PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a,b) = (z+1, z^6+z^5+z^4+z^2+z+1)$

GUO-NIU HAN* AND YINING HU*

Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a,b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a,b) in $\mathbb{F}_2[x] \setminus \mathbb{F}_2$. In this paper we prove the conjecture for $(a,b) = (z+1, z^6+z^5+z^4+z^2+z+1)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \ldots)$ of elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{\ddots + \cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a,b)-Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \ldots) = (a, b, b, a, b, a, a, b, \ldots)$$

that we shall denote

$$\mathrm{CF}(a,b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When* $(a,b) = (z+1, z^6+z^5+z^4+z^2+z+1)$*, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4 + p_3(z)y^3 + p_2(z)y^2 + p_1(z)y + p_0(z) = 0$$

where, for $\mathrm{CF}(a,b)$*,*

$$\begin{aligned}
p_0(z) &= z^{20}+z^{17}+z^{15}+z^{14}+z^{13}+z^9+z^8+z^7+z^3+1,\\
p_1(z) &= z^{23}+z^{22}+z^{18}+z^{17}+z^{16}+z^{15}+z^{14}+z^{12}+z^{11}+z^9+z^8+z^7+z^4\\
&\quad +z^2,\\
p_2(z) &= z^{24}+z^{22}+z^{16}+z^{14}+z^8+1,\\
p_3(z) &= z^{23}+z^{22}+z^{18}+z^{17}+z^{16}+z^{15}+z^{14}+z^{12}+z^{11}+z^9+z^8+z^7+z^4\\
&\quad +z^2,\\
p_4(z) &= z^{22}+z^{20}+z^{17}+z^{15}+z^{13}+z^{12}+z^9+z^7+z^4+z^3+1,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$*,*

$$\begin{aligned}
p_0(z) &= z^{17}+z^{15}+z^{14}+z^{13}+z^{12}+z^9+z^7+z^3+1,\\
p_1(z) &= z^{23}+z^{22}+z^{18}+z^{17}+z^{16}+z^{15}+z^{14}+z^{12}+z^{11}+z^9+z^8+z^7+z^4\\
&\quad +z^2,\\
p_2(z) &= z^{24}+z^{22}+z^{16}+z^{14}+z^8+1,\\
p_3(z) &= z^{23}+z^{22}+z^{18}+z^{17}+z^{16}+z^{15}+z^{14}+z^{12}+z^{11}+z^9+z^8+z^7+z^4\\
&\quad +z^2,\\
p_4(z) &= z^{22}+z^{17}+z^{15}+z^{13}+z^9+z^8+z^7+z^4+z^3+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z) \\ P_{n-1}(z) & Q_{n-1}(z) \end{pmatrix} := \begin{pmatrix} t_n(z) & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t_{n-1}(z) & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} t_0(z) & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} \in \mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x) &= x^{\deg(t_{2^n-1})} \begin{pmatrix} t_{2^n-1}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times x^{\deg(t_{2^n-2})} \begin{pmatrix} t_{2^n-2}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times \\
&\qquad \cdots \times x^{\deg(t_0)} \begin{pmatrix} t_0(1/x) & 1 \\ 1 & 0 \end{pmatrix},\\
W_n(x) &= x^{\deg(\bar t_{2^n-1})} \begin{pmatrix} \bar t_{2^n-1}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times x^{\deg(\bar t_{2^n-2})} \begin{pmatrix} \bar t_{2^n-2}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times \\
&\qquad \cdots \times x^{\deg(\bar t_0)} \begin{pmatrix} \bar t_0(1/x) & 1 \\ 1 & 0 \end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n \geq 0$

$$M_{n+1}(x) = W_n(x) \cdot M_n(x),$$
$$W_{n+1}(x) = M_n(x) \cdot W_n(x).$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \leq i,j \leq 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \leq i,j \leq 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \leq i,j \leq 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(175, 175, 175, 175, 175)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \geq 0$. If for some $n \geq 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \geq 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z\left(\phi(W^o,0,0)(x,z),\ z^{12}\cdot\phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 630 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 630, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z\left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ to order 770 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 770, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. □

Define

$$R^e = \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned}M^e[:14u] = {}& M^e[:7u] + x^{2u}M^e[:5u] + x^{3u}M^e[:4u] + x^{4u}M^e[:3u] + x^{5u}M^e[:2u]\\ &+ x^{6u}M^e[:u] + x^{2u}M^e[:7u] + x^{4u}M^e[:5u] + x^{5u}M^e[:4u]\\ &+ x^{6u}M^e[:3u] + x^{7u}M^e[:2u] + x^{8u}M^e[:u] + x^{3u}M^e[:7u]\\ &+ x^{5u}M^e[:5u] + x^{6u}M^e[:4u] + x^{7u}M^e[:3u] + x^{8u}M^e[:2u]\end{aligned}$$

$$
\begin{aligned}
&+ x^{9u}M^e[:u] + x^{5u}M^e[:7u] + x^{7u}M^e[:5u] + x^{8u}M^e[:4u] \\
&+ x^{9u}M^e[:3u] + x^{10u}M^e[:2u] + x^{11u}M^e[:u] + x^{6u}M^e[:7u] \\
&+ x^{8u}M^e[:5u] + x^{9u}M^e[:4u] + x^{10u}M^e[:3u] + x^{11u}M^e[:2u] \\
&+ x^{12u}M^e[:u] + x^{8u}M^e[:6u] + x^{10u}M^e[:4u] + x^{9u}M^e[:5u] \\
&+ x^{11u}M^e[:3u] + x^{13u}M^e[:u] \\
&+ (x^{7u} + x^{9u} + x^{10u} + x^{12u} + x^{13u})R^e,
\end{aligned}
$$

$$
\begin{aligned}
W^e[:14u] = {} & W^e[:7u] + x^{2u}W^e[:5u] + x^{3u}W^e[:4u] + x^{4u}W^e[:3u] + x^{5u}W^e[:2u] \\
&+ x^{6u}W^e[:u] + x^{2u}W^e[:7u] + x^{4u}W^e[:5u] + x^{5u}W^e[:4u] \\
&+ x^{6u}W^e[:3u] + x^{7u}W^e[:2u] + x^{8u}W^e[:u] + x^{3u}W^e[:7u] \\
&+ x^{5u}W^e[:5u] + x^{6u}W^e[:4u] + x^{7u}W^e[:3u] + x^{8u}W^e[:2u] \\
&+ x^{9u}W^e[:u] + x^{5u}W^e[:7u] + x^{7u}W^e[:5u] + x^{8u}W^e[:4u] \\
&+ x^{9u}W^e[:3u] + x^{10u}W^e[:2u] + x^{11u}W^e[:u] + x^{6u}W^e[:7u] \\
&+ x^{8u}W^e[:5u] + x^{9u}W^e[:4u] + x^{10u}W^e[:3u] + x^{11u}W^e[:2u] \\
&+ x^{12u}W^e[:u] + x^{8u}W^e[:6u] + x^{10u}W^e[:4u] + x^{9u}W^e[:5u] \\
&+ x^{11u}W^e[:3u] + x^{13u}W^e[:u] \\
&+ (x^{7u} + x^{9u} + x^{10u} + x^{12u} + x^{13u})R^e.
\end{aligned}
$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$
\begin{aligned}
M^o[:14u] = {} & M^o[:7u] + x^{2u}M^o[:5u] + x^{3u}M^o[:4u] + x^{4u}M^o[:3u] + x^{5u}M^o[:2u] \\
&+ x^{6u}M^o[:u] + x^{2u}M^o[:7u] + x^{4u}M^o[:5u] + x^{5u}M^o[:4u] \\
&+ x^{6u}M^o[:3u] + x^{7u}M^o[:2u] + x^{8u}M^o[:u] + x^{3u}M^o[:7u] \\
&+ x^{5u}M^o[:5u] + x^{6u}M^o[:4u] + x^{7u}M^o[:3u] + x^{8u}M^o[:2u] \\
&+ x^{9u}M^o[:u] + x^{5u}M^o[:7u] + x^{7u}M^o[:5u] + x^{8u}M^o[:4u] \\
&+ x^{9u}M^o[:3u] + x^{10u}M^o[:2u] + x^{11u}M^o[:u] + x^{6u}M^o[:7u] \\
&+ x^{8u}M^o[:5u] + x^{9u}M^o[:4u] + x^{10u}M^o[:3u] + x^{11u}M^o[:2u] \\
&+ x^{12u}M^o[:u] + x^{8u}M^o[:6u] + x^{10u}M^o[:4u] + x^{9u}M^o[:5u] \\
&+ x^{11u}M^o[:3u] + x^{13u}M^o[:u] \\
&+ (x^{7u} + x^{9u} + x^{10u} + x^{12u} + x^{13u})R^o,
\end{aligned}
$$

$$
\begin{aligned}
W^o[:14u] = {} & W^o[:7u] + x^{2u}W^o[:5u] + x^{3u}W^o[:4u] + x^{4u}W^o[:3u] + x^{5u}W^o[:2u] \\
&+ x^{6u}W^o[:u] + x^{2u}W^o[:7u] + x^{4u}W^o[:5u] + x^{5u}W^o[:4u] \\
&+ x^{6u}W^o[:3u] + x^{7u}W^o[:2u] + x^{8u}W^o[:u] + x^{3u}W^o[:7u] \\
&+ x^{5u}W^o[:5u] + x^{6u}W^o[:4u] + x^{7u}W^o[:3u] + x^{8u}W^o[:2u] \\
&+ x^{9u}W^o[:u] + x^{5u}W^o[:7u] + x^{7u}W^o[:5u] + x^{8u}W^o[:4u] \\
&+ x^{9u}W^o[:3u] + x^{10u}W^o[:2u] + x^{11u}W^o[:u] + x^{6u}W^o[:7u] \\
&+ x^{8u}W^o[:5u] + x^{9u}W^o[:4u] + x^{10u}W^o[:3u] + x^{11u}W^o[:2u] \\
&+ x^{12u}W^o[:u] + x^{8u}W^o[:6u] + x^{10u}W^o[:4u] + x^{9u}W^o[:5u]
\end{aligned}
$$

$$+ x^{11u}W^o[:3u] + x^{13u}W^o[:u]$$
$$+ (x^{7u} + x^{9u} + x^{10u} + x^{12u} + x^{13u})R^o.$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$\begin{aligned}
T[:14u] = {} & T[:7u] + x^{2u}T[:5u] + x^{3u}T[:4u] + x^{4u}T[:3u] + x^{5u}T[:2u] + x^{6u}T[:u] \\
& + x^{2u}T[:7u] + x^{4u}T[:5u] + x^{5u}T[:4u] + x^{6u}T[:3u] + x^{7u}T[:2u] \\
& + x^{8u}T[:u] + x^{3u}T[:7u] + x^{5u}T[:5u] + x^{6u}T[:4u] + x^{7u}T[:3u] \\
& + x^{8u}T[:2u] + x^{9u}T[:u] + x^{5u}T[:7u] + x^{7u}T[:5u] + x^{8u}T[:4u] \\
& + x^{9u}T[:3u] + x^{10u}T[:2u] + x^{11u}T[:u] + x^{6u}T[:7u] + x^{8u}T[:5u] \\
& + x^{9u}T[:4u] + x^{10u}T[:3u] + x^{11u}T[:2u] + x^{12u}T[:u] + x^{8u}T[:6u] \\
& + x^{10u}T[:4u] + x^{9u}T[:5u] + x^{11u}T[:3u] + x^{13u}T[:u] \\
& + (x^{7u} + x^{9u} + x^{10u} + x^{12u} + x^{13u}).
\end{aligned}$$

The proof of the other 15 cases are similar. We break down the above identity into 7 parts:

$$\begin{aligned}
x^7u &= x^{7u}T[:u] + x^{6u}T[u:2u] + x^{5u}T[2u:3u] + x^{4u}T[3u:4u] \\
&\quad + x^{3u}T[4u:5u] + x^{2u}T[5u:6u] + T[7u:8u], \\
0 &= x^{8u}T[:u] + x^{7u}T[u:2u] + x^{6u}T[2u:3u] + x^{5u}T[3u:4u] \\
&\quad + x^{4u}T[4u:5u] + x^{3u}T[5u:6u] + x^{2u}T[6u:7u] + T[8u:9u], \\
x^9u &= x^{3u}T[6u:7u] + T[9u:10u], \\
x^10u &= x^{10u}T[:u] + x^{9u}T[u:2u] + x^{8u}T[2u:3u] + x^{7u}T[3u:4u] \\
&\quad + x^{6u}T[4u:5u] + x^{5u}T[5u:6u] + T[10u:11u], \\
0 &= x^{11u}T[:u] + x^{10u}T[u:2u] + x^{9u}T[2u:3u] + x^{8u}T[3u:4u] \\
&\quad + x^{7u}T[4u:5u] + x^{6u}T[5u:6u] + x^{5u}T[6u:7u] + T[11u:12u], \\
x^12u &= x^{12u}T[:u] + x^{6u}T[6u:7u] + T[12u:13u], \\
x^13u &= x^{13u}T[:u] + x^{11u}T[2u:3u] + x^{10u}T[3u:4u] + x^{9u}T[4u:5u] \\
&\quad + x^{8u}T[5u:6u] + T[13u:14u],
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] \\
&\quad + T[[101w]_2] + T[[111w]_2], \qquad (2.7)\\
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] \\
&\quad + T[[101w]_2] + T[[110w]_2] + T[[1000w]_2], \qquad (2.8)\\
0 &= T[[110w]_2] + T[[1001w]_2], \qquad (2.9)\\
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] \\
&\quad + T[[101w]_2] + T[[1010w]_2], \qquad (2.10)
\end{aligned}$$

