

PROOF OF THE CONTINUED FRACTION CONJECTURE FOR

$$(a, b) = (z^5 + z^4 + z + 1, z^2 + z + 1)$$

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ABSTRACT. The continued fraction conjecture claims that the continued fraction defined by the (a, b) -Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a, b) in $\mathbb{F}_2[x] \setminus \mathbb{F}_2$. In this paper we prove the conjecture for $(a, b) = (z^5 + z^4 + z + 1, z^2 + z + 1)$.

1. INTRODUCTION

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \dots)$ of elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$, we define the infinite continued fraction

$$(1.1) \quad \text{CF}(\mathbf{a}(z)) = \frac{1}{a_0(z) + \frac{1}{a_1(z) + \frac{1}{a_2(z) + \frac{1}{\ddots}}}}$$

as the limit of the finite continued fractions

$$(1.2) \quad \text{CF}_n(\mathbf{a}(z)) = \frac{1}{a_0(z) + \frac{1}{a_1(z) + \frac{1}{\ddots + \frac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)).$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$(1.3) \quad c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \dots$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a, b) -Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \dots) = (a, b, b, a, b, a, a, b, \dots)$$

that we shall denote

$$(1.4) \quad \text{CF}(a, b) := \text{CF}(\mathbf{t}).$$

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Theorem 1.1. *When $(a, b) = (z^5 + z^4 + z + 1, z^2 + z + 1)$, the two power series $\text{CF}(a, b)$ and $\text{CF}(b, a)$ are algebraic over $\mathbb{F}_2(z)$, with minimal polynomials of the form*

$$p_4(z)y^4 + p_3(z)y^3 + p_2(z)y^2 + p_1(z)y + p_0(z) = 0$$

where, for $\text{CF}(a, b)$,

$$\begin{aligned} p_0(z) &= z^{21} + z^{18} + z^{14} + z^{11} + z^9 + z^8 + z^4 + z^3 + 1, \\ p_1(z) &= z^{26} + z^{25} + z^{22} + z^{20} + z^{16} + z^{15} + z^{14} + z^{13} + z^{12} + z^7 + z^4 + z^2, \\ p_2(z) &= z^{28} + z^{24} + z^{22} + z^{16} + z^{14} + z^8 + z^4 + 1, \\ p_3(z) &= z^{26} + z^{25} + z^{22} + z^{20} + z^{16} + z^{15} + z^{14} + z^{13} + z^{12} + z^7 + z^4 + z^2, \\ p_4(z) &= z^{24} + z^{22} + z^{21} + z^{20} + z^{18} + z^{16} + z^{12} + z^{11} + z^9 + z^8 + z^3 + 1, \end{aligned}$$

and for $\text{CF}(b, a)$,

$$\begin{aligned} p_0(z) &= z^{21} + z^{20} + z^{18} + z^{16} + z^{14} + z^{11} + z^9 + z^4 + z^3 + 1, \\ p_1(z) &= z^{26} + z^{25} + z^{22} + z^{20} + z^{16} + z^{15} + z^{14} + z^{13} + z^{12} + z^7 + z^4 + z^2, \\ p_2(z) &= z^{28} + z^{24} + z^{22} + z^{16} + z^{14} + z^8 + z^4 + 1, \\ p_3(z) &= z^{26} + z^{25} + z^{22} + z^{20} + z^{16} + z^{15} + z^{14} + z^{13} + z^{12} + z^7 + z^4 + z^2, \\ p_4(z) &= z^{24} + z^{22} + z^{21} + z^{18} + z^{12} + z^{11} + z^9 + z^3 + 1. \end{aligned}$$

2. PROOF

Define the sequences of polynomials $(P_n(z))_{n \geq 0}$ and $(Q_n(z))_{n \geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z) \\ P_{n-1}(z) & Q_{n-1}(z) \end{pmatrix} := \begin{pmatrix} t_n(z) & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t_{n-1}(z) & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} t_0(z) & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$(2.1) \quad \text{CF}_n(a, b) = \frac{P_n(z)}{Q_n(z)} \in \mathbb{F}_2((1/z)).$$

As formula (2.1) is not efficient for calculating $\text{CF}_n(a, b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned} M_n(x) &= x^{\deg(t_{2^n-1})} \begin{pmatrix} t_{2^n-1}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times x^{\deg(t_{2^n-2})} \begin{pmatrix} t_{2^n-2}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times \\ &\quad \cdots \times x^{\deg(t_0)} \begin{pmatrix} t_0(1/x) & 1 \\ 1 & 0 \end{pmatrix}, \\ W_n(x) &= x^{\deg(\bar{t}_{2^n-1})} \begin{pmatrix} \bar{t}_{2^n-1}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times x^{\deg(\bar{t}_{2^n-2})} \begin{pmatrix} \bar{t}_{2^n-2}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times \\ &\quad \cdots \times x^{\deg(\bar{t}_0)} \begin{pmatrix} \bar{t}_0(1/x) & 1 \\ 1 & 0 \end{pmatrix}, \end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b, a) -Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n \geq 0$

$$\begin{aligned} M_{n+1}(x) &= W_n(x) \cdot M_n(x), \\ W_{n+1}(x) &= M_n(x) \cdot W_n(x). \end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\text{CF}_n(a, b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$\begin{aligned} M_n(x)_{0,1} &= x^{d_n} \tilde{P}_{2^n-1}(x), \\ M_n(x)_{0,0} &= x^{d_n} \tilde{Q}_{2^n-1}(x), \end{aligned}$$

for some positive integer d_n . Hence,

$$(2.2) \quad \text{CF}_{2^{2^n}-1}(a, b) = \frac{\tilde{P}_{2^{2^n}-1}(x)}{\tilde{Q}_{2^{2^n}-1}(x)} = \frac{M_{2^n}(x)_{0,1}}{M_{2^n}(x)_{0,0}}.$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\text{CF}(a, b)$ will be established if it is shown that both $M_{2^n}(x)_{0,1}$ and $M_{2^n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \leq i, j \leq 1$ the four sequences $(M_{2^n}(x)_{i,j})_n$, $(M_{2^{n+1}}(x)_{i,j})_n$, $(W_{2^n}(x)_{i,j})_n$, and $(W_{2^{n+1}}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \leq i, j \leq 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T, i, j)$ under certain initial conditions; the polynomials $\phi(T, i, j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2^n}(x))_n$, $(M_{2^{n+1}}(x))_n$, $(W_{2^n}(x))_n$, and $(W_{2^{n+1}}(x))_n$.

Let us explain how the polynomials $\phi(T, i, j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \leq i, j \leq 1$, the coefficients of the polynomial $\phi(M^e, i, j)$ (resp. $\phi(M^o, i, j)$, $\phi(W^e, i, j)$, and $\phi(W^o, i, j)$) are the Padé-Hermite approximants of type

$$(175, 175, 175, 175, 175)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e, i, j)$ (resp. $\phi(M^o, i, j)$, $\phi(W^e, i, j)$, and $\phi(W^o, i, j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x, y) \in \mathbb{F}_2[x, y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \geq 0$. If for some $n \geq 0$ and $a_0, a_1, \dots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x, y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \geq 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o . The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$(2.3) \quad M^e = W^o \cdot M^o,$$

$$(2.4) \quad M^o = W^e \cdot M^e,$$

$$(2.5) \quad W^e = M^o \cdot W^o,$$

$$(2.6) \quad W^o = M^e \cdot W^e.$$

Proof. We give the proof of the identity

$$M_{0,0}^e = W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o,$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W_{0,0}^o M_{0,0}^o$ and $W_{0,1}^o M_{1,0}^o$. We know that

$$P(x, y) = \text{Res}_z (\phi(W^o, 0, 0)(x, z), z^{12} \cdot \phi(M^o, 0, 0)(x, y/z))$$

is an annihilating polynomial of $W_{0,0}^o M_{0,0}^o$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W_{0,0}^o M_{0,0}^o$, that will be called $\phi_0(x, y)$. To prove that $\phi_0(x, y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x, y)$ of multiplicity m and that $Q(x, y) := P(x, y)/\phi_0(x, y)^m$ is not an annihilating polynomial of $W_{0,0}^o M_{0,0}^o$. We verify the first point directly. For the second point, we truncate $W_{0,0}^o M_{0,0}^o$ to order 630 and substitute it for y in $Q(x, y)$. We get a series of valuation less than 630, which proves that $Q(x, y)$ is not an annihilating polynomial of $W_{0,0}^o M_{0,0}^o$. We find the minimal polynomial $\phi_1(x, y)$ of $W_{0,1}^o M_{1,0}^o$ in a similar way.

