PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a, b) = (z + 1, z^6 + z^5 + z^3 + z + 1)$

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Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a, b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a, b) in $\mathbb{F}_2[x] \setminus \mathbb{F}_2$. In this paper we prove the conjecture for $(a, b) = (z + 1, z^6 + z^5 + z^3 + z + 1)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \ldots)$ of elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{\ddots + \cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a, b)-Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \ldots) = (a, b, b, a, b, a, a, b, \ldots)$$

that we shall denote

$$\mathrm{CF}(a, b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When $(a,b)=(z+1, z^6+z^5+z^3+z+1)$, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

where, for $\mathrm{CF}(a,b)$*,*

$$\begin{aligned}
p_0(z) &= z^{24}+z^{21}+z^{20}+z^{18}+z^{14}+z^{13}+z^{12}+z^9+z^8+z^5+z^3+z^2+1,\\
p_1(z) &= z^{27}+z^{26}+z^{23}+z^{21}+z^{19}+z^{18}+z^{14}+z^{13}+z^{10}+z^8+z^7+z^3,\\
p_2(z) &= z^{28}+z^{26}+z^{24}+z^{22}+z^{12}+z^8+z^6+1,\\
p_3(z) &= z^{27}+z^{26}+z^{23}+z^{21}+z^{19}+z^{18}+z^{14}+z^{13}+z^{10}+z^8+z^7+z^3,\\
p_4(z) &= z^{26}+z^{24}+z^{22}+z^{21}+z^{20}+z^{18}+z^{16}+z^{14}+z^{13}+z^9+z^8+z^6+z^5\\
&\quad +z^3+z^2+1,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$*,*

$$\begin{aligned}
p_0(z) &= z^{21}+z^{18}+z^{14}+z^{13}+z^9+z^8+z^5+z^3+z^2+1,\\
p_1(z) &= z^{27}+z^{26}+z^{23}+z^{21}+z^{19}+z^{18}+z^{14}+z^{13}+z^{10}+z^8+z^7+z^3,\\
p_2(z) &= z^{28}+z^{26}+z^{24}+z^{22}+z^{12}+z^8+z^6+1,\\
p_3(z) &= z^{27}+z^{26}+z^{23}+z^{21}+z^{19}+z^{18}+z^{14}+z^{13}+z^{10}+z^8+z^7+z^3,\\
p_4(z) &= z^{26}+z^{22}+z^{21}+z^{18}+z^{16}+z^{14}+z^{13}+z^{12}+z^9+z^8+z^6+z^5+z^3\\
&\quad +z^2+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in \mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x) &= x^{\deg(t_{2^n-1})}\begin{pmatrix} t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad \cdots\times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x) &= x^{\deg(\bar t_{2^n-1})}\begin{pmatrix} \bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix} \bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad \cdots\times x^{\deg(\bar t_0)}\begin{pmatrix} \bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\geq 0$

$$M_{n+1}(x)=W_n(x)\cdot M_n(x),$$

$$W_{n+1}(x) = M_n(x) \cdot W_n(x).$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n} \tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n} \tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(175, 175, 175, 175, 175)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z\left(\phi(W^o,0,0)(x,z),\ z^{12}\cdot\phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 630 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 630, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z\left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ to order 770 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 770, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. □

Define

$$R^e = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Lemma 2.2. *For all integers $k \ge 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned} M^e[:14u] = {} & M^e[:7u] + x^{3u}M^e[:4u] + x^{6u}M^e[:u] + x^{3u}M^e[:7u] + x^{6u}M^e[:4u] \\ & + x^{9u}M^e[:u] + x^{8u}M^e[:6u] + (x^{7u}+x^{10u})R^e, \\ W^e[:14u] = {} & W^e[:7u] + x^{3u}W^e[:4u] + x^{6u}W^e[:u] + x^{3u}W^e[:7u] + x^{6u}W^e[:4u] \\ & + x^{9u}W^e[:u] + x^{8u}W^e[:6u] + (x^{7u}+x^{10u})R^e. \end{aligned}$$

For all integers $k \ge 2$ and $u = 2^{2k}$, the following identities hold.

$$M^o[:14u] = M^o[:7u] + x^{3u}M^o[:4u] + x^{6u}M^o[:u] + x^{3u}M^o[:7u] + x^{6u}M^o[:4u]$$

$$+ x^{9u}M^o[:u] + x^{8u}M^o[:6u] + (x^{7u} + x^{10u})R^o,$$

$$W^o[:14u] = W^o[:7u] + x^{3u}W^o[:4u] + x^{6u}W^o[:u] + x^{3u}W^o[:7u] + x^{6u}W^o[:4u] + x^{9u}W^o[:u] + x^{8u}W^o[:6u] + (x^{7u} + x^{10u})R^o.$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$T[:14u] = T[:7u] + x^{3u}T[:4u] + x^{6u}T[:u] + x^{3u}T[:7u] + x^{6u}T[:4u] + x^{9u}T[:u] + x^{8u}T[:6u] + (x^{7u} + x^{10u}).$$

The proof of the other 15 cases are similar. We break down the above identity into 7 parts:

$$x^7u = x^{6u}T[u:2u] + x^{3u}T[4u:5u] + T[7u:8u],$$
$$0 = x^{8u}T[:u] + x^{6u}T[2u:3u] + x^{3u}T[5u:6u] + T[8u:9u],$$
$$0 = x^{9u}T[:u] + x^{8u}T[u:2u] + x^{6u}T[3u:4u] + x^{3u}T[6u:7u] + T[9u:10u],$$
$$x^1 0u = x^{8u}T[2u:3u] + T[10u:11u],$$
$$0 = x^{8u}T[3u:4u] + T[11u:12u],$$
$$0 = x^{8u}T[4u:5u] + T[12u:13u],$$
$$0 = x^{8u}T[5u:6u] + T[13u:14u],$$

which can be rewritten as

$$0 = T[[1w]_2] + T[[100w]_2] + T[[111w]_2], \tag{2.7}$$
$$0 = T[[w]_2] + T[[10w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.8}$$
$$0 = T[[w]_2] + T[[1w]_2] + T[[11w]_2] + T[[110w]_2] + T[[1001w]_2], \tag{2.9}$$
$$0 = T[[10w]_2] + T[[1010w]_2], \tag{2.10}$$
$$0 = T[[11w]_2] + T[[1011w]_2], \tag{2.11}$$
$$0 = T[[100w]_2] + T[[1100w]_2], \tag{2.12}$$
$$0 = T[[101w]_2] + T[[1101w]_2], \tag{2.13}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$1 = T[[1w]_2] + T[[100w]_2] + T[[111w]_2], \tag{2.14}$$
$$0 = T[[w]_2] + T[[10w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.15}$$
$$0 = T[[w]_2] + T[[1w]_2] + T[[11w]_2] + T[[110w]_2] + T[[1001w]_2], \tag{2.16}$$
$$1 = T[[10w]_2] + T[[1010w]_2], \tag{2.17}$$
$$0 = T[[11w]_2] + T[[1011w]_2], \tag{2.18}$$
$$0 = T[[100w]_2] + T[[1100w]_2], \tag{2.19}$$
$$0 = T[[101w]_2] + T[[1101w]_2], \tag{2.20}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 613 states; its transition function and output function can be found in the annex. Let $A(s,w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i,w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.20) can be written as

$$0 = \tau(A(s,1)) + \tau(A(s,100)) + \tau(A(s,111)), \tag{2.21}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,101)) + \tau(A(s,1000)), \tag{2.22}$$

$$\begin{aligned} 0 = {}& \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,110)) \\ & + \tau(A(s,1001)), \end{aligned} \tag{2.23}$$

$$0 = \tau(A(s,10)) + \tau(A(s,1010)), \tag{2.24}$$

$$0 = \tau(A(s,11)) + \tau(A(s,1011)), \tag{2.25}$$

$$0 = \tau(A(s,100)) + \tau(A(s,1100)), \tag{2.26}$$

$$0 = \tau(A(s,101)) + \tau(A(s,1101)), \tag{2.27}$$

for all $s \in E_{2k}$, and

$$1 = \tau(A(s,1)) + \tau(A(s,100)) + \tau(A(s,111)), \tag{2.28}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,101)) + \tau(A(s,1000)), \tag{2.29}$$

$$\begin{aligned} 0 = {}& \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,110)) \\ & + \tau(A(s,1001)), \end{aligned} \tag{2.30}$$

$$1 = \tau(A(s,10)) + \tau(A(s,1010)), \tag{2.31}$$

$$0 = \tau(A(s,11)) + \tau(A(s,1011)), \tag{2.32}$$

$$0 = \tau(A(s,100)) + \tau(A(s,1100)), \tag{2.33}$$

$$0 = \tau(A(s,101)) + \tau(A(s,1101)), \tag{2.34}$$

for $s = A(i, 0^{2k})$. We find that $(A(i,0^{18}), E_{18}) = (A(i,0^{16}), E_{16})$, so that we only have to verify that identities (2.21) through (2.34) hold for $2 \le k \le 9$, which turns out to be true. $\square$

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \ge 2$, and $u = 2^{2k-1}$,*

$$M_{2k} = M^e[:7u] + x^{3u} M^e[:4u] + x^{6u} M^e[:u] + x^{7u} R^e,$$

$$W_{2k} = W^e[:7u] + x^{3u} W^e[:4u] + x^{6u} W^e[:u] + x^{7u} R^e.$$

For all integer $k \ge 2$, and $u = 2^{2k}$,

$$M_{2k+1} = M^o[:7u] + x^{3u} M^o[:4u] + x^{6u} M^o[:u] + x^{7u} R^o,$$

$$W_{2k+1} = W^o[:7u] + x^{3u} W^o[:4u] + x^{6u} W^o[:u] + x^{7u} R^o.$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \ge 2$, we claim that

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1} \wedge W_{2k+1},$$

$$M_{2k+1} \wedge W_{2k+1} \Rightarrow M_{2k+2} \wedge W_{2k+2}.$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$\begin{aligned} W_{2k}M_{2k} = &\big(W^e[:7v] + x^{3v}W^e[:4v] + x^{6v}W^e[:v] + x^{7u}R^e\big) \times \big(M^e[:7v] \\ &+ x^{3v}M^e[:4v] + x^{6v}M^e[:v] + x^{7u}R^e\big); \end{aligned} \tag{2.35}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{14v}R^o$. Therefore we only need to prove that their difference is $O(x^{14v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{14v}, to

$$W^e[:14v]M^e[:14v] + x^{6v}W^e[:8v]M^e[:8v] + x^{12v}W^e[:2v]M^e[:2v].$$

For all $n \le 14$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.35) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{14}, b_1, ..., b_{14}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{14})$, which completes the proof. □

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \le j, k \le 1$,

$$\begin{aligned} \lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k}, \\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k}, \\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k}, \\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}. \end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a, b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x, y) = \mathrm{Res}_t\left(\phi(M^e, 0, 1)(x, t), y^{12}\phi(M^e, 0, 1)(x, t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x,y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned}
q_0(x) &= x^{28} + x^{26} + x^{25} + x^{23} + x^{20} + x^{19} + x^{16} + x^{15} + x^{14} + x^{10} + x^8 + x^7 \\
&\quad + x^4, \\
q_1(x) &= x^{25} + x^{21} + x^{20} + x^{18} + x^{15} + x^{14} + x^{10} + x^9 + x^7 + x^5 + x^2 + x, \\
q_2(x) &= x^{28} + x^{22} + x^{20} + x^{16} + x^6 + x^4 + x^2 + 1, \\
q_3(x) &= x^{25} + x^{21} + x^{20} + x^{18} + x^{15} + x^{14} + x^{10} + x^9 + x^7 + x^5 + x^2 + x, \\
q_4(x) &= x^{28} + x^{26} + x^{25} + x^{23} + x^{22} + x^{20} + x^{19} + x^{15} + x^{14} + x^{12} + x^{10} + x^8 \\
&\quad + x^7 + x^6 + x^4 + x^2.
\end{aligned}$$

The polynomial $Q(x,y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and $R(x,y) := P(x,y)/Q(x,y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 216, and substitute it for y in $R(z,y)$, we get a series with valuation smaller than 216, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b) = f(1/z)$. □