$$
\begin{aligned}
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] \\
&\quad + T[[101w]_2] + T[[110w]_2] + T[[1011w]_2], &&(2.11)\\
0 &= T[[w]_2] + T[[110w]_2] + T[[1100w]_2], &&(2.12)\\
0 &= T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] \\
&\quad + T[[1101w]_2], &&(2.13)
\end{aligned}
$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$
\begin{aligned}
1 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] \\
&\quad + T[[101w]_2] + T[[111w]_2], &&(2.14)\\
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] \\
&\quad + T[[101w]_2] + T[[110w]_2] + T[[1000w]_2], &&(2.15)\\
1 &= T[[110w]_2] + T[[1001w]_2], &&(2.16)\\
1 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] \\
&\quad + T[[101w]_2] + T[[1010w]_2], &&(2.17)\\
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] \\
&\quad + T[[101w]_2] + T[[110w]_2] + T[[1011w]_2], &&(2.18)\\
1 &= T[[w]_2] + T[[110w]_2] + T[[1100w]_2], &&(2.19)\\
1 &= T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] \\
&\quad + T[[1101w]_2], &&(2.20)
\end{aligned}
$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 1042 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.20) can be written as

$$
\begin{aligned}
0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\
&\quad + \tau(A(s,101)) + \tau(A(s,111)), &&(2.21)\\
0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\
&\quad + \tau(A(s,101)) + \tau(A(s,110)) + \tau(A(s,1000)), &&(2.22)\\
0 &= \tau(A(s,110)) + \tau(A(s,1001)), &&(2.23)\\
0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\
&\quad + \tau(A(s,101)) + \tau(A(s,1010)), &&(2.24)\\
0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\
&\quad + \tau(A(s,101)) + \tau(A(s,110)) + \tau(A(s,1011)), &&(2.25)\\
0 &= \tau(A(s,\epsilon)) + \tau(A(s,110)) + \tau(A(s,1100)), &&(2.26)\\
0 &= \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\
&\quad + \tau(A(s,101)) + \tau(A(s,1101)), &&(2.27)
\end{aligned}
$$

for all $s \in E_{2k}$, and

$$\begin{aligned}1 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100))\\ &\quad + \tau(A(s,101)) + \tau(A(s,111)),\end{aligned} \tag{2.28}$$

$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100))\\ &\quad + \tau(A(s,101)) + \tau(A(s,110)) + \tau(A(s,1000)),\end{aligned} \tag{2.29}$$

$$1 = \tau(A(s,110)) + \tau(A(s,1001)), \tag{2.30}$$

$$\begin{aligned}1 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100))\\ &\quad + \tau(A(s,101)) + \tau(A(s,1010)),\end{aligned} \tag{2.31}$$

$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100))\\ &\quad + \tau(A(s,101)) + \tau(A(s,110)) + \tau(A(s,1011)),\end{aligned} \tag{2.32}$$

$$1 = \tau(A(s,\epsilon)) + \tau(A(s,110)) + \tau(A(s,1100)), \tag{2.33}$$

$$\begin{aligned}1 &= \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100))\\ &\quad + \tau(A(s,101)) + \tau(A(s,1101)),\end{aligned} \tag{2.34}$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{46}), E_{46}) = (A(i, 0^{18}), E_{18})$, so that we only have to verify that identities (2.21) through (2.34) hold for $2 \le k \le 23$, which turns out to be true. $\square$

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \ge 2$, and $u = 2^{2k-1}$,*

$$\begin{aligned}M_{2k} &= M^e[:7u] + x^{2u}M^e[:5u] + x^{3u}M^e[:4u] + x^{4u}M^e[:3u] + x^{5u}M^e[:2u]\\ &\quad + x^{6u}M^e[:u] + x^{7u}R^e,\\ W_{2k} &= W^e[:7u] + x^{2u}W^e[:5u] + x^{3u}W^e[:4u] + x^{4u}W^e[:3u] + x^{5u}W^e[:2u]\\ &\quad + x^{6u}W^e[:u] + x^{7u}R^e.\end{aligned}$$

For all integer $k \ge 2$, and $u = 2^{2k}$,

$$\begin{aligned}M_{2k+1} &= M^o[:7u] + x^{2u}M^o[:5u] + x^{3u}M^o[:4u] + x^{4u}M^o[:3u] + x^{5u}M^o[:2u]\\ &\quad + x^{6u}M^o[:u] + x^{7u}R^o,\\ W_{2k+1} &= W^o[:7u] + x^{2u}W^o[:5u] + x^{3u}W^o[:4u] + x^{4u}W^o[:3u] + x^{5u}W^o[:2u]\\ &\quad + x^{6u}W^o[:u] + x^{7u}R^o.\end{aligned}$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \ge 2$, we claim that

$$\begin{aligned}M_{2k} \wedge W_{2k} &\Rightarrow M_{2k+1} \wedge W_{2k+1},\\ M_{2k+1} \wedge W_{2k+1} &\Rightarrow M_{2k+2} \wedge W_{2k+2}.\end{aligned}$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$W_{2k}M_{2k} = \big(W^e[:7v] + x^{2v}W^e[:5v] + x^{3v}W^e[:4v] + x^{4v}W^e[:3v]$$

$$\begin{aligned}&+x^{5v}W^e[:2v]+x^{6v}W^e[:v]+x^{7u}R^e\big)\times\big(M^e[:7v]+x^{2v}M^e[:5v]\\&+x^{3v}M^e[:4v]+x^{4v}M^e[:3v]+x^{5v}M^e[:2v]+x^{6v}M^e[:v]+x^{7u}R^e\\&\big); \qquad (2.35)\end{aligned}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{14v}R^o$. Therefore we only need to prove that their difference is $O(x^{14v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{14v}, to

$$\begin{aligned}&W^e[:14v]M^e[:14v]+x^{4v}W^e[:10v]M^e[:10v]+x^{6v}W^e[:8v]M^e[:8v]\\&\qquad+x^{8v}W^e[:6v]M^e[:6v]+x^{10v}W^e[:4v]M^e[:4v]+x^{12v}W^e[:2v]M^e[:2v].\end{aligned}$$

For all $n\le 14$, replace the occurrences of $W^e[:n\cdot v]$ and $M^e[:n\cdot v]$ in the above expression by the reduction modulo $x^{n\cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$\begin{aligned}X&:=x^v,\\a_n&:=W^e[n\cdot v:(n+1)\cdot v]/X^n,\\b_n&:=M^e[n\cdot v:(n+1)\cdot v]/X^n,\\c&:=R^e.\end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.35) and rhs as polynomials in $\mathbb{F}_2[a_1,...,a_{14},b_1,...,b_{14},c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{14})$, which completes the proof. $\square$

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a,b)$; for $\mathrm{CF}(b,a)$, the proof is similar. By Lemma 2.3, we have For all $0\le j,k\le 1$,

$$\begin{aligned}\lim_{n\to\infty}M_{2n,j,k}&=M^e_{j,k},\\\lim_{n\to\infty}M_{2n+1,j,k}&=M^o_{j,k},\\\lim_{n\to\infty}W_{2n,j,k}&=W^e_{j,k},\\\lim_{n\to\infty}W_{2n+1,j,k}&=W^o_{j,k}.\end{aligned}$$

Let $z=1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a,b)=\frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e,0,1)$ and $\phi(M^e,0,0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x,y)=\mathrm{Res}_t\left(\phi(M^e,0,1)(x,t),y^{12}\phi(M^e,0,1)(x,t/y)\right)$$

is an annihilating polynomial of $f(x)=M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x,y)=q_4(x)y^4+q_3(x)y^3+q_2(x)y^2+q_1(x)y+q_0(x),$$

where

$$\begin{aligned}
q_0(x) &= x^{24}+x^{21}+x^{17}+x^{16}+x^{15}+x^{11}+x^{10}+x^9+x^7+x^4,\\
q_1(x) &= x^{22}+x^{20}+x^{17}+x^{16}+x^{15}+x^{13}+x^{12}+x^{10}+x^9+x^8+x^7+x^6+x^2\\
&\quad +x,\\
q_2(x) &= x^{24}+x^{16}+x^{10}+x^8+x^2+1,\\
q_3(x) &= x^{22}+x^{20}+x^{17}+x^{16}+x^{15}+x^{13}+x^{12}+x^{10}+x^9+x^8+x^7+x^6+x^2\\
&\quad +x,\\
q_4(x) &= x^{24}+x^{21}+x^{20}+x^{17}+x^{15}+x^{12}+x^{11}+x^9+x^7+x^4+x^2.
\end{aligned}$$

The polynomial $Q(x,y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and $R(x,y) := P(x,y)/Q(x,y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 216, and substitute it for y in $R(z,y)$, we get a series with valuation smaller than 216, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b)=f(1/z)$. □

3. Annexe

All 16 polynomials are of the form

$$p_0(x)+p_3(x)y^3+p_6(x)y^6+p_9(x)y^9+p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e,0,0)$:

$$\begin{aligned}
p_0(x) &= x^{144}+x^{138}+x^{136}+x^{132}+x^{130}+x^{128}+x^{124}+x^{122}+x^{116}+x^{114}\\
&\quad +x^{110}+x^{108}+x^{106}+x^{98}+x^{96}+x^{92}+x^{90}+x^{88}+x^{82}+x^{76}+x^{70}\\
&\quad +x^{68}+x^{66}+x^{62}+x^{60}+x^{56}+x^{54}+x^{52}+x^{44}+x^{36}+x^{34}+x^{26}\\
&\quad +x^{24}+x^{22}+x^{20}+x^{16}+x^{12},\\
p_3(x) &= x^{140}+x^{136}+x^{134}+x^{112}+x^{110}+x^{102}+x^{96}+x^{94}+x^{90}+x^{86}\\
&\quad +x^{72}+x^{66}+x^{64}+x^{60}+x^{58}+x^{56}+x^{52}+x^{46}+x^{42}+x^{36}+x^{32}\\
&\quad +x^{30}+x^{26}+x^{22}+x^{18}+x^8,\\
p_6(x) &= x^{138}+x^{134}+x^{132}+x^{128}+x^{120}+x^{114}+x^{110}+x^{106}+x^{100}+x^{98}\\
&\quad +x^{96}+x^{94}+x^{90}+x^{88}+x^{86}+x^{84}+x^{76}+x^{74}+x^{70}+x^{68}+x^{60}\\
&\quad +x^{56}+x^{52}+x^{48}+x^{44}+x^{42}+x^{38}+x^{34}+x^{30}+x^{22}+x^{20}+x^{14}\\
&\quad +x^{12}+x^{10}+x^8+x^6+x^2+1,\\
p_9(x) &= x^{140}+x^{130}+x^{126}+x^{122}+x^{114}+x^{112}+x^{110}+x^{94}+x^{90}+x^{84}\\
&\quad +x^{82}+x^{80}+x^{78}+x^{74}+x^{70}+x^{66}+x^{62}+x^{56}+x^{54}+x^{50}+x^{48}\\
&\quad +x^{46}+x^{38}+x^{34}+x^{30}+x^{24}+x^{22}+x^{20}+x^{18}+x^{16}+x^{14}+x^{12}\\
&\quad +x^{10}+x^2,
\end{aligned}$$

$$p_{12}(x) = x^{144} + x^{136} + x^{120} + x^{104} + x^{96} + x^{80} + x^{72} + x^{56} + x^{48} + x^{16} + x^{8} + 1,$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0].

For $\phi(M^e, 0, 1)$:

$$\begin{aligned}
p_0(x) = {} & x^{129} + x^{127} + x^{126} + x^{125} + x^{124} + x^{120} + x^{117} + x^{115} + x^{114} + x^{113} \\
& + x^{111} + x^{110} + x^{108} + x^{107} + x^{106} + x^{105} + x^{102} + x^{100} + x^{98} + x^{96} \\
& + x^{94} + x^{93} + x^{92} + x^{91} + x^{89} + x^{88} + x^{83} + x^{81} + x^{80} + x^{79} + x^{78} \\
& + x^{76} + x^{75} + x^{73} + x^{70} + x^{69} + x^{67} + x^{66} + x^{65} + x^{60} + x^{59} + x^{58} \\
& + x^{55} + x^{54} + x^{50} + x^{49} + x^{48} + x^{45} + x^{42} + x^{41} + x^{36} + x^{34} + x^{31} \\
& + x^{30} + x^{28} + x^{26} + x^{25} + x^{24} + x^{18}, \\
p_3(x) = {} & x^{111} + x^{108} + x^{106} + x^{105} + x^{103} + x^{100} + x^{98} + x^{94} + x^{92} + x^{91} \\
& + x^{89} + x^{86} + x^{84} + x^{83} + x^{81} + x^{78} + x^{76} + x^{75} + x^{73} + x^{71} + x^{70} \\
& + x^{67} + x^{66} + x^{62} + x^{60} + x^{59} + x^{49} + x^{47} + x^{46} + x^{43} + x^{42} + x^{41} \\
& + x^{39} + x^{36} + x^{34} + x^{33} + x^{31} + x^{28} + x^{26} + x^{23} + x^{22} + x^{19} + x^{18} \\
& + x^{15} + x^{14} + x^{11} + x^{10} + x^{9}, \\
p_6(x) = {} & x^{117} + x^{115} + x^{114} + x^{113} + x^{112} + x^{111} + x^{110} + x^{108} + x^{107} + x^{105} \\
& + x^{104} + x^{103} + x^{102} + x^{100} + x^{97} + x^{95} + x^{94} + x^{92} + x^{91} + x^{89} \\
& + x^{88} + x^{86} + x^{81} + x^{78} + x^{73} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} \\
& + x^{64} + x^{63} + x^{60} + x^{57} + x^{55} + x^{54} + x^{53} + x^{52} + x^{50} + x^{47} + x^{45} \\
& + x^{44} + x^{42} + x^{39} + x^{37} + x^{36} + x^{34} + x^{31} + x^{28} + x^{27} + x^{24} + x^{23} \\
& + x^{21} + x^{20} + x^{19} + x^{18} + x^{16} + x^{9} + x^{6}, \\
p_9(x) = {} & x^{123} + x^{120} + x^{119} + x^{118} + x^{117} + x^{116} + x^{114} + x^{111} + x^{110} + x^{107} \\
& + x^{106} + x^{105} + x^{103} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} \\
& + x^{93} + x^{92} + x^{91} + x^{90} + x^{88} + x^{84} + x^{82} + x^{79} + x^{78} + x^{77} + x^{67} \\
& + x^{65} + x^{64} + x^{63} + x^{55} + x^{52} + x^{51} + x^{50} + x^{48} + x^{47} + x^{43} + x^{41} \\
& + x^{40} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{30} + x^{27} + x^{26} + x^{25} + x^{21} \\
& + x^{19} + x^{18} + x^{17} + x^{15} + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6} + x^{4} \\
& + x^{3}, \\
p_{12}(x) = {} & x^{129} + x^{127} + x^{126} + x^{124} + x^{117} + x^{115} + x^{114} + x^{112} + x^{97} + x^{95} \\
& + x^{94} + x^{92} + x^{85} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{76} + x^{69} \\
& + x^{67} + x^{66} + x^{64} + x^{49} + x^{47} + x^{46} + x^{44} + x^{37} + x^{35} + x^{34} + x^{33} \\
& + x^{32} + x^{31} + x^{30} + x^{28} + x^{21} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} \\
& + x^{12} + x^{5} + x^{3} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [0, 1, 1, 1, 1, 0, 0, 0, 1, 0, 0, 1, 1, 1, 0, 1].