Now we prove that $\phi(M^e, 0, 0)$ is the minimal polynomial of $W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o$. We know that

$$S(x, y) = \text{Res}_z (\phi_0(x, z), \phi_1(x, y + z))$$

is an annihilating of $W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o$. We verify that $\phi(M^e, 0, 0)$ is an irreducible factor of $S(x, y)$ of multiplicity μ , and that $Q(x, y) := S(x, y)/\phi(M^e, 0, 0)^\mu$ is not an annihilating polynomial of $W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o$. To see the last point, we truncate $W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o$ to order 770 and substitute it for y in $Q(x, y)$. We get a series of valuation less than 770, and therefore $Q(x, y)$ is not an annihilating polynomial of $W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o$.

Finally, the first 16 terms of $M_{0,0}^e$ and $W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o$ coincide. As these first terms determine a unique solution of $\phi(M^e, 0, 0)$, we know that the two series are one and the same. $\square$

Define

$$R^e = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad R^o = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned} M^e[:14u] &= M^e[:7u] + x^{3u} M^e[:4u] + x^{6u} M^e[:u] + x^{3u} M^e[:7u] + x^{6u} M^e[:4u] \\ &\quad + x^{9u} M^e[:u] + x^{10u} M^e[:4u] + (x^{7u} + x^{10u}) R^e, \\ W^e[:14u] &= W^e[:7u] + x^{3u} W^e[:4u] + x^{6u} W^e[:u] + x^{3u} W^e[:7u] + x^{6u} W^e[:4u] \\ &\quad + x^{9u} W^e[:u] + x^{10u} W^e[:4u] + (x^{7u} + x^{10u}) R^e. \end{aligned}$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$M^o[:14u] = M^o[:7u] + x^{3u} M^o[:4u] + x^{6u} M^o[:u] + x^{3u} M^o[:7u] + x^{6u} M^o[:4u]$$

$$\begin{aligned}
& + x^{9u}M^o[:u] + x^{10u}M^o[:4u] + (x^{7u} + x^{10u})R^o, \\
W^o[:14u] &= W^o[:7u] + x^{3u}W^o[:4u] + x^{6u}W^o[:u] + x^{3u}W^o[:7u] + x^{6u}W^o[:4u] \\
& + x^{9u}W^o[:u] + x^{10u}W^o[:4u] + (x^{7u} + x^{10u})R^o.
\end{aligned}$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k 's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M_{0,0}^o$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$\begin{aligned}
T[:14u] &= T[:7u] + x^{3u}T[:4u] + x^{6u}T[:u] + x^{3u}T[:7u] + x^{6u}T[:4u] + x^{9u}T[:u] \\
& + x^{10u}T[:4u] + (x^{7u} + x^{10u}).
\end{aligned}$$

The proof of the other 15 cases are similar. We break down the above identity into 7 parts:

$$\begin{aligned}
x^7u &= x^{6u}T[u:2u] + x^{3u}T[4u:5u] + T[7u:8u], \\
0 &= x^{6u}T[2u:3u] + x^{3u}T[5u:6u] + T[8u:9u], \\
0 &= x^{9u}T[:u] + x^{6u}T[3u:4u] + x^{3u}T[6u:7u] + T[9u:10u], \\
x^{10}u &= x^{10u}T[:u] + T[10u:11u], \\
0 &= x^{10u}T[u:2u] + T[11u:12u], \\
0 &= x^{10u}T[2u:3u] + T[12u:13u], \\
0 &= x^{10u}T[3u:4u] + T[13u:14u],
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
(2.7) \quad 0 &= T[[1w]_2] + T[[100w]_2] + T[[111w]_2], \\
(2.8) \quad 0 &= T[[10w]_2] + T[[101w]_2] + T[[1000w]_2], \\
(2.9) \quad 0 &= T[[w]_2] + T[[11w]_2] + T[[110w]_2] + T[[1001w]_2], \\
(2.10) \quad 0 &= T[[w]_2] + T[[1010w]_2], \\
(2.11) \quad 0 &= T[[1w]_2] + T[[1011w]_2], \\
(2.12) \quad 0 &= T[[10w]_2] + T[[1100w]_2], \\
(2.13) \quad 0 &= T[[11w]_2] + T[[1101w]_2],
\end{aligned}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$\begin{aligned}
(2.14) \quad 1 &= T[[1w]_2] + T[[100w]_2] + T[[111w]_2], \\
(2.15) \quad 0 &= T[[10w]_2] + T[[101w]_2] + T[[1000w]_2], \\
(2.16) \quad 0 &= T[[w]_2] + T[[11w]_2] + T[[110w]_2] + T[[1001w]_2], \\
(2.17) \quad 1 &= T[[w]_2] + T[[1010w]_2], \\
(2.18) \quad 0 &= T[[1w]_2] + T[[1011w]_2], \\
(2.19) \quad 0 &= T[[10w]_2] + T[[1100w]_2], \\
(2.20) \quad 0 &= T[[11w]_2] + T[[1101w]_2],
\end{aligned}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 1514 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached

after reading w from right to left starting from the state s , and τ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.20) can be written as

$$(2.21) \quad 0 = \tau(A(s, 1)) + \tau(A(s, 100)) + \tau(A(s, 111)),$$

$$(2.22) \quad 0 = \tau(A(s, 10)) + \tau(A(s, 101)) + \tau(A(s, 1000)),$$

$$(2.23) \quad 0 = \tau(A(s, \epsilon)) + \tau(A(s, 11)) + \tau(A(s, 110)) + \tau(A(s, 1001)),$$

$$(2.24) \quad 0 = \tau(A(s, \epsilon)) + \tau(A(s, 1010)),$$

$$(2.25) \quad 0 = \tau(A(s, 1)) + \tau(A(s, 1011)),$$

$$(2.26) \quad 0 = \tau(A(s, 10)) + \tau(A(s, 1100)),$$

$$(2.27) \quad 0 = \tau(A(s, 11)) + \tau(A(s, 1101)),$$

for all $s \in E_{2k}$, and

$$(2.28) \quad 1 = \tau(A(s, 1)) + \tau(A(s, 100)) + \tau(A(s, 111)),$$

$$(2.29) \quad 0 = \tau(A(s, 10)) + \tau(A(s, 101)) + \tau(A(s, 1000)),$$

$$(2.30) \quad 0 = \tau(A(s, \epsilon)) + \tau(A(s, 11)) + \tau(A(s, 110)) + \tau(A(s, 1001)),$$

$$(2.31) \quad 1 = \tau(A(s, \epsilon)) + \tau(A(s, 1010)),$$

$$(2.32) \quad 0 = \tau(A(s, 1)) + \tau(A(s, 1011)),$$

$$(2.33) \quad 0 = \tau(A(s, 10)) + \tau(A(s, 1100)),$$

$$(2.34) \quad 0 = \tau(A(s, 11)) + \tau(A(s, 1101)),$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{28}), E_{28}) = (A(i, 0^{16}), E_{16})$, so that we only have to verify that identities (2.21) through (2.34) hold for $2 \leq k \leq 14$, which turns out to be true. $\square$

In the following lemma, we express M_{2k} , M_{2k+1} , W_{2k} , and W_{2k+1} in terms of M^e , M^o , W^e , and W^o .