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{162} + x^{160} + x^{158} + x^{154} + x^{152} + x^{150} + x^{140} + x^{138} + x^{136} + x^{132} \\
&\quad + x^{130} + x^{122} + x^{120} + x^{114} + x^{108} + x^{106} + x^{100} + x^{96} + x^{94} + x^{86} \\
&\quad + x^{78} + x^{74} + x^{62} + x^{60} + x^{54} + x^{52} + x^{50} + x^{48} + x^{46} + x^{44} + x^{40} \\
&\quad + x^{38} + x^{36} + x^{34} + x^{32} + x^{26} + x^{20} + x^{18} + x^{16} + x^{14} + x^{12}, \\
p_3(x) &= x^{156} + x^{154} + x^{152} + x^{148} + x^{144} + x^{142} + x^{138} + x^{136} + x^{130} + x^{126} \\
&\quad + x^{124} + x^{122} + x^{120} + x^{116} + x^{114} + x^{108} + x^{104} + x^{98} + x^{96} + x^{92} \\
&\quad + x^{90} + x^{86} + x^{84} + x^{80} + x^{72} + x^{64} + x^{60} + x^{58} + x^{54} + x^{50} + x^{48} \\
&\quad + x^{46} + x^{42} + x^{34} + x^{30} + x^{24} + x^{22} + x^{18} + x^{10} + x^8, \\
p_6(x) &= x^{158} + x^{154} + x^{150} + x^{144} + x^{142} + x^{136} + x^{134} + x^{132} + x^{128} + x^{124} \\
&\quad + x^{120} + x^{110} + x^{108} + x^{104} + x^{98} + x^{96} + x^{90} + x^{84} + x^{82} + x^{80} \\
&\quad + x^{78} + x^{76} + x^{74} + x^{72} + x^{64} + x^{60} + x^{58} + x^{56} + x^{54} + x^{46} + x^{42} \\
&\quad + x^{40} + x^{36} + x^{32} + x^{30} + x^{26} + x^{22} + x^{20} + x^{18} + x^6 + x^4 + 1, \\
p_9(x) &= x^{156} + x^{154} + x^{150} + x^{148} + x^{146} + x^{144} + x^{136} + x^{124} + x^{112} + x^{110}
\end{aligned}$$

$$+ x^{102} + x^{100} + x^{96} + x^{94} + x^{90} + x^{86} + x^{80} + x^{76} + x^{62} + x^{56} + x^{54} + x^{46} + x^{44} + x^{42} + x^{36} + x^{32} + x^{28} + x^{24} + x^{22} + x^{20} + x^{16} + x^{10} + x^{4} + x^{2},$$

$$p_{12}(x) = x^{162} + x^{160} + x^{154} + x^{152} + x^{138} + x^{136} + x^{130} + x^{128} + x^{114} + x^{112} + x^{106} + x^{104} + x^{98} + x^{96} + x^{82} + x^{80} + x^{74} + x^{72} + x^{66} + x^{64} + x^{18} + x^{16} + x^{10} + x^{8} + x^{2} + 1,$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0].

For $\phi(M^e, 0, 1)$:

$$p_0(x) = x^{147} + x^{145} + x^{144} + x^{142} + x^{141} + x^{135} + x^{133} + x^{132} + x^{131} + x^{126} + x^{125} + x^{124} + x^{123} + x^{122} + x^{119} + x^{118} + x^{117} + x^{114} + x^{113} + x^{112} + x^{108} + x^{107} + x^{103} + x^{102} + x^{98} + x^{92} + x^{91} + x^{88} + x^{82} + x^{81} + x^{80} + x^{78} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{67} + x^{66} + x^{65} + x^{64} + x^{62} + x^{61} + x^{60} + x^{59} + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{41} + x^{38} + x^{35} + x^{33} + x^{32} + x^{31} + x^{26} + x^{23} + x^{22} + x^{18},$$

$$p_3(x) = x^{120} + x^{118} + x^{117} + x^{114} + x^{111} + x^{110} + x^{109} + x^{108} + x^{106} + x^{105} + x^{98} + x^{96} + x^{95} + x^{94} + x^{91} + x^{89} + x^{87} + x^{86} + x^{85} + x^{84} + x^{82} + x^{81} + x^{80} + x^{78} + x^{77} + x^{70} + x^{68} + x^{67} + x^{58} + x^{56} + x^{55} + x^{54} + x^{51} + x^{49} + x^{47} + x^{46} + x^{45} + x^{44} + x^{42} + x^{41} + x^{40} + x^{38} + x^{37} + x^{30} + x^{28} + x^{27} + x^{18} + x^{15} + x^{13} + x^{11} + x^{10} + x^{9},$$

$$p_6(x) = x^{129} + x^{127} + x^{126} + x^{125} + x^{124} + x^{122} + x^{119} + x^{116} + x^{115} + x^{112} + x^{109} + x^{106} + x^{105} + x^{103} + x^{102} + x^{100} + x^{99} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{60} + x^{57} + x^{54} + x^{49} + x^{47} + x^{46} + x^{44} + x^{43} + x^{40} + x^{35} + x^{33} + x^{32} + x^{30} + x^{29} + x^{26} + x^{25} + x^{23} + x^{22} + x^{21} + x^{20} + x^{18} + x^{15} + x^{13} + x^{12} + x^{10} + x^{9} + x^{6},$$

$$p_9(x) = x^{138} + x^{136} + x^{135} + x^{132} + x^{130} + x^{129} + x^{128} + x^{125} + x^{124} + x^{123} + x^{122} + x^{119} + x^{117} + x^{115} + x^{113} + x^{112} + x^{111} + x^{108} + x^{105} + x^{104} + x^{103} + x^{100} + x^{98} + x^{97} + x^{92} + x^{90} + x^{89} + x^{88} + x^{85} + x^{83} + x^{82} + x^{81} + x^{78} + x^{76} + x^{75} + x^{74} + x^{71} + x^{70} + x^{69} + x^{66} + x^{64} + x^{63} + x^{60} + x^{58} + x^{57} + x^{54} + x^{51} + x^{50} + x^{49} + x^{48} + x^{45} + x^{44} + x^{43} + x^{38} + x^{36} + x^{35} + x^{32} + x^{29} + x^{28} + x^{27} + x^{26} + x^{23} + x^{21} + x^{20} + x^{19} + x^{16} + x^{13} + x^{12} + x^{11} + x^{6} + x^{4} + x^{3},$$

$$\begin{aligned}
p_{12}(x) = {} & x^{147} + x^{145} + x^{144} + x^{143} + x^{142} + x^{140} + x^{139} + x^{137} + x^{136} + x^{134} \\
& + x^{133} + x^{130} + x^{129} + x^{126} + x^{125} + x^{123} + x^{122} + x^{120} + x^{119} \\
& + x^{117} + x^{116} + x^{114} + x^{113} + x^{111} + x^{110} + x^{108} + x^{107} + x^{105} \\
& + x^{104} + x^{102} + x^{101} + x^{98} + x^{97} + x^{94} + x^{93} + x^{91} + x^{90} + x^{88} + x^{87} \\
& + x^{85} + x^{84} + x^{82} + x^{81} + x^{79} + x^{78} + x^{76} + x^{75} + x^{73} + x^{72} + x^{70} \\
& + x^{69} + x^{66} + x^{65} + x^{62} + x^{61} + x^{59} + x^{58} + x^{56} + x^{55} + x^{53} + x^{52} \\
& + x^{50} + x^{49} + x^{47} + x^{46} + x^{44} + x^{43} + x^{41} + x^{40} + x^{38} + x^{37} + x^{35} \\
& + x^{34} + x^{32} + x^{31} + x^{29} + x^{28} + x^{26} + x^{25} + x^{23} + x^{22} + x^{20} + x^{19} \\
& + x^{17} + x^{16} + x^{14} + x^{13} + x^{11} + x^{10} + x^{8} + x^{7} + x^{5} + x^{4} + x^{3} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [0, 1, 1, 1, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 1].

For $\phi(M^e, 1, 0)$:

$$\begin{aligned}
p_{0}(x) = {} & x^{147} + x^{145} + x^{144} + x^{142} + x^{141} + x^{135} + x^{133} + x^{132} + x^{131} + x^{126} \\
& + x^{125} + x^{124} + x^{123} + x^{122} + x^{119} + x^{118} + x^{117} + x^{114} + x^{113} \\
& + x^{112} + x^{108} + x^{107} + x^{103} + x^{102} + x^{98} + x^{92} + x^{91} + x^{88} + x^{82} \\
& + x^{81} + x^{80} + x^{78} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{67} + x^{66} \\
& + x^{65} + x^{64} + x^{62} + x^{61} + x^{60} + x^{59} + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} \\
& + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{41} + x^{38} + x^{35} + x^{33} + x^{32} \\
& + x^{31} + x^{26} + x^{23} + x^{22} + x^{18}, \\
p_{3}(x) = {} & x^{120} + x^{118} + x^{117} + x^{114} + x^{111} + x^{110} + x^{109} + x^{108} + x^{106} + x^{105} \\
& + x^{98} + x^{96} + x^{95} + x^{94} + x^{91} + x^{89} + x^{87} + x^{86} + x^{85} + x^{84} + x^{82} \\
& + x^{81} + x^{80} + x^{78} + x^{77} + x^{70} + x^{68} + x^{67} + x^{58} + x^{56} + x^{55} + x^{54} \\
& + x^{51} + x^{49} + x^{47} + x^{46} + x^{45} + x^{44} + x^{42} + x^{41} + x^{40} + x^{38} + x^{37} \\
& + x^{30} + x^{28} + x^{27} + x^{18} + x^{15} + x^{13} + x^{11} + x^{10} + x^{9}, \\
p_{6}(x) = {} & x^{129} + x^{127} + x^{126} + x^{125} + x^{124} + x^{122} + x^{119} + x^{116} + x^{115} + x^{112} \\
& + x^{109} + x^{106} + x^{105} + x^{103} + x^{102} + x^{100} + x^{99} + x^{97} + x^{96} + x^{95} \\
& + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} \\
& + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} \\
& + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} \\
& + x^{60} + x^{57} + x^{54} + x^{49} + x^{47} + x^{46} + x^{44} + x^{43} + x^{40} + x^{35} + x^{33} \\
& + x^{32} + x^{30} + x^{29} + x^{26} + x^{25} + x^{23} + x^{22} + x^{21} + x^{20} + x^{18} + x^{15} \\
& + x^{13} + x^{12} + x^{10} + x^{9} + x^{6}, \\
p_{9}(x) = {} & x^{138} + x^{136} + x^{135} + x^{132} + x^{130} + x^{129} + x^{128} + x^{125} + x^{124} + x^{123} \\
& + x^{122} + x^{119} + x^{117} + x^{115} + x^{113} + x^{112} + x^{111} + x^{108} + x^{105} \\
& + x^{104} + x^{103} + x^{100} + x^{98} + x^{97} + x^{92} + x^{90} + x^{89} + x^{88} + x^{85} + x^{83} \\
& + x^{82} + x^{81} + x^{78} + x^{76} + x^{75} + x^{74} + x^{71} + x^{70} + x^{69} + x^{66} + x^{64}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{63} + x^{60} + x^{58} + x^{57} + x^{54} + x^{51} + x^{50} + x^{49} + x^{48} + x^{45} + x^{44}\\
&+ x^{43} + x^{38} + x^{36} + x^{35} + x^{32} + x^{29} + x^{28} + x^{27} + x^{26} + x^{23} + x^{21}\\
&+ x^{20} + x^{19} + x^{16} + x^{13} + x^{12} + x^{11} + x^{6} + x^{4} + x^{3},\\
p_{12}(x) = {}& x^{147} + x^{145} + x^{144} + x^{143} + x^{142} + x^{140} + x^{139} + x^{137} + x^{136} + x^{134}\\
&+ x^{133} + x^{130} + x^{129} + x^{126} + x^{125} + x^{123} + x^{122} + x^{120} + x^{119}\\
&+ x^{117} + x^{116} + x^{114} + x^{113} + x^{111} + x^{110} + x^{108} + x^{107} + x^{105}\\
&+ x^{104} + x^{102} + x^{101} + x^{98} + x^{97} + x^{94} + x^{93} + x^{91} + x^{90} + x^{88} + x^{87}\\
&+ x^{85} + x^{84} + x^{82} + x^{81} + x^{79} + x^{78} + x^{76} + x^{75} + x^{73} + x^{72} + x^{70}\\
&+ x^{69} + x^{66} + x^{65} + x^{62} + x^{61} + x^{59} + x^{58} + x^{56} + x^{55} + x^{53} + x^{52}\\
&+ x^{50} + x^{49} + x^{47} + x^{46} + x^{44} + x^{43} + x^{41} + x^{40} + x^{38} + x^{37} + x^{35}\\
&+ x^{34} + x^{32} + x^{31} + x^{29} + x^{28} + x^{26} + x^{25} + x^{23} + x^{22} + x^{20} + x^{19}\\
&+ x^{17} + x^{16} + x^{14} + x^{13} + x^{11} + x^{10} + x^{8} + x^{7} + x^{5} + x^{4} + x^{3} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 1, 1, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 1].