For $\phi(M^e, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{129} + x^{127} + x^{126} + x^{125} + x^{124} + x^{120} + x^{117} + x^{115} + x^{114} + x^{113} \\
&\quad + x^{111} + x^{110} + x^{108} + x^{107} + x^{106} + x^{105} + x^{102} + x^{100} + x^{98} + x^{96} \\
&\quad + x^{94} + x^{93} + x^{92} + x^{91} + x^{89} + x^{88} + x^{83} + x^{81} + x^{80} + x^{79} + x^{78} \\
&\quad + x^{76} + x^{75} + x^{73} + x^{70} + x^{69} + x^{67} + x^{66} + x^{65} + x^{60} + x^{59} + x^{58} \\
&\quad + x^{55} + x^{54} + x^{50} + x^{49} + x^{48} + x^{45} + x^{42} + x^{41} + x^{36} + x^{34} + x^{31} \\
&\quad + x^{30} + x^{28} + x^{26} + x^{25} + x^{24} + x^{18}, \\
p_3(x) &= x^{111} + x^{108} + x^{106} + x^{105} + x^{103} + x^{100} + x^{98} + x^{94} + x^{92} + x^{91} \\
&\quad + x^{89} + x^{86} + x^{84} + x^{83} + x^{81} + x^{78} + x^{76} + x^{75} + x^{73} + x^{71} + x^{70} \\
&\quad + x^{67} + x^{66} + x^{62} + x^{60} + x^{59} + x^{49} + x^{47} + x^{46} + x^{43} + x^{42} + x^{41} \\
&\quad + x^{39} + x^{36} + x^{34} + x^{33} + x^{31} + x^{28} + x^{26} + x^{23} + x^{22} + x^{19} + x^{18} \\
&\quad + x^{15} + x^{14} + x^{11} + x^{10} + x^{9}, \\
p_6(x) &= x^{117} + x^{115} + x^{114} + x^{113} + x^{112} + x^{111} + x^{110} + x^{108} + x^{107} + x^{105} \\
&\quad + x^{104} + x^{103} + x^{102} + x^{100} + x^{97} + x^{95} + x^{94} + x^{92} + x^{91} + x^{89} \\
&\quad + x^{88} + x^{86} + x^{81} + x^{78} + x^{73} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} \\
&\quad + x^{64} + x^{63} + x^{60} + x^{57} + x^{55} + x^{54} + x^{53} + x^{52} + x^{50} + x^{47} + x^{45} \\
&\quad + x^{44} + x^{42} + x^{39} + x^{37} + x^{36} + x^{34} + x^{31} + x^{28} + x^{27} + x^{24} + x^{23} \\
&\quad + x^{21} + x^{20} + x^{19} + x^{18} + x^{16} + x^{9} + x^{6}, \\
p_9(x) &= x^{123} + x^{120} + x^{119} + x^{118} + x^{117} + x^{116} + x^{114} + x^{111} + x^{110} + x^{107} \\
&\quad + x^{106} + x^{105} + x^{103} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} \\
&\quad + x^{93} + x^{92} + x^{91} + x^{90} + x^{88} + x^{84} + x^{82} + x^{79} + x^{78} + x^{77} + x^{67} \\
&\quad + x^{65} + x^{64} + x^{63} + x^{55} + x^{52} + x^{51} + x^{50} + x^{48} + x^{47} + x^{43} + x^{41} \\
&\quad + x^{40} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{30} + x^{27} + x^{26} + x^{25} + x^{21} \\
&\quad + x^{19} + x^{18} + x^{17} + x^{15} + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6} + x^{4} \\
&\quad + x^{3}, \\
p_{12}(x) &= x^{129} + x^{127} + x^{126} + x^{124} + x^{117} + x^{115} + x^{114} + x^{112} + x^{97} + x^{95} \\
&\quad + x^{94} + x^{92} + x^{85} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{76} + x^{69} \\
&\quad + x^{67} + x^{66} + x^{64} + x^{49} + x^{47} + x^{46} + x^{44} + x^{37} + x^{35} + x^{34} + x^{33} \\
&\quad + x^{32} + x^{31} + x^{30} + x^{28} + x^{21} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} \\
&\quad + x^{12} + x^{5} + x^{3} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [0, 1, 1, 1, 1, 0, 0, 0, 1, 0, 0, 1, 1, 1, 0, 1].

For $\phi(M^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{114} + x^{110} + x^{100} + x^{94} + x^{88} + x^{84} + x^{80} + x^{78} + x^{76} + x^{70} + x^{66} \\
&\quad + x^{62} + x^{60} + x^{54} + x^{52} + x^{50} + x^{48} + x^{46} + x^{40} + x^{36} + x^{34} + x^{28} \\
&\quad + x^{24},
\end{aligned}$$

$$\begin{aligned}
p_3(x) &= x^{110} + x^{100} + x^{98} + x^{92} + x^{90} + x^{84} + x^{82} + x^{74} + x^{70} + x^{68} + x^{66} \\
&\quad + x^{60} + x^{54} + x^{50} + x^{46} + x^{36} + x^{26} + x^{18} + x^{14} + x^{6}, \\
p_6(x) &= x^{108} + x^{106} + x^{98} + x^{96} + x^{94} + x^{92} + x^{80} + x^{78} + x^{70} + x^{66} + x^{64} \\
&\quad + x^{58} + x^{56} + x^{54} + x^{52} + x^{48} + x^{46} + x^{44} + x^{40} + x^{38} + x^{36} + x^{28} \\
&\quad + x^{26} + x^{16} + x^{2} + 1, \\
p_9(x) &= x^{110} + x^{106} + x^{104} + x^{90} + x^{86} + x^{84} + x^{82} + x^{80} + x^{74} + x^{72} + x^{70} \\
&\quad + x^{68} + x^{66} + x^{64} + x^{62} + x^{60} + x^{58} + x^{56} + x^{48} + x^{46} + x^{44} + x^{42} \\
&\quad + x^{38} + x^{36} + x^{32} + x^{28} + x^{26} + x^{24} + x^{22} + x^{16} + x^{10} + x^{8} + x^{6} + x^{2}, \\
p_{12}(x) &= x^{114} + x^{110} + x^{108} + x^{106} + x^{104} + x^{102} + x^{100} + x^{98} + x^{96} + x^{94} \\
&\quad + x^{92} + x^{90} + x^{88} + x^{86} + x^{84} + x^{80} + x^{66} + x^{62} + x^{60} + x^{58} + x^{56} \\
&\quad + x^{54} + x^{52} + x^{50} + x^{48} + x^{46} + x^{44} + x^{42} + x^{40} + x^{38} + x^{36} + x^{32} \\
&\quad + x^{18} + x^{14} + x^{12} + x^{10} + x^{8} + x^{6} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0].

For $\phi(W^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{114} + x^{110} + x^{106} + x^{102} + x^{98} + x^{96} + x^{94} + x^{90} + x^{84} + x^{82} + x^{78} \\
&\quad + x^{76} + x^{72} + x^{70} + x^{66} + x^{64} + x^{62} + x^{60} + x^{48} + x^{46} + x^{44} + x^{42} \\
&\quad + x^{18} + x^{16} + x^{12}, \\
p_3(x) &= x^{110} + x^{102} + x^{100} + x^{98} + x^{96} + x^{94} + x^{88} + x^{82} + x^{78} + x^{76} + x^{68} \\
&\quad + x^{66} + x^{64} + x^{54} + x^{52} + x^{50} + x^{48} + x^{36} + x^{34} + x^{30} + x^{28} + x^{22} \\
&\quad + x^{16} + x^{14} + x^{10} + x^{8}, \\
p_6(x) &= x^{108} + x^{100} + x^{98} + x^{96} + x^{88} + x^{80} + x^{78} + x^{76} + x^{72} + x^{70} + x^{68} \\
&\quad + x^{64} + x^{56} + x^{52} + x^{50} + x^{48} + x^{40} + x^{36} + x^{34} + x^{32} + x^{26} + x^{24} \\
&\quad + x^{22} + x^{20} + x^{18} + x^{8} + x^{6} + x^{4} + x^{2} + 1, \\
p_9(x) &= x^{110} + x^{106} + x^{104} + x^{90} + x^{86} + x^{84} + x^{82} + x^{80} + x^{74} + x^{72} + x^{70} \\
&\quad + x^{68} + x^{66} + x^{64} + x^{62} + x^{60} + x^{58} + x^{56} + x^{48} + x^{46} + x^{44} + x^{42} \\
&\quad + x^{38} + x^{36} + x^{32} + x^{28} + x^{26} + x^{24} + x^{22} + x^{16} + x^{10} + x^{8} + x^{6} + x^{2}, \\
p_{12}(x) &= x^{114} + x^{110} + x^{108} + x^{106} + x^{104} + x^{102} + x^{100} + x^{98} + x^{96} + x^{94} \\
&\quad + x^{92} + x^{90} + x^{88} + x^{86} + x^{84} + x^{80} + x^{66} + x^{62} + x^{60} + x^{58} + x^{56} \\
&\quad + x^{54} + x^{52} + x^{50} + x^{48} + x^{46} + x^{44} + x^{42} + x^{40} + x^{38} + x^{36} + x^{32} \\
&\quad + x^{18} + x^{14} + x^{12} + x^{10} + x^{8} + x^{6} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{129} + x^{127} + x^{126} + x^{125} + x^{124} + x^{120} + x^{112} + x^{106} + x^{104} + x^{101} \\
&\quad + x^{100} + x^{98} + x^{97} + x^{92} + x^{91} + x^{90} + x^{89} + x^{87} + x^{86} + x^{84} + x^{83} \\
&\quad + x^{82} + x^{81} + x^{80} + x^{78} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{69} + x^{67}
\end{aligned}$$

$$
\begin{aligned}
&+x^{66}+x^{64}+x^{63}+x^{62}+x^{57}+x^{56}+x^{51}+x^{49}+x^{48}+x^{46}+x^{45}\\
&+x^{42}+x^{41}+x^{39}+x^{37}+x^{36}+x^{35}+x^{33}+x^{32}+x^{31}+x^{27},\\
p_3(x)&=x^{111}+x^{108}+x^{106}+x^{105}+x^{103}+x^{100}+x^{98}+x^{94}+x^{92}+x^{91}\\
&+x^{89}+x^{86}+x^{84}+x^{83}+x^{81}+x^{78}+x^{76}+x^{75}+x^{73}+x^{71}+x^{70}\\
&+x^{67}+x^{66}+x^{62}+x^{60}+x^{59}+x^{49}+x^{47}+x^{46}+x^{43}+x^{42}+x^{41}\\
&+x^{39}+x^{36}+x^{34}+x^{33}+x^{31}+x^{28}+x^{26}+x^{23}+x^{22}+x^{19}+x^{18}\\
&+x^{15}+x^{14}+x^{11}+x^{10}+x^{9},\\
p_6(x)&=x^{117}+x^{115}+x^{114}+x^{113}+x^{112}+x^{111}+x^{110}+x^{108}+x^{107}+x^{105}\\
&+x^{104}+x^{103}+x^{102}+x^{100}+x^{97}+x^{95}+x^{94}+x^{92}+x^{91}+x^{89}\\
&+x^{88}+x^{86}+x^{81}+x^{78}+x^{73}+x^{71}+x^{70}+x^{69}+x^{68}+x^{67}+x^{66}\\
&+x^{64}+x^{63}+x^{60}+x^{57}+x^{55}+x^{54}+x^{53}+x^{52}+x^{50}+x^{47}+x^{45}\\
&+x^{44}+x^{42}+x^{39}+x^{37}+x^{36}+x^{34}+x^{31}+x^{28}+x^{27}+x^{24}+x^{23}\\
&+x^{21}+x^{20}+x^{19}+x^{18}+x^{16}+x^{9}+x^{6},\\
p_9(x)&=x^{123}+x^{120}+x^{119}+x^{118}+x^{117}+x^{116}+x^{114}+x^{111}+x^{110}+x^{107}\\
&+x^{106}+x^{105}+x^{103}+x^{100}+x^{99}+x^{98}+x^{97}+x^{96}+x^{95}+x^{94}\\
&+x^{93}+x^{92}+x^{91}+x^{90}+x^{88}+x^{84}+x^{82}+x^{79}+x^{78}+x^{77}+x^{67}\\
&+x^{65}+x^{64}+x^{63}+x^{55}+x^{52}+x^{51}+x^{50}+x^{48}+x^{47}+x^{43}+x^{41}\\
&+x^{40}+x^{36}+x^{35}+x^{34}+x^{33}+x^{32}+x^{30}+x^{27}+x^{26}+x^{25}+x^{21}\\
&+x^{19}+x^{18}+x^{17}+x^{15}+x^{12}+x^{11}+x^{10}+x^{9}+x^{8}+x^{7}+x^{6}+x^{4}\\
&+x^{3},\\
p_{12}(x)&=x^{129}+x^{127}+x^{126}+x^{124}+x^{117}+x^{115}+x^{114}+x^{112}+x^{97}+x^{95}\\
&+x^{94}+x^{92}+x^{85}+x^{83}+x^{82}+x^{81}+x^{80}+x^{79}+x^{78}+x^{76}+x^{69}\\
&+x^{67}+x^{66}+x^{64}+x^{49}+x^{47}+x^{46}+x^{44}+x^{37}+x^{35}+x^{34}+x^{33}\\
&+x^{32}+x^{31}+x^{30}+x^{28}+x^{21}+x^{19}+x^{18}+x^{17}+x^{16}+x^{15}+x^{14}\\
&+x^{12}+x^{5}+x^{3}+x^{2}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 1, 0, 1, 1].