Lemma 2.3. *For all integer $k \geq 2$, and $u = 2^{2k-1}$,*

$$M_{2k} = M^e[:7u] + x^{3u}M^e[:4u] + x^{6u}M^e[:u] + x^{7u}R^e,$$

$$W_{2k} = W^e[:7u] + x^{3u}W^e[:4u] + x^{6u}W^e[:u] + x^{7u}R^e.$$

For all integer $k \geq 2$, and $u = 2^{2k}$,

$$M_{2k+1} = M^o[:7u] + x^{3u}M^o[:4u] + x^{6u}M^o[:u] + x^{7u}R^o,$$

$$W_{2k+1} = W^o[:7u] + x^{3u}W^o[:4u] + x^{6u}W^o[:u] + x^{7u}R^o.$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k} , W_{2k} , M_{2k+1} , and W_{2k+1} . For $n = 2$, the identities can be verified directly. For $n \geq 2$, we claim that

$$\begin{aligned} M_{2k} \wedge W_{2k} &\Rightarrow M_{2k+1} \wedge W_{2k+1}, \\ M_{2k+1} \wedge W_{2k+1} &\Rightarrow M_{2k+2} \wedge W_{2k+2}. \end{aligned}$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$(2.35) \quad \begin{aligned} W_{2k}M_{2k} = & (W^e[:7v] + x^{3v}W^e[:4v] + x^{6v}W^e[:v] + x^{7u}R^e) \times (M^e[:7v] \\ & + x^{3v}M^e[:4v] + x^{6v}M^e[:v] + x^{7u}R^e); \end{aligned}$$

Call this expression *lhs*. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{14v}R^o$. Therefore we only need to prove that their difference is $O(x^{14v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{14v} , to

$$W^e[:14v]M^e[:14v] + x^{6v}W^e[:8v]M^e[:8v] + x^{12v}W^e[:2v]M^e[:2v].$$

For all $n \leq 14$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call *rhs*. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v] / X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v] / X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions *lhs* (2.35) and *rhs* as polynomials in $\mathbb{F}_2[a_1, \dots, a_{14}, b_1, \dots, b_{14}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e , because in the expressions concerned, the products of W^e -terms and M^e -terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{14})$, which completes the proof. $\square$

Proof of Theorem 1.1. We prove the theorem for $\text{CF}(a, b)$; for $\text{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \leq j, k \leq 1$,

$$\begin{aligned} \lim_{n \rightarrow \infty} M_{2n,j,k} &= M_{j,k}^e, \\ \lim_{n \rightarrow \infty} M_{2n+1,j,k} &= M_{j,k}^o, \\ \lim_{n \rightarrow \infty} W_{2n,j,k} &= W_{j,k}^e, \\ \lim_{n \rightarrow \infty} W_{2n+1,j,k} &= W_{j,k}^o. \end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\text{CF}(a, b) = \frac{M_{0,1}^e(x)}{M_{0,0}^e(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M_{0,1}^e$ and $M_{0,0}^e$. Therefore

$$P(x, y) = \text{Res}_t (\phi(M^e, 0, 1)(x, t), y^{12}\phi(M^e, 0, 1)(x, t/y))$$

is an annihilating polynomial of $f(x) = M_{0,1}^e/M_{0,0}^e$.

Define

$$Q(x, y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$q_0(x) = x^{28} + x^{25} + x^{24} + x^{20} + x^{19} + x^{17} + x^{14} + x^{10} + x^7,$$

$$\begin{aligned}
q_1(x) &= x^{26} + x^{24} + x^{21} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^8 + x^6 + x^3 + x^2, \\
q_2(x) &= x^{28} + x^{24} + x^{20} + x^{14} + x^{12} + x^6 + x^4 + 1, \\
q_3(x) &= x^{26} + x^{24} + x^{21} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^8 + x^6 + x^3 + x^2, \\
q_4(x) &= x^{28} + x^{25} + x^{20} + x^{19} + x^{17} + x^{16} + x^{12} + x^{10} + x^8 + x^7 + x^6 + x^4.
\end{aligned}$$

The polynomial $Q(x, y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x, y)$ of multiplicity m and $R(x, y) := P(x, y)/Q(x, y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 216, and substitute it for y in $R(z, y)$, we get a series with valuation smaller than 216, which proves that $R(z, y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z, y)$ is the minimal polynomial of $\text{CF}(a, b) = f(1/z)$. $\square$

3. ANNEXE

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{138} + x^{136} + x^{132} + x^{130} + x^{126} + x^{124} + x^{122} + x^{120} + x^{116} + x^{112} \\
&\quad + x^{110} + x^{100} + x^{98} + x^{96} + x^{90} + x^{88} + x^{84} + x^{78} + x^{72} + x^{70} + x^{66} \\
&\quad + x^{60} + x^{58} + x^{54} + x^{42} + x^{40} + x^{36} + x^{32} + x^{28} + x^{26} + x^{24}, \\
p_3(x) &= x^{134} + x^{132} + x^{130} + x^{126} + x^{122} + x^{116} + x^{114} + x^{112} + x^{110} + x^{106} \\
&\quad + x^{104} + x^{102} + x^{96} + x^{94} + x^{90} + x^{86} + x^{74} + x^{62} + x^{56} + x^{46} + x^{44} \\
&\quad + x^{40} + x^{38} + x^{36} + x^{34} + x^{32} + x^{28} + x^{26} + x^{24} + x^{22} + x^{20} + x^{16}, \\
p_6(x) &= x^{132} + x^{130} + x^{128} + x^{120} + x^{116} + x^{98} + x^{92} + x^{84} + x^{82} + x^{80} + x^{78} \\
&\quad + x^{76} + x^{74} + x^{70} + x^{68} + x^{66} + x^{64} + x^{62} + x^{50} + x^{46} + x^{40} + x^{38} \\
&\quad + x^{36} + x^{32} + x^{30} + x^{28} + x^{26} + x^{22} + x^{18} + x^{16} + x^{14} + x^{12} + x^{10} \\
&\quad + x^4 + x^2 + 1, \\
p_9(x) &= x^{134} + x^{132} + x^{124} + x^{122} + x^{118} + x^{116} + x^{112} + x^{110} + x^{96} + x^{94} \\
&\quad + x^{92} + x^{86} + x^{78} + x^{70} + x^{68} + x^{66} + x^{64} + x^{62} + x^{56} + x^{50} + x^{48} \\
&\quad + x^{46} + x^{42} + x^{36} + x^{34} + x^{32} + x^{30} + x^{20} + x^{10} + x^4, \\
p_{12}(x) &= x^{138} + x^{136} + x^{114} + x^{112} + x^{106} + x^{104} + x^{90} + x^{88} + x^{82} + x^{80} \\
&\quad + x^{66} + x^{64} + x^{58} + x^{56} + x^{34} + x^{32} + x^{26} + x^{24} + x^2 + 1,
\end{aligned}$$

and the initial terms are $[1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0]$.

For $\phi(M^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{147} + x^{144} + x^{141} + x^{140} + x^{138} + x^{136} + x^{132} + x^{130} + x^{128} + x^{126} \\
&\quad + x^{124} + x^{121} + x^{120} + x^{115} + x^{113} + x^{112} + x^{109} + x^{108} + x^{107}
\end{aligned}$$

