PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a,b)=(z+1,z^6+z^4+z^3+z^2+z+1)$

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Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a,b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a,b) in $\mathbb{F}_2[x]\setminus\mathbb{F}_2$. In this paper we prove the conjecture for $(a,b)=(z+1,z^6+z^4+z^3+z^2+z+1)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a}=(a_0(z),a_1(z),a_2(z),\ldots)$ of elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z))=\cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{a_2(z)+\cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z))=\cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{\ddots+\cfrac{1}{a_n(z)}}}}\in\mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1}+c_{-2}z^{-2}+c_{-3}z^{-3}+\cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a,b)-Thue-Morse sequence

$$\mathbf{t}=(t_0(z),t_1(z),t_2(z),\ldots)=(a,b,b,a,b,a,a,b,\ldots)$$

that we shall denote

$$\mathrm{CF}(a,b):=\mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When* $(a,b)=(z+1,z^6+z^4+z^3+z^2+z+1)$*, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

where, for $\mathrm{CF}(a,b)$*,*

$$\begin{aligned}
p_0(z)&=z^{20}+z^{17}+z^{15}+z^{11}+z^{10}+z^8+z^7+z^5+z^4+1,\\
p_1(z)&=z^{23}+z^{22}+z^{19}+z^{15}+z^{12}+z^{11}+z^8+z^7+z^6+z^4+z^3+z^2,\\
p_2(z)&=z^{24}+z^{22}+z^{20}+z^{18}+z^{10}+z^8+z^6+1,\\
p_3(z)&=z^{23}+z^{22}+z^{19}+z^{15}+z^{12}+z^{11}+z^8+z^7+z^6+z^4+z^3+z^2,\\
p_4(z)&=z^{22}+z^{20}+z^{18}+z^{17}+z^{15}+z^{12}+z^{11}+z^7+z^6+z^5+1,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$*,*

$$\begin{aligned}
p_0(z)&=z^{17}+z^{16}+z^{15}+z^{11}+z^{10}+z^7+z^5+z^4+1,\\
p_1(z)&=z^{23}+z^{22}+z^{19}+z^{15}+z^{12}+z^{11}+z^8+z^7+z^6+z^4+z^3+z^2,\\
p_2(z)&=z^{24}+z^{22}+z^{20}+z^{18}+z^{10}+z^8+z^6+1,\\
p_3(z)&=z^{23}+z^{22}+z^{19}+z^{15}+z^{12}+z^{11}+z^8+z^7+z^6+z^4+z^3+z^2,\\
p_4(z)&=z^{22}+z^{18}+z^{17}+z^{16}+z^{15}+z^{12}+z^{11}+z^8+z^7+z^6+z^5+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix}P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix}:=\begin{pmatrix}t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix}t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x)=x^{\deg(t_{2^n-1})}&\begin{pmatrix}t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix}t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\cdots\times x^{\deg(t_0)}\begin{pmatrix}t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x)=x^{\deg(\bar t_{2^n-1})}&\begin{pmatrix}\bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix}\bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\cdots\times x^{\deg(\bar t_0)}\begin{pmatrix}\bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\geq 0$

$$\begin{aligned}
M_{n+1}(x)&=W_n(x)\cdot M_n(x),\\
W_{n+1}(x)&=M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(175, 175, 175, 175, 175)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z\left(\phi(W^o,0,0)(x,z),\ z^{12}\cdot\phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 630 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 630, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z\left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ to order 770 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 770, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. $\square$

Define

$$R^e = \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned}
M^e[:14u] = {} & M^e[:7u] + x^u M^e[:6u] + x^{3u}M^e[:4u] + x^{4u}M^e[:3u] + x^u M^e[:7u]\\
& + x^{2u}M^e[:6u] + x^{4u}M^e[:4u] + x^{5u}M^e[:3u] + x^{2u}M^e[:7u]\\
& + x^{3u}M^e[:6u] + x^{5u}M^e[:4u] + x^{6u}M^e[:3u] + x^{6u}M^e[:7u]\\
& + x^{7u}M^e[:6u] + x^{9u}M^e[:4u] + x^{10u}M^e[:3u] + x^{9u}M^e[:5u]\\
& + x^{12u}M^e[:2u] + x^{11u}M^e[:3u] + x^{13u}M^e[:u]\\
& + (x^{7u} + x^{8u} + x^{9u} + x^{13u})R^e,
\end{aligned}$$

$$\begin{aligned}
W^e[:14u] &= W^e[:7u] + x^u W^e[:6u] + x^{3u} W^e[:4u] + x^{4u} W^e[:3u] + x^u W^e[:7u] \\
&+ x^{2u} W^e[:6u] + x^{4u} W^e[:4u] + x^{5u} W^e[:3u] + x^{2u} W^e[:7u] \\
&+ x^{3u} W^e[:6u] + x^{5u} W^e[:4u] + x^{6u} W^e[:3u] + x^{6u} W^e[:7u] \\
&+ x^{7u} W^e[:6u] + x^{9u} W^e[:4u] + x^{10u} W^e[:3u] + x^{9u} W^e[:5u] \\
&+ x^{12u} W^e[:2u] + x^{11u} W^e[:3u] + x^{13u} W^e[:u] \\
&+ (x^{7u} + x^{8u} + x^{9u} + x^{13u}) R^e.
\end{aligned}$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$\begin{aligned}
M^o[:14u] &= M^o[:7u] + x^u M^o[:6u] + x^{3u} M^o[:4u] + x^{4u} M^o[:3u] + x^u M^o[:7u] \\
&+ x^{2u} M^o[:6u] + x^{4u} M^o[:4u] + x^{5u} M^o[:3u] + x^{2u} M^o[:7u] \\
&+ x^{3u} M^o[:6u] + x^{5u} M^o[:4u] + x^{6u} M^o[:3u] + x^{6u} M^o[:7u] \\
&+ x^{7u} M^o[:6u] + x^{9u} M^o[:4u] + x^{10u} M^o[:3u] + x^{9u} M^o[:5u] \\
&+ x^{12u} M^o[:2u] + x^{11u} M^o[:3u] + x^{13u} M^o[:u] \\
&+ (x^{7u} + x^{8u} + x^{9u} + x^{13u}) R^o, \\
W^o[:14u] &= W^o[:7u] + x^u W^o[:6u] + x^{3u} W^o[:4u] + x^{4u} W^o[:3u] + x^u W^o[:7u] \\
&+ x^{2u} W^o[:6u] + x^{4u} W^o[:4u] + x^{5u} W^o[:3u] + x^{2u} W^o[:7u] \\
&+ x^{3u} W^o[:6u] + x^{5u} W^o[:4u] + x^{6u} W^o[:3u] + x^{6u} W^o[:7u] \\
&+ x^{7u} W^o[:6u] + x^{9u} W^o[:4u] + x^{10u} W^o[:3u] + x^{9u} W^o[:5u] \\
&+ x^{12u} W^o[:2u] + x^{11u} W^o[:3u] + x^{13u} W^o[:u] \\
&+ (x^{7u} + x^{8u} + x^{9u} + x^{13u}) R^o.
\end{aligned}$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$\begin{aligned}
T[:14u] &= T[:7u] + x^u T[:6u] + x^{3u} T[:4u] + x^{4u} T[:3u] + x^u T[:7u] + x^{2u} T[:6u] \\
&+ x^{4u} T[:4u] + x^{5u} T[:3u] + x^{2u} T[:7u] + x^{3u} T[:6u] + x^{5u} T[:4u] \\
&+ x^{6u} T[:3u] + x^{6u} T[:7u] + x^{7u} T[:6u] + x^{9u} T[:4u] + x^{10u} T[:3u] \\
&+ x^{9u} T[:5u] + x^{12u} T[:2u] + x^{11u} T[:3u] + x^{13u} T[:u] \\
&+ (x^{7u} + x^{8u} + x^{9u} + x^{13u}).
\end{aligned}$$

The proof of the other 15 cases are similar. We break down the above identity into 7 parts:

$$\begin{aligned}
x^7 u &= x^{7u} T[:u] + x^{4u} T[3u:4u] + x^{3u} T[4u:5u] + x^u T[6u:7u] \\
&\quad + T[7u:8u], \\
x^8 u &= x^{7u} T[u:2u] + x^{5u} T[3u:4u] + x^{3u} T[5u:6u] + x^{2u} T[6u:7u] \\
&\quad + T[8u:9u], \\
x^9 u &= x^{7u} T[2u:3u] + x^{6u} T[3u:4u] + T[9u:10u], \\
0 &= x^{10u} T[:u] + x^{7u} T[3u:4u] + x^{6u} T[4u:5u] + T[10u:11u],
\end{aligned}$$

$$
\begin{aligned}
0 &= x^{11u}T[:u] + x^{10u}T[u:2u] + x^{7u}T[4u:5u] + x^{6u}T[5u:6u]\\
&\quad + T[11u:12u],\\
0 &= x^{12u}T[:u] + x^{11u}T[u:2u] + x^{10u}T[2u:3u] + x^{7u}T[5u:6u]\\
&\quad + x^{6u}T[6u:7u] + T[12u:13u],\\
x^1 3u &= x^{13u}T[:u] + x^{12u}T[u:2u] + x^{11u}T[2u:3u] + x^{9u}T[4u:5u]\\
&\quad + T[13u:14u],
\end{aligned}
$$

which can be rewritten as

$$0 = T[[w]_2] + T[[11w]_2] + T[[100w]_2] + T[[110w]_2] + T[[111w]_2], \tag{2.7}$$

$$0 = T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2] + T[[1000w]_2], \tag{2.8}$$

$$0 = T[[10w]_2] + T[[11w]_2] + T[[1001w]_2], \tag{2.9}$$

$$0 = T[[w]_2] + T[[11w]_2] + T[[100w]_2] + T[[1010w]_2], \tag{2.10}$$

$$0 = T[[w]_2] + T[[1w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1011w]_2], \tag{2.11}$$

$$\begin{aligned}0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[101w]_2] + T[[110w]_2]\\ &\quad + T[[1100w]_2],\end{aligned} \tag{2.12}$$

$$0 = T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[100w]_2] + T[[1101w]_2], \tag{2.13}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$1 = T[[w]_2] + T[[11w]_2] + T[[100w]_2] + T[[110w]_2] + T[[111w]_2], \tag{2.14}$$

$$1 = T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2] + T[[1000w]_2], \tag{2.15}$$

$$1 = T[[10w]_2] + T[[11w]_2] + T[[1001w]_2], \tag{2.16}$$

$$0 = T[[w]_2] + T[[11w]_2] + T[[100w]_2] + T[[1010w]_2], \tag{2.17}$$

$$0 = T[[w]_2] + T[[1w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1011w]_2], \tag{2.18}$$

$$\begin{aligned}0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[101w]_2] + T[[110w]_2]\\ &\quad + T[[1100w]_2],\end{aligned} \tag{2.19}$$

$$1 = T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[100w]_2] + T[[1101w]_2], \tag{2.20}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 693 states; its transition function and output function can be found in the annex. Let $A(s,w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i,w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.20) can be written as

$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,11)) + \tau(A(s,100)) + \tau(A(s,110))\\ &\quad + \tau(A(s,111)),\end{aligned} \tag{2.21}$$

$$\begin{aligned}0 &= \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,110))\\ &\quad + \tau(A(s,1000)),\end{aligned} \tag{2.22}$$

$$0 = \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,1001)), \tag{2.23}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,11)) + \tau(A(s,100)) + \tau(A(s,1010)), \tag{2.24}$$

$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,100)) + \tau(A(s,101)) \\ &\quad + \tau(A(s,1011)),\end{aligned} \tag{2.25}$$

$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,101)) \\ &\quad + \tau(A(s,110)) + \tau(A(s,1100)),\end{aligned} \tag{2.26}$$

$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,100)) \\ &\quad + \tau(A(s,1101)),\end{aligned} \tag{2.27}$$

for all $s \in E_{2k}$, and

$$\begin{aligned}1 &= \tau(A(s,\epsilon)) + \tau(A(s,11)) + \tau(A(s,100)) + \tau(A(s,110)) \\ &\quad + \tau(A(s,111)),\end{aligned} \tag{2.28}$$

$$\begin{aligned}1 &= \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,110)) \\ &\quad + \tau(A(s,1000)),\end{aligned} \tag{2.29}$$

$$1 = \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,1001)), \tag{2.30}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,11)) + \tau(A(s,100)) + \tau(A(s,1010)), \tag{2.31}$$