For $\phi(W^e,1,0)$:

$$
\begin{aligned}
p_0(x)&=x^{129}+x^{127}+x^{126}+x^{125}+x^{124}+x^{120}+x^{112}+x^{106}+x^{104}+x^{101}\\
&+x^{100}+x^{98}+x^{97}+x^{92}+x^{91}+x^{90}+x^{89}+x^{87}+x^{86}+x^{84}+x^{83}\\
&+x^{82}+x^{81}+x^{80}+x^{78}+x^{77}+x^{76}+x^{75}+x^{74}+x^{73}+x^{69}+x^{67}\\
&+x^{66}+x^{64}+x^{63}+x^{62}+x^{57}+x^{56}+x^{51}+x^{49}+x^{48}+x^{46}+x^{45}\\
&+x^{42}+x^{41}+x^{39}+x^{37}+x^{36}+x^{35}+x^{33}+x^{32}+x^{31}+x^{27},\\
p_3(x)&=x^{111}+x^{108}+x^{106}+x^{105}+x^{103}+x^{100}+x^{98}+x^{94}+x^{92}+x^{91}\\
&+x^{89}+x^{86}+x^{84}+x^{83}+x^{81}+x^{78}+x^{76}+x^{75}+x^{73}+x^{71}+x^{70}\\
&+x^{67}+x^{66}+x^{62}+x^{60}+x^{59}+x^{49}+x^{47}+x^{46}+x^{43}+x^{42}+x^{41}
\end{aligned}
$$

$$\begin{aligned}
&+ x^{39} + x^{36} + x^{34} + x^{33} + x^{31} + x^{28} + x^{26} + x^{23} + x^{22} + x^{19} + x^{18} \\
&+ x^{15} + x^{14} + x^{11} + x^{10} + x^{9}, \\
p_6(x) = {}& x^{117} + x^{115} + x^{114} + x^{113} + x^{112} + x^{111} + x^{110} + x^{108} + x^{107} + x^{105} \\
&+ x^{104} + x^{103} + x^{102} + x^{100} + x^{97} + x^{95} + x^{94} + x^{92} + x^{91} + x^{89} \\
&+ x^{88} + x^{86} + x^{81} + x^{78} + x^{73} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} \\
&+ x^{64} + x^{63} + x^{60} + x^{57} + x^{55} + x^{54} + x^{53} + x^{52} + x^{50} + x^{47} + x^{45} \\
&+ x^{44} + x^{42} + x^{39} + x^{37} + x^{36} + x^{34} + x^{31} + x^{28} + x^{27} + x^{24} + x^{23} \\
&+ x^{21} + x^{20} + x^{19} + x^{18} + x^{16} + x^{9} + x^{6}, \\
p_9(x) = {}& x^{123} + x^{120} + x^{119} + x^{118} + x^{117} + x^{116} + x^{114} + x^{111} + x^{110} + x^{107} \\
&+ x^{106} + x^{105} + x^{103} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} \\
&+ x^{93} + x^{92} + x^{91} + x^{90} + x^{88} + x^{84} + x^{82} + x^{79} + x^{78} + x^{77} + x^{67} \\
&+ x^{65} + x^{64} + x^{63} + x^{55} + x^{52} + x^{51} + x^{50} + x^{48} + x^{47} + x^{43} + x^{41} \\
&+ x^{40} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{30} + x^{27} + x^{26} + x^{25} + x^{21} \\
&+ x^{19} + x^{18} + x^{17} + x^{15} + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6} + x^{4} \\
&+ x^{3}, \\
p_{12}(x) = {}& x^{129} + x^{127} + x^{126} + x^{124} + x^{117} + x^{115} + x^{114} + x^{112} + x^{97} + x^{95} \\
&+ x^{94} + x^{92} + x^{85} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{76} + x^{69} \\
&+ x^{67} + x^{66} + x^{64} + x^{49} + x^{47} + x^{46} + x^{44} + x^{37} + x^{35} + x^{34} + x^{33} \\
&+ x^{32} + x^{31} + x^{30} + x^{28} + x^{21} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} \\
&+ x^{12} + x^{5} + x^{3} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 1, 0, 1, 1].

For $\phi(W^e, 1, 1)$:

$$\begin{aligned}
p_0(x) = {}& x^{144} + x^{138} + x^{132} + x^{130} + x^{120} + x^{106} + x^{104} + x^{102} + x^{96} + x^{94} \\
&+ x^{90} + x^{80} + x^{76} + x^{74} + x^{72} + x^{66} + x^{56} + x^{52} + x^{48} + x^{46} + x^{42}, \\
p_3(x) = {}& x^{140} + x^{136} + x^{134} + x^{132} + x^{128} + x^{122} + x^{120} + x^{118} + x^{116} + x^{114} \\
&+ x^{110} + x^{104} + x^{100} + x^{94} + x^{88} + x^{84} + x^{82} + x^{80} + x^{76} + x^{72} + x^{70} \\
&+ x^{68} + x^{64} + x^{62} + x^{58} + x^{56} + x^{54} + x^{50} + x^{44} + x^{42} + x^{38} + x^{34} \\
&+ x^{32} + x^{28} + x^{26} + x^{20} + x^{14} + x^{6}, \\
p_6(x) = {}& x^{138} + x^{136} + x^{134} + x^{122} + x^{118} + x^{112} + x^{110} + x^{106} + x^{104} + x^{102} \\
&+ x^{100} + x^{98} + x^{94} + x^{84} + x^{82} + x^{74} + x^{68} + x^{66} + x^{62} + x^{60} + x^{56} \\
&+ x^{54} + x^{52} + x^{50} + x^{48} + x^{46} + x^{44} + x^{42} + x^{20} + x^{18} + x^{16} + x^{10} \\
&+ x^{8} + x^{4} + x^{2} + 1, \\
p_9(x) = {}& x^{140} + x^{130} + x^{126} + x^{122} + x^{114} + x^{112} + x^{110} + x^{94} + x^{90} + x^{84} \\
&+ x^{82} + x^{80} + x^{78} + x^{74} + x^{70} + x^{66} + x^{62} + x^{56} + x^{54} + x^{50} + x^{48} \\
&+ x^{46} + x^{38} + x^{34} + x^{30} + x^{24} + x^{22} + x^{20} + x^{18} + x^{16} + x^{14} + x^{12}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{10} + x^2,\\
p_{12}(x) &= x^{144} + x^{136} + x^{120} + x^{104} + x^{96} + x^{80} + x^{72} + x^{56} + x^{48} + x^{16} + x^8\\
&+ 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0].

For $\phi(M^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{129} + x^{127} + x^{126} + x^{124} + x^{123} + x^{120} + x^{119} + x^{118} + x^{115} + x^{113}\\
&+ x^{112} + x^{111} + x^{110} + x^{108} + x^{106} + x^{103} + x^{102} + x^{100} + x^{99} + x^{98}\\
&+ x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{88} + x^{87} + x^{85}\\
&+ x^{83} + x^{82} + x^{80} + x^{79} + x^{77} + x^{72} + x^{68} + x^{66} + x^{63} + x^{62} + x^{61}\\
&+ x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{51} + x^{45} + x^{42} + x^{39}\\
&+ x^{35} + x^{31} + x^{30} + x^{25} + x^{24} + x^{23} + x^{22} + x^{19} + x^{15} + x^{12},\\
p_3(x) &= x^{121} + x^{119} + x^{118} + x^{116} + x^{115} + x^{113} + x^{112} + x^{109} + x^{108} + x^{107}\\
&+ x^{106} + x^{104} + x^{102} + x^{100} + x^{99} + x^{97} + x^{93} + x^{90} + x^{88} + x^{86}\\
&+ x^{85} + x^{83} + x^{82} + x^{81} + x^{79} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{70}\\
&+ x^{68} + x^{64} + x^{62} + x^{61} + x^{59} + x^{58} + x^{56} + x^{48} + x^{45} + x^{44} + x^{42}\\
&+ x^{39} + x^{38} + x^{35} + x^{33} + x^{32} + x^{31} + x^{29} + x^{27} + x^{26} + x^{23} + x^{17}\\
&+ x^{16} + x^{14} + x^{12} + x^{11} + x^8,\\
p_6(x) &= x^{123} + x^{121} + x^{120} + x^{116} + x^{113} + x^{111} + x^{109} + x^{107} + x^{106} + x^{103}\\
&+ x^{98} + x^{95} + x^{85} + x^{84} + x^{82} + x^{81} + x^{80} + x^{78} + x^{75} + x^{74} + x^{69}\\
&+ x^{67} + x^{65} + x^{63} + x^{60} + x^{56} + x^{55} + x^{53} + x^{51} + x^{49} + x^{47} + x^{45}\\
&+ x^{44} + x^{42} + x^{39} + x^{35} + x^{34} + x^{31} + x^{29} + x^{25} + x^{24} + x^{23} + x^{22}\\
&+ x^{21} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{10} + x^8 + x^7 + x^5 + x^4\\
&+ x^3 + x^2 + 1,\\
p_9(x) &= x^{121} + x^{119} + x^{118} + x^{116} + x^{113} + x^{111} + x^{110} + x^{109} + x^{108} + x^{107}\\
&+ x^{106} + x^{104} + x^{97} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88}\\
&+ x^{87} + x^{86} + x^{84} + x^{73} + x^{71} + x^{70} + x^{68} + x^{65} + x^{63} + x^{62} + x^{61}\\
&+ x^{60} + x^{59} + x^{58} + x^{56} + x^{49} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42}\\
&+ x^{41} + x^{40} + x^{39} + x^{38} + x^{36} + x^{25} + x^{23} + x^{22} + x^{20} + x^{17} + x^{15}\\
&+ x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^9 + x^8 + x^7 + x^6 + x^4,\\
p_{12}(x) &= x^{129} + x^{127} + x^{126} + x^{124} + x^{121} + x^{119} + x^{118} + x^{117} + x^{116} + x^{115}\\
&+ x^{114} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} + x^{107} + x^{106}\\
&+ x^{104} + x^{93} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83}\\
&+ x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{76} + x^{73} + x^{71} + x^{70} + x^{69} + x^{68}\\
&+ x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{56}\\
&+ x^{45} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34}
\end{aligned}
$$

$$+ x^{33} + x^{32} + x^{31} + x^{30} + x^{28} + x^{25} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19}$$
$$+ x^{18} + x^{16} + x^{13} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6} + x^{5} + x^{4} + x^{3} + x^{2}$$
$$+ 1,$$

and the initial terms are [1, 0, 1, 0, 1, 0, 1, 1, 0, 1, 1, 0, 0, 1, 0, 1].

For $\phi(M^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{120} + x^{114} + x^{113} + x^{110} + x^{108} + x^{107} + x^{103} + x^{102} + x^{101} + x^{100} \\
&\quad + x^{98} + x^{97} + x^{93} + x^{92} + x^{91} + x^{89} + x^{85} + x^{83} + x^{80} + x^{79} + x^{77} \\
&\quad + x^{75} + x^{74} + x^{71} + x^{68} + x^{67} + x^{65} + x^{64} + x^{61} + x^{60} + x^{56} + x^{54} \\
&\quad + x^{53} + x^{51} + x^{50} + x^{46} + x^{44} + x^{43} + x^{41} + x^{39} + x^{36} + x^{33} + x^{31} \\
&\quad + x^{26} + x^{25} + x^{22} + x^{21} + x^{18}, \\
p_3(x) &= x^{102} + x^{100} + x^{98} + x^{97} + x^{95} + x^{91} + x^{87} + x^{83} + x^{79} + x^{75} + x^{71} \\
&\quad + x^{67} + x^{63} + x^{62} + x^{60} + x^{59} + x^{58} + x^{57} + x^{54} + x^{52} + x^{50} + x^{49} \\
&\quad + x^{47} + x^{46} + x^{44} + x^{43} + x^{42} + x^{41} + x^{30} + x^{28} + x^{26} + x^{25} + x^{23} \\
&\quad + x^{19} + x^{15} + x^{14} + x^{12} + x^{11} + x^{10} + x^{9}, \\
p_6(x) &= x^{108} + x^{98} + x^{92} + x^{90} + x^{88} + x^{80} + x^{78} + x^{64} + x^{54} + x^{52} + x^{48} \\
&\quad + x^{46} + x^{44} + x^{42} + x^{40} + x^{36} + x^{32} + x^{30} + x^{28} + x^{26} + x^{24} + x^{14} \\
&\quad + x^{8} + x^{6}, \\
p_9(x) &= x^{114} + x^{112} + x^{109} + x^{108} + x^{107} + x^{106} + x^{105} + x^{104} + x^{100} + x^{99} \\
&\quad + x^{98} + x^{96} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{84} + x^{83} + x^{66} \\
&\quad + x^{64} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{52} + x^{51} + x^{50} + x^{48} \\
&\quad + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{36} + x^{35} + x^{18} + x^{16} + x^{13} \\
&\quad + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{4} + x^{3}, \\
p_{12}(x) &= x^{120} + x^{116} + x^{108} + x^{100} + x^{96} + x^{92} + x^{88} + x^{80} + x^{72} + x^{68} + x^{60} \\
&\quad + x^{52} + x^{48} + x^{44} + x^{40} + x^{32} + x^{24} + x^{20} + x^{12} + x^{8} + 1,
\end{aligned}$$

and the initial terms are [0, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 1, 1, 1].

