261	[432, 433]	527	[737, 738]	793	[619, 266]	1059	[839, 409]
262	[434, 404]	528	[739, 740]	794	[116, 928]	1060	[1033, 531]
263	[435, 436]	529	[93, 633]	795	[293, 239]	1061	[460, 838]
264	[159, 437]	530	[225, 671]	796	[743, 929]	1062	[897, 872]
265	[438, 439]	531	[73, 741]	797	[718, 337]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	178	1	356	1	534	1	712	0	890	0
1	0	179	1	357	1	535	1	713	0	891	1
2	0	180	0	358	0	536	0	714	1	892	1
3	0	181	1	359	1	537	1	715	0	893	1
4	0	182	0	360	0	538	0	716	1	894	1
5	0	183	1	361	0	539	1	717	0	895	0
6	1	184	1	362	1	540	0	718	0	896	1
7	0	185	1	363	1	541	0	719	0	897	1
8	1	186	0	364	1	542	0	720	0	898	1
9	0	187	0	365	1	543	0	721	1	899	0
10	0	188	0	366	1	544	0	722	0	900	1
11	0	189	1	367	0	545	0	723	0	901	0
12	0	190	0	368	0	546	1	724	0	902	0
13	1	191	1	369	0	547	0	725	1	903	1
14	1	192	1	370	0	548	1	726	0	904	0
15	0	193	1	371	1	549	0	727	0	905	0
16	1	194	0	372	0	550	1	728	1	906	0
17	1	195	1	373	1	551	0	729	0	907	0
18	1	196	0	374	0	552	1	730	1	908	1
19	0	197	0	375	0	553	0	731	1	909	1
20	1	198	1	376	0	554	0	732	0	910	0
21	0	199	1	377	1	555	0	733	1	911	0
22	0	200	1	378	0	556	0	734	0	912	1
23	0	201	0	379	1	557	0	735	1	913	0
24	1	202	1	380	1	558	1	736	1	914	1

25	0	203	0	381	0	559	0	737	1	915	1
26	0	204	1	382	1	560	0	738	1	916	1
27	1	205	0	383	1	561	0	739	0	917	0
28	1	206	1	384	0	562	1	740	0	918	0
29	1	207	1	385	1	563	0	741	0	919	1
30	1	208	0	386	1	564	0	742	0	920	0
31	0	209	1	387	0	565	1	743	1	921	0
32	1	210	1	388	1	566	0	744	1	922	1
33	1	211	0	389	1	567	1	745	1	923	1
34	0	212	0	390	0	568	1	746	1	924	1
35	1	213	1	391	1	569	1	747	0	925	1
36	0	214	1	392	0	570	1	748	0	926	0
37	1	215	1	393	1	571	0	749	0	927	0
38	1	216	1	394	1	572	0	750	0	928	0
39	0	217	0	395	1	573	1	751	0	929	1
40	0	218	0	396	1	574	1	752	0	930	1
41	1	219	1	397	1	575	0	753	0	931	0
42	1	220	0	398	0	576	0	754	1	932	0
43	0	221	0	399	1	577	1	755	1	933	1
44	1	222	0	400	1	578	1	756	1	934	1
45	0	223	1	401	1	579	0	757	0	935	0
46	1	224	1	402	0	580	0	758	1	936	0
47	0	225	1	403	0	581	1	759	1	937	0
48	1	226	1	404	0	582	1	760	0	938	1
49	1	227	0	405	0	583	1	761	1	939	1
50	1	228	1	406	1	584	0	762	0	940	0
51	0	229	1	407	1	585	0	763	0	941	1
52	1	230	1	408	1	586	0	764	1	942	1
53	0	231	1	409	0	587	0	765	1	943	0
54	0	232	1	410	0	588	1	766	0	944	1
55	1	233	1	411	0	589	1	767	0	945	0
56	1	234	1	412	1	590	1	768	1	946	0
57	1	235	0	413	1	591	0	769	0	947	1
58	1	236	1	414	1	592	0	770	0	948	0
59	0	237	1	415	1	593	1	771	1	949	0
60	1	238	1	416	0	594	1	772	1	950	0
61	1	239	0	417	0	595	1	773	1	951	0
62	0	240	0	418	0	596	1	774	0	952	1
63	0	241	1	419	1	597	1	775	1	953	0
64	1	242	1	420	1	598	0	776	0	954	1
65	1	243	0	421	1	599	0	777	0	955	1
66	1	244	0	422	1	600	0	778	1	956	1
67	0	245	1	423	0	601	1	779	0	957	1
68	0	246	1	424	0	602	0	780	0	958	1