$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,100)) + \tau(A(s,101)) \\ &\quad + \tau(A(s,1011)),\end{aligned} \tag{2.32}$$

$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,101)) \\ &\quad + \tau(A(s,110)) + \tau(A(s,1100)),\end{aligned} \tag{2.33}$$

$$\begin{aligned}1 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,100)) \\ &\quad + \tau(A(s,1101)),\end{aligned} \tag{2.34}$$

for $s = A(i, 0^{2k})$. We find that $(A(i,0^{24}), E_{24}) = (A(i,0^{12}), E_{12})$, so that we only have to verify that identities (2.21) through (2.34) hold for $2 \le k \le 12$, which turns out to be true. □

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \ge 2$, and $u = 2^{2k-1}$,*

$$\begin{aligned}M_{2k} &= M^e[:7u] + x^u M^e[:6u] + x^{3u} M^e[:4u] + x^{4u} M^e[:3u] + x^{7u} R^e, \\ W_{2k} &= W^e[:7u] + x^u W^e[:6u] + x^{3u} W^e[:4u] + x^{4u} W^e[:3u] + x^{7u} R^e.\end{aligned}$$

For all integer $k \ge 2$, and $u = 2^{2k}$,

$$\begin{aligned}M_{2k+1} &= M^o[:7u] + x^u M^o[:6u] + x^{3u} M^o[:4u] + x^{4u} M^o[:3u] + x^{7u} R^o, \\ W_{2k+1} &= W^o[:7u] + x^u W^o[:6u] + x^{3u} W^o[:4u] + x^{4u} W^o[:3u] + x^{7u} R^o.\end{aligned}$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \ge 2$, we claim that

$$\begin{aligned}M_{2k} \wedge W_{2k} &\Rightarrow M_{2k+1} \wedge W_{2k+1}, \\ M_{2k+1} \wedge W_{2k+1} &\Rightarrow M_{2k+2} \wedge W_{2k+2}.\end{aligned}$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$W_{2k}M_{2k} = \big(W^e[:7v] + x^v W^e[:6v] + x^{3v} W^e[:4v] + x^{4v} W^e[:3v] + x^{7u} R^e\big) \times \big($$
$$M^e[:7v] + x^v M^e[:6v] + x^{3v} M^e[:4v] + x^{4v} M^e[:3v] + x^{7u} R^e\big); \tag{2.35}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{14v}R^o$. Therefore we only need to prove that their difference is $O(x^{14v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{14v}, to

$$W^e[:14v]M^e[:14v] + x^{2v}W^e[:12v]M^e[:12v] + x^{6v}W^e[:8v]M^e[:8v] \\ + x^{8v}W^e[:6v]M^e[:6v].$$

For all $n \le 14$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.35) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{14}, b_1, ..., b_{14}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{14})$, which completes the proof. □

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \le j, k \le 1$,

$$\begin{aligned} \lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k}, \\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k}, \\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k}, \\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}. \end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a, b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x, y) = \mathrm{Res}_t\left(\phi(M^e, 0, 1)(x, t), y^{12}\phi(M^e, 0, 1)(x, t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x, y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$
\begin{aligned}
q_0(x) &= x^{24} + x^{20} + x^{19} + x^{17} + x^{16} + x^{14} + x^{13} + x^9 + x^7 + x^4,\\
q_1(x) &= x^{22} + x^{21} + x^{20} + x^{18} + x^{17} + x^{16} + x^{13} + x^{12} + x^9 + x^5 + x^2 + x,\\
q_2(x) &= x^{24} + x^{18} + x^{16} + x^{14} + x^6 + x^4 + x^2 + 1,\\
q_3(x) &= x^{22} + x^{21} + x^{20} + x^{18} + x^{17} + x^{16} + x^{13} + x^{12} + x^9 + x^5 + x^2 + x,\\
q_4(x) &= x^{24} + x^{19} + x^{18} + x^{17} + x^{13} + x^{12} + x^9 + x^7 + x^6 + x^4 + x^2.
\end{aligned}
$$

The polynomial $Q(x, y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x, y)$ of multiplicity m and $R(x, y) := P(x, y)/Q(x, y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 216, and substitute it for y in $R(z, y)$, we get a series with valuation smaller than 216, which proves that $R(z, y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z, y)$ is the minimal polynomial of $\mathrm{CF}(a, b) = f(1/z)$. □

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e, 0, 0)$:

$$
\begin{aligned}
p_0(x) = {} & x^{144} + x^{134} + x^{132} + x^{130} + x^{124} + x^{122} + x^{116} + x^{112} + x^{110} + x^{100}\\
& + x^{98} + x^{92} + x^{90} + x^{80} + x^{78} + x^{66} + x^{64} + x^{58} + x^{56} + x^{54} + x^{52}\\
& + x^{50} + x^{40} + x^{38} + x^{32} + x^{30} + x^{26} + x^{24} + x^{22} + x^{16} + x^{12},\\
p_3(x) = {} & x^{140} + x^{138} + x^{136} + x^{130} + x^{128} + x^{126} + x^{122} + x^{116} + x^{114} + x^{112}\\
& + x^{110} + x^{106} + x^{104} + x^{98} + x^{96} + x^{94} + x^{88} + x^{82} + x^{80} + x^{54} + x^{48}\\
& + x^{44} + x^{38} + x^{34} + x^{22} + x^{20} + x^{16} + x^8,\\
p_6(x) = {} & x^{136} + x^{134} + x^{132} + x^{128} + x^{120} + x^{118} + x^{108} + x^{104} + x^{102} + x^{98}\\
& + x^{96} + x^{92} + x^{88} + x^{84} + x^{82} + x^{80} + x^{76} + x^{70} + x^{66} + x^{64} + x^{58}\\
& + x^{56} + x^{54} + x^{50} + x^{46} + x^{44} + x^{42} + x^{30} + x^{28} + x^{26} + x^{24} + x^{22}\\
& + x^{16} + x^{14} + x^{12} + x^6 + x^2 + 1,\\
p_9(x) = {} & x^{140} + x^{138} + x^{130} + x^{124} + x^{122} + x^{116} + x^{106} + x^{92} + x^{90} + x^{82}\\
& + x^{76} + x^{74} + x^{66} + x^{60} + x^{58} + x^{50} + x^{20} + x^{18} + x^{12} + x^2,\\
p_{12}(x) = {} & x^{144} + x^{136} + x^{104} + x^{96} + x^{88} + x^{72} + x^{56} + x^{48} + x^{24} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0].

For $\phi(M^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) = {} & x^{129} + x^{127} + x^{124} + x^{121} + x^{119} + x^{113} + x^{110} + x^{108} + x^{107} + x^{105}\\
& + x^{103} + x^{99} + x^{98} + x^{97} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{88}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{87} + x^{85} + x^{82} + x^{81} + x^{80} + x^{77} + x^{76} + x^{73} + x^{71} + x^{65} + x^{63}\\
&+ x^{62} + x^{61} + x^{59} + x^{58} + x^{57} + x^{55} + x^{52} + x^{50} + x^{49} + x^{48} + x^{47}\\
&+ x^{43} + x^{37} + x^{36} + x^{34} + x^{28} + x^{24} + x^{21} + x^{20} + x^{19} + x^{18},\\
p_3(x) = {}& x^{111} + x^{110} + x^{108} + x^{104} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97}\\
&+ x^{96} + x^{95} + x^{94} + x^{90} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82}\\
&+ x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{69}\\
&+ x^{67} + x^{66} + x^{65} + x^{63} + x^{55} + x^{54} + x^{53} + x^{51} + x^{49} + x^{48} + x^{47}\\
&+ x^{45} + x^{43} + x^{42} + x^{41} + x^{38} + x^{37} + x^{35} + x^{33} + x^{32} + x^{31} + x^{28}\\
&+ x^{26} + x^{23} + x^{21} + x^{20} + x^{19} + x^{17} + x^{13} + x^{12} + x^{11} + x^{9},\\
p_6(x) = {}& x^{117} + x^{112} + x^{110} + x^{109} + x^{106} + x^{102} + x^{101} + x^{100} + x^{99} + x^{97}\\
&+ x^{95} + x^{94} + x^{91} + x^{87} + x^{86} + x^{85} + x^{84} + x^{82} + x^{81} + x^{80} + x^{79}\\
&+ x^{74} + x^{72} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{63} + x^{62} + x^{59} + x^{55}\\
&+ x^{54} + x^{53} + x^{52} + x^{50} + x^{49} + x^{48} + x^{47} + x^{42} + x^{40} + x^{39} + x^{37}\\
&+ x^{36} + x^{35} + x^{33} + x^{31} + x^{30} + x^{27} + x^{23} + x^{22} + x^{20} + x^{18} + x^{17}\\
&+ x^{15} + x^{14} + x^{13} + x^{8} + x^{7} + x^{6},\\
p_9(x) = {}& x^{123} + x^{122} + x^{121} + x^{119} + x^{115} + x^{114} + x^{112} + x^{111} + x^{110} + x^{109}\\
&+ x^{108} + x^{105} + x^{103} + x^{102} + x^{101} + x^{98} + x^{96} + x^{95} + x^{94} + x^{91}\\
&+ x^{89} + x^{88} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{76} + x^{75}\\
&+ x^{72} + x^{71} + x^{69} + x^{67} + x^{66} + x^{64} + x^{63} + x^{62} + x^{59} + x^{57} + x^{56}\\
&+ x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{44} + x^{43} + x^{40} + x^{39}\\
&+ x^{37} + x^{35} + x^{34} + x^{32} + x^{31} + x^{30} + x^{26} + x^{24} + x^{23} + x^{22} + x^{21}\\
&+ x^{20} + x^{17} + x^{15} + x^{14} + x^{13} + x^{11} + x^{9} + x^{8} + x^{6} + x^{3},\\
p_{12}(x) = {}& x^{129} + x^{127} + x^{125} + x^{124} + x^{113} + x^{111} + x^{108} + x^{107} + x^{104} + x^{103}\\
&+ x^{100} + x^{99} + x^{97} + x^{96} + x^{93} + x^{91} + x^{88} + x^{87} + x^{84} + x^{83} + x^{80}\\
&+ x^{79} + x^{77} + x^{76} + x^{61} + x^{59} + x^{56} + x^{55} + x^{52} + x^{51} + x^{48} + x^{47}\\
&+ x^{45} + x^{44} + x^{33} + x^{31} + x^{28} + x^{27} + x^{24} + x^{23} + x^{20} + x^{19} + x^{17}\\
&+ x^{16} + x^{13} + x^{11} + x^{8} + x^{7} + x^{4} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 1, 1, 1, 0, 0, 0, 1, 1, 1, 0, 0, 1, 0, 1].