$$\begin{aligned}
& + x^{105} + x^{103} + x^{101} + x^{100} + x^{98} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} \\
& + x^{90} + x^{89} + x^{88} + x^{87} + x^{83} + x^{82} + x^{81} + x^{80} + x^{78} + x^{72} + x^{69} \\
& + x^{68} + x^{65} + x^{64} + x^{59} + x^{57} + x^{56} + x^{54} + x^{53} + x^{52} + x^{46} + x^{44} \\
& + x^{43} + x^{40} + x^{37} + x^{36} + x^{35} + x^{33}, \\
p_3(x) = & x^{129} + x^{127} + x^{126} + x^{125} + x^{123} + x^{122} + x^{121} + x^{119} + x^{118} + x^{115} \\
& + x^{112} + x^{111} + x^{110} + x^{107} + x^{104} + x^{103} + x^{102} + x^{101} + x^{99} + x^{98} \\
& + x^{97} + x^{95} + x^{94} + x^{93} + x^{91} + x^{90} + x^{85} + x^{82} + x^{80} + x^{79} + x^{78} \\
& + x^{77} + x^{75} + x^{74} + x^{73} + x^{71} + x^{70} + x^{69} + x^{67} + x^{66} + x^{65} + x^{63} \\
& + x^{62} + x^{59} + x^{56} + x^{55} + x^{54} + x^{51} + x^{48} + x^{47} + x^{46} + x^{45} + x^{43} \\
& + x^{42} + x^{41} + x^{39} + x^{38} + x^{37} + x^{35} + x^{34} + x^{29} + x^{26} + x^{24} + x^{23} \\
& + x^{22} + x^{21} + x^{19} + x^{18}, \\
p_6(x) = & x^{135} + x^{132} + x^{125} + x^{123} + x^{122} + x^{121} + x^{120} + x^{118} + x^{117} + x^{114} \\
& + x^{113} + x^{111} + x^{110} + x^{108} + x^{105} + x^{103} + x^{102} + x^{101} + x^{100} \\
& + x^{98} + x^{97} + x^{95} + x^{94} + x^{93} + x^{92} + x^{90} + x^{83} + x^{81} + x^{80} + x^{79} \\
& + x^{78} + x^{77} + x^{76} + x^{74} + x^{67} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} \\
& + x^{59} + x^{58} + x^{56} + x^{47} + x^{44} + x^{43} + x^{40} + x^{39} + x^{37} + x^{36} + x^{35} \\
& + x^{34} + x^{32} + x^{31} + x^{29} + x^{28} + x^{26} + x^{25} + x^{22} + x^{17} + x^{15} + x^{14} \\
& + x^{12}, \\
p_9(x) = & x^{141} + x^{139} + x^{138} + x^{131} + x^{128} + x^{127} + x^{126} + x^{117} + x^{114} + x^{113} \\
& + x^{112} + x^{109} + x^{107} + x^{106} + x^{105} + x^{102} + x^{101} + x^{100} + x^{99} + x^{97} \\
& + x^{96} + x^{95} + x^{92} + x^{90} + x^{89} + x^{88} + x^{85} + x^{82} + x^{80} + x^{78} + x^{77} \\
& + x^{76} + x^{75} + x^{72} + x^{71} + x^{70} + x^{65} + x^{62} + x^{61} + x^{60} + x^{53} + x^{50} \\
& + x^{48} + x^{47} + x^{46} + x^{43} + x^{41} + x^{40} + x^{35} + x^{33} + x^{32} + x^{31} + x^{29} \\
& + x^{28} + x^{25} + x^{22} + x^{20} + x^{19} + x^{18} + x^{13} + x^{10} + x^8 + x^7 + x^6, \\
p_{12}(x) = & x^{147} + x^{144} + x^{143} + x^{140} + x^{135} + x^{132} + x^{127} + x^{124} + x^{119} + x^{116} \\
& + x^{111} + x^{108} + x^{103} + x^{100} + x^{99} + x^{96} + x^{83} + x^{80} + x^{79} + x^{76} \\
& + x^{71} + x^{68} + x^{63} + x^{60} + x^{55} + x^{52} + x^{47} + x^{44} + x^{39} + x^{36} + x^{31} \\
& + x^{28} + x^{23} + x^{20} + x^{15} + x^{12} + x^7 + x^4 + x^3 + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 1, 0, 0, 1]$.

For $\phi(M^e, 1, 0)$:

$$\begin{aligned}
p_0(x) = & x^{147} + x^{144} + x^{141} + x^{140} + x^{138} + x^{136} + x^{132} + x^{130} + x^{128} + x^{126} \\
& + x^{124} + x^{121} + x^{120} + x^{115} + x^{113} + x^{112} + x^{109} + x^{108} + x^{107} \\
& + x^{105} + x^{103} + x^{101} + x^{100} + x^{98} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} \\
& + x^{90} + x^{89} + x^{88} + x^{87} + x^{83} + x^{82} + x^{81} + x^{80} + x^{78} + x^{72} + x^{69} \\
& + x^{68} + x^{65} + x^{64} + x^{59} + x^{57} + x^{56} + x^{54} + x^{53} + x^{52} + x^{46} + x^{44}
\end{aligned}$$

$$\begin{aligned}
& + x^{43} + x^{40} + x^{37} + x^{36} + x^{35} + x^{33}, \\
p_3(x) = & x^{129} + x^{127} + x^{126} + x^{125} + x^{123} + x^{122} + x^{121} + x^{119} + x^{118} + x^{115} \\
& + x^{112} + x^{111} + x^{110} + x^{107} + x^{104} + x^{103} + x^{102} + x^{101} + x^{99} + x^{98} \\
& + x^{97} + x^{95} + x^{94} + x^{93} + x^{91} + x^{90} + x^{85} + x^{82} + x^{80} + x^{79} + x^{78} \\
& + x^{77} + x^{75} + x^{74} + x^{73} + x^{71} + x^{70} + x^{69} + x^{67} + x^{66} + x^{65} + x^{63} \\
& + x^{62} + x^{59} + x^{56} + x^{55} + x^{54} + x^{51} + x^{48} + x^{47} + x^{46} + x^{45} + x^{43} \\
& + x^{42} + x^{41} + x^{39} + x^{38} + x^{37} + x^{35} + x^{34} + x^{29} + x^{26} + x^{24} + x^{23} \\
& + x^{22} + x^{21} + x^{19} + x^{18}, \\
p_6(x) = & x^{135} + x^{132} + x^{125} + x^{123} + x^{122} + x^{121} + x^{120} + x^{118} + x^{117} + x^{114} \\
& + x^{113} + x^{111} + x^{110} + x^{108} + x^{105} + x^{103} + x^{102} + x^{101} + x^{100} \\
& + x^{98} + x^{97} + x^{95} + x^{94} + x^{93} + x^{92} + x^{90} + x^{83} + x^{81} + x^{80} + x^{79} \\
& + x^{78} + x^{77} + x^{76} + x^{74} + x^{67} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} \\
& + x^{59} + x^{58} + x^{56} + x^{47} + x^{44} + x^{43} + x^{40} + x^{39} + x^{37} + x^{36} + x^{35} \\
& + x^{34} + x^{32} + x^{31} + x^{29} + x^{28} + x^{26} + x^{25} + x^{22} + x^{17} + x^{15} + x^{14} \\
& + x^{12}, \\
p_9(x) = & x^{141} + x^{139} + x^{138} + x^{131} + x^{128} + x^{127} + x^{126} + x^{117} + x^{114} + x^{113} \\
& + x^{112} + x^{109} + x^{107} + x^{106} + x^{105} + x^{102} + x^{101} + x^{100} + x^{99} + x^{97} \\
& + x^{96} + x^{95} + x^{92} + x^{90} + x^{89} + x^{88} + x^{85} + x^{82} + x^{80} + x^{78} + x^{77} \\
& + x^{76} + x^{75} + x^{72} + x^{71} + x^{70} + x^{65} + x^{62} + x^{61} + x^{60} + x^{53} + x^{50} \\
& + x^{48} + x^{47} + x^{46} + x^{43} + x^{41} + x^{40} + x^{35} + x^{33} + x^{32} + x^{31} + x^{29} \\
& + x^{28} + x^{25} + x^{22} + x^{20} + x^{19} + x^{18} + x^{13} + x^{10} + x^8 + x^7 + x^6, \\
p_{12}(x) = & x^{147} + x^{144} + x^{143} + x^{140} + x^{135} + x^{132} + x^{127} + x^{124} + x^{119} + x^{116} \\
& + x^{111} + x^{108} + x^{103} + x^{100} + x^{99} + x^{96} + x^{83} + x^{80} + x^{79} + x^{76} \\
& + x^{71} + x^{68} + x^{63} + x^{60} + x^{55} + x^{52} + x^{47} + x^{44} + x^{39} + x^{36} + x^{31} \\
& + x^{28} + x^{23} + x^{20} + x^{15} + x^{12} + x^7 + x^4 + x^3 + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 1, 0, 0, 1]$.

For $\phi(M^e, 1, 1)$:

$$\begin{aligned}
p_0(x) = & x^{156} + x^{154} + x^{152} + x^{150} + x^{148} + x^{146} + x^{144} + x^{140} + x^{136} + x^{128} \\
& + x^{126} + x^{122} + x^{118} + x^{114} + x^{110} + x^{108} + x^{104} + x^{98} + x^{94} + x^{92} \\
& + x^{88} + x^{86} + x^{82} + x^{80} + x^{78} + x^{70} + x^{66} + x^{62} + x^{60} + x^{58} + x^{56} \\
& + x^{48} + x^{46} + x^{44} + x^{42}, \\
p_3(x) = & x^{152} + x^{150} + x^{142} + x^{140} + x^{138} + x^{128} + x^{122} + x^{116} + x^{114} + x^{112} \\
& + x^{108} + x^{106} + x^{104} + x^{102} + x^{96} + x^{94} + x^{86} + x^{84} + x^{80} + x^{74} \\
& + x^{72} + x^{70} + x^{68} + x^{66} + x^{60} + x^{58} + x^{54} + x^{52} + x^{50} + x^{48} + x^{40} \\
& + x^{38} + x^{32} + x^{26} + x^{18} + x^{12},
\end{aligned}$$