69	0	247	0	425	1	603	1	781	0	959	1
70	1	248	1	426	1	604	1	782	1	960	0
71	1	249	1	427	1	605	1	783	1	961	0
72	0	250	1	428	1	606	0	784	1	962	0
73	0	251	1	429	0	607	1	785	0	963	0
74	1	252	0	430	1	608	1	786	1	964	0
75	1	253	0	431	1	609	1	787	1	965	0
76	1	254	1	432	1	610	0	788	0	966	1
77	0	255	1	433	0	611	1	789	0	967	1
78	0	256	1	434	1	612	0	790	0	968	1
79	0	257	0	435	1	613	0	791	0	969	0
80	1	258	0	436	1	614	1	792	1	970	0
81	1	259	1	437	1	615	0	793	1	971	0
82	1	260	1	438	0	616	1	794	0	972	0
83	1	261	1	439	1	617	1	795	0	973	0
84	0	262	1	440	0	618	1	796	1	974	0
85	0	263	1	441	0	619	1	797	0	975	1
86	1	264	1	442	0	620	0	798	0	976	0
87	1	265	0	443	0	621	1	799	1	977	0
88	0	266	1	444	0	622	1	800	0	978	1
89	0	267	0	445	1	623	0	801	0	979	0
90	1	268	1	446	0	624	0	802	0	980	1
91	1	269	1	447	0	625	1	803	1	981	1
92	0	270	0	448	0	626	0	804	1	982	0
93	1	271	0	449	0	627	1	805	0	983	1
94	1	272	1	450	0	628	1	806	0	984	1
95	1	273	0	451	0	629	0	807	0	985	1
96	1	274	0	452	0	630	1	808	0	986	0
97	1	275	0	453	0	631	0	809	1	987	1
98	1	276	0	454	0	632	1	810	0	988	1
99	1	277	0	455	0	633	1	811	0	989	1
100	0	278	0	456	0	634	1	812	0	990	0
101	1	279	1	457	0	635	1	813	1	991	0
102	1	280	0	458	0	636	1	814	1	992	0
103	1	281	0	459	1	637	1	815	0	993	0
104	0	282	1	460	0	638	0	816	0	994	0
105	0	283	0	461	1	639	0	817	1	995	1
106	0	284	1	462	0	640	0	818	1	996	1
107	1	285	0	463	1	641	0	819	0	997	1
108	1	286	1	464	1	642	0	820	1	998	1
109	0	287	1	465	0	643	1	821	0	999	0
110	1	288	0	466	0	644	1	822	1	1000	0
111	1	289	1	467	1	645	1	823	1	1001	0
112	1	290	1	468	0	646	0	824	1	1002	1

113	1	291	1	469	1	647	0	825	0	1003	0
114	1	292	1	470	0	648	0	826	0	1004	1
115	1	293	0	471	1	649	0	827	0	1005	0
116	0	294	0	472	0	650	0	828	0	1006	0
117	1	295	1	473	1	651	0	829	1	1007	1
118	1	296	1	474	1	652	0	830	0	1008	0
119	1	297	0	475	0	653	0	831	1	1009	1
120	0	298	0	476	1	654	0	832	1	1010	1
121	0	299	1	477	1	655	0	833	1	1011	0
122	0	300	1	478	0	656	0	834	1	1012	0
123	0	301	1	479	0	657	0	835	1	1013	1
124	1	302	0	480	1	658	0	836	1	1014	0
125	1	303	0	481	0	659	0	837	1	1015	1
126	1	304	1	482	0	660	0	838	0	1016	0
127	1	305	0	483	0	661	0	839	0	1017	1
128	1	306	0	484	1	662	0	840	1	1018	1
129	1	307	1	485	0	663	1	841	0	1019	0
130	1	308	0	486	0	664	0	842	0	1020	1
131	0	309	0	487	0	665	1	843	0	1021	1
132	1	310	0	488	0	666	0	844	0	1022	1
133	0	311	0	489	0	667	0	845	0	1023	1
134	1	312	1	490	0	668	0	846	0	1024	1
135	0	313	1	491	1	669	0	847	0	1025	1
136	1	314	1	492	1	670	0	848	0	1026	1
137	1	315	1	493	1	671	1	849	0	1027	1
138	1	316	0	494	0	672	1	850	0	1028	0
139	1	317	0	495	0	673	1	851	0	1029	0
140	1	318	0	496	0	674	0	852	0	1030	1
141	0	319	1	497	0	675	0	853	0	1031	0
142	1	320	1	498	0	676	0	854	1	1032	0
143	0	321	0	499	1	677	1	855	0	1033	0
144	1	322	1	500	1	678	0	856	0	1034	1
145	1	323	0	501	1	679	0	857	0	1035	0
146	1	324	1	502	0	680	0	858	1	1036	0
147	1	325	0	503	0	681	1	859	0	1037	0
148	1	326	0	504	0	682	1	860	0	1038	0
149	1	327	0	505	0	683	1	861	0	1039	0
150	1	328	1	506	1	684	0	862	0	1040	1
151	0	329	1	507	1	685	1	863	0	1041	0
152	1	330	1	508	0	686	0	864	1	1042	1
153	0	331	0	509	1	687	0	865	1	1043	0
154	0	332	1	510	0	688	0	866	1	1044	1
155	0	333	0	511	0	689	1	867	1	1045	1
156	0	334	1	512	0	690	1	868	0	1046	1

157	1	335	1	513	0	691	1	869	1	1047	0
158	1	336	1	514	1	692	0	870	0	1048	1
159	1	337	0	515	1	693	0	871	0	1049	0
160	0	338	0	516	1	694	0	872	0	1050	1
161	1	339	0	517	0	695	1	873	1	1051	0
162	1	340	1	518	0	696	0	874	1	1052	0
163	1	341	1	519	1	697	0	875	1	1053	0
164	0	342	1	520	0	698	1	876	1	1054	0
165	0	343	1	521	1	699	0	877	1	1055	0
166	1	344	1	522	1	700	0	878	1	1056	0
167	1	345	0	523	1	701	0	879	1	1057	0
168	0	346	0	524	0	702	0	880	1	1058	0
169	1	347	1	525	1	703	0	881	1	1059	0
170	0	348	0	526	1	704	0	882	0	1060	0
171	1	349	1	527	1	705	0	883	0	1061	0
172	0	350	1	528	0	706	1	884	0	1062	1
173	1	351	0	529	1	707	1	885	1		
174	0	352	1	530	1	708	0	886	0		
175	0	353	1	531	0	709	1	887	0		
176	0	354	1	532	0	710	0	888	1		
177	1	355	0	533	0	711	1	889	0		

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