For $\phi(M^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) = {}& x^{129} + x^{127} + x^{124} + x^{121} + x^{119} + x^{113} + x^{110} + x^{108} + x^{107} + x^{105}\\
&+ x^{103} + x^{99} + x^{98} + x^{97} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{88}\\
&+ x^{87} + x^{85} + x^{82} + x^{81} + x^{80} + x^{77} + x^{76} + x^{73} + x^{71} + x^{65} + x^{63}\\
&+ x^{62} + x^{61} + x^{59} + x^{58} + x^{57} + x^{55} + x^{52} + x^{50} + x^{49} + x^{48} + x^{47}\\
&+ x^{43} + x^{37} + x^{36} + x^{34} + x^{28} + x^{24} + x^{21} + x^{20} + x^{19} + x^{18},\\
p_3(x) = {}& x^{111} + x^{110} + x^{108} + x^{104} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{96} + x^{95} + x^{94} + x^{90} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82}\\
&+ x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{69}\\
&+ x^{67} + x^{66} + x^{65} + x^{63} + x^{55} + x^{54} + x^{53} + x^{51} + x^{49} + x^{48} + x^{47}\\
&+ x^{45} + x^{43} + x^{42} + x^{41} + x^{38} + x^{37} + x^{35} + x^{33} + x^{32} + x^{31} + x^{28}\\
&+ x^{26} + x^{23} + x^{21} + x^{20} + x^{19} + x^{17} + x^{13} + x^{12} + x^{11} + x^{9},\\
p_6(x) &= x^{117} + x^{112} + x^{110} + x^{109} + x^{106} + x^{102} + x^{101} + x^{100} + x^{99} + x^{97}\\
&+ x^{95} + x^{94} + x^{91} + x^{87} + x^{86} + x^{85} + x^{84} + x^{82} + x^{81} + x^{80} + x^{79}\\
&+ x^{74} + x^{72} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{63} + x^{62} + x^{59} + x^{55}\\
&+ x^{54} + x^{53} + x^{52} + x^{50} + x^{49} + x^{48} + x^{47} + x^{42} + x^{40} + x^{39} + x^{37}\\
&+ x^{36} + x^{35} + x^{33} + x^{31} + x^{30} + x^{27} + x^{23} + x^{22} + x^{20} + x^{18} + x^{17}\\
&+ x^{15} + x^{14} + x^{13} + x^{8} + x^{7} + x^{6},\\
p_9(x) &= x^{123} + x^{122} + x^{121} + x^{119} + x^{115} + x^{114} + x^{112} + x^{111} + x^{110} + x^{109}\\
&+ x^{108} + x^{105} + x^{103} + x^{102} + x^{101} + x^{98} + x^{96} + x^{95} + x^{94} + x^{91}\\
&+ x^{89} + x^{88} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{76} + x^{75}\\
&+ x^{72} + x^{71} + x^{69} + x^{67} + x^{66} + x^{64} + x^{63} + x^{62} + x^{59} + x^{57} + x^{56}\\
&+ x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{44} + x^{43} + x^{40} + x^{39}\\
&+ x^{37} + x^{35} + x^{34} + x^{32} + x^{31} + x^{30} + x^{26} + x^{24} + x^{23} + x^{22} + x^{21}\\
&+ x^{20} + x^{17} + x^{15} + x^{14} + x^{13} + x^{11} + x^{9} + x^{8} + x^{6} + x^{3},\\
p_{12}(x) &= x^{129} + x^{127} + x^{125} + x^{124} + x^{113} + x^{111} + x^{108} + x^{107} + x^{104} + x^{103}\\
&+ x^{100} + x^{99} + x^{97} + x^{96} + x^{93} + x^{91} + x^{88} + x^{87} + x^{84} + x^{83} + x^{80}\\
&+ x^{79} + x^{77} + x^{76} + x^{61} + x^{59} + x^{56} + x^{55} + x^{52} + x^{51} + x^{48} + x^{47}\\
&+ x^{45} + x^{44} + x^{33} + x^{31} + x^{28} + x^{27} + x^{24} + x^{23} + x^{20} + x^{19} + x^{17}\\
&+ x^{16} + x^{13} + x^{11} + x^{8} + x^{7} + x^{4} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 1, 1, 1, 0, 0, 0, 1, 1, 1, 0, 0, 1, 0, 1].

For $\phi(M^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{114} + x^{110} + x^{102} + x^{96} + x^{84} + x^{82} + x^{78} + x^{76} + x^{74} + x^{72} + x^{70}\\
&+ x^{68} + x^{60} + x^{56} + x^{54} + x^{52} + x^{48} + x^{46} + x^{38} + x^{36} + x^{34} + x^{26}\\
&+ x^{24},\\
p_3(x) &= x^{110} + x^{108} + x^{104} + x^{102} + x^{96} + x^{92} + x^{90} + x^{86} + x^{84} + x^{82} + x^{78}\\
&+ x^{74} + x^{70} + x^{66} + x^{60} + x^{56} + x^{52} + x^{50} + x^{40} + x^{38} + x^{32} + x^{30}\\
&+ x^{28} + x^{24} + x^{22} + x^{20} + x^{16} + x^{14} + x^{12} + x^{10} + x^{8} + x^{6},\\
p_6(x) &= x^{104} + x^{102} + x^{98} + x^{96} + x^{92} + x^{84} + x^{80} + x^{78} + x^{76} + x^{72} + x^{64}\\
&+ x^{58} + x^{54} + x^{44} + x^{42} + x^{36} + x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{24}\\
&+ x^{18} + x^{16} + x^{8} + 1,\\
p_9(x) &= x^{110} + x^{108} + x^{106} + x^{104} + x^{102} + x^{100} + x^{98} + x^{96} + x^{92} + x^{90}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{86}+x^{84}+x^{82}+x^{80}+x^{70}+x^{64}+x^{60}+x^{58}+x^{54}+x^{52}+x^{50}\\
&+x^{48}+x^{38}+x^{32}+x^{28}+x^{26}+x^{22}+x^{20}+x^{18}+x^{16}+x^{14}+x^{12}\\
&+x^{10}+x^{8}+x^{4}+x^{2},\\
p_{12}(x)=&x^{114}+x^{110}+x^{104}+x^{102}+x^{96}+x^{94}+x^{88}+x^{86}+x^{82}+x^{80}+x^{66}\\
&+x^{62}+x^{56}+x^{54}+x^{50}+x^{48}+x^{34}+x^{30}+x^{24}+x^{22}+x^{16}+x^{14}\\
&+x^{8}+x^{6}+x^{2}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0].

For $\phi(W^e, 0, 0)$:

$$
\begin{aligned}
p_0(x)=&x^{114}+x^{110}+x^{106}+x^{102}+x^{98}+x^{94}+x^{92}+x^{86}+x^{82}+x^{80}+x^{78}\\
&+x^{76}+x^{72}+x^{66}+x^{60}+x^{54}+x^{42}+x^{34}+x^{30}+x^{28}+x^{18}+x^{14}\\
&+x^{12},\\
p_3(x)=&x^{110}+x^{108}+x^{104}+x^{100}+x^{98}+x^{96}+x^{94}+x^{92}+x^{88}+x^{86}+x^{82}\\
&+x^{72}+x^{66}+x^{64}+x^{62}+x^{60}+x^{54}+x^{52}+x^{50}+x^{48}+x^{44}+x^{40}\\
&+x^{34}+x^{30}+x^{24}+x^{22}+x^{18}+x^{16}+x^{14}+x^{8},\\
p_6(x)=&x^{106}+x^{104}+x^{100}+x^{94}+x^{88}+x^{84}+x^{82}+x^{80}+x^{74}+x^{68}+x^{66}\\
&+x^{64}+x^{60}+x^{54}+x^{52}+x^{44}+x^{38}+x^{34}+x^{30}+x^{28}+x^{26}+x^{24}\\
&+x^{22}+x^{18}+x^{14}+x^{10}+x^{4}+1,\\
p_9(x)=&x^{110}+x^{108}+x^{106}+x^{104}+x^{102}+x^{100}+x^{98}+x^{96}+x^{92}+x^{90}\\
&+x^{86}+x^{84}+x^{82}+x^{80}+x^{70}+x^{64}+x^{60}+x^{58}+x^{54}+x^{52}+x^{50}\\
&+x^{48}+x^{38}+x^{32}+x^{28}+x^{26}+x^{22}+x^{20}+x^{18}+x^{16}+x^{14}+x^{12}\\
&+x^{10}+x^{8}+x^{4}+x^{2},\\
p_{12}(x)=&x^{114}+x^{110}+x^{104}+x^{102}+x^{96}+x^{94}+x^{88}+x^{86}+x^{82}+x^{80}+x^{66}\\
&+x^{62}+x^{56}+x^{54}+x^{50}+x^{48}+x^{34}+x^{30}+x^{24}+x^{22}+x^{16}+x^{14}\\
&+x^{8}+x^{6}+x^{2}+1,
\end{aligned}
$$

and the initial terms are [1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 1)$:

$$
\begin{aligned}
p_0(x)=&x^{129}+x^{127}+x^{124}+x^{121}+x^{119}+x^{117}+x^{113}+x^{112}+x^{109}+x^{108}\\
&+x^{107}+x^{106}+x^{102}+x^{98}+x^{97}+x^{96}+x^{94}+x^{93}+x^{89}+x^{88}+x^{87}\\
&+x^{85}+x^{83}+x^{82}+x^{80}+x^{78}+x^{75}+x^{74}+x^{71}+x^{70}+x^{65}+x^{64}\\
&+x^{63}+x^{62}+x^{61}+x^{60}+x^{59}+x^{58}+x^{53}+x^{51}+x^{46}+x^{42}+x^{41}\\
&+x^{39}+x^{37}+x^{34}+x^{32}+x^{29}+x^{27},\\
p_3(x)=&x^{111}+x^{110}+x^{108}+x^{104}+x^{102}+x^{101}+x^{100}+x^{99}+x^{98}+x^{97}\\
&+x^{96}+x^{95}+x^{94}+x^{90}+x^{88}+x^{87}+x^{86}+x^{85}+x^{84}+x^{83}+x^{82}\\
&+x^{81}+x^{80}+x^{79}+x^{78}+x^{77}+x^{76}+x^{75}+x^{74}+x^{73}+x^{72}+x^{69}\\
&+x^{67}+x^{66}+x^{65}+x^{63}+x^{55}+x^{54}+x^{53}+x^{51}+x^{49}+x^{48}+x^{47}
\end{aligned}
$$

$$+ x^{45} + x^{43} + x^{42} + x^{41} + x^{38} + x^{37} + x^{35} + x^{33} + x^{32} + x^{31} + x^{28}$$
$$+ x^{26} + x^{23} + x^{21} + x^{20} + x^{19} + x^{17} + x^{13} + x^{12} + x^{11} + x^{9},$$
$$p_6(x) = x^{117} + x^{112} + x^{110} + x^{109} + x^{106} + x^{102} + x^{101} + x^{100} + x^{99} + x^{97}$$
$$+ x^{95} + x^{94} + x^{91} + x^{87} + x^{86} + x^{85} + x^{84} + x^{82} + x^{81} + x^{80} + x^{79}$$
$$+ x^{74} + x^{72} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{63} + x^{62} + x^{59} + x^{55}$$
$$+ x^{54} + x^{53} + x^{52} + x^{50} + x^{49} + x^{48} + x^{47} + x^{42} + x^{40} + x^{39} + x^{37}$$
$$+ x^{36} + x^{35} + x^{33} + x^{31} + x^{30} + x^{27} + x^{23} + x^{22} + x^{20} + x^{18} + x^{17}$$
$$+ x^{15} + x^{14} + x^{13} + x^{8} + x^{7} + x^{6},$$
$$p_9(x) = x^{123} + x^{122} + x^{121} + x^{119} + x^{115} + x^{114} + x^{112} + x^{111} + x^{110} + x^{109}$$
$$+ x^{108} + x^{105} + x^{103} + x^{102} + x^{101} + x^{98} + x^{96} + x^{95} + x^{94} + x^{91}$$
$$+ x^{89} + x^{88} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{76} + x^{75}$$
$$+ x^{72} + x^{71} + x^{69} + x^{67} + x^{66} + x^{64} + x^{63} + x^{62} + x^{59} + x^{57} + x^{56}$$
$$+ x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{44} + x^{43} + x^{40} + x^{39}$$
$$+ x^{37} + x^{35} + x^{34} + x^{32} + x^{31} + x^{30} + x^{26} + x^{24} + x^{23} + x^{22} + x^{21}$$
$$+ x^{20} + x^{17} + x^{15} + x^{14} + x^{13} + x^{11} + x^{9} + x^{8} + x^{6} + x^{3},$$
$$p_{12}(x) = x^{129} + x^{127} + x^{125} + x^{124} + x^{113} + x^{111} + x^{108} + x^{107} + x^{104} + x^{103}$$
$$+ x^{100} + x^{99} + x^{97} + x^{96} + x^{93} + x^{91} + x^{88} + x^{87} + x^{84} + x^{83} + x^{80}$$
$$+ x^{79} + x^{77} + x^{76} + x^{61} + x^{59} + x^{56} + x^{55} + x^{52} + x^{51} + x^{48} + x^{47}$$
$$+ x^{45} + x^{44} + x^{33} + x^{31} + x^{28} + x^{27} + x^{24} + x^{23} + x^{20} + x^{19} + x^{17}$$
$$+ x^{16} + x^{13} + x^{11} + x^{8} + x^{7} + x^{4} + x^{3} + x + 1,$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 0, 1, 1, 1, 0].