For $\phi(M^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{144} + x^{140} + x^{138} + x^{137} + x^{136} + x^{133} + x^{131} + x^{129} + x^{125} + x^{121} \\
&\quad + x^{119} + x^{116} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{107} + x^{106} \\
&\quad + x^{103} + x^{102} + x^{100} + x^{98} + x^{95} + x^{93} + x^{92} + x^{90} + x^{88} + x^{85} + x^{84} \\
&\quad + x^{82} + x^{81} + x^{80} + x^{78} + x^{77} + x^{73} + x^{72} + x^{71} + x^{69} + x^{66} + x^{61} \\
&\quad + x^{58} + x^{57} + x^{56} + x^{51} + x^{50} + x^{49} + x^{47} + x^{38} + x^{34} + x^{32} + x^{30} \\
&\quad + x^{27}, \\
p_3(x) &= x^{126} + x^{124} + x^{121} + x^{120} + x^{119} + x^{117} + x^{116} + x^{114} + x^{113} + x^{111} \\
&\quad + x^{110} + x^{108} + x^{107} + x^{105} + x^{104} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97} \\
&\quad + x^{86} + x^{84} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{74} + x^{72} + x^{69} + x^{68}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{67} + x^{66} + x^{65} + x^{62} + x^{60} + x^{57} + x^{56} + x^{55} + x^{53} + x^{52} + x^{50}\\
&+ x^{49} + x^{47} + x^{43} + x^{39} + x^{35} + x^{31} + x^{27} + x^{23} + x^{22} + x^{20} + x^{19}\\
&+ x^{17} + x^{16} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9},\\
p_6(x) =\ & x^{132} + x^{128} + x^{124} + x^{122} + x^{118} + x^{112} + x^{110} + x^{108} + x^{98} + x^{96}\\
&+ x^{90} + x^{88} + x^{86} + x^{82} + x^{76} + x^{74} + x^{72} + x^{62} + x^{56} + x^{54} + x^{44}\\
&+ x^{36} + x^{34} + x^{28} + x^{18} + x^{16} + x^{12} + x^{10} + x^{8} + x^{6},\\
p_9(x) =\ & x^{138} + x^{136} + x^{134} + x^{133} + x^{131} + x^{128} + x^{127} + x^{126} + x^{124} + x^{122}\\
&+ x^{121} + x^{119} + x^{116} + x^{115} + x^{106} + x^{104} + x^{102} + x^{101} + x^{99} + x^{96}\\
&+ x^{95} + x^{94} + x^{92} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{80} + x^{79}\\
&+ x^{78} + x^{76} + x^{74} + x^{73} + x^{71} + x^{68} + x^{67} + x^{58} + x^{56} + x^{54} + x^{53}\\
&+ x^{51} + x^{48} + x^{47} + x^{46} + x^{44} + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36}\\
&+ x^{32} + x^{31} + x^{30} + x^{28} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{16}\\
&+ x^{15} + x^{14} + x^{12} + x^{10} + x^{9} + x^{7} + x^{4} + x^{3},\\
p_{12}(x) =\ & x^{144} + x^{116} + x^{96} + x^{84} + x^{80} + x^{68} + x^{48} + x^{36} + x^{20} + x^{16} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 1, 1, 0, 1, 1, 1].

For $\phi(M^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) =\ & x^{129} + x^{127} + x^{126} + x^{124} + x^{123} + x^{121} + x^{120} + x^{117} + x^{116} + x^{114}\\
&+ x^{109} + x^{107} + x^{104} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{93} + x^{92}\\
&+ x^{90} + x^{89} + x^{88} + x^{86} + x^{85} + x^{79} + x^{77} + x^{76} + x^{75} + x^{71} + x^{70}\\
&+ x^{69} + x^{67} + x^{63} + x^{62} + x^{58} + x^{54} + x^{53} + x^{52} + x^{48} + x^{45} + x^{44}\\
&+ x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{33},\\
p_3(x) =\ & x^{121} + x^{119} + x^{118} + x^{116} + x^{115} + x^{112} + x^{111} + x^{110} + x^{105} + x^{103}\\
&+ x^{101} + x^{99} + x^{98} + x^{95} + x^{94} + x^{92} + x^{91} + x^{89} + x^{87} + x^{84} + x^{80}\\
&+ x^{78} + x^{73} + x^{72} + x^{70} + x^{68} + x^{65} + x^{64} + x^{63} + x^{60} + x^{59} + x^{56}\\
&+ x^{51} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{37} + x^{36} + x^{34} + x^{30}\\
&+ x^{28} + x^{27} + x^{23} + x^{17} + x^{16} + x^{14} + x^{12},\\
p_6(x) =\ & x^{123} + x^{120} + x^{119} + x^{118} + x^{117} + x^{115} + x^{114} + x^{112} + x^{110} + x^{108}\\
&+ x^{107} + x^{103} + x^{101} + x^{99} + x^{97} + x^{96} + x^{94} + x^{92} + x^{91} + x^{88} + x^{87}\\
&+ x^{85} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{73} + x^{72} + x^{71}\\
&+ x^{70} + x^{68} + x^{66} + x^{65} + x^{64} + x^{61} + x^{59} + x^{58} + x^{55} + x^{51} + x^{50}\\
&+ x^{49} + x^{37} + x^{36} + x^{35} + x^{33} + x^{31} + x^{30} + x^{26} + x^{24} + x^{23} + x^{21}\\
&+ x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{10} + x^{8} + x^{5} + x^{3} + x^{2} + 1,\\
p_9(x) =\ & x^{121} + x^{119} + x^{118} + x^{116} + x^{113} + x^{111} + x^{110} + x^{109} + x^{108} + x^{107}\\
&+ x^{106} + x^{104} + x^{97} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88}\\
&+ x^{87} + x^{86} + x^{84} + x^{73} + x^{71} + x^{70} + x^{68} + x^{65} + x^{63} + x^{62} + x^{61}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{60} + x^{59} + x^{58} + x^{56} + x^{49} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42}\\
&+ x^{41} + x^{40} + x^{39} + x^{38} + x^{36} + x^{25} + x^{23} + x^{22} + x^{20} + x^{17} + x^{15}\\
&+ x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6} + x^{4},\\
p_{12}(x) = {}& x^{129} + x^{127} + x^{126} + x^{124} + x^{121} + x^{119} + x^{118} + x^{117} + x^{116} + x^{115}\\
&+ x^{114} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} + x^{107} + x^{106}\\
&+ x^{104} + x^{93} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83}\\
&+ x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{76} + x^{73} + x^{71} + x^{70} + x^{69} + x^{68}\\
&+ x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{56}\\
&+ x^{45} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34}\\
&+ x^{33} + x^{32} + x^{31} + x^{30} + x^{28} + x^{25} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19}\\
&+ x^{18} + x^{16} + x^{13} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6} + x^{5} + x^{4} + x^{3} + x^{2}\\
&+ 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 0, 1, 1, 1, 1].

For $\phi(W^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) = {}& x^{129} + x^{127} + x^{126} + x^{124} + x^{123} + x^{120} + x^{119} + x^{118} + x^{115} + x^{113}\\
&+ x^{112} + x^{111} + x^{110} + x^{108} + x^{106} + x^{103} + x^{102} + x^{100} + x^{99} + x^{98}\\
&+ x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{88} + x^{87} + x^{85}\\
&+ x^{83} + x^{82} + x^{80} + x^{79} + x^{77} + x^{72} + x^{68} + x^{66} + x^{63} + x^{62} + x^{61}\\
&+ x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{51} + x^{45} + x^{42} + x^{39}\\
&+ x^{35} + x^{31} + x^{30} + x^{25} + x^{24} + x^{23} + x^{22} + x^{19} + x^{15} + x^{12},\\
p_3(x) = {}& x^{121} + x^{119} + x^{118} + x^{116} + x^{115} + x^{113} + x^{112} + x^{109} + x^{108} + x^{107}\\
&+ x^{106} + x^{104} + x^{102} + x^{100} + x^{99} + x^{97} + x^{93} + x^{90} + x^{88} + x^{86}\\
&+ x^{85} + x^{83} + x^{82} + x^{81} + x^{79} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{70}\\
&+ x^{68} + x^{64} + x^{62} + x^{61} + x^{59} + x^{58} + x^{56} + x^{48} + x^{45} + x^{44} + x^{42}\\
&+ x^{39} + x^{38} + x^{35} + x^{33} + x^{32} + x^{31} + x^{29} + x^{27} + x^{26} + x^{23} + x^{17}\\
&+ x^{16} + x^{14} + x^{12} + x^{11} + x^{8},\\
p_6(x) = {}& x^{123} + x^{121} + x^{120} + x^{116} + x^{113} + x^{111} + x^{109} + x^{107} + x^{106} + x^{103}\\
&+ x^{98} + x^{95} + x^{85} + x^{84} + x^{82} + x^{81} + x^{80} + x^{78} + x^{75} + x^{74} + x^{69}\\
&+ x^{67} + x^{65} + x^{63} + x^{60} + x^{56} + x^{55} + x^{53} + x^{51} + x^{49} + x^{47} + x^{45}\\
&+ x^{44} + x^{42} + x^{39} + x^{35} + x^{34} + x^{31} + x^{29} + x^{25} + x^{24} + x^{23} + x^{22}\\
&+ x^{21} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{10} + x^{8} + x^{7} + x^{5} + x^{4}\\
&+ x^{3} + x^{2} + 1,\\
p_9(x) = {}& x^{121} + x^{119} + x^{118} + x^{116} + x^{113} + x^{111} + x^{110} + x^{109} + x^{108} + x^{107}\\
&+ x^{106} + x^{104} + x^{97} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88}\\
&+ x^{87} + x^{86} + x^{84} + x^{73} + x^{71} + x^{70} + x^{68} + x^{65} + x^{63} + x^{62} + x^{61}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{60} + x^{59} + x^{58} + x^{56} + x^{49} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42}\\
&+ x^{41} + x^{40} + x^{39} + x^{38} + x^{36} + x^{25} + x^{23} + x^{22} + x^{20} + x^{17} + x^{15}\\
&+ x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6} + x^{4},\\
p_{12}(x) = {}& x^{129} + x^{127} + x^{126} + x^{124} + x^{121} + x^{119} + x^{118} + x^{117} + x^{116} + x^{115}\\
&+ x^{114} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} + x^{107} + x^{106}\\
&+ x^{104} + x^{93} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83}\\
&+ x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{76} + x^{73} + x^{71} + x^{70} + x^{69} + x^{68}\\
&+ x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{56}\\
&+ x^{45} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34}\\
&+ x^{33} + x^{32} + x^{31} + x^{30} + x^{28} + x^{25} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19}\\
&+ x^{18} + x^{16} + x^{13} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6} + x^{5} + x^{4} + x^{3} + x^{2}\\
&+ 1,
\end{aligned}
$$

and the initial terms are [1, 0, 1, 0, 1, 0, 1, 1, 0, 1, 1, 0, 0, 1, 0, 1].

For $\phi(W^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) = {}& x^{144} + x^{140} + x^{138} + x^{137} + x^{136} + x^{133} + x^{131} + x^{129} + x^{125} + x^{121}\\
&+ x^{119} + x^{116} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{107} + x^{106}\\
&+ x^{103} + x^{102} + x^{100} + x^{98} + x^{95} + x^{93} + x^{92} + x^{90} + x^{88} + x^{85} + x^{84}\\
&+ x^{82} + x^{81} + x^{80} + x^{78} + x^{77} + x^{73} + x^{72} + x^{71} + x^{69} + x^{66} + x^{61}\\
&+ x^{58} + x^{57} + x^{56} + x^{51} + x^{50} + x^{49} + x^{47} + x^{38} + x^{34} + x^{32} + x^{30}\\
&+ x^{27},\\
p_3(x) = {}& x^{126} + x^{124} + x^{121} + x^{120} + x^{119} + x^{117} + x^{116} + x^{114} + x^{113} + x^{111}\\
&+ x^{110} + x^{108} + x^{107} + x^{105} + x^{104} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97}\\
&+ x^{86} + x^{84} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{74} + x^{72} + x^{69} + x^{68}\\
&+ x^{67} + x^{66} + x^{65} + x^{62} + x^{60} + x^{57} + x^{56} + x^{55} + x^{53} + x^{52} + x^{50}\\
&+ x^{49} + x^{47} + x^{43} + x^{39} + x^{35} + x^{31} + x^{27} + x^{23} + x^{22} + x^{20} + x^{19}\\
&+ x^{17} + x^{16} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9},\\
p_6(x) = {}& x^{132} + x^{128} + x^{124} + x^{122} + x^{118} + x^{112} + x^{110} + x^{108} + x^{98} + x^{96}\\
&+ x^{90} + x^{88} + x^{86} + x^{82} + x^{76} + x^{74} + x^{72} + x^{62} + x^{56} + x^{54} + x^{44}\\
&+ x^{36} + x^{34} + x^{28} + x^{18} + x^{16} + x^{12} + x^{10} + x^{8} + x^{6},\\
p_9(x) = {}& x^{138} + x^{136} + x^{134} + x^{133} + x^{131} + x^{128} + x^{127} + x^{126} + x^{124} + x^{122}\\
&+ x^{121} + x^{119} + x^{116} + x^{115} + x^{106} + x^{104} + x^{102} + x^{101} + x^{99} + x^{96}\\
&+ x^{95} + x^{94} + x^{92} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{80} + x^{79}\\
&+ x^{78} + x^{76} + x^{74} + x^{73} + x^{71} + x^{68} + x^{67} + x^{58} + x^{56} + x^{54} + x^{53}\\
&+ x^{51} + x^{48} + x^{47} + x^{46} + x^{44} + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36}\\
&+ x^{32} + x^{31} + x^{30} + x^{28} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{16}
\end{aligned}
$$

$$+ x^{15} + x^{14} + x^{12} + x^{10} + x^{9} + x^{7} + x^{4} + x^{3},$$
$$p_{12}(x) = x^{144} + x^{116} + x^{96} + x^{84} + x^{80} + x^{68} + x^{48} + x^{36} + x^{20} + x^{16} + x^{4} + 1,$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 1, 1, 0, 1, 1, 1].