$$\begin{aligned}
p_6(x) &= x^{150} + x^{146} + x^{142} + x^{140} + x^{136} + x^{134} + x^{126} + x^{124} + x^{114} + x^{110} \\
&\quad + x^{106} + x^{104} + x^{98} + x^{96} + x^{94} + x^{90} + x^{86} + x^{84} + x^{82} + x^{78} + x^{74} \\
&\quad + x^{70} + x^{64} + x^{60} + x^{44} + x^{42} + x^{38} + x^{36} + x^{34} + x^{30} + x^{28} + x^{24} \\
&\quad + x^{22} + x^{16} + x^{12} + x^8 + x^2 + 1, \\
p_9(x) &= x^{152} + x^{150} + x^{148} + x^{144} + x^{138} + x^{136} + x^{128} + x^{126} + x^{122} + x^{120} \\
&\quad + x^{116} + x^{112} + x^{106} + x^{104} + x^{94} + x^{92} + x^{88} + x^{86} + x^{82} + x^{80} \\
&\quad + x^{78} + x^{76} + x^{74} + x^{68} + x^{64} + x^{50} + x^{46} + x^{44} + x^{42} + x^{40} + x^{28} \\
&\quad + x^{26} + x^{24} + x^{16} + x^8 + x^4, \\
p_{12}(x) &= x^{156} + x^{154} + x^{152} + x^{148} + x^{146} + x^{144} + x^{108} + x^{106} + x^{104} + x^{100} \\
&\quad + x^{98} + x^{96} + x^{92} + x^{90} + x^{88} + x^{84} + x^{82} + x^{80} + x^{12} + x^{10} + x^8 \\
&\quad + x^4 + x^2 + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0]$.

For $\phi(W^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{156} + x^{154} + x^{152} + x^{150} + x^{138} + x^{136} + x^{130} + x^{126} + x^{124} + x^{120} \\
&\quad + x^{118} + x^{114} + x^{112} + x^{106} + x^{102} + x^{98} + x^{96} + x^{94} + x^{92} + x^{90} \\
&\quad + x^{78} + x^{76} + x^{74} + x^{72} + x^{70} + x^{68} + x^{66} + x^{62} + x^{60} + x^{58} + x^{44} \\
&\quad + x^{42} + x^{34} + x^{26} + x^{24}, \\
p_3(x) &= x^{152} + x^{150} + x^{144} + x^{140} + x^{132} + x^{128} + x^{124} + x^{118} + x^{112} + x^{106} \\
&\quad + x^{104} + x^{102} + x^{100} + x^{98} + x^{94} + x^{92} + x^{88} + x^{84} + x^{82} + x^{78} + x^{76} \\
&\quad + x^{68} + x^{66} + x^{64} + x^{56} + x^{54} + x^{52} + x^{44} + x^{34} + x^{32} + x^{28} + x^{26} \\
&\quad + x^{20} + x^{16}, \\
p_6(x) &= x^{150} + x^{148} + x^{140} + x^{138} + x^{136} + x^{134} + x^{130} + x^{124} + x^{122} + x^{120} \\
&\quad + x^{116} + x^{114} + x^{112} + x^{110} + x^{104} + x^{102} + x^{100} + x^{98} + x^{90} + x^{88} \\
&\quad + x^{82} + x^{76} + x^{70} + x^{68} + x^{60} + x^{50} + x^{46} + x^{44} + x^{42} + x^{40} + x^{36} \\
&\quad + x^{30} + x^{28} + x^{26} + x^{24} + x^{20} + x^{14} + x^{10} + x^2 + 1, \\
p_9(x) &= x^{152} + x^{150} + x^{148} + x^{144} + x^{138} + x^{136} + x^{128} + x^{126} + x^{122} + x^{120} \\
&\quad + x^{116} + x^{112} + x^{106} + x^{104} + x^{94} + x^{92} + x^{88} + x^{86} + x^{82} + x^{80} \\
&\quad + x^{78} + x^{76} + x^{74} + x^{68} + x^{64} + x^{50} + x^{46} + x^{44} + x^{42} + x^{40} + x^{28} \\
&\quad + x^{26} + x^{24} + x^{16} + x^8 + x^4, \\
p_{12}(x) &= x^{156} + x^{154} + x^{152} + x^{148} + x^{146} + x^{144} + x^{108} + x^{106} + x^{104} + x^{100} \\
&\quad + x^{98} + x^{96} + x^{92} + x^{90} + x^{88} + x^{84} + x^{82} + x^{80} + x^{12} + x^{10} + x^8 \\
&\quad + x^4 + x^2 + 1,
\end{aligned}$$

and the initial terms are $[1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0]$.

For $\phi(W^e, 0, 1)$:

$$p_0(x) = x^{147} + x^{144} + x^{141} + x^{140} + x^{138} + x^{136} + x^{135} + x^{130} + x^{128} + x^{126}$$

$$\begin{aligned}
& + x^{125} + x^{124} + x^{123} + x^{122} + x^{121} + x^{116} + x^{112} + x^{111} + x^{110} \\
& + x^{109} + x^{108} + x^{107} + x^{106} + x^{102} + x^{101} + x^{96} + x^{95} + x^{94} + x^{93} \\
& + x^{90} + x^{84} + x^{82} + x^{79} + x^{78} + x^{76} + x^{72} + x^{68} + x^{67} + x^{65} + x^{63} \\
& + x^{62} + x^{59} + x^{58} + x^{57} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} \\
& + x^{45} + x^{44} + x^{43} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{34} + x^{33}, \\
p_3(x) = & x^{129} + x^{127} + x^{126} + x^{125} + x^{123} + x^{122} + x^{121} + x^{119} + x^{118} + x^{115} \\
& + x^{112} + x^{111} + x^{110} + x^{107} + x^{104} + x^{103} + x^{102} + x^{101} + x^{99} + x^{98} \\
& + x^{97} + x^{95} + x^{94} + x^{93} + x^{91} + x^{90} + x^{85} + x^{82} + x^{80} + x^{79} + x^{78} \\
& + x^{77} + x^{75} + x^{74} + x^{73} + x^{71} + x^{70} + x^{69} + x^{67} + x^{66} + x^{65} + x^{63} \\
& + x^{62} + x^{59} + x^{56} + x^{55} + x^{54} + x^{51} + x^{48} + x^{47} + x^{46} + x^{45} + x^{43} \\
& + x^{42} + x^{41} + x^{39} + x^{38} + x^{37} + x^{35} + x^{34} + x^{29} + x^{26} + x^{24} + x^{23} \\
& + x^{22} + x^{21} + x^{19} + x^{18}, \\
p_6(x) = & x^{135} + x^{132} + x^{125} + x^{123} + x^{122} + x^{121} + x^{120} + x^{118} + x^{117} + x^{114} \\
& + x^{113} + x^{111} + x^{110} + x^{108} + x^{105} + x^{103} + x^{102} + x^{101} + x^{100} \\
& + x^{98} + x^{97} + x^{95} + x^{94} + x^{93} + x^{92} + x^{90} + x^{83} + x^{81} + x^{80} + x^{79} \\
& + x^{78} + x^{77} + x^{76} + x^{74} + x^{67} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} \\
& + x^{59} + x^{58} + x^{56} + x^{47} + x^{44} + x^{43} + x^{40} + x^{39} + x^{37} + x^{36} + x^{35} \\
& + x^{34} + x^{32} + x^{31} + x^{29} + x^{28} + x^{26} + x^{25} + x^{22} + x^{17} + x^{15} + x^{14} \\
& + x^{12}, \\
p_9(x) = & x^{141} + x^{139} + x^{138} + x^{131} + x^{128} + x^{127} + x^{126} + x^{117} + x^{114} + x^{113} \\
& + x^{112} + x^{109} + x^{107} + x^{106} + x^{105} + x^{102} + x^{101} + x^{100} + x^{99} + x^{97} \\
& + x^{96} + x^{95} + x^{92} + x^{90} + x^{89} + x^{88} + x^{85} + x^{82} + x^{80} + x^{78} + x^{77} \\
& + x^{76} + x^{75} + x^{72} + x^{71} + x^{70} + x^{65} + x^{62} + x^{61} + x^{60} + x^{53} + x^{50} \\
& + x^{48} + x^{47} + x^{46} + x^{43} + x^{41} + x^{40} + x^{35} + x^{33} + x^{32} + x^{31} + x^{29} \\
& + x^{28} + x^{25} + x^{22} + x^{20} + x^{19} + x^{18} + x^{13} + x^{10} + x^8 + x^7 + x^6, \\
p_{12}(x) = & x^{147} + x^{144} + x^{143} + x^{140} + x^{135} + x^{132} + x^{127} + x^{124} + x^{119} + x^{116} \\
& + x^{111} + x^{108} + x^{103} + x^{100} + x^{99} + x^{96} + x^{83} + x^{80} + x^{79} + x^{76} \\
& + x^{71} + x^{68} + x^{63} + x^{60} + x^{55} + x^{52} + x^{47} + x^{44} + x^{39} + x^{36} + x^{31} \\
& + x^{28} + x^{23} + x^{20} + x^{15} + x^{12} + x^7 + x^4 + x^3 + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 1, 1, 0, 1, 1, 0, 1, 0, 0, 0, 1, 1, 1]$.