For $\phi(W^e, 1, 0)$:

$$p_0(x) = x^{129} + x^{127} + x^{124} + x^{121} + x^{119} + x^{117} + x^{113} + x^{112} + x^{109} + x^{108}$$
$$+ x^{107} + x^{106} + x^{102} + x^{98} + x^{97} + x^{96} + x^{94} + x^{93} + x^{89} + x^{88} + x^{87}$$
$$+ x^{85} + x^{83} + x^{82} + x^{80} + x^{78} + x^{75} + x^{74} + x^{71} + x^{70} + x^{65} + x^{64}$$
$$+ x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{53} + x^{51} + x^{46} + x^{42} + x^{41}$$
$$+ x^{39} + x^{37} + x^{34} + x^{32} + x^{29} + x^{27},$$
$$p_3(x) = x^{111} + x^{110} + x^{108} + x^{104} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97}$$
$$+ x^{96} + x^{95} + x^{94} + x^{90} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82}$$
$$+ x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{69}$$
$$+ x^{67} + x^{66} + x^{65} + x^{63} + x^{55} + x^{54} + x^{53} + x^{51} + x^{49} + x^{48} + x^{47}$$
$$+ x^{45} + x^{43} + x^{42} + x^{41} + x^{38} + x^{37} + x^{35} + x^{33} + x^{32} + x^{31} + x^{28}$$
$$+ x^{26} + x^{23} + x^{21} + x^{20} + x^{19} + x^{17} + x^{13} + x^{12} + x^{11} + x^{9},$$
$$p_6(x) = x^{117} + x^{112} + x^{110} + x^{109} + x^{106} + x^{102} + x^{101} + x^{100} + x^{99} + x^{97}$$
$$+ x^{95} + x^{94} + x^{91} + x^{87} + x^{86} + x^{85} + x^{84} + x^{82} + x^{81} + x^{80} + x^{79}$$

$$
\begin{aligned}
&+ x^{74} + x^{72} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{63} + x^{62} + x^{59} + x^{55}\\
&+ x^{54} + x^{53} + x^{52} + x^{50} + x^{49} + x^{48} + x^{47} + x^{42} + x^{40} + x^{39} + x^{37}\\
&+ x^{36} + x^{35} + x^{33} + x^{31} + x^{30} + x^{27} + x^{23} + x^{22} + x^{20} + x^{18} + x^{17}\\
&+ x^{15} + x^{14} + x^{13} + x^{8} + x^{7} + x^{6},\\
p_9(x) &= x^{123} + x^{122} + x^{121} + x^{119} + x^{115} + x^{114} + x^{112} + x^{111} + x^{110} + x^{109}\\
&+ x^{108} + x^{105} + x^{103} + x^{102} + x^{101} + x^{98} + x^{96} + x^{95} + x^{94} + x^{91}\\
&+ x^{89} + x^{88} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{76} + x^{75}\\
&+ x^{72} + x^{71} + x^{69} + x^{67} + x^{66} + x^{64} + x^{63} + x^{62} + x^{59} + x^{57} + x^{56}\\
&+ x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{44} + x^{43} + x^{40} + x^{39}\\
&+ x^{37} + x^{35} + x^{34} + x^{32} + x^{31} + x^{30} + x^{26} + x^{24} + x^{23} + x^{22} + x^{21}\\
&+ x^{20} + x^{17} + x^{15} + x^{14} + x^{13} + x^{11} + x^{9} + x^{8} + x^{6} + x^{3},\\
p_{12}(x) &= x^{129} + x^{127} + x^{125} + x^{124} + x^{113} + x^{111} + x^{108} + x^{107} + x^{104} + x^{103}\\
&+ x^{100} + x^{99} + x^{97} + x^{96} + x^{93} + x^{91} + x^{88} + x^{87} + x^{84} + x^{83} + x^{80}\\
&+ x^{79} + x^{77} + x^{76} + x^{61} + x^{59} + x^{56} + x^{55} + x^{52} + x^{51} + x^{48} + x^{47}\\
&+ x^{45} + x^{44} + x^{33} + x^{31} + x^{28} + x^{27} + x^{24} + x^{23} + x^{20} + x^{19} + x^{17}\\
&+ x^{16} + x^{13} + x^{11} + x^{8} + x^{7} + x^{4} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 0, 1, 1, 1, 0].

For $\phi(W^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{144} + x^{136} + x^{134} + x^{130} + x^{128} + x^{122} + x^{120} + x^{118} + x^{114} + x^{112}\\
&+ x^{100} + x^{98} + x^{96} + x^{90} + x^{84} + x^{78} + x^{66} + x^{64} + x^{60} + x^{58} + x^{56}\\
&+ x^{52} + x^{48} + x^{44} + x^{42},\\
p_3(x) &= x^{140} + x^{138} + x^{136} + x^{132} + x^{128} + x^{118} + x^{116} + x^{114} + x^{112} + x^{108}\\
&+ x^{100} + x^{98} + x^{96} + x^{94} + x^{92} + x^{88} + x^{86} + x^{78} + x^{72} + x^{64} + x^{62}\\
&+ x^{60} + x^{58} + x^{52} + x^{48} + x^{36} + x^{34} + x^{32} + x^{26} + x^{24} + x^{20} + x^{6},\\
p_6(x) &= x^{134} + x^{132} + x^{130} + x^{128} + x^{126} + x^{124} + x^{122} + x^{110} + x^{106} + x^{102}\\
&+ x^{98} + x^{96} + x^{88} + x^{86} + x^{84} + x^{80} + x^{78} + x^{76} + x^{72} + x^{70} + x^{66}\\
&+ x^{62} + x^{60} + x^{52} + x^{50} + x^{46} + x^{42} + x^{40} + x^{38} + x^{30} + x^{24} + x^{14}\\
&+ x^{12} + x^{4} + x^{2} + 1,\\
p_9(x) &= x^{140} + x^{138} + x^{130} + x^{124} + x^{122} + x^{116} + x^{106} + x^{92} + x^{90} + x^{82}\\
&+ x^{76} + x^{74} + x^{66} + x^{60} + x^{58} + x^{50} + x^{20} + x^{18} + x^{12} + x^{2},\\
p_{12}(x) &= x^{144} + x^{136} + x^{104} + x^{96} + x^{88} + x^{72} + x^{56} + x^{48} + x^{24} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0].

For $\phi(M^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{129} + x^{127} + x^{125} + x^{124} + x^{119} + x^{116} + x^{114} + x^{112} + x^{111} + x^{106}\\
&+ x^{101} + x^{100} + x^{99} + x^{97} + x^{95} + x^{94} + x^{92} + x^{88} + x^{83} + x^{82} + x^{80}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{78} + x^{77} + x^{74} + x^{72} + x^{70} + x^{68} + x^{66} + x^{63} + x^{62} + x^{60} + x^{56}\\
&+ x^{55} + x^{53} + x^{52} + x^{50} + x^{49} + x^{45} + x^{44} + x^{43} + x^{42} + x^{40} + x^{39}\\
&+ x^{37} + x^{36} + x^{35} + x^{30} + x^{28} + x^{24} + x^{23} + x^{21} + x^{20} + x^{14} + x^{13}\\
&+ x^{12},\\
p_3(x) =& x^{121} + x^{119} + x^{116} + x^{115} + x^{112} + x^{109} + x^{107} + x^{106} + x^{105} + x^{103}\\
&+ x^{102} + x^{101} + x^{100} + x^{99} + x^{96} + x^{94} + x^{93} + x^{92} + x^{88} + x^{87} + x^{83}\\
&+ x^{81} + x^{80} + x^{78} + x^{75} + x^{73} + x^{72} + x^{70} + x^{68} + x^{59} + x^{57} + x^{54}\\
&+ x^{52} + x^{51} + x^{48} + x^{47} + x^{46} + x^{45} + x^{41} + x^{37} + x^{35} + x^{33} + x^{32}\\
&+ x^{31} + x^{30} + x^{28} + x^{27} + x^{25} + x^{22} + x^{21} + x^{19} + x^{16} + x^{15} + x^{13}\\
&+ x^{12} + x^{10} + x^{9} + x^{8},\\
p_6(x) =& x^{121} + x^{115} + x^{114} + x^{113} + x^{108} + x^{107} + x^{105} + x^{103} + x^{102} + x^{100}\\
&+ x^{97} + x^{95} + x^{92} + x^{88} + x^{85} + x^{84} + x^{81} + x^{80} + x^{79} + x^{77} + x^{75}\\
&+ x^{69} + x^{68} + x^{67} + x^{64} + x^{63} + x^{60} + x^{56} + x^{53} + x^{52} + x^{49} + x^{48}\\
&+ x^{47} + x^{45} + x^{43} + x^{37} + x^{36} + x^{35} + x^{32} + x^{31} + x^{28} + x^{25} + x^{24}\\
&+ x^{21} + x^{20} + x^{19} + x^{18} + x^{16} + x^{15} + x^{13} + x^{12} + x^{9} + x^{7} + x^{6} + x^{5}\\
&+ x^{3} + x + 1,\\
p_9(x) =& x^{121} + x^{119} + x^{116} + x^{115} + x^{112} + x^{111} + x^{109} + x^{108} + x^{101} + x^{99}\\
&+ x^{96} + x^{95} + x^{93} + x^{92} + x^{89} + x^{87} + x^{85} + x^{84} + x^{73} + x^{71} + x^{68}\\
&+ x^{67} + x^{64} + x^{63} + x^{61} + x^{60} + x^{57} + x^{55} + x^{53} + x^{52} + x^{41} + x^{39}\\
&+ x^{36} + x^{35} + x^{32} + x^{31} + x^{29} + x^{28} + x^{21} + x^{19} + x^{16} + x^{15} + x^{13}\\
&+ x^{12} + x^{9} + x^{7} + x^{5} + x^{4},\\
p_{12}(x) =& x^{129} + x^{127} + x^{125} + x^{124} + x^{121} + x^{119} + x^{116} + x^{115} + x^{113} + x^{112}\\
&+ x^{109} + x^{107} + x^{104} + x^{103} + x^{101} + x^{100} + x^{97} + x^{95} + x^{92} + x^{91}\\
&+ x^{89} + x^{88} + x^{85} + x^{83} + x^{80} + x^{79} + x^{77} + x^{76} + x^{73} + x^{71} + x^{68}\\
&+ x^{67} + x^{64} + x^{63} + x^{60} + x^{59} + x^{57} + x^{56} + x^{53} + x^{51} + x^{48} + x^{47}\\
&+ x^{45} + x^{44} + x^{41} + x^{39} + x^{36} + x^{35} + x^{33} + x^{32} + x^{29} + x^{27} + x^{24}\\
&+ x^{23} + x^{21} + x^{20} + x^{17} + x^{15} + x^{12} + x^{11} + x^{9} + x^{8} + x^{5} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 1, 0, 1, 0, 1, 1, 1, 1, 0, 0, 1, 1, 1, 0].

For $\phi(M^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) =& x^{122} + x^{120} + x^{116} + x^{114} + x^{113} + x^{111} + x^{110} + x^{108} + x^{106} + x^{105}\\
&+ x^{104} + x^{103} + x^{100} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{90}\\
&+ x^{88} + x^{87} + x^{86} + x^{84} + x^{82} + x^{80} + x^{78} + x^{77} + x^{76} + x^{74} + x^{69}\\
&+ x^{68} + x^{62} + x^{60} + x^{59} + x^{57} + x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48}\\
&+ x^{45} + x^{44} + x^{42} + x^{37} + x^{33} + x^{32} + x^{31} + x^{26} + x^{24} + x^{23} + x^{22}\\
&+ x^{21} + x^{20} + x^{18},
\end{aligned}
$$

$$\begin{aligned}
p_3(x) &= x^{104}+x^{103}+x^{101}+x^{100}+x^{98}+x^{97}+x^{94}+x^{90}+x^{89}+x^{64}+x^{63}\\
&\quad +x^{61}+x^{60}+x^{58}+x^{57}+x^{54}+x^{50}+x^{49}+x^{48}+x^{47}+x^{45}+x^{44}\\
&\quad +x^{42}+x^{41}+x^{40}+x^{39}+x^{38}+x^{37}+x^{36}+x^{32}+x^{31}+x^{30}+x^{29}\\
&\quad +x^{28}+x^{24}+x^{23}+x^{22}+x^{21}+x^{20}+x^{14}+x^{10}+x^{9},\\
p_6(x) &= x^{110}+x^{104}+x^{102}+x^{98}+x^{96}+x^{94}+x^{84}+x^{80}+x^{72}+x^{66}+x^{64}\\
&\quad +x^{62}+x^{52}+x^{48}+x^{40}+x^{34}+x^{32}+x^{30}+x^{20}+x^{16}+x^{14}+x^{6},\\
p_9(x) &= x^{116}+x^{115}+x^{114}+x^{107}+x^{105}+x^{103}+x^{102}+x^{101}+x^{98}+x^{91}\\
&\quad +x^{89}+x^{87}+x^{86}+x^{85}+x^{84}+x^{83}+x^{68}+x^{67}+x^{66}+x^{59}+x^{57}\\
&\quad +x^{55}+x^{54}+x^{53}+x^{52}+x^{51}+x^{36}+x^{35}+x^{34}+x^{27}+x^{25}+x^{23}\\
&\quad +x^{22}+x^{21}+x^{18}+x^{11}+x^{9}+x^{7}+x^{6}+x^{5}+x^{4}+x^{3},\\
p_{12}(x) &= x^{122}+x^{120}+x^{118}+x^{116}+x^{110}+x^{108}+x^{106}+x^{104}+x^{98}+x^{96}\\
&\quad +x^{94}+x^{92}+x^{86}+x^{84}+x^{82}+x^{80}+x^{74}+x^{72}+x^{70}+x^{68}+x^{62}\\
&\quad +x^{60}+x^{54}+x^{52}+x^{50}+x^{48}+x^{42}+x^{40}+x^{38}+x^{36}+x^{30}+x^{28}\\
&\quad +x^{26}+x^{24}+x^{18}+x^{16}+x^{14}+x^{12}+x^{6}+x^{4}+x^{2}+1,
\end{aligned}$$

and the initial terms are [0, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 1, 1, 0].