For $\phi(W^o, 1, 0)$:

$$\begin{aligned}
p_0(x) = {} & x^{120} + x^{114} + x^{113} + x^{110} + x^{108} + x^{107} + x^{103} + x^{102} + x^{101} + x^{100} \\
& + x^{98} + x^{97} + x^{93} + x^{92} + x^{91} + x^{89} + x^{85} + x^{83} + x^{80} + x^{79} + x^{77} \\
& + x^{75} + x^{74} + x^{71} + x^{68} + x^{67} + x^{65} + x^{64} + x^{61} + x^{60} + x^{56} + x^{54} \\
& + x^{53} + x^{51} + x^{50} + x^{46} + x^{44} + x^{43} + x^{41} + x^{39} + x^{36} + x^{33} + x^{31} \\
& + x^{26} + x^{25} + x^{22} + x^{21} + x^{18}, \\
p_3(x) = {} & x^{102} + x^{100} + x^{98} + x^{97} + x^{95} + x^{91} + x^{87} + x^{83} + x^{79} + x^{75} + x^{71} \\
& + x^{67} + x^{63} + x^{62} + x^{60} + x^{59} + x^{58} + x^{57} + x^{54} + x^{52} + x^{50} + x^{49} \\
& + x^{47} + x^{46} + x^{44} + x^{43} + x^{42} + x^{41} + x^{30} + x^{28} + x^{26} + x^{25} + x^{23} \\
& + x^{19} + x^{15} + x^{14} + x^{12} + x^{11} + x^{10} + x^{9}, \\
p_6(x) = {} & x^{108} + x^{98} + x^{92} + x^{90} + x^{88} + x^{80} + x^{78} + x^{64} + x^{54} + x^{52} + x^{48} \\
& + x^{46} + x^{44} + x^{42} + x^{40} + x^{36} + x^{32} + x^{30} + x^{28} + x^{26} + x^{24} + x^{14} \\
& + x^{8} + x^{6}, \\
p_9(x) = {} & x^{114} + x^{112} + x^{109} + x^{108} + x^{107} + x^{106} + x^{105} + x^{104} + x^{100} + x^{99} \\
& + x^{98} + x^{96} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{84} + x^{83} + x^{66} \\
& + x^{64} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{52} + x^{51} + x^{50} + x^{48} \\
& + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{36} + x^{35} + x^{18} + x^{16} + x^{13} \\
& + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{4} + x^{3}, \\
p_{12}(x) = {} & x^{120} + x^{116} + x^{108} + x^{100} + x^{96} + x^{92} + x^{88} + x^{80} + x^{72} + x^{68} + x^{60} \\
& + x^{52} + x^{48} + x^{44} + x^{40} + x^{32} + x^{24} + x^{20} + x^{12} + x^{8} + 1,
\end{aligned}$$

and the initial terms are [0, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 1, 1, 1].

For $\phi(W^o, 1, 1)$:

$$\begin{aligned}
p_0(x) = {} & x^{129} + x^{127} + x^{126} + x^{124} + x^{123} + x^{121} + x^{120} + x^{117} + x^{116} + x^{114} \\
& + x^{109} + x^{107} + x^{104} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{93} + x^{92} \\
& + x^{90} + x^{89} + x^{88} + x^{86} + x^{85} + x^{79} + x^{77} + x^{76} + x^{75} + x^{71} + x^{70} \\
& + x^{69} + x^{67} + x^{63} + x^{62} + x^{58} + x^{54} + x^{53} + x^{52} + x^{48} + x^{45} + x^{44} \\
& + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{33}, \\
p_3(x) = {} & x^{121} + x^{119} + x^{118} + x^{116} + x^{115} + x^{112} + x^{111} + x^{110} + x^{105} + x^{103} \\
& + x^{101} + x^{99} + x^{98} + x^{95} + x^{94} + x^{92} + x^{91} + x^{89} + x^{87} + x^{84} + x^{80} \\
& + x^{78} + x^{73} + x^{72} + x^{70} + x^{68} + x^{65} + x^{64} + x^{63} + x^{60} + x^{59} + x^{56} \\
& + x^{51} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{37} + x^{36} + x^{34} + x^{30} \\
& + x^{28} + x^{27} + x^{23} + x^{17} + x^{16} + x^{14} + x^{12},
\end{aligned}$$

$$\begin{aligned}
p_6(x) = {} & x^{123} + x^{120} + x^{119} + x^{118} + x^{117} + x^{115} + x^{114} + x^{112} + x^{110} + x^{108} \\
& + x^{107} + x^{103} + x^{101} + x^{99} + x^{97} + x^{96} + x^{94} + x^{92} + x^{91} + x^{88} + x^{87} \\
& + x^{85} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{73} + x^{72} + x^{71} \\
& + x^{70} + x^{68} + x^{66} + x^{65} + x^{64} + x^{61} + x^{59} + x^{58} + x^{55} + x^{51} + x^{50} \\
& + x^{49} + x^{37} + x^{36} + x^{35} + x^{33} + x^{31} + x^{30} + x^{26} + x^{24} + x^{23} + x^{21} \\
& + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{10} + x^{8} + x^{5} + x^{3} + x^{2} + 1, \\
p_9(x) = {} & x^{121} + x^{119} + x^{118} + x^{116} + x^{113} + x^{111} + x^{110} + x^{109} + x^{108} + x^{107} \\
& + x^{106} + x^{104} + x^{97} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} \\
& + x^{87} + x^{86} + x^{84} + x^{73} + x^{71} + x^{70} + x^{68} + x^{65} + x^{63} + x^{62} + x^{61} \\
& + x^{60} + x^{59} + x^{58} + x^{56} + x^{49} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} \\
& + x^{41} + x^{40} + x^{39} + x^{38} + x^{36} + x^{25} + x^{23} + x^{22} + x^{20} + x^{17} + x^{15} \\
& + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6} + x^{4}, \\
p_{12}(x) = {} & x^{129} + x^{127} + x^{126} + x^{124} + x^{121} + x^{119} + x^{118} + x^{117} + x^{116} + x^{115} \\
& + x^{114} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} + x^{107} + x^{106} \\
& + x^{104} + x^{93} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} \\
& + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{76} + x^{73} + x^{71} + x^{70} + x^{69} + x^{68} \\
& + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{56} \\
& + x^{45} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} \\
& + x^{33} + x^{32} + x^{31} + x^{30} + x^{28} + x^{25} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} \\
& + x^{18} + x^{16} + x^{13} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6} + x^{5} + x^{4} + x^{3} + x^{2} \\
& + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 0, 1, 1, 1, 1].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	261	[448, 38]	522	[786, 787]	783	[493, 808]
1	[3, 4]	262	[449, 397]	523	[715, 610]	784	[189, 493]
2	[5, 6]	263	[447, 398]	524	[788, 789]	785	[107, 177]
3	[7, 8]	264	[141, 450]	525	[790, 463]	786	[847, 37]
4	[9, 10]	265	[446, 143]	526	[791, 111]	787	[397, 825]
5	[8, 11]	266	[415, 417]	527	[792, 394]	788	[372, 805]
6	[12, 13]	267	[451, 416]	528	[793, 433]	789	[438, 454]
7	[14, 15]	268	[452, 382]	529	[494, 153]	790	[882, 883]
8	[16, 17]	269	[405, 453]	530	[794, 494]	791	[884, 885]
9	[18, 19]	270	[454, 455]	531	[414, 8]	792	[886, 887]
10	[20, 21]	271	[456, 160]	532	[521, 463]	793	[277, 888]
11	[22, 23]	272	[457, 429]	533	[509, 110]	794	[533, 282]

12	[24, 25]	273	[405, 376]	534	[159, 433]	795	[889, 890]
13	[26, 27]	274	[458, 459]	535	[388, 459]	796	[891, 892]
14	[28, 29]	275	[440, 123]	536	[365, 795]	797	[893, 603]
15	[30, 31]	276	[178, 460]	537	[465, 183]	798	[609, 894]
16	[32, 33]	277	[403, 360]	538	[12, 796]	799	[895, 896]
17	[34, 35]	278	[396, 422]	539	[497, 482]	800	[897, 78]
18	[36, 37]	279	[461, 462]	540	[126, 399]	801	[898, 899]
19	[38, 39]	280	[39, 122]	541	[797, 401]	802	[900, 901]
20	[40, 41]	281	[140, 448]	542	[376, 798]	803	[69, 338]
21	[42, 43]	282	[463, 49]	543	[108, 51]	804	[902, 709]
22	[34, 44]	283	[110, 116]	544	[162, 170]	805	[303, 903]
23	[40, 45]	284	[425, 135]	545	[117, 394]	806	[904, 905]
24	[46, 47]	285	[464, 186]	546	[799, 148]	807	[906, 907]
25	[48, 49]	286	[465, 392]	547	[9, 43]	808	[908, 909]
26	[50, 51]	287	[446, 448]	548	[800, 487]	809	[910, 886]
27	[52, 53]	288	[466, 467]	549	[801, 44]	810	[911, 655]
28	[54, 55]	289	[468, 469]	550	[12, 154]	811	[909, 912]
29	[56, 57]	290	[470, 471]	551	[34, 471]	812	[913, 914]
30	[58, 59]	291	[418, 472]	552	[497, 505]	813	[883, 240]
31	[60, 61]	292	[53, 443]	553	[174, 165]	814	[915, 916]
32	[62, 63]	293	[457, 473]	554	[423, 459]	815	[917, 918]
33	[64, 65]	294	[147, 179]	555	[370, 176]	816	[919, 920]
34	[66, 67]	295	[428, 474]	556	[802, 499]	817	[921, 922]
35	[68, 69]	296	[475, 476]	557	[385, 503]	818	[923, 328]
36	[70, 71]	297	[475, 477]	558	[500, 31]	819	[604, 924]
37	[72, 73]	298	[478, 479]	559	[466, 803]	820	[560, 925]
38	[74, 75]	299	[480, 358]	560	[801, 497]	821	[926, 927]
39	[76, 77]	300	[379, 372]	561	[427, 804]	822	[647, 928]
40	[78, 79]	301	[480, 181]	562	[168, 805]	823	[688, 929]
41	[80, 81]	302	[381, 481]	563	[355, 123]	824	[930, 931]
42	[82, 83]	303	[44, 482]	564	[164, 413]	825	[77, 932]
43	[84, 85]	304	[483, 484]	565	[384, 412]	826	[933, 934]
44	[86, 87]	305	[485, 355]	566	[369, 415]	827	[935, 316]
45	[88, 89]	306	[470, 35]	567	[487, 806]	828	[194, 936]
46	[90, 91]	307	[482, 486]	568	[494, 807]	829	[559, 743]
47	[92, 93]	308	[487, 488]	569	[387, 808]	830	[937, 298]
48	[94, 95]	309	[489, 355]	570	[35, 466]	831	[938, 939]
49	[96, 97]	310	[419, 191]	571	[517, 431]	832	[940, 941]
50	[98, 99]	311	[480, 445]	572	[809, 810]	833	[942, 943]
51	[100, 101]	312	[378, 490]	573	[451, 8]	834	[944, 570]
52	[102, 103]	313	[167, 491]	574	[471, 466]	835	[945, 942]
53	[104, 105]	314	[492, 493]	575	[811, 474]	836	[309, 946]
54	[106, 107]	315	[494, 12]	576	[372, 169]	837	[947, 948]
55	[108, 109]	316	[439, 487]	577	[812, 127]	838	[949, 702]