For $\phi(W^e, 1, 0)$:

$$\begin{aligned}
p_0(x) = & x^{147} + x^{144} + x^{141} + x^{140} + x^{138} + x^{136} + x^{135} + x^{130} + x^{128} + x^{126} \\
& + x^{125} + x^{124} + x^{123} + x^{122} + x^{121} + x^{116} + x^{112} + x^{111} + x^{110} \\
& + x^{109} + x^{108} + x^{107} + x^{106} + x^{102} + x^{101} + x^{96} + x^{95} + x^{94} + x^{93} \\
& + x^{90} + x^{84} + x^{82} + x^{79} + x^{78} + x^{76} + x^{72} + x^{68} + x^{67} + x^{65} + x^{63}
\end{aligned}$$

$$\begin{aligned}
& + x^{62} + x^{59} + x^{58} + x^{57} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} \\
& + x^{45} + x^{44} + x^{43} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{34} + x^{33}, \\
p_3(x) = & x^{129} + x^{127} + x^{126} + x^{125} + x^{123} + x^{122} + x^{121} + x^{119} + x^{118} + x^{115} \\
& + x^{112} + x^{111} + x^{110} + x^{107} + x^{104} + x^{103} + x^{102} + x^{101} + x^{99} + x^{98} \\
& + x^{97} + x^{95} + x^{94} + x^{93} + x^{91} + x^{90} + x^{85} + x^{82} + x^{80} + x^{79} + x^{78} \\
& + x^{77} + x^{75} + x^{74} + x^{73} + x^{71} + x^{70} + x^{69} + x^{67} + x^{66} + x^{65} + x^{63} \\
& + x^{62} + x^{59} + x^{56} + x^{55} + x^{54} + x^{51} + x^{48} + x^{47} + x^{46} + x^{45} + x^{43} \\
& + x^{42} + x^{41} + x^{39} + x^{38} + x^{37} + x^{35} + x^{34} + x^{29} + x^{26} + x^{24} + x^{23} \\
& + x^{22} + x^{21} + x^{19} + x^{18}, \\
p_6(x) = & x^{135} + x^{132} + x^{125} + x^{123} + x^{122} + x^{121} + x^{120} + x^{118} + x^{117} + x^{114} \\
& + x^{113} + x^{111} + x^{110} + x^{108} + x^{105} + x^{103} + x^{102} + x^{101} + x^{100} \\
& + x^{98} + x^{97} + x^{95} + x^{94} + x^{93} + x^{92} + x^{90} + x^{83} + x^{81} + x^{80} + x^{79} \\
& + x^{78} + x^{77} + x^{76} + x^{74} + x^{67} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} \\
& + x^{59} + x^{58} + x^{56} + x^{47} + x^{44} + x^{43} + x^{40} + x^{39} + x^{37} + x^{36} + x^{35} \\
& + x^{34} + x^{32} + x^{31} + x^{29} + x^{28} + x^{26} + x^{25} + x^{22} + x^{17} + x^{15} + x^{14} \\
& + x^{12}, \\
p_9(x) = & x^{141} + x^{139} + x^{138} + x^{131} + x^{128} + x^{127} + x^{126} + x^{117} + x^{114} + x^{113} \\
& + x^{112} + x^{109} + x^{107} + x^{106} + x^{105} + x^{102} + x^{101} + x^{100} + x^{99} + x^{97} \\
& + x^{96} + x^{95} + x^{92} + x^{90} + x^{89} + x^{88} + x^{85} + x^{82} + x^{80} + x^{78} + x^{77} \\
& + x^{76} + x^{75} + x^{72} + x^{71} + x^{70} + x^{65} + x^{62} + x^{61} + x^{60} + x^{53} + x^{50} \\
& + x^{48} + x^{47} + x^{46} + x^{43} + x^{41} + x^{40} + x^{35} + x^{33} + x^{32} + x^{31} + x^{29} \\
& + x^{28} + x^{25} + x^{22} + x^{20} + x^{19} + x^{18} + x^{13} + x^{10} + x^8 + x^7 + x^6, \\
p_{12}(x) = & x^{147} + x^{144} + x^{143} + x^{140} + x^{135} + x^{132} + x^{127} + x^{124} + x^{119} + x^{116} \\
& + x^{111} + x^{108} + x^{103} + x^{100} + x^{99} + x^{96} + x^{83} + x^{80} + x^{79} + x^{76} \\
& + x^{71} + x^{68} + x^{63} + x^{60} + x^{55} + x^{52} + x^{47} + x^{44} + x^{39} + x^{36} + x^{31} \\
& + x^{28} + x^{23} + x^{20} + x^{15} + x^{12} + x^7 + x^4 + x^3 + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 1, 1, 0, 1, 1, 0, 1, 0, 0, 0, 1, 1, 1]$.

For $\phi(W^e, 1, 1)$:

$$\begin{aligned}
p_0(x) = & x^{138} + x^{136} + x^{132} + x^{128} + x^{126} + x^{124} + x^{122} + x^{120} + x^{118} + x^{116} \\
& + x^{114} + x^{112} + x^{110} + x^{106} + x^{98} + x^{96} + x^{94} + x^{92} + x^{90} + x^{86} \\
& + x^{82} + x^{80} + x^{78} + x^{58} + x^{54} + x^{48} + x^{42}, \\
p_3(x) = & x^{134} + x^{132} + x^{130} + x^{124} + x^{120} + x^{112} + x^{108} + x^{104} + x^{98} + x^{96} \\
& + x^{94} + x^{92} + x^{90} + x^{88} + x^{84} + x^{78} + x^{72} + x^{70} + x^{64} + x^{56} + x^{54} \\
& + x^{50} + x^{48} + x^{46} + x^{42} + x^{40} + x^{32} + x^{26} + x^{16} + x^{12}, \\
p_6(x) = & x^{132} + x^{126} + x^{124} + x^{120} + x^{114} + x^{104} + x^{102} + x^{98} + x^{96} + x^{90}
\end{aligned}$$

$$\begin{aligned}
& + x^{86} + x^{78} + x^{74} + x^{68} + x^{64} + x^{62} + x^{60} + x^{52} + x^{48} + x^{46} + x^{44} \\
& + x^{42} + x^{40} + x^{38} + x^{36} + x^{34} + x^{32} + x^{30} + x^{20} + x^{18} + x^{16} + x^{12} \\
& + x^8 + x^4 + x^2 + 1, \\
p_9(x) &= x^{134} + x^{132} + x^{124} + x^{122} + x^{118} + x^{116} + x^{112} + x^{110} + x^{96} + x^{94} \\
& + x^{92} + x^{86} + x^{78} + x^{70} + x^{68} + x^{66} + x^{64} + x^{62} + x^{56} + x^{50} + x^{48} \\
& + x^{46} + x^{42} + x^{36} + x^{34} + x^{32} + x^{30} + x^{20} + x^{10} + x^4, \\
p_{12}(x) &= x^{138} + x^{136} + x^{114} + x^{112} + x^{106} + x^{104} + x^{90} + x^{88} + x^{82} + x^{80} \\
& + x^{66} + x^{64} + x^{58} + x^{56} + x^{34} + x^{32} + x^{26} + x^{24} + x^2 + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0]$.