For $\phi(M^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{144}+x^{140}+x^{138}+x^{136}+x^{135}+x^{134}+x^{132}+x^{131}+x^{124}+x^{119}\\
&\quad +x^{118}+x^{116}+x^{115}+x^{113}+x^{110}+x^{109}+x^{107}+x^{106}+x^{105}\\
&\quad +x^{104}+x^{103}+x^{102}+x^{101}+x^{100}+x^{98}+x^{96}+x^{95}+x^{94}+x^{93}\\
&\quad +x^{92}+x^{91}+x^{87}+x^{86}+x^{85}+x^{84}+x^{83}+x^{82}+x^{81}+x^{80}+x^{78}\\
&\quad +x^{76}+x^{71}+x^{70}+x^{69}+x^{67}+x^{65}+x^{64}+x^{63}+x^{61}+x^{60}+x^{58}\\
&\quad +x^{56}+x^{53}+x^{52}+x^{48}+x^{47}+x^{44}+x^{41}+x^{40}+x^{37}+x^{35}+x^{32}\\
&\quad +x^{29}+x^{28}+x^{27},\\
p_3(x) &= x^{126}+x^{125}+x^{124}+x^{122}+x^{121}+x^{119}+x^{113}+x^{112}+x^{110}+x^{109}\\
&\quad +x^{106}+x^{102}+x^{101}+x^{100}+x^{99}+x^{97}+x^{95}+x^{93}+x^{92}+x^{91}\\
&\quad +x^{90}+x^{89}+x^{86}+x^{85}+x^{84}+x^{82}+x^{81}+x^{79}+x^{73}+x^{72}+x^{68}\\
&\quad +x^{65}+x^{63}+x^{59}+x^{58}+x^{57}+x^{56}+x^{53}+x^{51}+x^{50}+x^{49}+x^{48}\\
&\quad +x^{47}+x^{46}+x^{45}+x^{43}+x^{41}+x^{40}+x^{37}+x^{33}+x^{32}+x^{31}+x^{29}\\
&\quad +x^{23}+x^{22}+x^{18}+x^{15}+x^{13}+x^{12}+x^{11}+x^{10}+x^{9},\\
p_6(x) &= x^{132}+x^{130}+x^{126}+x^{124}+x^{120}+x^{118}+x^{116}+x^{114}+x^{96}+x^{94}\\
&\quad +x^{90}+x^{86}+x^{84}+x^{78}+x^{76}+x^{74}+x^{72}+x^{70}+x^{64}+x^{62}+x^{58}\\
&\quad +x^{54}+x^{52}+x^{46}+x^{44}+x^{42}+x^{40}+x^{38}+x^{36}+x^{34}+x^{32}+x^{28}\\
&\quad +x^{26}+x^{24}+x^{18}+x^{14}+x^{12}+x^{10}+x^{8}+x^{6},\\
p_9(x) &= x^{138}+x^{137}+x^{135}+x^{134}+x^{130}+x^{127}+x^{126}+x^{124}+x^{121}+x^{120}\\
&\quad +x^{119}+x^{118}+x^{117}+x^{116}+x^{115}+x^{113}+x^{112}+x^{110}+x^{109}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{106} + x^{104} + x^{103} + x^{101} + x^{98} + x^{97} + x^{96} + x^{95} + x^{93} + x^{92} \\
&+ x^{90} + x^{89} + x^{86} + x^{84} + x^{83} + x^{82} + x^{79} + x^{78} + x^{76} + x^{74} + x^{72} \\
&+ x^{69} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} + x^{58} + x^{57} \\
&+ x^{54} + x^{52} + x^{51} + x^{50} + x^{47} + x^{46} + x^{44} + x^{41} + x^{40} + x^{39} + x^{38} \\
&+ x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{30} + x^{29} + x^{26} + x^{24} + x^{23} + x^{21} \\
&+ x^{18} + x^{17} + x^{16} + x^{15} + x^{13} + x^{12} + x^{7} + x^{4} + x^{3}, \\
p_{12}(x) &= x^{144} + x^{132} + x^{124} + x^{116} + x^{112} + x^{104} + x^{100} + x^{92} + x^{88} + x^{76} \\
&+ x^{64} + x^{60} + x^{56} + x^{48} + x^{44} + x^{36} + x^{32} + x^{24} + x^{20} + x^{16} + x^{12} \\
&+ x^{8} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 1, 0, 0].

For $\phi(M^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{129} + x^{127} + x^{125} + x^{124} + x^{121} + x^{117} + x^{115} + x^{114} + x^{113} + x^{112} \\
&+ x^{111} + x^{110} + x^{104} + x^{100} + x^{98} + x^{97} + x^{96} + x^{95} + x^{93} + x^{89} \\
&+ x^{88} + x^{87} + x^{85} + x^{83} + x^{80} + x^{77} + x^{74} + x^{73} + x^{71} + x^{70} + x^{69} \\
&+ x^{66} + x^{62} + x^{61} + x^{60} + x^{59} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} \\
&+ x^{50} + x^{49} + x^{48} + x^{47} + x^{44} + x^{43} + x^{42} + x^{40} + x^{39} + x^{37} + x^{36} \\
&+ x^{35} + x^{33}, \\
p_3(x) &= x^{121} + x^{119} + x^{116} + x^{115} + x^{113} + x^{112} + x^{111} + x^{109} + x^{108} + x^{107} \\
&+ x^{106} + x^{105} + x^{104} + x^{101} + x^{100} + x^{99} + x^{96} + x^{95} + x^{94} + x^{93} \\
&+ x^{92} + x^{90} + x^{89} + x^{87} + x^{83} + x^{81} + x^{80} + x^{78} + x^{75} + x^{72} + x^{71} \\
&+ x^{70} + x^{64} + x^{63} + x^{62} + x^{59} + x^{54} + x^{51} + x^{50} + x^{48} + x^{47} + x^{45} \\
&+ x^{44} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{27} \\
&+ x^{26} + x^{21} + x^{19} + x^{18} + x^{17} + x^{13} + x^{12}, \\
p_6(x) &= x^{119} + x^{117} + x^{116} + x^{114} + x^{110} + x^{109} + x^{108} + x^{107} + x^{106} + x^{105} \\
&+ x^{103} + x^{101} + x^{99} + x^{94} + x^{92} + x^{91} + x^{88} + x^{87} + x^{86} + x^{85} + x^{83} \\
&+ x^{82} + x^{80} + x^{79} + x^{77} + x^{76} + x^{74} + x^{69} + x^{68} + x^{65} + x^{64} + x^{62} \\
&+ x^{60} + x^{59} + x^{56} + x^{55} + x^{54} + x^{53} + x^{51} + x^{50} + x^{48} + x^{47} + x^{45} \\
&+ x^{44} + x^{42} + x^{37} + x^{36} + x^{33} + x^{32} + x^{30} + x^{28} + x^{27} + x^{24} + x^{22} \\
&+ x^{20} + x^{19} + x^{16} + x^{15} + x^{14} + x^{11} + x^{9} + x^{7} + x^{4} + x^{3} + x + 1, \\
p_9(x) &= x^{121} + x^{119} + x^{116} + x^{115} + x^{112} + x^{111} + x^{109} + x^{108} + x^{101} + x^{99} \\
&+ x^{96} + x^{95} + x^{93} + x^{92} + x^{89} + x^{87} + x^{85} + x^{84} + x^{73} + x^{71} + x^{68} \\
&+ x^{67} + x^{64} + x^{63} + x^{61} + x^{60} + x^{57} + x^{55} + x^{53} + x^{52} + x^{41} + x^{39} \\
&+ x^{36} + x^{35} + x^{32} + x^{31} + x^{29} + x^{28} + x^{21} + x^{19} + x^{16} + x^{15} + x^{13} \\
&+ x^{12} + x^{9} + x^{7} + x^{5} + x^{4}, \\
p_{12}(x) &= x^{129} + x^{127} + x^{125} + x^{124} + x^{121} + x^{119} + x^{116} + x^{115} + x^{113} + x^{112}
\end{aligned}
$$

$$+ x^{109} + x^{107} + x^{104} + x^{103} + x^{101} + x^{100} + x^{97} + x^{95} + x^{92} + x^{91} + x^{89} + x^{88} + x^{85} + x^{83} + x^{80} + x^{79} + x^{77} + x^{76} + x^{73} + x^{71} + x^{68} + x^{67} + x^{64} + x^{63} + x^{60} + x^{59} + x^{57} + x^{56} + x^{53} + x^{51} + x^{48} + x^{47} + x^{45} + x^{44} + x^{41} + x^{39} + x^{36} + x^{35} + x^{33} + x^{32} + x^{29} + x^{27} + x^{24} + x^{23} + x^{21} + x^{20} + x^{17} + x^{15} + x^{12} + x^{11} + x^{9} + x^{8} + x^{5} + x^{3} + x + 1,$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 1, 1, 0].

For $\phi(W^o, 0, 0)$:

$$\begin{aligned}
p_0(x) = {}& x^{129} + x^{127} + x^{125} + x^{124} + x^{119} + x^{116} + x^{114} + x^{112} + x^{111} + x^{106} \\
&+ x^{101} + x^{100} + x^{99} + x^{97} + x^{95} + x^{94} + x^{92} + x^{88} + x^{83} + x^{82} + x^{80} \\
&+ x^{78} + x^{77} + x^{74} + x^{72} + x^{70} + x^{68} + x^{66} + x^{63} + x^{62} + x^{60} + x^{56} \\
&+ x^{55} + x^{53} + x^{52} + x^{50} + x^{49} + x^{45} + x^{44} + x^{43} + x^{42} + x^{40} + x^{39} \\
&+ x^{37} + x^{36} + x^{35} + x^{30} + x^{28} + x^{24} + x^{23} + x^{21} + x^{20} + x^{14} + x^{13} \\
&+ x^{12}, \\
p_3(x) = {}& x^{121} + x^{119} + x^{116} + x^{115} + x^{112} + x^{109} + x^{107} + x^{106} + x^{105} + x^{103} \\
&+ x^{102} + x^{101} + x^{100} + x^{99} + x^{96} + x^{94} + x^{93} + x^{92} + x^{88} + x^{87} + x^{83} \\
&+ x^{81} + x^{80} + x^{78} + x^{75} + x^{73} + x^{72} + x^{70} + x^{68} + x^{59} + x^{57} + x^{54} \\
&+ x^{52} + x^{51} + x^{48} + x^{47} + x^{46} + x^{45} + x^{41} + x^{37} + x^{35} + x^{33} + x^{32} \\
&+ x^{31} + x^{30} + x^{28} + x^{27} + x^{25} + x^{22} + x^{21} + x^{19} + x^{16} + x^{15} + x^{13} \\
&+ x^{12} + x^{10} + x^{9} + x^{8}, \\
p_6(x) = {}& x^{121} + x^{115} + x^{114} + x^{113} + x^{108} + x^{107} + x^{105} + x^{103} + x^{102} + x^{100} \\
&+ x^{97} + x^{95} + x^{92} + x^{88} + x^{85} + x^{84} + x^{81} + x^{80} + x^{79} + x^{77} + x^{75} \\
&+ x^{69} + x^{68} + x^{67} + x^{64} + x^{63} + x^{60} + x^{56} + x^{53} + x^{52} + x^{49} + x^{48} \\
&+ x^{47} + x^{45} + x^{43} + x^{37} + x^{36} + x^{35} + x^{32} + x^{31} + x^{28} + x^{25} + x^{24} \\
&+ x^{21} + x^{20} + x^{19} + x^{18} + x^{16} + x^{15} + x^{13} + x^{12} + x^{9} + x^{7} + x^{6} + x^{5} \\
&+ x^{3} + x + 1, \\
p_9(x) = {}& x^{121} + x^{119} + x^{116} + x^{115} + x^{112} + x^{111} + x^{109} + x^{108} + x^{101} + x^{99} \\
&+ x^{96} + x^{95} + x^{93} + x^{92} + x^{89} + x^{87} + x^{85} + x^{84} + x^{73} + x^{71} + x^{68} \\
&+ x^{67} + x^{64} + x^{63} + x^{61} + x^{60} + x^{57} + x^{55} + x^{53} + x^{52} + x^{41} + x^{39} \\
&+ x^{36} + x^{35} + x^{32} + x^{31} + x^{29} + x^{28} + x^{21} + x^{19} + x^{16} + x^{15} + x^{13} \\
&+ x^{12} + x^{9} + x^{7} + x^{5} + x^{4}, \\
p_{12}(x) = {}& x^{129} + x^{127} + x^{125} + x^{124} + x^{121} + x^{119} + x^{116} + x^{115} + x^{113} + x^{112} \\
&+ x^{109} + x^{107} + x^{104} + x^{103} + x^{101} + x^{100} + x^{97} + x^{95} + x^{92} + x^{91} \\
&+ x^{89} + x^{88} + x^{85} + x^{83} + x^{80} + x^{79} + x^{77} + x^{76} + x^{73} + x^{71} + x^{68} \\
&+ x^{67} + x^{64} + x^{63} + x^{60} + x^{59} + x^{57} + x^{56} + x^{53} + x^{51} + x^{48} + x^{47} \\
&+ x^{45} + x^{44} + x^{41} + x^{39} + x^{36} + x^{35} + x^{33} + x^{32} + x^{29} + x^{27} + x^{24}
\end{aligned}$$

$$+ x^{23} + x^{21} + x^{20} + x^{17} + x^{15} + x^{12} + x^{11} + x^{9} + x^{8} + x^{5} + x^{3} + x + 1,$$

and the initial terms are [1, 0, 1, 0, 1, 0, 1, 1, 1, 1, 0, 0, 1, 1, 1, 0].

For $\phi(W^o, 0, 1)$:

$$\begin{aligned}
p_0(x) = {} & x^{144} + x^{140} + x^{138} + x^{136} + x^{135} + x^{134} + x^{132} + x^{131} + x^{124} + x^{119} \\
& + x^{118} + x^{116} + x^{115} + x^{113} + x^{110} + x^{109} + x^{107} + x^{106} + x^{105} \\
& + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{98} + x^{96} + x^{95} + x^{94} + x^{93} \\
& + x^{92} + x^{91} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{78} \\
& + x^{76} + x^{71} + x^{70} + x^{69} + x^{67} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} + x^{58} \\
& + x^{56} + x^{53} + x^{52} + x^{48} + x^{47} + x^{44} + x^{41} + x^{40} + x^{37} + x^{35} + x^{32} \\
& + x^{29} + x^{28} + x^{27}, \\
p_3(x) = {} & x^{126} + x^{125} + x^{124} + x^{122} + x^{121} + x^{119} + x^{113} + x^{112} + x^{110} + x^{109} \\
& + x^{106} + x^{102} + x^{101} + x^{100} + x^{99} + x^{97} + x^{95} + x^{93} + x^{92} + x^{91} \\
& + x^{90} + x^{89} + x^{86} + x^{85} + x^{84} + x^{82} + x^{81} + x^{79} + x^{73} + x^{72} + x^{68} \\
& + x^{65} + x^{63} + x^{59} + x^{58} + x^{57} + x^{56} + x^{53} + x^{51} + x^{50} + x^{49} + x^{48} \\
& + x^{47} + x^{46} + x^{45} + x^{43} + x^{41} + x^{40} + x^{37} + x^{33} + x^{32} + x^{31} + x^{29} \\
& + x^{23} + x^{22} + x^{18} + x^{15} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9}, \\
p_6(x) = {} & x^{132} + x^{130} + x^{126} + x^{124} + x^{120} + x^{118} + x^{116} + x^{114} + x^{96} + x^{94} \\
& + x^{90} + x^{86} + x^{84} + x^{78} + x^{76} + x^{74} + x^{72} + x^{70} + x^{64} + x^{62} + x^{58} \\
& + x^{54} + x^{52} + x^{46} + x^{44} + x^{42} + x^{40} + x^{38} + x^{36} + x^{34} + x^{32} + x^{28} \\
& + x^{26} + x^{24} + x^{18} + x^{14} + x^{12} + x^{10} + x^{8} + x^{6}, \\
p_9(x) = {} & x^{138} + x^{137} + x^{135} + x^{134} + x^{130} + x^{127} + x^{126} + x^{124} + x^{121} + x^{120} \\
& + x^{119} + x^{118} + x^{117} + x^{116} + x^{115} + x^{113} + x^{112} + x^{110} + x^{109} \\
& + x^{106} + x^{104} + x^{103} + x^{101} + x^{98} + x^{97} + x^{96} + x^{95} + x^{93} + x^{92} \\
& + x^{90} + x^{89} + x^{86} + x^{84} + x^{83} + x^{82} + x^{79} + x^{78} + x^{76} + x^{74} + x^{72} \\
& + x^{69} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} + x^{58} + x^{57} \\
& + x^{54} + x^{52} + x^{51} + x^{50} + x^{47} + x^{46} + x^{44} + x^{41} + x^{40} + x^{39} + x^{38} \\
& + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{30} + x^{29} + x^{26} + x^{24} + x^{23} + x^{21} \\
& + x^{18} + x^{17} + x^{16} + x^{15} + x^{13} + x^{12} + x^{7} + x^{4} + x^{3}, \\
p_{12}(x) = {} & x^{144} + x^{132} + x^{124} + x^{116} + x^{112} + x^{104} + x^{100} + x^{92} + x^{88} + x^{76} \\
& + x^{64} + x^{60} + x^{56} + x^{48} + x^{44} + x^{36} + x^{32} + x^{24} + x^{20} + x^{16} + x^{12} \\
& + x^{8} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 1, 0, 0].

For $\phi(W^o, 1, 0)$:

$$\begin{aligned}
p_0(x) = {} & x^{122} + x^{120} + x^{116} + x^{114} + x^{113} + x^{111} + x^{110} + x^{108} + x^{106} + x^{105} \\
& + x^{104} + x^{103} + x^{100} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{90}
\end{aligned}$$

$$
\begin{aligned}
&+x^{88}+x^{87}+x^{86}+x^{84}+x^{82}+x^{80}+x^{78}+x^{77}+x^{76}+x^{74}+x^{69}\\
&+x^{68}+x^{62}+x^{60}+x^{59}+x^{57}+x^{55}+x^{53}+x^{52}+x^{51}+x^{49}+x^{48}\\
&+x^{45}+x^{44}+x^{42}+x^{37}+x^{33}+x^{32}+x^{31}+x^{26}+x^{24}+x^{23}+x^{22}\\
&+x^{21}+x^{20}+x^{18},\\
p_3(x)={}&x^{104}+x^{103}+x^{101}+x^{100}+x^{98}+x^{97}+x^{94}+x^{90}+x^{89}+x^{64}+x^{63}\\
&+x^{61}+x^{60}+x^{58}+x^{57}+x^{54}+x^{50}+x^{49}+x^{48}+x^{47}+x^{45}+x^{44}\\
&+x^{42}+x^{41}+x^{40}+x^{39}+x^{38}+x^{37}+x^{36}+x^{32}+x^{31}+x^{30}+x^{29}\\
&+x^{28}+x^{24}+x^{23}+x^{22}+x^{21}+x^{20}+x^{14}+x^{10}+x^{9},\\
p_6(x)={}&x^{110}+x^{104}+x^{102}+x^{98}+x^{96}+x^{94}+x^{84}+x^{80}+x^{72}+x^{66}+x^{64}\\
&+x^{62}+x^{52}+x^{48}+x^{40}+x^{34}+x^{32}+x^{30}+x^{20}+x^{16}+x^{14}+x^{6},\\
p_9(x)={}&x^{116}+x^{115}+x^{114}+x^{107}+x^{105}+x^{103}+x^{102}+x^{101}+x^{98}+x^{91}\\
&+x^{89}+x^{87}+x^{86}+x^{85}+x^{84}+x^{83}+x^{68}+x^{67}+x^{66}+x^{59}+x^{57}\\
&+x^{55}+x^{54}+x^{53}+x^{52}+x^{51}+x^{36}+x^{35}+x^{34}+x^{27}+x^{25}+x^{23}\\
&+x^{22}+x^{21}+x^{18}+x^{11}+x^{9}+x^{7}+x^{6}+x^{5}+x^{4}+x^{3},\\
p_{12}(x)={}&x^{122}+x^{120}+x^{118}+x^{116}+x^{110}+x^{108}+x^{106}+x^{104}+x^{98}+x^{96}\\
&+x^{94}+x^{92}+x^{86}+x^{84}+x^{82}+x^{80}+x^{74}+x^{72}+x^{70}+x^{68}+x^{62}\\
&+x^{60}+x^{54}+x^{52}+x^{50}+x^{48}+x^{42}+x^{40}+x^{38}+x^{36}+x^{30}+x^{28}\\
&+x^{26}+x^{24}+x^{18}+x^{16}+x^{14}+x^{12}+x^{6}+x^{4}+x^{2}+1,
\end{aligned}
$$

and the initial terms are [0, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 1, 1, 0].

For $\phi(W^o, 1, 1)$:

$$
\begin{aligned}
p_0(x)={}&x^{129}+x^{127}+x^{125}+x^{124}+x^{121}+x^{117}+x^{115}+x^{114}+x^{113}+x^{112}\\
&+x^{111}+x^{110}+x^{104}+x^{100}+x^{98}+x^{97}+x^{96}+x^{95}+x^{93}+x^{89}\\
&+x^{88}+x^{87}+x^{85}+x^{83}+x^{80}+x^{77}+x^{74}+x^{73}+x^{71}+x^{70}+x^{69}\\
&+x^{66}+x^{62}+x^{61}+x^{60}+x^{59}+x^{57}+x^{56}+x^{55}+x^{54}+x^{53}+x^{52}\\
&+x^{50}+x^{49}+x^{48}+x^{47}+x^{44}+x^{43}+x^{42}+x^{40}+x^{39}+x^{37}+x^{36}\\
&+x^{35}+x^{33},\\
p_3(x)={}&x^{121}+x^{119}+x^{116}+x^{115}+x^{113}+x^{112}+x^{111}+x^{109}+x^{108}+x^{107}\\
&+x^{106}+x^{105}+x^{104}+x^{101}+x^{100}+x^{99}+x^{96}+x^{95}+x^{94}+x^{93}\\
&+x^{92}+x^{90}+x^{89}+x^{87}+x^{83}+x^{81}+x^{80}+x^{78}+x^{75}+x^{72}+x^{71}\\
&+x^{70}+x^{64}+x^{63}+x^{62}+x^{59}+x^{54}+x^{51}+x^{50}+x^{48}+x^{47}+x^{45}\\
&+x^{44}+x^{40}+x^{39}+x^{38}+x^{37}+x^{36}+x^{35}+x^{34}+x^{33}+x^{32}+x^{27}\\
&+x^{26}+x^{21}+x^{19}+x^{18}+x^{17}+x^{13}+x^{12},\\
p_6(x)={}&x^{119}+x^{117}+x^{116}+x^{114}+x^{110}+x^{109}+x^{108}+x^{107}+x^{106}+x^{105}\\
&+x^{103}+x^{101}+x^{99}+x^{94}+x^{92}+x^{91}+x^{88}+x^{87}+x^{86}+x^{85}+x^{83}\\
&+x^{82}+x^{80}+x^{79}+x^{77}+x^{76}+x^{74}+x^{69}+x^{68}+x^{65}+x^{64}+x^{62}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{60}+x^{59}+x^{56}+x^{55}+x^{54}+x^{53}+x^{51}+x^{50}+x^{48}+x^{47}+x^{45}\\
&+x^{44}+x^{42}+x^{37}+x^{36}+x^{33}+x^{32}+x^{30}+x^{28}+x^{27}+x^{24}+x^{22}\\
&+x^{20}+x^{19}+x^{16}+x^{15}+x^{14}+x^{11}+x^{9}+x^{7}+x^{4}+x^{3}+x+1,\\
p_9(x)=&\,x^{121}+x^{119}+x^{116}+x^{115}+x^{112}+x^{111}+x^{109}+x^{108}+x^{101}+x^{99}\\
&+x^{96}+x^{95}+x^{93}+x^{92}+x^{89}+x^{87}+x^{85}+x^{84}+x^{73}+x^{71}+x^{68}\\
&+x^{67}+x^{64}+x^{63}+x^{61}+x^{60}+x^{57}+x^{55}+x^{53}+x^{52}+x^{41}+x^{39}\\
&+x^{36}+x^{35}+x^{32}+x^{31}+x^{29}+x^{28}+x^{21}+x^{19}+x^{16}+x^{15}+x^{13}\\
&+x^{12}+x^{9}+x^{7}+x^{5}+x^{4},\\
p_{12}(x)=&\,x^{129}+x^{127}+x^{125}+x^{124}+x^{121}+x^{119}+x^{116}+x^{115}+x^{113}+x^{112}\\
&+x^{109}+x^{107}+x^{104}+x^{103}+x^{101}+x^{100}+x^{97}+x^{95}+x^{92}+x^{91}\\
&+x^{89}+x^{88}+x^{85}+x^{83}+x^{80}+x^{79}+x^{77}+x^{76}+x^{73}+x^{71}+x^{68}\\
&+x^{67}+x^{64}+x^{63}+x^{60}+x^{59}+x^{57}+x^{56}+x^{53}+x^{51}+x^{48}+x^{47}\\
&+x^{45}+x^{44}+x^{41}+x^{39}+x^{36}+x^{35}+x^{33}+x^{32}+x^{29}+x^{27}+x^{24}\\
&+x^{23}+x^{21}+x^{20}+x^{17}+x^{15}+x^{12}+x^{11}+x^{9}+x^{8}+x^{5}+x^{3}+x+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 1, 1, 0].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	139	[249, 250]	278	[115, 438]	417	[294, 208]	556	[358, 647]
1	[3, 4]	140	[251, 252]	279	[353, 439]	418	[575, 512]	557	[648, 598]
2	[5, 6]	141	[104, 253]	280	[440, 441]	419	[576, 305]	558	[649, 443]
3	[7, 8]	142	[254, 255]	281	[364, 442]	420	[521, 208]	559	[477, 650]
4	[9, 10]	143	[256, 257]	282	[443, 444]	421	[209, 524]	560	[200, 651]
5	[11, 12]	144	[258, 259]	283	[445, 446]	422	[98, 318]	561	[468, 652]
6	[13, 14]	145	[260, 261]	284	[193, 447]	423	[552, 577]	562	[356, 653]
7	[15, 16]	146	[262, 250]	285	[189, 394]	424	[567, 578]	563	[171, 654]
8	[17, 18]	147	[254, 263]	286	[448, 181]	425	[61, 563]	564	[144, 655]
9	[19, 20]	148	[264, 265]	287	[449, 399]	426	[269, 255]	565	[644, 656]
10	[21, 22]	149	[249, 266]	288	[49, 450]	427	[308, 81]	566	[630, 657]
11	[23, 24]	150	[267, 237]	289	[146, 451]	428	[540, 293]	567	[378, 658]
12	[25, 26]	151	[268, 84]	290	[136, 452]	429	[557, 317]	568	[501, 659]
13	[27, 28]	152	[269, 270]	291	[453, 454]	430	[579, 544]	569	[54, 161]
14	[29, 30]	153	[90, 261]	292	[455, 456]	431	[292, 97]	570	[455, 660]
15	[31, 32]	154	[256, 271]	293	[448, 457]	432	[517, 577]	571	[393, 352]
16	[33, 34]	155	[272, 273]	294	[458, 137]	433	[516, 325]	572	[439, 661]
17	[35, 36]	156	[274, 275]	295	[459, 369]	434	[519, 223]	573	[365, 610]
18	[37, 38]	157	[276, 277]	296	[460, 461]	435	[79, 310]	574	[642, 662]
19	[34, 39]	158	[278, 279]	297	[462, 403]	436	[580, 329]	575	[620, 349]
20	[40, 41]	159	[280, 281]	298	[162, 463]	437	[323, 71]	576	[389, 663]

21	[42, 43]	160	[282, 283]	299	[12, 425]	438	[225, 581]	577	[649, 664]
22	[34, 44]	161	[284, 226]	300	[359, 186]	439	[582, 557]	578	[380, 665]
23	[45, 46]	162	[103, 285]	301	[464, 465]	440	[510, 22]	579	[634, 619]
24	[47, 48]	163	[286, 97]	302	[13, 147]	441	[105, 583]	580	[380, 416]
25	[49, 50]	164	[287, 283]	303	[466, 467]	442	[86, 218]	581	[365, 666]
26	[51, 52]	165	[274, 288]	304	[468, 469]	443	[261, 271]	582	[186, 667]
27	[53, 51]	166	[289, 234]	305	[470, 389]	444	[68, 63]	583	[668, 669]
28	[54, 55]	167	[290, 291]	306	[471, 472]	445	[584, 320]	584	[668, 670]
29	[56, 57]	168	[292, 106]	307	[150, 473]	446	[508, 281]	585	[148, 471]
30	[58, 59]	169	[85, 293]	308	[47, 474]	447	[98, 508]	586	[120, 671]
31	[60, 61]	170	[294, 295]	309	[407, 475]	448	[585, 305]	587	[466, 672]
32	[62, 63]	171	[296, 297]	310	[172, 476]	449	[91, 27]	588	[673, 674]
33	[64, 65]	172	[215, 298]	311	[408, 477]	450	[285, 321]	589	[165, 675]
34	[66, 67]	173	[232, 87]	312	[478, 479]	451	[586, 24]	590	[413, 676]
35	[68, 69]	174	[299, 291]	313	[399, 174]	452	[271, 28]	591	[145, 677]
36	[70, 71]	175	[244, 262]	314	[441, 480]	453	[586, 515]	592	[192, 678]
37	[72, 73]	176	[300, 301]	315	[425, 481]	454	[580, 29]	593	[140, 679]
38	[74, 75]	177	[208, 105]	316	[144, 482]	455	[576, 81]	594	[680, 271]
39	[76, 16]	178	[302, 303]	317	[483, 484]	456	[279, 529]	595	[681, 314]
40	[77, 78]	179	[113, 253]	318	[485, 486]	457	[289, 531]	596	[321, 252]
41	[79, 80]	180	[304, 22]	319	[369, 196]	458	[287, 247]	597	[95, 111]
42	[81, 82]	181	[273, 86]	320	[358, 487]	459	[29, 513]	598	[21, 682]
43	[83, 84]	182	[305, 306]	321	[488, 124]	460	[587, 533]	599	[72, 683]
44	[85, 20]	183	[307, 62]	322	[489, 490]	461	[237, 219]	600	[217, 572]
45	[86, 87]	184	[265, 257]	323	[436, 354]	462	[545, 251]	601	[211, 293]
46	[88, 89]	185	[213, 66]	324	[491, 492]	463	[93, 112]	602	[100, 325]
47	[90, 91]	186	[308, 309]	325	[493, 494]	464	[326, 248]	603	[556, 75]
48	[92, 62]	187	[214, 310]	326	[495, 496]	465	[525, 518]	604	[521, 295]
49	[93, 94]	188	[311, 84]	327	[405, 497]	466	[278, 588]	605	[213, 508]
50	[95, 96]	189	[216, 238]	328	[361, 498]	467	[585, 78]	606	[684, 258]
51	[97, 98]	190	[312, 223]	329	[499, 500]	468	[309, 306]	607	[584, 560]
52	[99, 100]	191	[313, 314]	330	[501, 502]	469	[311, 505]	608	[285, 223]
53	[19, 101]	192	[291, 315]	331	[503, 283]	470	[302, 255]	609	[241, 24]
54	[102, 103]	193	[16, 316]	332	[504, 328]	471	[30, 543]	610	[298, 567]
55	[104, 105]	194	[295, 229]	333	[26, 30]	472	[265, 27]	611	[539, 534]
56	[106, 107]	195	[279, 217]	334	[324, 220]	473	[86, 572]	612	[29, 249]
57	[108, 109]	196	[92, 258]	335	[83, 505]	474	[513, 266]	613	[504, 541]
58	[110, 111]	197	[256, 61]	336	[236, 274]	475	[260, 91]	614	[213, 318]
59	[112, 113]	198	[317, 318]	337	[285, 506]	476	[587, 505]	615	[88, 277]
60	[114, 115]	199	[319, 320]	338	[17, 507]	477	[238, 535]	616	[519, 321]
61	[116, 117]	200	[321, 322]	339	[508, 509]	478	[237, 274]	617	[681, 18]
62	[118, 119]	201	[323, 324]	340	[325, 247]	479	[533, 566]	618	[590, 515]
63	[120, 121]	202	[269, 263]	341	[510, 511]	480	[110, 574]	619	[107, 280]
64	[122, 123]	203	[325, 326]	342	[512, 232]	481	[26, 249]	620	[573, 524]

65	[124, 125]	204	[327, 328]	343	[513, 514]	482	[257, 28]	621	[113, 229]
66	[126, 127]	205	[329, 249]	344	[23, 515]	483	[588, 529]	622	[579, 581]
67	[128, 129]	206	[5, 330]	345	[99, 516]	484	[309, 538]	623	[571, 216]
68	[130, 131]	207	[331, 332]	346	[517, 518]	485	[304, 511]	624	[83, 539]
69	[45, 132]	208	[333, 334]	347	[210, 519]	486	[253, 73]	625	[238, 275]
70	[133, 134]	209	[335, 336]	348	[520, 521]	487	[522, 544]	626	[537, 685]
71	[135, 136]	210	[337, 338]	349	[522, 523]	488	[313, 18]	627	[512, 217]
72	[137, 138]	211	[339, 340]	350	[524, 519]	489	[505, 589]	628	[545, 506]
73	[122, 139]	212	[177, 341]	351	[76, 247]	490	[219, 578]	629	[271, 528]
74	[140, 141]	213	[342, 128]	352	[525, 526]	491	[590, 591]	630	[219, 275]
75	[36, 142]	214	[343, 344]	353	[527, 282]	492	[239, 329]	631	[91, 257]
76	[143, 144]	215	[345, 346]	354	[257, 528]	493	[265, 61]	632	[569, 588]
77	[145, 146]	216	[347, 348]	355	[329, 262]	494	[62, 69]	633	[287, 326]
78	[147, 148]	217	[349, 350]	356	[529, 233]	495	[592, 593]	634	[247, 542]
79	[149, 135]	218	[168, 351]	357	[530, 531]	496	[280, 509]	635	[552, 549]
80	[150, 151]	219	[352, 353]	358	[506, 532]	497	[576, 78]	636	[61, 240]
81	[152, 153]	220	[354, 355]	359	[533, 534]	498	[548, 577]	637	[107, 66]
82	[149, 154]	221	[131, 356]	360	[274, 535]	499	[212, 317]	638	[553, 685]
83	[155, 156]	222	[357, 150]	361	[283, 536]	500	[547, 516]	639	[236, 219]
84	[157, 158]	223	[338, 358]	362	[317, 66]	501	[271, 240]	640	[312, 506]
85	[159, 160]	224	[359, 360]	363	[537, 89]	502	[244, 513]	641	[23, 591]
86	[161, 162]	225	[361, 362]	364	[305, 538]	503	[594, 35]	642	[290, 5]
87	[163, 164]	226	[188, 363]	365	[268, 539]	504	[489, 448]	643	[329, 513]
88	[165, 166]	227	[364, 365]	366	[540, 101]	505	[388, 418]	644	[253, 683]
89	[12, 167]	228	[159, 366]	367	[327, 541]	506	[595, 596]	645	[299, 324]
90	[168, 169]	229	[367, 368]	368	[326, 542]	507	[127, 597]	646	[540, 570]
91	[170, 171]	230	[369, 370]	369	[71, 6]	508	[598, 599]	647	[113, 72]
92	[172, 173]	231	[146, 371]	370	[513, 543]	509	[42, 600]	648	[104, 72]
93	[174, 175]	232	[372, 373]	371	[261, 61]	510	[128, 460]	649	[684, 68]
94	[39, 176]	233	[374, 375]	372	[225, 544]	511	[339, 601]	650	[279, 86]
95	[124, 177]	234	[131, 376]	373	[210, 545]	512	[602, 374]	651	[579, 523]
96	[174, 178]	235	[377, 378]	374	[546, 212]	513	[603, 604]	652	[529, 87]
97	[179, 180]	236	[379, 380]	375	[325, 16]	514	[393, 605]	653	[236, 567]
98	[181, 182]	237	[381, 382]	376	[311, 533]	515	[148, 606]	654	[295, 72]
99	[183, 184]	238	[383, 384]	377	[547, 100]	516	[447, 607]	655	[323, 5]
100	[185, 33]	239	[385, 386]	378	[548, 549]	517	[127, 608]	656	[556, 96]
101	[186, 187]	240	[115, 387]	379	[297, 312]	518	[196, 609]	657	[588, 232]
102	[188, 189]	241	[356, 388]	380	[550, 294]	519	[610, 611]	658	[100, 287]
103	[190, 191]	242	[389, 390]	381	[524, 545]	520	[36, 612]	659	[302, 270]
104	[175, 192]	243	[348, 391]	382	[551, 94]	521	[160, 613]	660	[530, 234]
105	[176, 193]	244	[392, 393]	383	[552, 518]	522	[464, 614]	661	[282, 326]
106	[194, 126]	245	[394, 395]	384	[109, 76]	523	[460, 615]	662	[568, 270]
107	[195, 42]	246	[193, 396]	385	[551, 112]	524	[616, 617]	663	[249, 514]
108	[196, 197]	247	[397, 398]	386	[110, 75]	525	[441, 462]	664	[586, 242]