For $\phi(M^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) = {}& x^{132} + x^{130} + x^{128} + x^{126} + x^{124} + x^{122} + x^{116} + x^{112} + x^{110} + x^{106}\\
&+ x^{104} + x^{102} + x^{98} + x^{94} + x^{92} + x^{88} + x^{86} + x^{84} + x^{82} + x^{80} + x^{78}\\
&+ x^{76} + x^{74} + x^{72} + x^{70} + x^{68} + x^{66} + x^{64} + x^{60} + x^{58} + x^{56} + x^{52}\\
&+ x^{50} + x^{48} + x^{42} + x^{40} + x^{38} + x^{34} + x^{32} + x^{26} + x^{24},\\
p_3(x) = {}& x^{126} + x^{124} + x^{122} + x^{120} + x^{118} + x^{112} + x^{106} + x^{104} + x^{102} + x^{100}\\
&+ x^{98} + x^{96} + x^{94} + x^{92} + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{80} + x^{78}\\
&+ x^{76} + x^{74} + x^{72} + x^{68} + x^{66} + x^{64} + x^{56} + x^{54} + x^{52} + x^{48} + x^{42}\\
&+ x^{38} + x^{34} + x^{32} + x^{30} + x^{20} + x^{18} + x^{16} + x^{14} + x^{12} + x^{10} + x^{8}\\
&+ x^{6},\\
p_6(x) = {}& x^{128} + x^{124} + x^{122} + x^{120} + x^{114} + x^{108} + x^{104} + x^{100} + x^{82} + x^{78}\\
&+ x^{72} + x^{62} + x^{60} + x^{56} + x^{52} + x^{50} + x^{46} + x^{42} + x^{36} + x^{34} + x^{32}\\
&+ x^{30} + x^{28} + x^{26} + x^{24} + x^{22} + x^{20} + x^{12} + x^{8} + 1,\\
p_9(x) = {}& x^{126} + x^{124} + x^{118} + x^{114} + x^{108} + x^{104} + x^{102} + x^{98} + x^{86} + x^{76}\\
&+ x^{74} + x^{72} + x^{70} + x^{68} + x^{64} + x^{60} + x^{58} + x^{56} + x^{50} + x^{44} + x^{38}\\
&+ x^{36} + x^{34} + x^{26} + x^{24} + x^{22} + x^{20} + x^{18} + x^{14} + x^{12} + x^{10} + x^{8}\\
&+ x^{4} + x^{2},\\
p_{12}(x) = {}& x^{132} + x^{130} + x^{126} + x^{124} + x^{120} + x^{118} + x^{114} + x^{112} + x^{100} + x^{98}\\
&+ x^{94} + x^{92} + x^{88} + x^{86} + x^{82} + x^{80} + x^{68} + x^{66} + x^{62} + x^{60} + x^{56}\\
&+ x^{54} + x^{50} + x^{48} + x^{36} + x^{34} + x^{30} + x^{28} + x^{24} + x^{22} + x^{20} + x^{16}\\
&+ x^{14} + x^{12} + x^{8} + x^{6} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0].

For $\phi(W^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{132} + x^{130} + x^{128} + x^{126} + x^{124} + x^{122} + x^{120} + x^{118} + x^{116} + x^{114} \\
&\quad + x^{112} + x^{108} + x^{106} + x^{102} + x^{100} + x^{98} + x^{96} + x^{90} + x^{88} + x^{86} \\
&\quad + x^{84} + x^{82} + x^{80} + x^{78} + x^{76} + x^{72} + x^{68} + x^{64} + x^{58} + x^{56} + x^{54} \\
&\quad + x^{48} + x^{46} + x^{44} + x^{42} + x^{34} + x^{30} + x^{28} + x^{18} + x^{14} + x^{12}, \\
p_3(x) &= x^{126} + x^{124} + x^{122} + x^{120} + x^{118} + x^{114} + x^{108} + x^{106} + x^{104} + x^{100} \\
&\quad + x^{92} + x^{90} + x^{86} + x^{78} + x^{76} + x^{74} + x^{70} + x^{68} + x^{66} + x^{62} + x^{58} \\
&\quad + x^{56} + x^{50} + x^{48} + x^{44} + x^{42} + x^{40} + x^{36} + x^{28} + x^{16} + x^{14} + x^{8}, \\
p_6(x) &= x^{128} + x^{124} + x^{122} + x^{118} + x^{114} + x^{108} + x^{106} + x^{102} + x^{100} + x^{98} \\
&\quad + x^{92} + x^{86} + x^{84} + x^{82} + x^{78} + x^{76} + x^{74} + x^{66} + x^{60} + x^{58} + x^{56} \\
&\quad + x^{54} + x^{52} + x^{50} + x^{48} + x^{46} + x^{38} + x^{32} + x^{30} + x^{28} + x^{26} + x^{14} \\
&\quad + x^{12} + x^{10} + x^{4} + 1, \\
p_9(x) &= x^{126} + x^{124} + x^{118} + x^{114} + x^{108} + x^{104} + x^{102} + x^{98} + x^{86} + x^{76} \\
&\quad + x^{74} + x^{72} + x^{70} + x^{68} + x^{64} + x^{60} + x^{58} + x^{56} + x^{50} + x^{44} + x^{38} \\
&\quad + x^{36} + x^{34} + x^{26} + x^{24} + x^{22} + x^{20} + x^{18} + x^{14} + x^{12} + x^{10} + x^{8} \\
&\quad + x^{4} + x^{2}, \\
p_{12}(x) &= x^{132} + x^{130} + x^{126} + x^{124} + x^{120} + x^{118} + x^{114} + x^{112} + x^{100} + x^{98} \\
&\quad + x^{94} + x^{92} + x^{88} + x^{86} + x^{82} + x^{80} + x^{68} + x^{66} + x^{62} + x^{60} + x^{56} \\
&\quad + x^{54} + x^{50} + x^{48} + x^{36} + x^{34} + x^{30} + x^{28} + x^{24} + x^{22} + x^{20} + x^{16} \\
&\quad + x^{14} + x^{12} + x^{8} + x^{6} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{147} + x^{145} + x^{144} + x^{142} + x^{141} + x^{135} + x^{133} + x^{132} + x^{131} + x^{129} \\
&\quad + x^{127} + x^{123} + x^{121} + x^{119} + x^{117} + x^{114} + x^{113} + x^{112} + x^{108} \\
&\quad + x^{107} + x^{103} + x^{98} + x^{97} + x^{95} + x^{94} + x^{92} + x^{91} + x^{90} + x^{84} + x^{83} \\
&\quad + x^{82} + x^{80} + x^{78} + x^{77} + x^{76} + x^{75} + x^{74} + x^{69} + x^{68} + x^{67} + x^{65} \\
&\quad + x^{63} + x^{61} + x^{60} + x^{58} + x^{52} + x^{51} + x^{50} + x^{48} + x^{46} + x^{42} + x^{39} \\
&\quad + x^{38} + x^{37} + x^{35} + x^{33} + x^{32} + x^{29} + x^{27}, \\
p_3(x) &= x^{120} + x^{118} + x^{117} + x^{114} + x^{111} + x^{110} + x^{109} + x^{108} + x^{106} + x^{105} \\
&\quad + x^{98} + x^{96} + x^{95} + x^{94} + x^{91} + x^{89} + x^{87} + x^{86} + x^{85} + x^{84} + x^{82} \\
&\quad + x^{81} + x^{80} + x^{78} + x^{77} + x^{70} + x^{68} + x^{67} + x^{58} + x^{56} + x^{55} + x^{54} \\
&\quad + x^{51} + x^{49} + x^{47} + x^{46} + x^{45} + x^{44} + x^{42} + x^{41} + x^{40} + x^{38} + x^{37} \\
&\quad + x^{30} + x^{28} + x^{27} + x^{18} + x^{15} + x^{13} + x^{11} + x^{10} + x^{9}, \\
p_6(x) &= x^{129} + x^{127} + x^{126} + x^{125} + x^{124} + x^{122} + x^{119} + x^{116} + x^{115} + x^{112} \\
&\quad + x^{109} + x^{106} + x^{105} + x^{103} + x^{102} + x^{100} + x^{99} + x^{97} + x^{96} + x^{95}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} \\
&+ x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} \\
&+ x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} \\
&+ x^{60} + x^{57} + x^{54} + x^{49} + x^{47} + x^{46} + x^{44} + x^{43} + x^{40} + x^{35} + x^{33} \\
&+ x^{32} + x^{30} + x^{29} + x^{26} + x^{25} + x^{23} + x^{22} + x^{21} + x^{20} + x^{18} + x^{15} \\
&+ x^{13} + x^{12} + x^{10} + x^{9} + x^{6}, \\
p_9(x) = {} & x^{138} + x^{136} + x^{135} + x^{132} + x^{130} + x^{129} + x^{128} + x^{125} + x^{124} + x^{123} \\
&+ x^{122} + x^{119} + x^{117} + x^{115} + x^{113} + x^{112} + x^{111} + x^{108} + x^{105} \\
&+ x^{104} + x^{103} + x^{100} + x^{98} + x^{97} + x^{92} + x^{90} + x^{89} + x^{88} + x^{85} + x^{83} \\
&+ x^{82} + x^{81} + x^{78} + x^{76} + x^{75} + x^{74} + x^{71} + x^{70} + x^{69} + x^{66} + x^{64} \\
&+ x^{63} + x^{60} + x^{58} + x^{57} + x^{54} + x^{51} + x^{50} + x^{49} + x^{48} + x^{45} + x^{44} \\
&+ x^{43} + x^{38} + x^{36} + x^{35} + x^{32} + x^{29} + x^{28} + x^{27} + x^{26} + x^{23} + x^{21} \\
&+ x^{20} + x^{19} + x^{16} + x^{13} + x^{12} + x^{11} + x^{6} + x^{4} + x^{3}, \\
p_{12}(x) = {} & x^{147} + x^{145} + x^{144} + x^{143} + x^{142} + x^{140} + x^{139} + x^{137} + x^{136} + x^{134} \\
&+ x^{133} + x^{130} + x^{129} + x^{126} + x^{125} + x^{123} + x^{122} + x^{120} + x^{119} \\
&+ x^{117} + x^{116} + x^{114} + x^{113} + x^{111} + x^{110} + x^{108} + x^{107} + x^{105} \\
&+ x^{104} + x^{102} + x^{101} + x^{98} + x^{97} + x^{94} + x^{93} + x^{91} + x^{90} + x^{88} + x^{87} \\
&+ x^{85} + x^{84} + x^{82} + x^{81} + x^{79} + x^{78} + x^{76} + x^{75} + x^{73} + x^{72} + x^{70} \\
&+ x^{69} + x^{66} + x^{65} + x^{62} + x^{61} + x^{59} + x^{58} + x^{56} + x^{55} + x^{53} + x^{52} \\
&+ x^{50} + x^{49} + x^{47} + x^{46} + x^{44} + x^{43} + x^{41} + x^{40} + x^{38} + x^{37} + x^{35} \\
&+ x^{34} + x^{32} + x^{31} + x^{29} + x^{28} + x^{26} + x^{25} + x^{23} + x^{22} + x^{20} + x^{19} \\
&+ x^{17} + x^{16} + x^{14} + x^{13} + x^{11} + x^{10} + x^{8} + x^{7} + x^{5} + x^{4} + x^{3} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 0, 0, 1, 1].