56	[110, 111]	317	[495, 496]	578	[48, 813]	839	[950, 207]
57	[112, 113]	318	[44, 495]	579	[395, 462]	840	[727, 573]
58	[114, 46]	319	[470, 497]	580	[814, 360]	841	[760, 722]
59	[115, 116]	320	[498, 499]	581	[815, 816]	842	[951, 270]
60	[117, 118]	321	[500, 494]	582	[794, 31]	843	[952, 953]
61	[50, 119]	322	[501, 362]	583	[812, 399]	844	[954, 955]
62	[120, 121]	323	[502, 380]	584	[817, 520]	845	[956, 957]
63	[122, 38]	324	[503, 406]	585	[161, 818]	846	[958, 242]
64	[123, 124]	325	[367, 504]	586	[107, 405]	847	[959, 605]
65	[32, 125]	326	[418, 493]	587	[157, 819]	848	[285, 960]
66	[126, 127]	327	[35, 505]	588	[462, 357]	849	[961, 962]
67	[128, 113]	328	[506, 436]	589	[820, 410]	850	[963, 964]
68	[129, 130]	329	[150, 476]	590	[120, 821]	851	[965, 669]
69	[131, 109]	330	[488, 507]	591	[822, 420]	852	[966, 967]
70	[132, 133]	331	[508, 488]	592	[823, 809]	853	[968, 969]
71	[134, 135]	332	[509, 115]	593	[501, 824]	854	[970, 549]
72	[136, 137]	333	[454, 510]	594	[142, 39]	855	[971, 972]
73	[138, 139]	334	[511, 512]	595	[152, 140]	856	[275, 973]
74	[38, 140]	335	[513, 464]	596	[825, 39]	857	[974, 975]
75	[141, 141]	336	[411, 514]	597	[448, 152]	858	[629, 944]
76	[122, 142]	337	[50, 109]	598	[387, 826]	859	[896, 976]
77	[143, 142]	338	[130, 137]	599	[823, 481]	860	[977, 978]
78	[144, 145]	339	[515, 158]	600	[379, 502]	861	[979, 980]
79	[46, 146]	340	[516, 481]	601	[472, 808]	862	[962, 981]
80	[147, 51]	341	[517, 153]	602	[827, 171]	863	[982, 983]
81	[148, 149]	342	[518, 519]	603	[502, 805]	864	[984, 985]
82	[150, 151]	343	[191, 406]	604	[828, 376]	865	[986, 987]
83	[143, 152]	344	[114, 520]	605	[131, 119]	866	[988, 667]
84	[153, 154]	345	[521, 399]	606	[829, 824]	867	[989, 940]
85	[155, 156]	346	[497, 466]	607	[830, 364]	868	[990, 991]
86	[157, 158]	347	[385, 420]	608	[827, 415]	869	[654, 979]
87	[159, 160]	348	[522, 409]	609	[386, 406]	870	[992, 993]
88	[161, 135]	349	[523, 386]	610	[831, 402]	871	[994, 995]
89	[162, 163]	350	[515, 464]	611	[144, 832]	872	[996, 997]
90	[164, 165]	351	[108, 179]	612	[107, 828]	873	[998, 999]
91	[166, 167]	352	[48, 400]	613	[475, 151]	874	[1000, 1001]
92	[168, 169]	353	[399, 524]	614	[142, 140]	875	[1002, 1003]
93	[170, 168]	354	[525, 526]	615	[807, 833]	876	[1004, 1005]
94	[171, 172]	355	[527, 528]	616	[834, 429]	877	[321, 314]
95	[173, 174]	356	[529, 530]	617	[187, 358]	878	[1006, 278]
96	[175, 176]	357	[531, 246]	618	[453, 382]	879	[1007, 294]
97	[177, 178]	358	[532, 533]	619	[835, 807]	880	[15, 1008]
98	[50, 179]	359	[534, 535]	620	[375, 836]	881	[1009, 340]
99	[180, 181]	360	[536, 537]	621	[794, 835]	882	[1010, 422]

100	[182, 183]	361	[538, 539]	622	[807, 796]	883	[453, 377]
101	[184, 185]	362	[540, 541]	623	[835, 153]	884	[174, 413]
102	[186, 178]	363	[542, 543]	624	[822, 503]	885	[492, 387]
103	[187, 188]	364	[544, 545]	625	[837, 816]	886	[853, 517]
104	[189, 190]	365	[323, 248]	626	[30, 835]	887	[166, 164]
105	[191, 192]	366	[229, 546]	627	[153, 833]	888	[458, 389]
106	[193, 194]	367	[547, 548]	628	[503, 192]	889	[148, 422]
107	[195, 196]	368	[549, 550]	629	[838, 35]	890	[881, 107]
108	[197, 198]	369	[331, 551]	630	[839, 112]	891	[359, 404]
109	[199, 200]	370	[274, 542]	631	[840, 118]	892	[136, 379]
110	[201, 202]	371	[552, 553]	632	[478, 818]	893	[464, 828]
111	[203, 204]	372	[554, 555]	633	[420, 518]	894	[415, 172]
112	[205, 206]	373	[556, 557]	634	[523, 191]	895	[34, 497]
113	[207, 208]	374	[558, 559]	635	[823, 373]	896	[173, 375]
114	[209, 210]	375	[21, 560]	636	[503, 518]	897	[842, 46]
115	[211, 212]	376	[561, 562]	637	[487, 123]	898	[817, 144]
116	[213, 214]	377	[563, 564]	638	[839, 128]	899	[127, 524]
117	[215, 216]	378	[253, 565]	639	[814, 512]	900	[363, 366]
118	[214, 217]	379	[566, 567]	640	[834, 473]	901	[448, 142]
119	[218, 219]	380	[568, 569]	641	[182, 37]	902	[867, 865]
120	[220, 221]	381	[570, 558]	642	[29, 841]	903	[481, 374]
121	[222, 223]	382	[571, 572]	643	[494, 431]	904	[112, 402]
122	[224, 225]	383	[573, 574]	644	[806, 45]	905	[115, 111]
123	[226, 227]	384	[575, 576]	645	[517, 807]	906	[401, 113]
124	[228, 229]	385	[577, 578]	646	[842, 520]	907	[831, 1011]
125	[230, 231]	386	[579, 580]	647	[843, 844]	908	[800, 355]
126	[232, 233]	387	[581, 582]	648	[443, 455]	909	[431, 13]
127	[234, 235]	388	[23, 318]	649	[115, 845]	910	[872, 156]
128	[236, 237]	389	[583, 584]	650	[520, 47]	911	[419, 503]
129	[238, 239]	390	[312, 585]	651	[793, 160]	912	[355, 40]
130	[240, 241]	391	[586, 587]	652	[444, 188]	913	[118, 442]
131	[242, 243]	392	[588, 589]	653	[846, 185]	914	[372, 490]
132	[244, 245]	393	[101, 590]	654	[847, 848]	915	[134, 479]
133	[246, 247]	394	[204, 575]	655	[123, 507]	916	[811, 860]
134	[248, 249]	395	[591, 592]	656	[503, 29]	917	[184, 821]
135	[250, 251]	396	[93, 593]	657	[849, 791]	918	[447, 450]
136	[252, 253]	397	[594, 595]	658	[850, 48]	919	[29, 878]
137	[239, 254]	398	[596, 597]	659	[139, 357]	920	[498, 1012]
138	[255, 256]	399	[598, 599]	660	[840, 52]	921	[158, 177]
139	[257, 258]	400	[233, 600]	661	[851, 438]	922	[452, 798]
140	[259, 260]	401	[601, 602]	662	[852, 136]	923	[879, 391]
141	[261, 262]	402	[603, 604]	663	[166, 375]	924	[361, 435]
142	[263, 224]	403	[605, 606]	664	[513, 158]	925	[381, 367]
143	[264, 265]	404	[329, 607]	665	[440, 806]	926	[819, 186]

144	[266, 267]	405	[608, 609]	666	[853, 835]	927	[877, 130]
145	[268, 269]	406	[610, 611]	667	[854, 373]	928	[520, 146]
146	[270, 257]	407	[55, 612]	668	[508, 40]	929	[485, 508]
147	[271, 272]	408	[613, 614]	669	[829, 855]	930	[817, 865]
148	[273, 274]	409	[615, 556]	670	[175, 856]	931	[842, 865]
149	[256, 275]	410	[97, 616]	671	[482, 803]	932	[142, 446]
150	[276, 277]	411	[617, 618]	672	[471, 505]	933	[385, 191]
151	[278, 279]	412	[619, 620]	673	[857, 831]	934	[505, 467]
152	[280, 281]	413	[621, 622]	674	[857, 128]	935	[471, 495]
153	[282, 283]	414	[623, 624]	675	[38, 449]	936	[355, 488]
154	[284, 285]	415	[625, 591]	676	[449, 448]	937	[443, 514]
155	[286, 287]	416	[231, 626]	677	[122, 825]	938	[375, 165]
156	[288, 289]	417	[624, 627]	678	[191, 518]	939	[516, 809]
157	[290, 291]	418	[628, 629]	679	[30, 517]	940	[132, 371]
158	[292, 293]	419	[630, 56]	680	[361, 855]	941	[859, 370]
159	[294, 295]	420	[631, 632]	681	[452, 377]	942	[456, 861]
160	[296, 297]	421	[633, 634]	682	[457, 474]	943	[462, 859]
161	[298, 299]	422	[635, 636]	683	[388, 804]	944	[192, 841]
162	[300, 301]	423	[317, 637]	684	[373, 368]	945	[799, 462]
163	[302, 303]	424	[638, 66]	685	[161, 426]	946	[488, 41]
164	[304, 305]	425	[639, 640]	686	[501, 855]	947	[365, 437]
165	[306, 307]	426	[537, 641]	687	[505, 496]	948	[39, 397]
166	[308, 309]	427	[642, 643]	688	[386, 192]	949	[849, 844]
167	[85, 310]	428	[644, 645]	689	[153, 13]	950	[136, 163]
168	[311, 312]	429	[646, 647]	690	[843, 791]	951	[851, 442]
169	[254, 313]	430	[258, 648]	691	[849, 115]	952	[819, 828]
170	[314, 315]	431	[649, 650]	692	[493, 858]	953	[384, 514]
171	[289, 255]	432	[651, 652]	693	[30, 494]	954	[190, 826]
172	[316, 317]	433	[653, 654]	694	[414, 171]	955	[421, 164]
173	[318, 319]	434	[655, 656]	695	[859, 132]	956	[384, 455]
174	[320, 321]	435	[657, 658]	696	[428, 860]	957	[186, 452]
175	[322, 323]	436	[659, 660]	697	[481, 368]	958	[453, 798]
176	[324, 325]	437	[661, 662]	698	[441, 472]	959	[168, 490]
177	[326, 327]	438	[663, 619]	699	[163, 168]	960	[792, 851]
178	[196, 70]	439	[541, 226]	700	[452, 460]	961	[139, 149]
179	[328, 329]	440	[664, 228]	701	[159, 861]	962	[378, 169]
180	[330, 331]	441	[665, 666]	702	[844, 845]	963	[396, 859]
181	[332, 58]	442	[667, 668]	703	[844, 862]	964	[453, 460]
182	[333, 334]	443	[669, 670]	704	[134, 818]	965	[149, 175]
183	[27, 335]	444	[671, 672]	705	[182, 392]	966	[485, 440]
184	[336, 337]	445	[673, 674]	706	[450, 447]	967	[369, 416]
185	[338, 339]	446	[225, 675]	707	[505, 486]	968	[857, 401]
186	[340, 341]	447	[676, 596]	708	[863, 33]	969	[401, 1011]
187	[342, 343]	448	[281, 677]	709	[864, 112]	970	[517, 12]

188	[344, 345]	449	[595, 280]	710	[865, 832]	971	[864, 128]
189	[346, 347]	450	[450, 450]	711	[866, 394]	972	[868, 791]
190	[348, 349]	451	[678, 679]	712	[511, 393]	973	[190, 1013]
191	[350, 351]	452	[680, 681]	713	[489, 508]	974	[462, 149]
192	[352, 353]	453	[682, 333]	714	[797, 831]	975	[443, 510]
193	[354, 355]	454	[683, 336]	715	[867, 144]	976	[497, 495]
194	[356, 167]	455	[668, 684]	716	[868, 115]	977	[850, 463]
195	[357, 132]	456	[685, 686]	717	[839, 831]	978	[812, 463]
196	[180, 358]	457	[687, 688]	718	[384, 510]	979	[871, 795]
197	[359, 360]	458	[689, 633]	719	[793, 869]	980	[846, 121]
198	[361, 362]	459	[690, 691]	720	[118, 438]	981	[149, 430]
199	[363, 364]	460	[327, 692]	721	[866, 118]	982	[830, 366]
200	[365, 366]	461	[693, 694]	722	[410, 170]	983	[152, 446]
201	[367, 368]	462	[695, 696]	723	[129, 162]	984	[52, 383]
202	[369, 8]	463	[697, 698]	724	[108, 119]	985	[430, 133]
203	[370, 371]	464	[699, 561]	725	[423, 804]	986	[809, 374]
204	[137, 372]	465	[700, 701]	726	[870, 484]	987	[854, 367]
205	[373, 374]	466	[702, 703]	727	[838, 471]	988	[800, 508]
206	[356, 375]	467	[704, 659]	728	[127, 49]	989	[148, 859]
207	[376, 377]	468	[705, 706]	729	[865, 145]	990	[130, 163]
208	[163, 378]	469	[707, 708]	730	[871, 437]	991	[370, 856]
209	[162, 379]	470	[709, 710]	731	[141, 398]	992	[846, 821]
210	[168, 380]	471	[711, 712]	732	[518, 407]	993	[140, 143]
211	[8, 172]	472	[708, 713]	733	[872, 873]	994	[430, 856]
212	[381, 373]	473	[714, 715]	734	[868, 110]	995	[814, 404]
213	[178, 382]	474	[716, 717]	735	[112, 874]	996	[879, 477]
214	[383, 384]	475	[718, 719]	736	[159, 869]	997	[398, 141]
215	[385, 386]	476	[720, 721]	737	[819, 177]	998	[12, 833]
216	[189, 387]	477	[722, 723]	738	[138, 148]	999	[837, 10]
217	[388, 389]	478	[724, 725]	739	[463, 813]	1000	[430, 176]
218	[390, 391]	479	[607, 218]	740	[791, 862]	1001	[53, 411]
219	[36, 392]	480	[622, 623]	741	[487, 40]	1002	[390, 477]
220	[372, 380]	481	[726, 727]	742	[171, 11]	1003	[449, 122]
221	[359, 393]	482	[728, 729]	743	[31, 12]	1004	[431, 154]
222	[394, 53]	483	[730, 731]	744	[800, 440]	1005	[522, 1014]
223	[395, 396]	484	[732, 733]	745	[875, 469]	1006	[814, 393]
224	[397, 38]	485	[734, 735]	746	[470, 44]	1007	[370, 133]
225	[398, 398]	486	[736, 737]	747	[852, 410]	1008	[29, 407]
226	[399, 400]	487	[738, 80]	748	[359, 512]	1009	[794, 517]
227	[401, 402]	488	[526, 739]	749	[127, 400]	1010	[1015, 322]
228	[403, 404]	489	[658, 740]	750	[394, 442]	1011	[1016, 195]
229	[158, 405]	490	[741, 742]	751	[461, 148]	1012	[1017, 581]
230	[406, 407]	491	[634, 342]	752	[134, 426]	1013	[666, 1018]
231	[408, 409]	492	[743, 744]	753	[506, 121]	1014	[1019, 1020]