For $\phi(M^o, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{147} + x^{144} + x^{143} + x^{141} + x^{140} + x^{139} + x^{138} + x^{137} + x^{135} + x^{134} \\
& + x^{133} + x^{131} + x^{130} + x^{128} + x^{125} + x^{120} + x^{119} + x^{117} + x^{115} \\
& + x^{112} + x^{111} + x^{107} + x^{106} + x^{103} + x^{102} + x^{101} + x^{99} + x^{98} + x^{97} \\
& + x^{93} + x^{92} + x^{91} + x^{90} + x^{86} + x^{82} + x^{81} + x^{77} + x^{75} + x^{74} + x^{73} \\
& + x^{72} + x^{71} + x^{69} + x^{66} + x^{64} + x^{62} + x^{60} + x^{57} + x^{56} + x^{53} + x^{48} \\
& + x^{47} + x^{46} + x^{44} + x^{43} + x^{41} + x^{36} + x^{35} + x^{34} + x^{32} + x^{28} + x^{27} \\
& + x^{24}, \\
p_3(x) &= x^{139} + x^{136} + x^{135} + x^{133} + x^{132} + x^{130} + x^{129} + x^{128} + x^{127} + x^{126} \\
& + x^{123} + x^{117} + x^{113} + x^{112} + x^{110} + x^{109} + x^{104} + x^{103} + x^{101} \\
& + x^{98} + x^{96} + x^{95} + x^{93} + x^{91} + x^{90} + x^{88} + x^{87} + x^{85} + x^{83} + x^{81} \\
& + x^{79} + x^{78} + x^{73} + x^{72} + x^{71} + x^{70} + x^{65} + x^{64} + x^{62} + x^{61} + x^{56} \\
& + x^{51} + x^{50} + x^{49} + x^{46} + x^{45} + x^{44} + x^{42} + x^{40} + x^{36} + x^{35} + x^{32} \\
& + x^{31} + x^{26} + x^{24} + x^{23} + x^{19} + x^{16}, \\
p_6(x) &= x^{141} + x^{138} + x^{137} + x^{136} + x^{134} + x^{132} + x^{131} + x^{125} + x^{123} + x^{119} \\
& + x^{114} + x^{107} + x^{104} + x^{103} + x^{102} + x^{101} + x^{99} + x^{98} + x^{97} + x^{96} \\
& + x^{95} + x^{92} + x^{81} + x^{80} + x^{79} + x^{77} + x^{76} + x^{74} + x^{73} + x^{71} + x^{70} \\
& + x^{69} + x^{68} + x^{67} + x^{66} + x^{62} + x^{60} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} \\
& + x^{52} + x^{51} + x^{50} + x^{49} + x^{46} + x^{45} + x^{42} + x^{40} + x^{39} + x^{37} + x^{36} \\
& + x^{31} + x^{30} + x^{23} + x^{22} + x^{20} + x^{19} + x^{18} + x^{16} + x^{15} + x^{11} + x^8 \\
& + x^7 + x^4 + x^3 + 1, \\
p_9(x) &= x^{139} + x^{136} + x^{135} + x^{132} + x^{131} + x^{128} + x^{123} + x^{120} + x^{107} + x^{104} \\
& + x^{103} + x^{100} + x^{99} + x^{96} + x^{87} + x^{84} + x^{83} + x^{80} + x^{75} + x^{72} + x^{59} \\
& + x^{56} + x^{55} + x^{52} + x^{51} + x^{48} + x^{43} + x^{40} + x^{27} + x^{24} + x^{23} + x^{20} \\
& + x^{19} + x^{16} + x^{11} + x^8, \\
p_{12}(x) &= x^{147} + x^{144} + x^{143} + x^{140} + x^{139} + x^{136} + x^{131} + x^{128} + x^{127} + x^{124}
\end{aligned}$$

$$\begin{aligned}
& + x^{123} + x^{120} + x^{119} + x^{116} + x^{111} + x^{108} + x^{107} + x^{104} + x^{87} + x^{84} \\
& + x^{79} + x^{76} + x^{75} + x^{72} + x^{71} + x^{68} + x^{63} + x^{60} + x^{59} + x^{56} + x^{51} \\
& + x^{48} + x^{47} + x^{44} + x^{43} + x^{40} + x^{39} + x^{36} + x^{31} + x^{28} + x^{27} + x^{24} \\
& + x^{19} + x^{16} + x^{15} + x^{12} + x^{11} + x^8 + x^7 + x^4 + x^3 + 1,
\end{aligned}$$

and the initial terms are $[1, 0, 0, 0, 1, 0, 1, 1, 0, 0, 1, 0, 0, 0, 0, 0]$.

For $\phi(M^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{160} + x^{154} + x^{153} + x^{152} + x^{150} + x^{147} + x^{146} + x^{144} + x^{143} + x^{142} \\
&+ x^{141} + x^{139} + x^{138} + x^{137} + x^{135} + x^{127} + x^{126} + x^{124} + x^{119} \\
&+ x^{116} + x^{115} + x^{114} + x^{113} + x^{111} + x^{109} + x^{108} + x^{106} + x^{105} \\
&+ x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{98} + x^{94} + x^{92} + x^{90} + x^{89} \\
&+ x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{77} + x^{76} + x^{74} \\
&+ x^{73} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{62} + x^{61} + x^{60} + x^{57} + x^{54} \\
&+ x^{50} + x^{49} + x^{46} + x^{45} + x^{41} + x^{40} + x^{38} + x^{35} + x^{33}, \\
p_3(x) &= x^{142} + x^{140} + x^{137} + x^{134} + x^{131} + x^{130} + x^{102} + x^{100} + x^{97} + x^{92} \\
&+ x^{91} + x^{90} + x^{89} + x^{86} + x^{83} + x^{82} + x^{46} + x^{44} + x^{41} + x^{36} + x^{35} \\
&+ x^{34} + x^{33} + x^{28} + x^{27} + x^{26} + x^{25} + x^{22} + x^{19} + x^{18}, \\
p_6(x) &= x^{148} + x^{138} + x^{136} + x^{134} + x^{132} + x^{130} + x^{126} + x^{124} + x^{122} + x^{120} \\
&+ x^{116} + x^{106} + x^{102} + x^{100} + x^{98} + x^{96} + x^{72} + x^{64} + x^{62} + x^{58} \\
&+ x^{52} + x^{50} + x^{48} + x^{42} + x^{40} + x^{38} + x^{36} + x^{34} + x^{32} + x^{30} + x^{26} \\
&+ x^{22} + x^{14} + x^{12}, \\
p_9(x) &= x^{154} + x^{152} + x^{150} + x^{149} + x^{148} + x^{145} + x^{143} + x^{141} + x^{140} + x^{137} \\
&+ x^{135} + x^{133} + x^{132} + x^{129} + x^{127} + x^{125} + x^{124} + x^{121} + x^{119} \\
&+ x^{117} + x^{116} + x^{113} + x^{111} + x^{109} + x^{108} + x^{106} + x^{105} + x^{104} \\
&+ x^{103} + x^{102} + x^{90} + x^{88} + x^{86} + x^{85} + x^{84} + x^{81} + x^{79} + x^{77} + x^{76} \\
&+ x^{73} + x^{71} + x^{69} + x^{68} + x^{65} + x^{63} + x^{61} + x^{60} + x^{57} + x^{55} + x^{53} \\
&+ x^{52} + x^{49} + x^{47} + x^{45} + x^{44} + x^{41} + x^{39} + x^{37} + x^{36} + x^{33} + x^{31} \\
&+ x^{29} + x^{28} + x^{25} + x^{23} + x^{21} + x^{20} + x^{17} + x^{15} + x^{13} + x^{12} + x^{10} \\
&+ x^9 + x^8 + x^7 + x^6, \\
p_{12}(x) &= x^{160} + x^{156} + x^{152} + x^{132} + x^{128} + x^{116} + x^{108} + x^{104} + x^{100} + x^{92} \\
&+ x^{88} + x^{68} + x^{64} + x^{52} + x^{48} + x^{36} + x^{32} + x^{20} + x^{12} + x^8 + x^4 + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 1]$.