For $\phi(W^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) = {} & x^{147} + x^{145} + x^{144} + x^{142} + x^{141} + x^{135} + x^{133} + x^{132} + x^{131} + x^{129} \\
&+ x^{127} + x^{123} + x^{121} + x^{119} + x^{117} + x^{114} + x^{113} + x^{112} + x^{108} \\
&+ x^{107} + x^{103} + x^{98} + x^{97} + x^{95} + x^{94} + x^{92} + x^{91} + x^{90} + x^{84} + x^{83} \\
&+ x^{82} + x^{80} + x^{78} + x^{77} + x^{76} + x^{75} + x^{74} + x^{69} + x^{68} + x^{67} + x^{65} \\
&+ x^{63} + x^{61} + x^{60} + x^{58} + x^{52} + x^{51} + x^{50} + x^{48} + x^{46} + x^{42} + x^{39} \\
&+ x^{38} + x^{37} + x^{35} + x^{33} + x^{32} + x^{29} + x^{27}, \\
p_3(x) = {} & x^{120} + x^{118} + x^{117} + x^{114} + x^{111} + x^{110} + x^{109} + x^{108} + x^{106} + x^{105} \\
&+ x^{98} + x^{96} + x^{95} + x^{94} + x^{91} + x^{89} + x^{87} + x^{86} + x^{85} + x^{84} + x^{82} \\
&+ x^{81} + x^{80} + x^{78} + x^{77} + x^{70} + x^{68} + x^{67} + x^{58} + x^{56} + x^{55} + x^{54} \\
&+ x^{51} + x^{49} + x^{47} + x^{46} + x^{45} + x^{44} + x^{42} + x^{41} + x^{40} + x^{38} + x^{37} \\
&+ x^{30} + x^{28} + x^{27} + x^{18} + x^{15} + x^{13} + x^{11} + x^{10} + x^{9},
\end{aligned}
$$

$$\begin{aligned}
p_6(x) &= x^{129} + x^{127} + x^{126} + x^{125} + x^{124} + x^{122} + x^{119} + x^{116} + x^{115} + x^{112}\\
&\quad + x^{109} + x^{106} + x^{105} + x^{103} + x^{102} + x^{100} + x^{99} + x^{97} + x^{96} + x^{95}\\
&\quad + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84}\\
&\quad + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73}\\
&\quad + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62}\\
&\quad + x^{60} + x^{57} + x^{54} + x^{49} + x^{47} + x^{46} + x^{44} + x^{43} + x^{40} + x^{35} + x^{33}\\
&\quad + x^{32} + x^{30} + x^{29} + x^{26} + x^{25} + x^{23} + x^{22} + x^{21} + x^{20} + x^{18} + x^{15}\\
&\quad + x^{13} + x^{12} + x^{10} + x^{9} + x^{6},\\
p_9(x) &= x^{138} + x^{136} + x^{135} + x^{132} + x^{130} + x^{129} + x^{128} + x^{125} + x^{124} + x^{123}\\
&\quad + x^{122} + x^{119} + x^{117} + x^{115} + x^{113} + x^{112} + x^{111} + x^{108} + x^{105}\\
&\quad + x^{104} + x^{103} + x^{100} + x^{98} + x^{97} + x^{92} + x^{90} + x^{89} + x^{88} + x^{85} + x^{83}\\
&\quad + x^{82} + x^{81} + x^{78} + x^{76} + x^{75} + x^{74} + x^{71} + x^{70} + x^{69} + x^{66} + x^{64}\\
&\quad + x^{63} + x^{60} + x^{58} + x^{57} + x^{54} + x^{51} + x^{50} + x^{49} + x^{48} + x^{45} + x^{44}\\
&\quad + x^{43} + x^{38} + x^{36} + x^{35} + x^{32} + x^{29} + x^{28} + x^{27} + x^{26} + x^{23} + x^{21}\\
&\quad + x^{20} + x^{19} + x^{16} + x^{13} + x^{12} + x^{11} + x^{6} + x^{4} + x^{3},\\
p_{12}(x) &= x^{147} + x^{145} + x^{144} + x^{143} + x^{142} + x^{140} + x^{139} + x^{137} + x^{136} + x^{134}\\
&\quad + x^{133} + x^{130} + x^{129} + x^{126} + x^{125} + x^{123} + x^{122} + x^{120} + x^{119}\\
&\quad + x^{117} + x^{116} + x^{114} + x^{113} + x^{111} + x^{110} + x^{108} + x^{107} + x^{105}\\
&\quad + x^{104} + x^{102} + x^{101} + x^{98} + x^{97} + x^{94} + x^{93} + x^{91} + x^{90} + x^{88} + x^{87}\\
&\quad + x^{85} + x^{84} + x^{82} + x^{81} + x^{79} + x^{78} + x^{76} + x^{75} + x^{73} + x^{72} + x^{70}\\
&\quad + x^{69} + x^{66} + x^{65} + x^{62} + x^{61} + x^{59} + x^{58} + x^{56} + x^{55} + x^{53} + x^{52}\\
&\quad + x^{50} + x^{49} + x^{47} + x^{46} + x^{44} + x^{43} + x^{41} + x^{40} + x^{38} + x^{37} + x^{35}\\
&\quad + x^{34} + x^{32} + x^{31} + x^{29} + x^{28} + x^{26} + x^{25} + x^{23} + x^{22} + x^{20} + x^{19}\\
&\quad + x^{17} + x^{16} + x^{14} + x^{13} + x^{11} + x^{10} + x^{8} + x^{7} + x^{5} + x^{4} + x^{3} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 0, 0, 1, 1].

For $\phi(W^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{162} + x^{160} + x^{158} + x^{154} + x^{152} + x^{148} + x^{140} + x^{138} + x^{136} + x^{134}\\
&\quad + x^{132} + x^{128} + x^{126} + x^{124} + x^{122} + x^{118} + x^{112} + x^{110} + x^{106}\\
&\quad + x^{102} + x^{92} + x^{88} + x^{86} + x^{84} + x^{78} + x^{76} + x^{74} + x^{72} + x^{66} + x^{64}\\
&\quad + x^{60} + x^{56} + x^{54} + x^{52} + x^{48} + x^{44} + x^{42},\\
p_3(x) &= x^{156} + x^{154} + x^{152} + x^{148} + x^{138} + x^{136} + x^{134} + x^{130} + x^{126} + x^{120}\\
&\quad + x^{114} + x^{102} + x^{88} + x^{86} + x^{74} + x^{72} + x^{70} + x^{64} + x^{60} + x^{58} + x^{56}\\
&\quad + x^{50} + x^{46} + x^{42} + x^{40} + x^{36} + x^{26} + x^{24} + x^{22} + x^{20} + x^{16} + x^{14}\\
&\quad + x^{8} + x^{6},\\
p_6(x) &= x^{158} + x^{154} + x^{148} + x^{142} + x^{140} + x^{136} + x^{134} + x^{130} + x^{122} + x^{120}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{116} + x^{114} + x^{112} + x^{108} + x^{102} + x^{94} + x^{88} + x^{86} + x^{84} + x^{82}\\
&+ x^{78} + x^{76} + x^{74} + x^{68} + x^{64} + x^{62} + x^{58} + x^{48} + x^{44} + x^{40} + x^{30}\\
&+ x^{28} + x^{20} + x^{18} + x^{12} + x^{8} + x^{6} + 1,\\
p_9(x) &= x^{156} + x^{154} + x^{150} + x^{148} + x^{146} + x^{144} + x^{136} + x^{124} + x^{112} + x^{110}\\
&+ x^{102} + x^{100} + x^{96} + x^{94} + x^{90} + x^{86} + x^{80} + x^{76} + x^{62} + x^{56} + x^{54}\\
&+ x^{46} + x^{44} + x^{42} + x^{36} + x^{32} + x^{28} + x^{24} + x^{22} + x^{20} + x^{16} + x^{10}\\
&+ x^{4} + x^{2},\\
p_{12}(x) &= x^{162} + x^{160} + x^{154} + x^{152} + x^{138} + x^{136} + x^{130} + x^{128} + x^{114} + x^{112}\\
&+ x^{106} + x^{104} + x^{98} + x^{96} + x^{82} + x^{80} + x^{74} + x^{72} + x^{66} + x^{64} + x^{18}\\
&+ x^{16} + x^{10} + x^{8} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0].

For $\phi(M^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{147} + x^{145} + x^{144} + x^{142} + x^{135} + x^{131} + x^{128} + x^{127} + x^{126} + x^{125}\\
&+ x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{117} + x^{115} + x^{113} + x^{112}\\
&+ x^{111} + x^{110} + x^{106} + x^{101} + x^{100} + x^{97} + x^{91} + x^{90} + x^{88} + x^{85}\\
&+ x^{80} + x^{76} + x^{74} + x^{73} + x^{72} + x^{70} + x^{69} + x^{67} + x^{66} + x^{64} + x^{62}\\
&+ x^{61} + x^{59} + x^{58} + x^{57} + x^{54} + x^{53} + x^{52} + x^{49} + x^{47} + x^{45} + x^{44}\\
&+ x^{42} + x^{40} + x^{39} + x^{35} + x^{34} + x^{33} + x^{31} + x^{28} + x^{27} + x^{25} + x^{24}\\
&+ x^{23} + x^{18} + x^{16} + x^{15} + x^{12},\\
p_3(x) &= x^{135} + x^{133} + x^{132} + x^{130} + x^{127} + x^{125} + x^{124} + x^{122} + x^{121} + x^{120}\\
&+ x^{119} + x^{118} + x^{116} + x^{114} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109}\\
&+ x^{106} + x^{99} + x^{98} + x^{96} + x^{95} + x^{94} + x^{93} + x^{88} + x^{87} + x^{85} + x^{83}\\
&+ x^{82} + x^{79} + x^{76} + x^{73} + x^{69} + x^{66} + x^{65} + x^{63} + x^{62} + x^{61} + x^{60}\\
&+ x^{57} + x^{55} + x^{53} + x^{48} + x^{44} + x^{43} + x^{37} + x^{35} + x^{34} + x^{32} + x^{30}\\
&+ x^{29} + x^{26} + x^{25} + x^{22} + x^{19} + x^{18} + x^{17} + x^{15} + x^{11} + x^{8},\\
p_6(x) &= x^{143} + x^{140} + x^{135} + x^{133} + x^{130} + x^{126} + x^{124} + x^{116} + x^{114} + x^{111}\\
&+ x^{107} + x^{104} + x^{101} + x^{100} + x^{99} + x^{98} + x^{95} + x^{93} + x^{92} + x^{91}\\
&+ x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{76}\\
&+ x^{75} + x^{71} + x^{69} + x^{68} + x^{65} + x^{64} + x^{62} + x^{61} + x^{56} + x^{52} + x^{51}\\
&+ x^{49} + x^{45} + x^{39} + x^{38} + x^{36} + x^{35} + x^{32} + x^{30} + x^{29} + x^{28} + x^{27}\\
&+ x^{24} + x^{23} + x^{21} + x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{10} + x^{8}\\
&+ x^{5} + x^{3} + x^{2} + 1,\\
p_9(x) &= x^{135} + x^{133} + x^{132} + x^{131} + x^{130} + x^{128} + x^{127} + x^{125} + x^{124} + x^{122}\\
&+ x^{121} + x^{119} + x^{118} + x^{116} + x^{103} + x^{101} + x^{100} + x^{99} + x^{98} + x^{96}\\
&+ x^{95} + x^{93} + x^{92} + x^{90} + x^{89} + x^{87} + x^{86} + x^{84} + x^{71} + x^{69} + x^{68}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{67}+x^{66}+x^{64}+x^{63}+x^{61}+x^{60}+x^{58}+x^{57}+x^{55}+x^{54}+x^{52}\\
&+x^{39}+x^{37}+x^{36}+x^{35}+x^{34}+x^{32}+x^{31}+x^{29}+x^{28}+x^{26}+x^{25}\\
&+x^{22}+x^{21}+x^{19}+x^{18}+x^{16}+x^{15}+x^{13}+x^{12}+x^{10}+x^{9}+x^{7}+x^{6}\\
&+x^{4},\\
p_{12}(x)=\ &x^{147}+x^{145}+x^{144}+x^{143}+x^{142}+x^{140}+x^{139}+x^{137}+x^{136}+x^{135}\\
&+x^{134}+x^{132}+x^{131}+x^{129}+x^{128}+x^{127}+x^{126}+x^{124}+x^{123}\\
&+x^{121}+x^{120}+x^{118}+x^{117}+x^{114}+x^{113}+x^{111}+x^{110}+x^{108}\\
&+x^{107}+x^{105}+x^{104}+x^{103}+x^{102}+x^{100}+x^{99}+x^{97}+x^{96}+x^{95}\\
&+x^{94}+x^{92}+x^{91}+x^{89}+x^{88}+x^{86}+x^{85}+x^{82}+x^{81}+x^{79}+x^{78}\\
&+x^{76}+x^{75}+x^{73}+x^{72}+x^{71}+x^{70}+x^{68}+x^{67}+x^{65}+x^{64}+x^{63}\\
&+x^{62}+x^{60}+x^{59}+x^{57}+x^{56}+x^{54}+x^{53}+x^{50}+x^{49}+x^{47}+x^{46}\\
&+x^{44}+x^{43}+x^{41}+x^{40}+x^{39}+x^{38}+x^{36}+x^{23}+x^{21}+x^{20}+x^{18}\\
&+x^{17}+x^{15}+x^{14}+x^{12}+x^{11}+x^{9}+x^{8}+x^{6}+x^{5}+x^{3}+x^{2}+1,
\end{aligned}
$$

and the initial terms are [1, 0, 1, 0, 1, 0, 1, 1, 0, 0, 1, 0, 1, 0, 0, 1].