232	[410, 379]	493	[745, 746]	754	[825, 449]	1015	[1021, 502]
233	[411, 412]	494	[747, 748]	755	[806, 41]	1016	[443, 412]
234	[167, 413]	495	[67, 749]	756	[875, 876]	1017	[192, 878]
235	[414, 415]	496	[712, 750]	757	[521, 48]	1018	[806, 124]
236	[416, 417]	497	[751, 752]	758	[406, 519]	1019	[482, 467]
237	[418, 190]	498	[753, 754]	759	[877, 410]	1020	[9, 1022]
238	[419, 420]	499	[755, 756]	760	[432, 869]	1021	[1023, 568]
239	[421, 167]	500	[757, 24]	761	[485, 487]	1022	[1024, 1025]
240	[422, 175]	501	[758, 665]	762	[406, 878]	1023	[1026, 387]
241	[423, 424]	502	[616, 276]	763	[865, 47]	1024	[495, 486]
242	[425, 426]	503	[759, 760]	764	[128, 874]	1025	[483, 1027]
243	[427, 424]	504	[761, 762]	765	[793, 861]	1026	[1028, 908]
244	[428, 429]	505	[763, 764]	766	[820, 130]	1027	[1029, 726]
245	[430, 371]	506	[670, 271]	767	[161, 479]	1028	[1030, 431]
246	[31, 431]	507	[765, 661]	768	[394, 383]	1029	[488, 124]
247	[8, 417]	508	[766, 767]	769	[191, 29]	1030	[1031, 26]
248	[432, 433]	509	[768, 213]	770	[472, 826]	1031	[1032, 52]
249	[434, 435]	510	[769, 770]	771	[187, 445]	1032	[1033, 104]
250	[120, 436]	511	[334, 771]	772	[36, 848]	1033	[1034, 191]
251	[363, 437]	512	[772, 773]	773	[879, 151]	1034	[1035, 352]
252	[438, 384]	513	[349, 608]	774	[471, 482]	1035	[1036, 399]
253	[434, 362]	514	[774, 775]	775	[416, 880]	1036	[1037, 788]
254	[420, 29]	515	[713, 776]	776	[356, 174]	1037	[1038, 438]
255	[439, 440]	516	[777, 778]	777	[355, 806]	1038	[1039, 683]
256	[441, 190]	517	[779, 284]	778	[419, 386]	1039	[1040, 411]
257	[442, 443]	518	[780, 781]	779	[881, 464]	1040	[1041, 774]
258	[444, 445]	519	[632, 782]	780	[520, 145]	1041	[7, 416]
259	[446, 122]	520	[783, 784]	781	[791, 845]		
260	[447, 447]	521	[785, 96]	782	[118, 383]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	174	1	348	1	522	1	696	0	870	1
1	0	175	1	349	0	523	0	697	1	871	0
2	0	176	0	350	1	524	0	698	0	872	1
3	0	177	0	351	0	525	0	699	0	873	0
4	0	178	1	352	1	526	1	700	0	874	0
5	0	179	1	353	0	527	1	701	0	875	0
6	0	180	1	354	0	528	1	702	1	876	0
7	0	181	0	355	1	529	1	703	1	877	0
8	0	182	1	356	1	530	0	704	1	878	1
9	0	183	1	357	0	531	0	705	1	879	1
10	1	184	0	358	0	532	0	706	0	880	1
11	0	185	1	359	0	533	0	707	1	881	0

12	0	186	1	360	1	534	0	708	1	882	0
13	1	187	0	361	0	535	1	709	1	883	1
14	0	188	1	362	0	536	1	710	1	884	1
15	1	189	1	363	0	537	0	711	0	885	1
16	0	190	1	364	1	538	0	712	0	886	1
17	0	191	1	365	1	539	1	713	1	887	0
18	0	192	1	366	1	540	0	714	0	888	1
19	1	193	0	367	0	541	0	715	0	889	0
20	1	194	1	368	1	542	0	716	1	890	0
21	1	195	0	369	1	543	0	717	0	891	0
22	0	196	1	370	1	544	1	718	0	892	0
23	1	197	0	371	1	545	1	719	1	893	0
24	0	198	0	372	0	546	0	720	1	894	1
25	1	199	0	373	0	547	0	721	0	895	0
26	1	200	1	374	0	548	0	722	0	896	1
27	1	201	0	375	1	549	1	723	0	897	0
28	0	202	1	376	0	550	0	724	0	898	1
29	0	203	1	377	1	551	0	725	0	899	0
30	1	204	0	378	0	552	1	726	1	900	0
31	1	205	0	379	1	553	1	727	0	901	1
32	0	206	1	380	1	554	0	728	0	902	0
33	0	207	0	381	0	555	1	729	1	903	1
34	0	208	0	382	0	556	0	730	0	904	0
35	0	209	1	383	1	557	1	731	1	905	0
36	0	210	1	384	0	558	0	732	0	906	1
37	0	211	0	385	1	559	1	733	1	907	1
38	1	212	0	386	1	560	1	734	1	908	0
39	0	213	1	387	0	561	0	735	0	909	0
40	1	214	1	388	1	562	1	736	0	910	1
41	0	215	1	389	1	563	1	737	1	911	0
42	1	216	1	390	0	564	0	738	0	912	1
43	1	217	1	391	0	565	0	739	1	913	1
44	1	218	0	392	1	566	1	740	1	914	0
45	0	219	0	393	0	567	0	741	0	915	1
46	0	220	0	394	0	568	1	742	1	916	0
47	1	221	0	395	1	569	0	743	1	917	0
48	1	222	0	396	1	570	0	744	0	918	0
49	1	223	1	397	0	571	0	745	0	919	0
50	1	224	0	398	1	572	1	746	1	920	1
51	1	225	1	399	0	573	1	747	1	921	1
52	1	226	0	400	0	574	0	748	0	922	0
53	1	227	1	401	1	575	0	749	0	923	1
54	0	228	1	402	1	576	0	750	0	924	0
55	0	229	1	403	1	577	1	751	1	925	0

56	0	230	1	404	1	578	1	752	1	926	1
57	0	231	0	405	0	579	1	753	1	927	0
58	1	232	0	406	1	580	1	754	1	928	0
59	0	233	0	407	0	581	0	755	0	929	1
60	1	234	0	408	0	582	0	756	0	930	1
61	1	235	0	409	1	583	1	757	0	931	0
62	0	236	0	410	0	584	1	758	1	932	0
63	0	237	0	411	0	585	0	759	0	933	1
64	0	238	0	412	0	586	0	760	1	934	1
65	0	239	0	413	0	587	1	761	1	935	0
66	0	240	1	414	0	588	1	762	1	936	1
67	0	241	0	415	1	589	1	763	1	937	1
68	0	242	1	416	0	590	0	764	0	938	1
69	1	243	0	417	1	591	1	765	1	939	1
70	0	244	0	418	0	592	1	766	1	940	0
71	1	245	0	419	0	593	1	767	0	941	1
72	0	246	1	420	0	594	0	768	0	942	0
73	0	247	0	421	0	595	0	769	1	943	1
74	1	248	1	422	1	596	1	770	1	944	1
75	1	249	0	423	0	597	1	771	0	945	0
76	0	250	0	424	0	598	0	772	0	946	1
77	1	251	0	425	1	599	1	773	1	947	1
78	1	252	0	426	0	600	1	774	0	948	0
79	0	253	0	427	0	601	1	775	0	949	0
80	0	254	0	428	0	602	0	776	1	950	0
81	0	255	0	429	0	603	1	777	1	951	0
82	1	256	0	430	0	604	1	778	0	952	1
83	1	257	0	431	0	605	1	779	0	953	0
84	1	258	0	432	1	606	1	780	0	954	1
85	0	259	1	433	1	607	1	781	1	955	0
86	1	260	0	434	0	608	0	782	1	956	0
87	0	261	1	435	0	609	1	783	0	957	1
88	0	262	0	436	0	610	1	784	1	958	1
89	1	263	0	437	0	611	1	785	0	959	1
90	0	264	1	438	0	612	0	786	1	960	1
91	0	265	1	439	0	613	0	787	0	961	0
92	1	266	1	440	0	614	0	788	0	962	0
93	1	267	1	441	0	615	1	789	0	963	1
94	1	268	0	442	0	616	1	790	0	964	1
95	1	269	0	443	1	617	0	791	1	965	0
96	1	270	1	444	0	618	1	792	1	966	1
97	0	271	0	445	1	619	0	793	1	967	1
98	1	272	1	446	1	620	1	794	0	968	1
99	1	273	0	447	0	621	0	795	0	969	1

100	1	274	1	448	1	622	1	796	0	970	0
101	0	275	0	449	0	623	0	797	0	971	1
102	1	276	1	450	0	624	1	798	1	972	1
103	0	277	1	451	1	625	1	799	0	973	1
104	1	278	1	452	0	626	1	800	0	974	1
105	1	279	1	453	1	627	1	801	1	975	1
106	0	280	0	454	1	628	0	802	0	976	1
107	0	281	1	455	1	629	0	803	1	977	1
108	0	282	1	456	0	630	0	804	0	978	1
109	0	283	0	457	1	631	0	805	1	979	0
110	0	284	1	458	1	632	0	806	0	980	1
111	1	285	0	459	1	633	0	807	1	981	0
112	0	286	0	460	0	634	0	808	0	982	1
113	0	287	1	461	1	635	1	809	1	983	0
114	1	288	1	462	1	636	0	810	0	984	1
115	0	289	1	463	1	637	0	811	0	985	0
116	1	290	1	464	0	638	0	812	1	986	1
117	1	291	0	465	0	639	1	813	1	987	0
118	1	292	1	466	1	640	1	814	1	988	0
119	0	293	1	467	1	641	1	815	0	989	0
120	0	294	0	468	1	642	0	816	0	990	1
121	0	295	0	469	1	643	1	817	1	991	1
122	0	296	0	470	1	644	0	818	1	992	1
123	0	297	0	471	0	645	0	819	1	993	1
124	1	298	0	472	1	646	0	820	1	994	0
125	1	299	1	473	0	647	1	821	1	995	1
126	0	300	1	474	1	648	1	822	1	996	1
127	0	301	1	475	0	649	0	823	1	997	1
128	0	302	0	476	1	650	0	824	1	998	0
129	0	303	1	477	0	651	1	825	1	999	1
130	1	304	0	478	0	652	0	826	1	1000	0
131	1	305	1	479	1	653	1	827	0	1001	1
132	0	306	1	480	1	654	1	828	1	1002	0
133	1	307	0	481	1	655	0	829	1	1003	0
134	1	308	0	482	0	656	0	830	1	1004	0
135	0	309	1	483	0	657	0	831	1	1005	1
136	0	310	0	484	0	658	1	832	0	1006	1
137	0	311	1	485	1	659	0	833	0	1007	1
138	0	312	0	486	0	660	0	834	1	1008	0
139	0	313	0	487	0	661	0	835	0	1009	0
140	1	314	1	488	1	662	1	836	1	1010	0
141	1	315	1	489	1	663	0	837	1	1011	1
142	0	316	0	490	0	664	0	838	0	1012	1
143	1	317	0	491	0	665	0	839	0	1013	1

144	1	318	1	492	1	666	1	840	0	1014	0
145	0	319	1	493	0	667	0	841	1	1015	0
146	1	320	1	494	1	668	1	842	0	1016	1
147	0	321	0	495	0	669	1	843	1	1017	1
148	0	322	1	496	0	670	1	844	1	1018	0
149	0	323	1	497	1	671	0	845	0	1019	0
150	1	324	0	498	1	672	0	846	1	1020	0
151	1	325	0	499	0	673	1	847	1	1021	0
152	0	326	0	500	0	674	1	848	0	1022	0
153	1	327	0	501	1	675	1	849	0	1023	0
154	1	328	1	502	1	676	0	850	1	1024	0
155	0	329	1	503	0	677	0	851	0	1025	0
156	1	330	1	504	1	678	1	852	1	1026	0
157	1	331	1	505	1	679	1	853	1	1027	1
158	1	332	0	506	1	680	0	854	0	1028	0
159	0	333	1	507	1	681	0	855	1	1029	1
160	0	334	0	508	1	682	1	856	0	1030	0
161	0	335	0	509	0	683	1	857	1	1031	0
162	1	336	0	510	1	684	0	858	0	1032	0
163	0	337	1	511	0	685	0	859	1	1033	0
164	0	338	1	512	0	686	1	860	1	1034	0
165	1	339	1	513	0	687	1	861	0	1035	0
166	0	340	1	514	0	688	1	862	0	1036	0
167	0	341	0	515	1	689	1	863	1	1037	0
168	1	342	0	516	1	690	1	864	1	1038	0
169	0	343	1	517	0	691	0	865	1	1039	0
170	1	344	1	518	0	692	0	866	0	1040	0
171	1	345	0	519	0	693	1	867	0	1041	0
172	0	346	1	520	0	694	0	868	1		
173	1	347	1	521	0	695	1	869	1		

Université de Strasbourg, CNRS, IRMA UMR 7501, F-67000 Strasbourg, France
Email address: guoniu.han@unistra.fr

School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan, PR China
Email address: huyining@hust.edu.cn