109	[198, 199]	248	[399, 400]	387	[208, 253]	526	[183, 618]	665	[312, 251]
110	[200, 194]	249	[50, 401]	388	[553, 554]	527	[619, 495]	666	[276, 554]
111	[201, 202]	250	[377, 402]	389	[555, 256]	528	[620, 621]	667	[588, 217]
112	[203, 204]	251	[403, 404]	390	[221, 68]	529	[622, 379]	668	[324, 330]
113	[205, 206]	252	[405, 406]	391	[556, 111]	530	[471, 491]	669	[30, 266]
114	[207, 208]	253	[332, 361]	392	[557, 213]	531	[623, 624]	670	[684, 62]
115	[209, 210]	254	[354, 407]	393	[558, 527]	532	[394, 625]	671	[537, 277]
116	[211, 20]	255	[408, 409]	394	[84, 559]	533	[626, 155]	672	[219, 288]
117	[212, 213]	256	[401, 54]	395	[77, 81]	534	[424, 627]	673	[516, 287]
118	[214, 80]	257	[410, 56]	396	[517, 526]	535	[385, 628]	674	[286, 106]
119	[215, 216]	258	[411, 412]	397	[319, 560]	536	[35, 629]	675	[273, 529]
120	[217, 218]	259	[413, 414]	398	[318, 561]	537	[630, 118]	676	[276, 685]
121	[216, 219]	260	[374, 116]	399	[222, 562]	538	[343, 631]	677	[307, 222]
122	[5, 220]	261	[59, 415]	400	[27, 563]	539	[46, 632]	678	[307, 68]
123	[221, 222]	262	[416, 417]	401	[94, 104]	540	[398, 633]	679	[284, 581]
124	[223, 224]	263	[418, 419]	402	[317, 280]	541	[634, 635]	680	[686, 620]
125	[225, 226]	264	[163, 53]	403	[564, 565]	542	[32, 636]	681	[32, 687]
126	[227, 228]	265	[420, 421]	404	[251, 322]	543	[57, 637]	682	[398, 688]
127	[229, 230]	266	[345, 422]	405	[539, 566]	544	[335, 638]	683	[192, 689]
128	[78, 231]	267	[57, 423]	406	[567, 288]	545	[639, 489]	684	[623, 120]
129	[232, 233]	268	[418, 424]	407	[568, 303]	546	[40, 342]	685	[136, 690]
130	[79, 234]	269	[425, 426]	408	[569, 279]	547	[123, 449]	686	[691, 113]
131	[235, 236]	270	[155, 427]	409	[77, 309]	548	[37, 640]	687	[290, 324]
132	[237, 238]	271	[428, 429]	410	[85, 570]	549	[123, 641]	688	[19, 570]
133	[239, 29]	272	[171, 430]	411	[289, 310]	550	[201, 333]	689	[262, 514]
134	[27, 240]	273	[384, 431]	412	[571, 298]	551	[642, 643]	690	[70, 5]
135	[241, 242]	274	[432, 433]	413	[232, 572]	552	[644, 616]	691	[692, 367]
136	[243, 244]	275	[348, 434]	414	[298, 274]	553	[477, 357]	692	[15, 326]
137	[245, 246]	276	[424, 435]	415	[573, 210]	554	[133, 645]		
138	[247, 248]	277	[436, 437]	416	[74, 574]	555	[439, 646]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	100	0	200	0	300	0	400	0	500	0	600	1
1	0	101	1	201	0	301	1	401	0	501	1	601	1
2	0	102	0	202	1	302	0	402	1	502	0	602	0
3	0	103	1	203	1	303	1	403	1	503	0	603	1
4	0	104	0	204	1	304	1	404	1	504	0	604	1
5	0	105	0	205	1	305	0	405	1	505	0	605	0
6	0	106	0	206	0	306	0	406	1	506	1	606	0
7	0	107	0	207	0	307	1	407	1	507	1	607	1
8	0	108	1	208	1	308	1	408	0	508	1	608	0
9	0	109	1	209	1	309	1	409	0	509	1	609	0
10	1	110	0	210	0	310	1	410	0	510	0	610	1

11	0	111	0	211	1	311	0	411	1	511	1	611	1
12	0	112	1	212	1	312	1	412	0	512	0	612	0
13	0	113	1	213	0	313	0	413	1	513	1	613	0
14	0	114	0	214	1	314	0	414	1	514	0	614	0
15	0	115	1	215	1	315	1	415	0	515	0	615	1
16	0	116	1	216	0	316	1	416	1	516	1	616	1
17	0	117	1	217	1	317	1	417	0	517	1	617	1
18	1	118	1	218	1	318	1	418	0	518	1	618	1
19	0	119	1	219	0	319	0	419	1	519	1	619	0
20	0	120	1	220	0	320	1	420	1	520	1	620	0
21	1	121	0	221	0	321	0	421	1	521	1	621	1
22	0	122	0	222	0	322	0	422	1	522	1	622	0
23	0	123	0	223	0	323	0	423	0	523	1	623	0
24	1	124	0	224	0	324	1	424	1	524	1	624	1
25	0	125	1	225	1	325	1	425	1	525	0	625	0
26	1	126	0	226	0	326	1	426	1	526	1	626	0
27	0	127	1	227	0	327	1	427	1	527	0	627	0
28	0	128	0	228	0	328	1	428	1	528	0	628	0
29	0	129	1	229	1	329	1	429	0	529	0	629	1
30	0	130	0	230	0	330	1	430	0	530	0	630	0
31	0	131	0	231	1	331	0	431	1	531	0	631	0
32	1	132	1	232	1	332	0	432	1	532	1	632	0
33	0	133	1	233	0	333	1	433	1	533	0	633	0
34	0	134	0	234	0	334	1	434	1	534	1	634	0
35	0	135	0	235	0	335	1	435	0	535	1	635	0
36	1	136	1	236	0	336	0	436	0	536	0	636	1
37	1	137	1	237	1	337	0	437	0	537	0	637	0
38	1	138	0	238	0	338	0	438	1	538	1	638	0
39	0	139	0	239	1	339	1	439	1	539	1	639	0
40	0	140	1	240	1	340	1	440	0	540	1	640	1
41	0	141	0	241	0	341	0	441	0	541	0	641	0
42	1	142	0	242	1	342	0	442	0	542	1	642	1
43	1	143	0	243	1	343	1	443	1	543	1	643	1
44	0	144	1	244	0	344	0	444	0	544	1	644	0
45	0	145	0	245	1	345	1	445	1	545	0	645	0
46	1	146	1	246	0	346	1	446	1	546	0	646	1
47	1	147	0	247	0	347	0	447	1	547	0	647	1
48	1	148	0	248	0	348	1	448	0	548	1	648	0
49	0	149	0	249	0	349	1	449	0	549	0	649	0
50	0	150	1	250	0	350	1	450	0	550	0	650	0
51	1	151	0	251	1	351	0	451	1	551	1	651	0
52	1	152	1	252	1	352	0	452	1	552	0	652	0
53	0	153	1	253	0	353	0	453	1	553	0	653	0
54	0	154	0	254	0	354	0	454	0	554	1	654	0

55	0	155	1	255	0	355	1	455	1	555	1	655	0
56	0	156	1	256	0	356	0	456	0	556	1	656	1
57	1	157	1	257	0	357	0	457	1	557	0	657	1
58	0	158	1	258	1	358	1	458	0	558	0	658	0
59	1	159	0	259	1	359	0	459	0	559	0	659	0
60	0	160	1	260	0	360	1	460	1	560	0	660	0
61	1	161	0	261	1	361	1	461	1	561	1	661	1
62	1	162	1	262	1	362	1	462	0	562	0	662	1
63	1	163	0	263	0	363	0	463	0	563	1	663	0
64	0	164	0	264	0	364	0	464	1	564	1	664	1
65	0	165	1	265	1	365	0	465	0	565	0	665	1
66	0	166	1	266	1	366	1	466	1	566	0	666	1
67	0	167	1	267	1	367	1	467	0	567	1	667	1
68	0	168	1	268	0	368	1	468	1	568	1	668	1
69	0	169	0	269	1	369	0	469	0	569	0	669	0
70	1	170	0	270	1	370	1	470	0	570	1	670	0
71	0	171	1	271	1	371	1	471	0	571	0	671	0
72	1	172	1	272	1	372	1	472	1	572	1	672	0
73	0	173	1	273	1	373	0	473	0	573	0	673	1
74	1	174	0	274	1	374	0	474	1	574	1	674	0
75	1	175	0	275	1	375	1	475	0	575	0	675	1
76	0	176	0	276	1	376	0	476	1	576	1	676	1
77	0	177	1	277	0	377	0	477	0	577	0	677	1
78	0	178	0	278	1	378	1	478	1	578	0	678	1
79	0	179	1	279	0	379	0	479	0	579	0	679	0
80	1	180	1	280	0	380	0	480	0	580	0	680	0
81	1	181	1	281	0	381	1	481	1	581	0	681	1
82	0	182	0	282	1	382	1	482	0	582	1	682	1
83	1	183	1	283	1	383	0	483	1	583	1	683	1
84	1	184	1	284	0	384	1	484	1	584	1	684	0
85	0	185	0	285	0	385	1	485	1	585	0	685	1
86	0	186	1	286	0	386	0	486	0	586	1	686	0
87	0	187	1	287	0	387	1	487	1	587	1	687	1
88	1	188	0	288	0	388	0	488	0	588	1	688	0
89	0	189	0	289	1	389	1	489	0	589	1	689	1
90	1	190	1	290	1	390	0	490	0	590	1	690	1
91	0	191	0	291	1	391	1	491	1	591	0	691	0
92	1	192	1	292	1	392	0	492	1	592	1	692	0
93	0	193	0	293	0	393	0	493	1	593	1		
94	0	194	0	294	0	394	1	494	1	594	0		
95	0	195	0	295	0	395	0	495	1	595	1		
96	0	196	1	296	1	396	1	496	0	596	0		
97	1	197	0	297	0	397	0	497	1	597	0		
98	1	198	1	298	1	398	1	498	1	598	1		

99	1	199	0	299	0	399	0	499	1	599	1		

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