For $\phi(M^o, 0, 1)$:

$$
\begin{aligned}
p_0(x)=\ &x^{144}+x^{135}+x^{134}+x^{133}+x^{131}+x^{126}+x^{125}+x^{124}+x^{122}+x^{120}\\
&+x^{119}+x^{118}+x^{117}+x^{115}+x^{113}+x^{112}+x^{110}+x^{107}+x^{105}\\
&+x^{103}+x^{101}+x^{100}+x^{97}+x^{93}+x^{90}+x^{88}+x^{84}+x^{83}+x^{82}+x^{81}\\
&+x^{74}+x^{72}+x^{70}+x^{69}+x^{68}+x^{65}+x^{64}+x^{63}+x^{62}+x^{61}+x^{58}\\
&+x^{57}+x^{54}+x^{44}+x^{42}+x^{41}+x^{40}+x^{31}+x^{27}+x^{23}+x^{22}+x^{21}\\
&+x^{18},\\
p_3(x)=\ &x^{117}+x^{112}+x^{111}+x^{110}+x^{109}+x^{108}+x^{107}+x^{106}+x^{105}+x^{104}\\
&+x^{103}+x^{102}+x^{101}+x^{100}+x^{99}+x^{98}+x^{97}+x^{96}+x^{95}+x^{94}\\
&+x^{93}+x^{92}+x^{91}+x^{90}+x^{89}+x^{88}+x^{87}+x^{86}+x^{85}+x^{84}+x^{83}\\
&+x^{82}+x^{81}+x^{80}+x^{79}+x^{78}+x^{76}+x^{75}+x^{74}+x^{73}+x^{69}+x^{64}\\
&+x^{63}+x^{62}+x^{61}+x^{60}+x^{59}+x^{58}+x^{57}+x^{56}+x^{55}+x^{54}+x^{53}\\
&+x^{52}+x^{51}+x^{50}+x^{49}+x^{48}+x^{47}+x^{46}+x^{45}+x^{44}+x^{43}+x^{42}\\
&+x^{41}+x^{40}+x^{39}+x^{38}+x^{36}+x^{35}+x^{34}+x^{33}+x^{29}+x^{24}+x^{23}\\
&+x^{22}+x^{21}+x^{20}+x^{19}+x^{18}+x^{17}+x^{16}+x^{15}+x^{14}+x^{12}+x^{11}\\
&+x^{10}+x^{9},\\
p_6(x)=\ &x^{126}+x^{122}+x^{116}+x^{114}+x^{110}+x^{100}+x^{96}+x^{92}+x^{90}+x^{86}\\
&+x^{84}+x^{80}+x^{78}+x^{74}+x^{72}+x^{68}+x^{66}+x^{62}+x^{60}+x^{58}+x^{56}\\
&+x^{54}+x^{46}+x^{36}+x^{34}+x^{32}+x^{30}+x^{28}+x^{24}+x^{22}+x^{20}+x^{14}\\
&+x^{8}+x^{6},\\
p_9(x)=\ &x^{135}+x^{130}+x^{129}+x^{128}+x^{126}+x^{124}+x^{121}+x^{120}+x^{116}+x^{115}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{103} + x^{98} + x^{97} + x^{96} + x^{94} + x^{92} + x^{89} + x^{88} + x^{84} + x^{83} + x^{71}\\
&+ x^{66} + x^{65} + x^{64} + x^{62} + x^{60} + x^{57} + x^{56} + x^{52} + x^{51} + x^{39} + x^{34}\\
&+ x^{33} + x^{32} + x^{30} + x^{28} + x^{25} + x^{24} + x^{23} + x^{20} + x^{19} + x^{18} + x^{17}\\
&+ x^{16} + x^{14} + x^{12} + x^{9} + x^{8} + x^{4} + x^{3},\\
p_{12}(x) = {}& x^{144} + x^{140} + x^{136} + x^{132} + x^{128} + x^{124} + x^{120} + x^{108} + x^{104} + x^{100}\\
&+ x^{96} + x^{92} + x^{88} + x^{76} + x^{72} + x^{68} + x^{64} + x^{60} + x^{56} + x^{44} + x^{40}\\
&+ x^{36} + x^{20} + x^{12} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 1, 1, 1].

For $\phi(M^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) = {}& x^{164} + x^{155} + x^{154} + x^{153} + x^{152} + x^{151} + x^{148} + x^{145} + x^{144} + x^{143}\\
&+ x^{142} + x^{141} + x^{140} + x^{139} + x^{138} + x^{136} + x^{134} + x^{132} + x^{131}\\
&+ x^{130} + x^{126} + x^{125} + x^{123} + x^{122} + x^{121} + x^{118} + x^{117} + x^{116}\\
&+ x^{115} + x^{114} + x^{113} + x^{111} + x^{110} + x^{108} + x^{104} + x^{103} + x^{102}\\
&+ x^{101} + x^{100} + x^{96} + x^{93} + x^{92} + x^{89} + x^{84} + x^{83} + x^{81} + x^{77} + x^{73}\\
&+ x^{71} + x^{70} + x^{69} + x^{65} + x^{63} + x^{62} + x^{61} + x^{60} + x^{57} + x^{52} + x^{50}\\
&+ x^{47} + x^{46} + x^{44} + x^{42} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{31} + x^{30}\\
&+ x^{29} + x^{27},\\
p_3(x) = {}& x^{137} + x^{132} + x^{131} + x^{130} + x^{129} + x^{128} + x^{127} + x^{126} + x^{124} + x^{123}\\
&+ x^{122} + x^{116} + x^{115} + x^{114} + x^{112} + x^{111} + x^{110} + x^{105} + x^{104}\\
&+ x^{103} + x^{102} + x^{96} + x^{95} + x^{94} + x^{92} + x^{91} + x^{90} + x^{89} + x^{73} + x^{68}\\
&+ x^{67} + x^{66} + x^{64} + x^{63} + x^{62} + x^{56} + x^{55} + x^{54} + x^{52} + x^{51} + x^{50}\\
&+ x^{49} + x^{25} + x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{12} + x^{11}\\
&+ x^{10} + x^{9},\\
p_6(x) = {}& x^{146} + x^{142} + x^{136} + x^{130} + x^{126} + x^{124} + x^{116} + x^{114} + x^{106} + x^{96}\\
&+ x^{94} + x^{92} + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{80} + x^{76} + x^{74} + x^{72}\\
&+ x^{66} + x^{62} + x^{60} + x^{58} + x^{54} + x^{52} + x^{48} + x^{42} + x^{40} + x^{38} + x^{34}\\
&+ x^{32} + x^{30} + x^{28} + x^{26} + x^{18} + x^{14} + x^{8} + x^{6},\\
p_9(x) = {}& x^{155} + x^{150} + x^{149} + x^{148} + x^{146} + x^{144} + x^{143} + x^{141} + x^{140} + x^{139}\\
&+ x^{138} + x^{137} + x^{135} + x^{133} + x^{131} + x^{130} + x^{129} + x^{127} + x^{126}\\
&+ x^{124} + x^{123} + x^{121} + x^{120} + x^{118} + x^{117} + x^{115} + x^{114} + x^{112}\\
&+ x^{111} + x^{109} + x^{108} + x^{107} + x^{106} + x^{105} + x^{103} + x^{101} + x^{99} + x^{98}\\
&+ x^{97} + x^{95} + x^{94} + x^{92} + x^{91} + x^{89} + x^{88} + x^{86} + x^{85} + x^{83} + x^{82}\\
&+ x^{80} + x^{79} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{71} + x^{69} + x^{67} + x^{66}\\
&+ x^{65} + x^{63} + x^{62} + x^{60} + x^{59} + x^{57} + x^{56} + x^{54} + x^{53} + x^{51} + x^{50}\\
&+ x^{48} + x^{47} + x^{45} + x^{44} + x^{42} + x^{41} + x^{39} + x^{38} + x^{36} + x^{35} + x^{33}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{32} + x^{30} + x^{29} + x^{27} + x^{26} + x^{24} + x^{23} + x^{21} + x^{20} + x^{18} + x^{17}\\
&+ x^{15} + x^{14} + x^{12} + x^{9} + x^{8} + x^{4} + x^{3},\\
p_{12}(x) = {}& x^{164} + x^{160} + x^{156} + x^{148} + x^{144} + x^{132} + x^{128} + x^{124} + x^{120} + x^{116}\\
&+ x^{100} + x^{96} + x^{92} + x^{88} + x^{84} + x^{68} + x^{64} + x^{60} + x^{56} + x^{48} + x^{44}\\
&+ x^{36} + x^{32} + x^{24} + x^{16} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 0, 0, 0, 1, 0].

For $\phi(M^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) = {}& x^{147} + x^{145} + x^{144} + x^{142} + x^{133} + x^{132} + x^{130} + x^{129} + x^{121} + x^{119}\\
&+ x^{117} + x^{116} + x^{115} + x^{113} + x^{112} + x^{110} + x^{109} + x^{107} + x^{105}\\
&+ x^{104} + x^{103} + x^{100} + x^{98} + x^{97} + x^{93} + x^{92} + x^{91} + x^{90} + x^{88} + x^{83}\\
&+ x^{82} + x^{78} + x^{77} + x^{75} + x^{73} + x^{72} + x^{71} + x^{65} + x^{62} + x^{58} + x^{57}\\
&+ x^{56} + x^{52} + x^{43} + x^{39} + x^{38} + x^{36} + x^{35} + x^{33},\\
p_3(x) = {}& x^{135} + x^{133} + x^{132} + x^{130} + x^{127} + x^{125} + x^{124} + x^{123} + x^{122} + x^{117}\\
&+ x^{113} + x^{112} + x^{111} + x^{110} + x^{107} + x^{104} + x^{101} + x^{95} + x^{94} + x^{91}\\
&+ x^{90} + x^{87} + x^{83} + x^{81} + x^{78} + x^{71} + x^{70} + x^{68} + x^{65} + x^{63} + x^{62}\\
&+ x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{51} + x^{50} + x^{45} + x^{44} + x^{43}\\
&+ x^{42} + x^{41} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31}\\
&+ x^{28} + x^{25} + x^{22} + x^{21} + x^{19} + x^{14} + x^{12},\\
p_6(x) = {}& x^{143} + x^{140} + x^{132} + x^{131} + x^{129} + x^{128} + x^{124} + x^{123} + x^{120} + x^{119}\\
&+ x^{114} + x^{113} + x^{110} + x^{109} + x^{108} + x^{106} + x^{103} + x^{99} + x^{97} + x^{94}\\
&+ x^{90} + x^{84} + x^{78} + x^{77} + x^{74} + x^{73} + x^{72} + x^{70} + x^{66} + x^{65} + x^{64}\\
&+ x^{63} + x^{62} + x^{60} + x^{58} + x^{56} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{45}\\
&+ x^{43} + x^{40} + x^{38} + x^{37} + x^{35} + x^{34} + x^{33} + x^{32} + x^{28} + x^{27} + x^{26}\\
&+ x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18} + x^{16} + x^{15} + x^{14}\\
&+ x^{13} + x^{11} + x^{7} + x^{5} + x^{4} + x^{3} + x^{2} + 1,\\
p_9(x) = {}& x^{135} + x^{133} + x^{132} + x^{131} + x^{130} + x^{128} + x^{127} + x^{125} + x^{124} + x^{122}\\
&+ x^{121} + x^{119} + x^{118} + x^{116} + x^{103} + x^{101} + x^{100} + x^{99} + x^{98} + x^{96}\\
&+ x^{95} + x^{93} + x^{92} + x^{90} + x^{89} + x^{87} + x^{86} + x^{84} + x^{71} + x^{69} + x^{68}\\
&+ x^{67} + x^{66} + x^{64} + x^{63} + x^{61} + x^{60} + x^{58} + x^{57} + x^{55} + x^{54} + x^{52}\\
&+ x^{39} + x^{37} + x^{36} + x^{35} + x^{34} + x^{32} + x^{31} + x^{29} + x^{28} + x^{26} + x^{25}\\
&+ x^{22} + x^{21} + x^{19} + x^{18} + x^{16} + x^{15} + x^{13} + x^{12} + x^{10} + x^{9} + x^{7} + x^{6}\\
&+ x^{4},\\
p_{12}(x) = {}& x^{147} + x^{145} + x^{144} + x^{143} + x^{142} + x^{140} + x^{139} + x^{137} + x^{136} + x^{135}\\
&+ x^{134} + x^{132} + x^{131} + x^{129} + x^{128} + x^{127} + x^{126} + x^{124} + x^{123}\\
&+ x^{121} + x^{120} + x^{118} + x^{117} + x^{114} + x^{113} + x^{111} + x^{110} + x^{108}
\end{aligned}
$$

$$+ x^{107} + x^{105} + x^{104} + x^{103} + x^{102} + x^{100} + x^{99} + x^{97} + x^{96} + x^{95} + x^{94} + x^{92} + x^{91} + x^{89} + x^{88} + x^{86} + x^{85} + x^{82} + x^{81} + x^{79} + x^{78} + x^{76} + x^{75} + x^{73} + x^{72} + x^{71} + x^{70} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{62} + x^{60} + x^{59} + x^{57} + x^{56} + x^{54} + x^{53} + x^{50} + x^{49} + x^{47} + x^{46} + x^{44} + x^{43} + x^{41} + x^{40} + x^{39} + x^{38} + x^{36} + x^{23} + x^{21} + x^{20} + x^{18} + x^{17} + x^{15} + x^{14} + x^{12} + x^{11} + x^{9} + x^{8} + x^{6} + x^{5} + x^{3} + x^{2} + 1,$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 1].

For $\phi(W^o, 0, 0)$:

$$p_0(x) = x^{147} + x^{145} + x^{144} + x^{142} + x^{135} + x^{131} + x^{128} + x^{127} + x^{126} + x^{125} + x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{117} + x^{115} + x^{113} + x^{112} + x^{111} + x^{110} + x^{106} + x^{101} + x^{100} + x^{97} + x^{91} + x^{90} + x^{88} + x^{85} + x^{80} + x^{76} + x^{74} + x^{73} + x^{72} + x^{70} + x^{69} + x^{67} + x^{66} + x^{64} + x^{62} + x^{61} + x^{59} + x^{58} + x^{57} + x^{54} + x^{53} + x^{52} + x^{49} + x^{47} + x^{45} + x^{44} + x^{42} + x^{40} + x^{39} + x^{35} + x^{34} + x^{33} + x^{31} + x^{28} + x^{27} + x^{25} + x^{24} + x^{23} + x^{18} + x^{16} + x^{15} + x^{12},$$

$$p_3(x) = x^{135} + x^{133} + x^{132} + x^{130} + x^{127} + x^{125} + x^{124} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{116} + x^{114} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{106} + x^{99} + x^{98} + x^{96} + x^{95} + x^{94} + x^{93} + x^{88} + x^{87} + x^{85} + x^{83} + x^{82} + x^{79} + x^{76} + x^{73} + x^{69} + x^{66} + x^{65} + x^{63} + x^{62} + x^{61} + x^{60} + x^{57} + x^{55} + x^{53} + x^{48} + x^{44} + x^{43} + x^{37} + x^{35} + x^{34} + x^{32} + x^{30} + x^{29} + x^{26} + x^{25} + x^{22} + x^{19} + x^{18} + x^{17} + x^{15} + x^{11} + x^{8},$$

$$p_6(x) = x^{143} + x^{140} + x^{135} + x^{133} + x^{130} + x^{126} + x^{124} + x^{116} + x^{114} + x^{111} + x^{107} + x^{104} + x^{101} + x^{100} + x^{99} + x^{98} + x^{95} + x^{93} + x^{92} + x^{91} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{76} + x^{75} + x^{71} + x^{69} + x^{68} + x^{65} + x^{64} + x^{62} + x^{61} + x^{56} + x^{52} + x^{51} + x^{49} + x^{45} + x^{39} + x^{38} + x^{36} + x^{35} + x^{32} + x^{30} + x^{29} + x^{28} + x^{27} + x^{24} + x^{23} + x^{21} + x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{10} + x^{8} + x^{5} + x^{3} + x^{2} + 1,$$

$$p_9(x) = x^{135} + x^{133} + x^{132} + x^{131} + x^{130} + x^{128} + x^{127} + x^{125} + x^{124} + x^{122} + x^{121} + x^{119} + x^{118} + x^{116} + x^{103} + x^{101} + x^{100} + x^{99} + x^{98} + x^{96} + x^{95} + x^{93} + x^{92} + x^{90} + x^{89} + x^{87} + x^{86} + x^{84} + x^{71} + x^{69} + x^{68} + x^{67} + x^{66} + x^{64} + x^{63} + x^{61} + x^{60} + x^{58} + x^{57} + x^{55} + x^{54} + x^{52} + x^{39} + x^{37} + x^{36} + x^{35} + x^{34} + x^{32} + x^{31} + x^{29} + x^{28} + x^{26} + x^{25} + x^{22} + x^{21} + x^{19} + x^{18} + x^{16} + x^{15} + x^{13} + x^{12} + x^{10} + x^{9} + x^{7} + x^{6} + x^{4},$$

$$\begin{aligned}p_{12}(x) = {} & x^{147} + x^{145} + x^{144} + x^{143} + x^{142} + x^{140} + x^{139} + x^{137} + x^{136} + x^{135} \\ & + x^{134} + x^{132} + x^{131} + x^{129} + x^{128} + x^{127} + x^{126} + x^{124} + x^{123} \\ & + x^{121} + x^{120} + x^{118} + x^{117} + x^{114} + x^{113} + x^{111} + x^{110} + x^{108} \\ & + x^{107} + x^{105} + x^{104} + x^{103} + x^{102} + x^{100} + x^{99} + x^{97} + x^{96} + x^{95} \\ & + x^{94} + x^{92} + x^{91} + x^{89} + x^{88} + x^{86} + x^{85} + x^{82} + x^{81} + x^{79} + x^{78} \\ & + x^{76} + x^{75} + x^{73} + x^{72} + x^{71} + x^{70} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} \\ & + x^{62} + x^{60} + x^{59} + x^{57} + x^{56} + x^{54} + x^{53} + x^{50} + x^{49} + x^{47} + x^{46} \\ & + x^{44} + x^{43} + x^{41} + x^{40} + x^{39} + x^{38} + x^{36} + x^{23} + x^{21} + x^{20} + x^{18} \\ & + x^{17} + x^{15} + x^{14} + x^{12} + x^{11} + x^{9} + x^{8} + x^{6} + x^{5} + x^{3} + x^{2} + 1,\end{aligned}$$

and the initial terms are [1, 0, 1, 0, 1, 0, 1, 1, 0, 0, 1, 0, 1, 0, 0, 1].

For $\phi(W^o, 0, 1)$:

$$\begin{aligned}p_{0}(x) = {} & x^{164} + x^{155} + x^{154} + x^{153} + x^{152} + x^{151} + x^{148} + x^{145} + x^{144} + x^{143} \\ & + x^{142} + x^{141} + x^{140} + x^{139} + x^{138} + x^{136} + x^{134} + x^{132} + x^{131} \\ & + x^{130} + x^{126} + x^{125} + x^{123} + x^{122} + x^{121} + x^{118} + x^{117} + x^{116} \\ & + x^{115} + x^{114} + x^{113} + x^{111} + x^{110} + x^{108} + x^{104} + x^{103} + x^{102} \\ & + x^{101} + x^{100} + x^{96} + x^{93} + x^{92} + x^{89} + x^{84} + x^{83} + x^{81} + x^{77} + x^{73} \\ & + x^{71} + x^{70} + x^{69} + x^{65} + x^{63} + x^{62} + x^{61} + x^{60} + x^{57} + x^{52} + x^{50} \\ & + x^{47} + x^{46} + x^{44} + x^{42} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{31} + x^{30} \\ & + x^{29} + x^{27}, \\ p_{3}(x) = {} & x^{137} + x^{132} + x^{131} + x^{130} + x^{129} + x^{128} + x^{127} + x^{126} + x^{124} + x^{123} \\ & + x^{122} + x^{116} + x^{115} + x^{114} + x^{112} + x^{111} + x^{110} + x^{105} + x^{104} \\ & + x^{103} + x^{102} + x^{96} + x^{95} + x^{94} + x^{92} + x^{91} + x^{90} + x^{89} + x^{73} + x^{68} \\ & + x^{67} + x^{66} + x^{64} + x^{63} + x^{62} + x^{56} + x^{55} + x^{54} + x^{52} + x^{51} + x^{50} \\ & + x^{49} + x^{25} + x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{12} + x^{11} \\ & + x^{10} + x^{9}, \\ p_{6}(x) = {} & x^{146} + x^{142} + x^{136} + x^{130} + x^{126} + x^{124} + x^{116} + x^{114} + x^{106} + x^{96} \\ & + x^{94} + x^{92} + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{80} + x^{76} + x^{74} + x^{72} \\ & + x^{66} + x^{62} + x^{60} + x^{58} + x^{54} + x^{52} + x^{48} + x^{42} + x^{40} + x^{38} + x^{34} \\ & + x^{32} + x^{30} + x^{28} + x^{26} + x^{18} + x^{14} + x^{8} + x^{6}, \\ p_{9}(x) = {} & x^{155} + x^{150} + x^{149} + x^{148} + x^{146} + x^{144} + x^{143} + x^{141} + x^{140} + x^{139} \\ & + x^{138} + x^{137} + x^{135} + x^{133} + x^{131} + x^{130} + x^{129} + x^{127} + x^{126} \\ & + x^{124} + x^{123} + x^{121} + x^{120} + x^{118} + x^{117} + x^{115} + x^{114} + x^{112} \\ & + x^{111} + x^{109} + x^{108} + x^{107} + x^{106} + x^{105} + x^{103} + x^{101} + x^{99} + x^{98} \\ & + x^{97} + x^{95} + x^{94} + x^{92} + x^{91} + x^{89} + x^{88} + x^{86} + x^{85} + x^{83} + x^{82} \\ & + x^{80} + x^{79} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{71} + x^{69} + x^{67} + x^{66}\end{aligned}$$

$$
\begin{aligned}
&+ x^{65} + x^{63} + x^{62} + x^{60} + x^{59} + x^{57} + x^{56} + x^{54} + x^{53} + x^{51} + x^{50}\\
&+ x^{48} + x^{47} + x^{45} + x^{44} + x^{42} + x^{41} + x^{39} + x^{38} + x^{36} + x^{35} + x^{33}\\
&+ x^{32} + x^{30} + x^{29} + x^{27} + x^{26} + x^{24} + x^{23} + x^{21} + x^{20} + x^{18} + x^{17}\\
&+ x^{15} + x^{14} + x^{12} + x^{9} + x^{8} + x^{4} + x^{3},\\
p_{12}(x) &= x^{164} + x^{160} + x^{156} + x^{148} + x^{144} + x^{132} + x^{128} + x^{124} + x^{120} + x^{116}\\
&+ x^{100} + x^{96} + x^{92} + x^{88} + x^{84} + x^{68} + x^{64} + x^{60} + x^{56} + x^{48} + x^{44}\\
&+ x^{36} + x^{32} + x^{24} + x^{16} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 0, 0, 0, 1, 0].

For $\phi(W^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{144} + x^{135} + x^{134} + x^{133} + x^{131} + x^{126} + x^{125} + x^{124} + x^{122} + x^{120}\\
&+ x^{119} + x^{118} + x^{117} + x^{115} + x^{113} + x^{112} + x^{110} + x^{107} + x^{105}\\
&+ x^{103} + x^{101} + x^{100} + x^{97} + x^{93} + x^{90} + x^{88} + x^{84} + x^{83} + x^{82} + x^{81}\\
&+ x^{74} + x^{72} + x^{70} + x^{69} + x^{68} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{58}\\
&+ x^{57} + x^{54} + x^{44} + x^{42} + x^{41} + x^{40} + x^{31} + x^{27} + x^{23} + x^{22} + x^{21}\\
&+ x^{18},\\
p_3(x) &= x^{117} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} + x^{107} + x^{106} + x^{105} + x^{104}\\
&+ x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94}\\
&+ x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83}\\
&+ x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{76} + x^{75} + x^{74} + x^{73} + x^{69} + x^{64}\\
&+ x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53}\\
&+ x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42}\\
&+ x^{41} + x^{40} + x^{39} + x^{38} + x^{36} + x^{35} + x^{34} + x^{33} + x^{29} + x^{24} + x^{23}\\
&+ x^{22} + x^{21} + x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{12} + x^{11}\\
&+ x^{10} + x^{9},\\
p_6(x) &= x^{126} + x^{122} + x^{116} + x^{114} + x^{110} + x^{100} + x^{96} + x^{92} + x^{90} + x^{86}\\
&+ x^{84} + x^{80} + x^{78} + x^{74} + x^{72} + x^{68} + x^{66} + x^{62} + x^{60} + x^{58} + x^{56}\\
&+ x^{54} + x^{46} + x^{36} + x^{34} + x^{32} + x^{30} + x^{28} + x^{24} + x^{22} + x^{20} + x^{14}\\
&+ x^{8} + x^{6},\\
p_9(x) &= x^{135} + x^{130} + x^{129} + x^{128} + x^{126} + x^{124} + x^{121} + x^{120} + x^{116} + x^{115}\\
&+ x^{103} + x^{98} + x^{97} + x^{96} + x^{94} + x^{92} + x^{89} + x^{88} + x^{84} + x^{83} + x^{71}\\
&+ x^{66} + x^{65} + x^{64} + x^{62} + x^{60} + x^{57} + x^{56} + x^{52} + x^{51} + x^{39} + x^{34}\\
&+ x^{33} + x^{32} + x^{30} + x^{28} + x^{25} + x^{24} + x^{23} + x^{20} + x^{19} + x^{18} + x^{17}\\
&+ x^{16} + x^{14} + x^{12} + x^{9} + x^{8} + x^{4} + x^{3},\\
p_{12}(x) &= x^{144} + x^{140} + x^{136} + x^{132} + x^{128} + x^{124} + x^{120} + x^{108} + x^{104} + x^{100}\\
&+ x^{96} + x^{92} + x^{88} + x^{76} + x^{72} + x^{68} + x^{64} + x^{60} + x^{56} + x^{44} + x^{40}
\end{aligned}
$$

$$+ x^{36} + x^{20} + x^{12} + x^{8} + 1,$$

and the initial terms are [0, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 1, 1, 1].

For $\phi(W^o, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{147} + x^{145} + x^{144} + x^{142} + x^{133} + x^{132} + x^{130} + x^{129} + x^{121} + x^{119} \\
&+ x^{117} + x^{116} + x^{115} + x^{113} + x^{112} + x^{110} + x^{109} + x^{107} + x^{105} \\
&+ x^{104} + x^{103} + x^{100} + x^{98} + x^{97} + x^{93} + x^{92} + x^{91} + x^{90} + x^{88} + x^{83} \\
&+ x^{82} + x^{78} + x^{77} + x^{75} + x^{73} + x^{72} + x^{71} + x^{65} + x^{62} + x^{58} + x^{57} \\
&+ x^{56} + x^{52} + x^{43} + x^{39} + x^{38} + x^{36} + x^{35} + x^{33}, \\
p_3(x) &= x^{135} + x^{133} + x^{132} + x^{130} + x^{127} + x^{125} + x^{124} + x^{123} + x^{122} + x^{117} \\
&+ x^{113} + x^{112} + x^{111} + x^{110} + x^{107} + x^{104} + x^{101} + x^{95} + x^{94} + x^{91} \\
&+ x^{90} + x^{87} + x^{83} + x^{81} + x^{78} + x^{71} + x^{70} + x^{68} + x^{65} + x^{63} + x^{62} \\
&+ x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{51} + x^{50} + x^{45} + x^{44} + x^{43} \\
&+ x^{42} + x^{41} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} \\
&+ x^{28} + x^{25} + x^{22} + x^{21} + x^{19} + x^{14} + x^{12}, \\
p_6(x) &= x^{143} + x^{140} + x^{132} + x^{131} + x^{129} + x^{128} + x^{124} + x^{123} + x^{120} + x^{119} \\
&+ x^{114} + x^{113} + x^{110} + x^{109} + x^{108} + x^{106} + x^{103} + x^{99} + x^{97} + x^{94} \\
&+ x^{90} + x^{84} + x^{78} + x^{77} + x^{74} + x^{73} + x^{72} + x^{70} + x^{66} + x^{65} + x^{64} \\
&+ x^{63} + x^{62} + x^{60} + x^{58} + x^{56} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{45} \\
&+ x^{43} + x^{40} + x^{38} + x^{37} + x^{35} + x^{34} + x^{33} + x^{32} + x^{28} + x^{27} + x^{26} \\
&+ x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18} + x^{16} + x^{15} + x^{14} \\
&+ x^{13} + x^{11} + x^{7} + x^{5} + x^{4} + x^{3} + x^{2} + 1, \\
p_9(x) &= x^{135} + x^{133} + x^{132} + x^{131} + x^{130} + x^{128} + x^{127} + x^{125} + x^{124} + x^{122} \\
&+ x^{121} + x^{119} + x^{118} + x^{116} + x^{103} + x^{101} + x^{100} + x^{99} + x^{98} + x^{96} \\
&+ x^{95} + x^{93} + x^{92} + x^{90} + x^{89} + x^{87} + x^{86} + x^{84} + x^{71} + x^{69} + x^{68} \\
&+ x^{67} + x^{66} + x^{64} + x^{63} + x^{61} + x^{60} + x^{58} + x^{57} + x^{55} + x^{54} + x^{52} \\
&+ x^{39} + x^{37} + x^{36} + x^{35} + x^{34} + x^{32} + x^{31} + x^{29} + x^{28} + x^{26} + x^{25} \\
&+ x^{22} + x^{21} + x^{19} + x^{18} + x^{16} + x^{15} + x^{13} + x^{12} + x^{10} + x^{9} + x^{7} + x^{6} \\
&+ x^{4}, \\
p_{12}(x) &= x^{147} + x^{145} + x^{144} + x^{143} + x^{142} + x^{140} + x^{139} + x^{137} + x^{136} + x^{135} \\
&+ x^{134} + x^{132} + x^{131} + x^{129} + x^{128} + x^{127} + x^{126} + x^{124} + x^{123} \\
&+ x^{121} + x^{120} + x^{118} + x^{117} + x^{114} + x^{113} + x^{111} + x^{110} + x^{108} \\
&+ x^{107} + x^{105} + x^{104} + x^{103} + x^{102} + x^{100} + x^{99} + x^{97} + x^{96} + x^{95} \\
&+ x^{94} + x^{92} + x^{91} + x^{89} + x^{88} + x^{86} + x^{85} + x^{82} + x^{81} + x^{79} + x^{78} \\
&+ x^{76} + x^{75} + x^{73} + x^{72} + x^{71} + x^{70} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} \\
&+ x^{62} + x^{60} + x^{59} + x^{57} + x^{56} + x^{54} + x^{53} + x^{50} + x^{49} + x^{47} + x^{46}
\end{aligned}$$

$$+x^{44}+x^{43}+x^{41}+x^{40}+x^{39}+x^{38}+x^{36}+x^{23}+x^{21}+x^{20}+x^{18}$$
$$+x^{17}+x^{15}+x^{14}+x^{12}+x^{11}+x^{9}+x^{8}+x^{6}+x^{5}+x^{3}+x^{2}+1,$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 1].

Below is the transition function and output function of an 2-automaton that generates $T = M^{o}_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	123	[206, 207]	246	[348, 349]	369	[241, 210]	492	[531, 159]
1	[3, 4]	124	[59, 208]	247	[350, 305]	370	[482, 21]	493	[544, 372]
2	[5, 6]	125	[209, 210]	248	[351, 345]	371	[483, 484]	494	[31, 334]
3	[7, 8]	126	[211, 212]	249	[36, 166]	372	[485, 486]	495	[179, 361]
4	[9, 10]	127	[213, 214]	250	[115, 131]	373	[487, 457]	496	[169, 545]
5	[11, 12]	128	[196, 215]	251	[308, 43]	374	[488, 489]	497	[382, 52]
6	[13, 14]	129	[216, 85]	252	[131, 57]	375	[432, 490]	498	[152, 56]
7	[15, 16]	130	[217, 218]	253	[352, 43]	376	[491, 492]	499	[546, 167]
8	[4, 17]	131	[185, 219]	254	[56, 132]	377	[493, 494]	500	[340, 342]
9	[18, 19]	132	[186, 187]	255	[353, 354]	378	[495, 496]	501	[343, 114]
10	[20, 21]	133	[220, 221]	256	[316, 355]	379	[497, 443]	502	[348, 159]
11	[22, 23]	134	[222, 223]	257	[356, 138]	380	[498, 206]	503	[547, 548]
12	[24, 25]	135	[224, 199]	258	[357, 358]	381	[499, 500]	504	[538, 49]
13	[26, 27]	136	[200, 225]	259	[128, 53]	382	[501, 216]	505	[338, 126]
14	[28, 29]	137	[226, 227]	260	[359, 162]	383	[502, 503]	506	[377, 545]
15	[30, 31]	138	[228, 229]	261	[360, 34]	384	[504, 435]	507	[549, 464]
16	[32, 33]	139	[230, 231]	262	[32, 361]	385	[505, 506]	508	[94, 494]
17	[8, 34]	140	[140, 140]	263	[112, 343]	386	[302, 182]	509	[550, 450]
18	[8, 35]	141	[232, 233]	264	[163, 182]	387	[303, 135]	510	[551, 236]
19	[36, 37]	142	[234, 235]	265	[44, 33]	388	[161, 507]	511	[198, 552]
20	[38, 39]	143	[236, 237]	266	[8, 362]	389	[158, 508]	512	[553, 486]
21	[40, 41]	144	[238, 239]	267	[340, 363]	390	[367, 365]	513	[554, 555]
22	[42, 43]	145	[240, 241]	268	[364, 365]	391	[509, 52]	514	[556, 557]
23	[44, 45]	146	[242, 18]	269	[175, 115]	392	[128, 368]	515	[83, 558]
24	[46, 47]	147	[243, 211]	270	[56, 114]	393	[328, 51]	516	[559, 418]
25	[48, 49]	148	[244, 245]	271	[366, 319]	394	[510, 511]	517	[560, 561]
26	[50, 51]	149	[90, 227]	272	[367, 123]	395	[512, 147]	518	[562, 418]
27	[52, 53]	150	[246, 247]	273	[52, 368]	396	[167, 10]	519	[563, 102]
28	[54, 55]	151	[248, 249]	274	[127, 369]	397	[310, 31]	520	[564, 565]
29	[56, 57]	152	[250, 93]	275	[370, 315]	398	[335, 326]	521	[566, 249]
30	[58, 59]	153	[251, 252]	276	[157, 363]	399	[139, 141]	522	[567, 287]
31	[60, 61]	154	[253, 254]	277	[364, 121]	400	[366, 39]	523	[568, 405]
32	[62, 63]	155	[255, 256]	278	[139, 327]	401	[153, 513]	524	[569, 570]
33	[64, 65]	156	[188, 257]	279	[371, 372]	402	[514, 333]	525	[571, 572]
34	[16, 66]	157	[258, 259]	280	[182, 124]	403	[167, 149]	526	[573, 283]
35	[67, 68]	158	[260, 261]	281	[373, 174]	404	[515, 516]	527	[574, 575]

36	[69, 70]	159	[262, 263]	282	[314, 374]	405	[140, 141]	528	[576, 483]
37	[71, 72]	160	[264, 265]	283	[356, 315]	406	[316, 164]	529	[413, 577]
38	[73, 74]	161	[266, 267]	284	[183, 136]	407	[180, 517]	530	[578, 280]
39	[75, 23]	162	[268, 104]	285	[38, 319]	408	[383, 318]	531	[92, 579]
40	[76, 77]	163	[269, 270]	286	[375, 376]	409	[114, 176]	532	[301, 414]
41	[78, 79]	164	[271, 272]	287	[377, 174]	410	[120, 365]	533	[500, 580]
42	[80, 81]	165	[273, 274]	288	[356, 378]	411	[154, 513]	534	[265, 442]
43	[82, 83]	166	[275, 202]	289	[310, 182]	412	[518, 307]	535	[581, 98]
44	[84, 85]	167	[89, 276]	290	[176, 131]	413	[13, 166]	536	[582, 583]
45	[86, 87]	168	[277, 278]	291	[323, 162]	414	[314, 138]	537	[584, 475]
46	[88, 89]	169	[279, 280]	292	[379, 380]	415	[519, 312]	538	[585, 437]
47	[79, 90]	170	[281, 282]	293	[381, 382]	416	[137, 315]	539	[586, 456]
48	[91, 92]	171	[283, 284]	294	[383, 117]	417	[125, 355]	540	[587, 388]
49	[93, 94]	172	[285, 218]	295	[163, 303]	418	[127, 329]	541	[61, 406]
50	[95, 96]	173	[286, 263]	296	[375, 384]	419	[143, 369]	542	[588, 589]
51	[97, 98]	174	[287, 288]	297	[377, 170]	420	[130, 354]	543	[590, 403]
52	[99, 100]	175	[289, 107]	298	[130, 385]	421	[520, 362]	544	[239, 591]
53	[101, 102]	176	[290, 189]	299	[176, 56]	422	[157, 347]	545	[552, 592]
54	[103, 104]	177	[291, 292]	300	[183, 118]	423	[154, 319]	546	[593, 568]
55	[105, 106]	178	[293, 80]	301	[31, 183]	424	[306, 380]	547	[594, 266]
56	[107, 108]	179	[294, 295]	302	[386, 224]	425	[521, 522]	548	[233, 549]
57	[109, 110]	180	[296, 66]	303	[387, 264]	426	[143, 329]	549	[122, 365]
58	[111, 112]	181	[297, 298]	304	[388, 389]	427	[337, 142]	550	[124, 118]
59	[113, 114]	182	[299, 269]	305	[77, 390]	428	[148, 523]	551	[333, 161]
60	[115, 56]	183	[300, 301]	306	[391, 392]	429	[173, 170]	552	[339, 524]
61	[114, 112]	184	[302, 31]	307	[219, 203]	430	[118, 303]	553	[307, 157]
62	[116, 117]	185	[113, 175]	308	[393, 394]	431	[147, 167]	554	[311, 354]
63	[118, 119]	186	[175, 112]	309	[395, 396]	432	[157, 322]	555	[125, 317]
64	[120, 121]	187	[183, 163]	310	[397, 300]	433	[180, 524]	556	[307, 340]
65	[122, 123]	188	[112, 56]	311	[398, 399]	434	[345, 13]	557	[360, 35]
66	[31, 124]	189	[118, 182]	312	[400, 401]	435	[158, 349]	558	[595, 526]
67	[125, 126]	190	[131, 114]	313	[402, 403]	436	[327, 139]	559	[596, 382]
68	[112, 113]	191	[303, 183]	314	[404, 405]	437	[323, 507]	560	[373, 170]
69	[127, 51]	192	[114, 115]	315	[406, 407]	438	[525, 526]	561	[311, 597]
70	[128, 129]	193	[304, 35]	316	[408, 409]	439	[144, 522]	562	[145, 143]
71	[130, 55]	194	[146, 305]	317	[410, 411]	440	[527, 528]	563	[527, 151]
72	[131, 132]	195	[306, 307]	318	[412, 413]	441	[529, 384]	564	[359, 507]
73	[133, 134]	196	[36, 171]	319	[414, 415]	442	[182, 334]	565	[531, 530]
74	[135, 136]	197	[57, 115]	320	[416, 417]	443	[52, 358]	566	[546, 148]
75	[137, 138]	198	[131, 175]	321	[418, 419]	444	[52, 129]	567	[124, 163]
76	[40, 40]	199	[182, 135]	322	[420, 74]	445	[41, 40]	568	[122, 121]
77	[139, 140]	200	[118, 31]	323	[421, 422]	446	[367, 121]	569	[598, 344]
78	[141, 139]	201	[308, 309]	324	[423, 399]	447	[54, 354]	570	[356, 374]
79	[142, 142]	202	[135, 310]	325	[223, 424]	448	[183, 310]	571	[522, 127]

80	[143, 51]	203	[311, 55]	326	[425, 426]	449	[334, 163]	572	[128, 358]
81	[144, 145]	204	[44, 312]	327	[427, 232]	450	[339, 344]	573	[119, 183]
82	[146, 147]	205	[133, 313]	328	[249, 428]	451	[314, 378]	574	[304, 362]
83	[148, 149]	206	[314, 315]	329	[429, 430]	452	[9, 149]	575	[514, 548]
84	[150, 151]	207	[316, 317]	330	[431, 432]	453	[158, 530]	576	[595, 345]
85	[114, 152]	208	[112, 131]	331	[110, 433]	454	[370, 138]	577	[540, 599]
86	[153, 39]	209	[116, 318]	332	[434, 435]	455	[531, 508]	578	[535, 164]
87	[154, 155]	210	[175, 152]	333	[436, 423]	456	[44, 361]	579	[8, 534]
88	[156, 157]	211	[153, 319]	334	[270, 60]	457	[115, 113]	580	[167, 523]
89	[158, 159]	212	[122, 320]	335	[437, 438]	458	[532, 36]	581	[150, 528]
90	[154, 39]	213	[321, 322]	336	[439, 69]	459	[360, 362]	582	[533, 358]
91	[160, 8]	214	[323, 172]	337	[235, 76]	460	[509, 128]	583	[144, 511]
92	[161, 162]	215	[323, 324]	338	[440, 430]	461	[533, 368]	584	[371, 600]
93	[163, 31]	216	[169, 174]	339	[441, 442]	462	[381, 331]	585	[541, 8]
94	[125, 164]	217	[325, 326]	340	[443, 444]	463	[515, 178]	586	[132, 115]
95	[165, 166]	218	[140, 327]	341	[445, 446]	464	[40, 337]	587	[49, 158]
96	[167, 168]	219	[31, 135]	342	[447, 448]	465	[519, 33]	588	[598, 524]
97	[169, 170]	220	[328, 329]	343	[449, 250]	466	[339, 517]	589	[130, 597]
98	[163, 119]	221	[330, 331]	344	[450, 451]	467	[153, 155]	590	[49, 531]
99	[165, 171]	222	[332, 333]	345	[208, 71]	468	[177, 516]	591	[601, 599]
100	[161, 172]	223	[148, 10]	346	[452, 453]	469	[321, 342]	592	[311, 385]
101	[173, 174]	224	[334, 118]	347	[454, 252]	470	[148, 168]	593	[327, 140]
102	[175, 176]	225	[135, 163]	348	[403, 455]	471	[13, 37]	594	[156, 340]
103	[177, 178]	226	[335, 336]	349	[456, 457]	472	[360, 534]	595	[602, 273]
104	[142, 41]	227	[142, 337]	350	[458, 459]	473	[367, 320]	596	[603, 413]
105	[179, 33]	228	[338, 164]	351	[460, 426]	474	[535, 126]	597	[604, 605]
106	[180, 181]	229	[339, 181]	352	[461, 462]	475	[115, 343]	598	[606, 68]
107	[56, 175]	230	[141, 140]	353	[463, 464]	476	[352, 309]	599	[607, 488]
108	[182, 183]	231	[139, 139]	354	[465, 466]	477	[353, 55]	600	[608, 89]
109	[57, 112]	232	[40, 142]	355	[390, 467]	478	[519, 45]	601	[609, 291]
110	[135, 118]	233	[142, 40]	356	[468, 278]	479	[536, 134]	602	[145, 127]
111	[184, 185]	234	[337, 40]	357	[469, 470]	480	[536, 313]	603	[341, 323]
112	[108, 186]	235	[140, 139]	358	[433, 63]	481	[537, 524]	604	[316, 126]
113	[187, 188]	236	[340, 322]	359	[435, 471]	482	[325, 336]	605	[169, 542]
114	[189, 190]	237	[161, 324]	360	[396, 472]	483	[13, 171]	606	[544, 600]
115	[191, 192]	238	[341, 161]	361	[446, 473]	484	[538, 539]	607	[136, 182]
116	[193, 194]	239	[157, 342]	362	[474, 475]	485	[540, 541]	608	[610, 541]
117	[195, 196]	240	[343, 175]	363	[257, 254]	486	[531, 349]	609	[611, 158]
118	[197, 198]	241	[180, 344]	364	[476, 284]	487	[529, 376]	610	[612, 83]
119	[199, 200]	242	[345, 36]	365	[477, 478]	488	[519, 361]	611	[192, 465]
120	[201, 202]	243	[41, 142]	366	[479, 448]	489	[377, 542]	612	[541, 360]
121	[203, 204]	244	[346, 159]	367	[480, 29]	490	[48, 539]		
122	[205, 72]	245	[340, 347]	368	[481, 295]	491	[543, 49]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	88	0	176	0	264	0	352	0	440	0	528	0
1	0	89	1	177	1	265	1	353	0	441	0	529	0
2	0	90	0	178	0	266	0	354	0	442	0	530	1
3	0	91	0	179	1	267	0	355	0	443	0	531	0
4	0	92	0	180	1	268	0	356	1	444	0	532	1
5	0	93	0	181	1	269	0	357	1	445	1	533	0
6	0	94	0	182	0	270	0	358	1	446	0	534	1
7	0	95	0	183	1	271	0	359	1	447	1	535	1
8	0	96	1	184	0	272	0	360	1	448	1	536	0
9	0	97	0	185	1	273	0	361	0	449	0	537	0
10	1	98	0	186	0	274	1	362	1	450	0	538	1
11	0	99	0	187	1	275	0	363	1	451	0	539	0
12	0	100	0	188	0	276	1	364	0	452	0	540	0
13	0	101	1	189	1	277	0	365	0	453	1	541	1
14	1	102	0	190	1	278	1	366	0	454	0	542	1
15	0	103	1	191	1	279	0	367	0	455	0	543	0
16	0	104	0	192	1	280	0	368	0	456	1	544	1
17	0	105	1	193	0	281	0	369	1	457	1	545	0
18	0	106	1	194	0	282	0	370	0	458	1	546	0
19	1	107	0	195	1	283	1	371	0	459	1	547	0
20	1	108	0	196	1	284	1	372	0	460	1	548	0
21	1	109	1	197	1	285	1	373	0	461	0	549	1
22	0	110	0	198	1	286	1	374	0	462	0	550	1
23	1	111	0	199	0	287	1	375	1	463	0	551	0
24	0	112	0	200	1	288	1	376	0	464	1	552	0
25	0	113	1	201	1	289	0	377	1	465	0	553	1
26	0	114	1	202	0	290	0	378	1	466	0	554	1
27	0	115	1	203	1	291	1	379	0	467	1	555	0
28	1	116	0	204	1	292	0	380	1	468	1	556	1
29	0	117	1	205	1	293	0	381	0	469	1	557	1
30	0	118	1	206	0	294	1	382	0	470	0	558	0
31	1	119	0	207	1	295	0	383	1	471	0	559	1
32	0	120	1	208	0	296	1	384	1	472	1	560	0
33	1	121	1	209	0	297	1	385	0	473	0	561	1
34	0	122	1	210	0	298	0	386	0	474	1	562	0
35	0	123	0	211	1	299	0	387	1	475	1	563	0
36	1	124	1	212	1	300	1	388	0	476	0	564	1
37	0	125	0	213	1	301	1	389	1	477	0	565	0
38	1	126	1	214	1	302	0	390	0	478	0	566	0
39	1	127	1	215	1	303	1	391	1	479	0	567	1
40	1	128	1	216	0	304	0	392	1	480	0	568	1

41	1	129	0	217	0	305	1	393	1	481	0	569	1
42	0	130	0	218	0	306	1	394	0	482	0	570	1
43	0	131	1	219	1	307	1	395	1	483	0	571	1
44	1	132	0	220	1	308	1	396	1	484	1	572	1
45	1	133	1	221	1	309	1	397	0	485	0	573	0
46	0	134	0	222	0	310	0	398	1	486	0	574	0
47	0	135	0	223	0	311	1	399	1	487	0	575	1
48	0	136	1	224	0	312	0	400	0	488	0	576	0
49	0	137	1	225	0	313	1	401	1	489	1	577	0
50	0	138	0	226	1	314	0	402	1	490	0	578	1
51	0	139	1	227	0	315	1	403	1	491	0	579	0
52	0	140	0	228	0	316	1	404	0	492	0	580	1
53	1	141	1	229	0	317	1	405	0	493	1	581	1
54	1	142	0	230	1	318	0	406	1	494	1	582	0
55	1	143	0	231	1	319	0	407	1	495	1	583	1
56	0	144	1	232	1	320	1	408	1	496	0	584	0
57	1	145	0	233	0	321	1	409	1	497	0	585	1
58	0	146	0	234	0	322	0	410	1	498	1	586	0
59	1	147	1	235	0	323	1	411	0	499	0	587	0
60	1	148	0	236	0	324	0	412	0	500	0	588	1
61	1	149	0	237	0	325	0	413	0	501	0	589	0
62	0	150	1	238	1	326	0	414	0	502	1	590	0
63	1	151	1	239	1	327	0	415	0	503	0	591	1
64	1	152	1	240	0	328	1	416	1	504	1	592	1
65	1	153	1	241	1	329	1	417	0	505	0	593	0
66	1	154	0	242	0	330	1	418	1	506	1	594	0
67	0	155	0	243	1	331	0	419	0	507	1	595	0
68	0	156	0	244	0	332	0	420	0	508	0	596	1
69	1	157	1	245	0	333	0	421	1	509	1	597	1
70	1	158	1	246	1	334	0	422	1	510	0	598	1
71	0	159	0	247	1	335	1	423	0	511	1	599	1
72	1	160	0	248	1	336	1	424	1	512	1	600	1
73	1	161	0	249	1	337	0	425	0	513	1	601	1
74	0	162	0	250	1	338	0	426	0	514	1	602	0
75	1	163	0	251	1	339	0	427	0	515	0	603	1
76	1	164	0	252	1	340	0	428	0	516	1	604	1
77	1	165	0	253	0	341	1	429	1	517	0	605	0
78	1	166	0	254	0	342	1	430	1	518	0	606	1
79	0	167	1	255	0	343	0	431	1	519	0	607	1
80	0	168	0	256	1	344	0	432	1	520	1	608	1
81	1	169	0	257	1	345	0	433	1	521	0	609	1
82	0	170	0	258	1	346	0	434	0	522	1	610	1
83	0	171	1	259	1	347	0	435	1	523	1	611	1
84	1	172	1	260	1	348	1	436	0	524	1	612	1

85	1	173	1	261	1	349	1	437	1	525	1		
86	1	174	1	262	0	350	1	438	1	526	0		
87	0	175	0	263	0	351	1	439	1	527	0		

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