For $\phi(M^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{148} + x^{142} + x^{141} + x^{138} + x^{135} + x^{133} + x^{132} + x^{131} + x^{130} + x^{125} \\
&+ x^{124} + x^{123} + x^{122} + x^{120} + x^{118} + x^{117} + x^{113} + x^{111} + x^{108} \\
&+ x^{106} + x^{104} + x^{100} + x^{98} + x^{92} + x^{91} + x^{89} + x^{88} + x^{86} + x^{84} + x^{78}
\end{aligned}$$

$$\begin{aligned}
& + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{67} + x^{66} + x^{65} + x^{63} + x^{62} \\
& + x^{59} + x^{58} + x^{57} + x^{51} + x^{50} + x^{48} + x^{47} + x^{46} + x^{43} + x^{40} + x^{37} \\
& + x^{34} + x^{33}, \\
p_3(x) &= x^{130} + x^{128} + x^{125} + x^{120} + x^{119} + x^{117} + x^{116} + x^{114} + x^{113} + x^{111} \\
& + x^{110} + x^{108} + x^{107} + x^{106} + x^{105} + x^{102} + x^{99} + x^{96} + x^{93} + x^{90} \\
& + x^{87} + x^{84} + x^{81} + x^{78} + x^{75} + x^{72} + x^{69} + x^{64} + x^{63} + x^{61} + x^{60} \\
& + x^{58} + x^{57} + x^{55} + x^{54} + x^{52} + x^{51} + x^{50} + x^{49} + x^{46} + x^{43} + x^{40} \\
& + x^{37} + x^{34} + x^{31} + x^{28} + x^{25} + x^{22} + x^{19} + x^{18}, \\
p_6(x) &= x^{136} + x^{128} + x^{126} + x^{122} + x^{120} + x^{114} + x^{112} + x^{110} + x^{106} + x^{102} \\
& + x^{100} + x^{96} + x^{94} + x^{90} + x^{88} + x^{82} + x^{78} + x^{74} + x^{72} + x^{66} + x^{62} \\
& + x^{60} + x^{58} + x^{56} + x^{48} + x^{44} + x^{38} + x^{36} + x^{24} + x^{20} + x^{14} + x^{12}, \\
p_9(x) &= x^{142} + x^{140} + x^{138} + x^{137} + x^{136} + x^{134} + x^{133} + x^{132} + x^{131} + x^{128} \\
& + x^{126} + x^{125} + x^{124} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{110} \\
& + x^{108} + x^{106} + x^{105} + x^{104} + x^{102} + x^{101} + x^{100} + x^{99} + x^{96} + x^{93} \\
& + x^{87} + x^{85} + x^{84} + x^{83} + x^{80} + x^{78} + x^{77} + x^{76} + x^{74} + x^{73} + x^{72} \\
& + x^{71} + x^{70} + x^{62} + x^{60} + x^{58} + x^{57} + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} \\
& + x^{48} + x^{46} + x^{45} + x^{44} + x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{30} + x^{28} \\
& + x^{26} + x^{25} + x^{24} + x^{22} + x^{21} + x^{20} + x^{19} + x^{16} + x^{14} + x^{13} + x^{12} \\
& + x^{10} + x^9 + x^8 + x^7 + x^6, \\
p_{12}(x) &= x^{148} + x^{144} + x^{124} + x^{100} + x^{96} + x^{92} + x^{84} + x^{80} + x^{76} + x^{44} + x^{12} \\
& + x^4 + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 1, 1, 0, 1, 0, 1, 0, 1, 1, 0, 1, 1, 0, 1]$.

For $\phi(M^o, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{147} + x^{144} + x^{143} + x^{141} + x^{140} + x^{138} + x^{137} + x^{136} + x^{135} + x^{134} \\
& + x^{133} + x^{131} + x^{130} + x^{128} + x^{127} + x^{125} + x^{124} + x^{114} + x^{113} \\
& + x^{112} + x^{111} + x^{110} + x^{109} + x^{106} + x^{105} + x^{103} + x^{101} + x^{100} \\
& + x^{99} + x^{97} + x^{92} + x^{88} + x^{85} + x^{84} + x^{80} + x^{79} + x^{77} + x^{75} + x^{74} \\
& + x^{70} + x^{68} + x^{67} + x^{66} + x^{61} + x^{57} + x^{52} + x^{51} + x^{50} + x^{47} + x^{46} \\
& + x^{44} + x^{43} + x^{42}, \\
p_3(x) &= x^{139} + x^{136} + x^{135} + x^{133} + x^{132} + x^{131} + x^{130} + x^{129} + x^{127} + x^{126} \\
& + x^{123} + x^{119} + x^{116} + x^{115} + x^{114} + x^{111} + x^{108} + x^{107} + x^{106} \\
& + x^{105} + x^{103} + x^{102} + x^{101} + x^{98} + x^{96} + x^{95} + x^{93} + x^{90} + x^{84} \\
& + x^{83} + x^{82} + x^{79} + x^{73} + x^{72} + x^{71} + x^{70} + x^{65} + x^{64} + x^{63} + x^{62} \\
& + x^{60} + x^{59} + x^{58} + x^{57} + x^{55} + x^{54} + x^{53} + x^{52} + x^{48} + x^{45} + x^{44} \\
& + x^{42} + x^{40} + x^{36} + x^{29} + x^{28} + x^{25} + x^{24} + x^{23} + x^{22},
\end{aligned}$$

$$\begin{aligned}
p_6(x) &= x^{141} + x^{139} + x^{138} + x^{137} + x^{134} + x^{132} + x^{131} + x^{127} + x^{124} + x^{122} \\
&\quad + x^{121} + x^{120} + x^{119} + x^{118} + x^{117} + x^{115} + x^{113} + x^{112} + x^{110} \\
&\quad + x^{107} + x^{105} + x^{104} + x^{100} + x^{95} + x^{94} + x^{93} + x^{92} + x^{90} + x^{83} \\
&\quad + x^{78} + x^{73} + x^{71} + x^{70} + x^{69} + x^{68} + x^{66} + x^{65} + x^{64} + x^{61} + x^{60} \\
&\quad + x^{58} + x^{57} + x^{56} + x^{54} + x^{53} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} \\
&\quad + x^{44} + x^{43} + x^{42} + x^{34} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} \\
&\quad + x^{23} + x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{14} + x^{12} + x^7 + x^4 + x^3 + 1, \\
p_9(x) &= x^{139} + x^{136} + x^{135} + x^{132} + x^{131} + x^{128} + x^{123} + x^{120} + x^{107} + x^{104} \\
&\quad + x^{103} + x^{100} + x^{99} + x^{96} + x^{87} + x^{84} + x^{83} + x^{80} + x^{75} + x^{72} + x^{59} \\
&\quad + x^{56} + x^{55} + x^{52} + x^{51} + x^{48} + x^{43} + x^{40} + x^{27} + x^{24} + x^{23} + x^{20} \\
&\quad + x^{19} + x^{16} + x^{11} + x^8, \\
p_{12}(x) &= x^{147} + x^{144} + x^{143} + x^{140} + x^{139} + x^{136} + x^{131} + x^{128} + x^{127} + x^{124} \\
&\quad + x^{123} + x^{120} + x^{119} + x^{116} + x^{111} + x^{108} + x^{107} + x^{104} + x^{87} + x^{84} \\
&\quad + x^{79} + x^{76} + x^{75} + x^{72} + x^{71} + x^{68} + x^{63} + x^{60} + x^{59} + x^{56} + x^{51} \\
&\quad + x^{48} + x^{47} + x^{44} + x^{43} + x^{40} + x^{39} + x^{36} + x^{31} + x^{28} + x^{27} + x^{24} \\
&\quad + x^{19} + x^{16} + x^{15} + x^{12} + x^{11} + x^8 + x^7 + x^4 + x^3 + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0]$.

For $\phi(W^o, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{147} + x^{144} + x^{143} + x^{141} + x^{140} + x^{139} + x^{138} + x^{137} + x^{135} + x^{134} \\
&\quad + x^{133} + x^{131} + x^{130} + x^{128} + x^{125} + x^{120} + x^{119} + x^{117} + x^{115} \\
&\quad + x^{112} + x^{111} + x^{107} + x^{106} + x^{103} + x^{102} + x^{101} + x^{99} + x^{98} + x^{97} \\
&\quad + x^{93} + x^{92} + x^{91} + x^{90} + x^{86} + x^{82} + x^{81} + x^{77} + x^{75} + x^{74} + x^{73} \\
&\quad + x^{72} + x^{71} + x^{69} + x^{66} + x^{64} + x^{62} + x^{60} + x^{57} + x^{56} + x^{53} + x^{48} \\
&\quad + x^{47} + x^{46} + x^{44} + x^{43} + x^{41} + x^{36} + x^{35} + x^{34} + x^{32} + x^{28} + x^{27} \\
&\quad + x^{24}, \\
p_3(x) &= x^{139} + x^{136} + x^{135} + x^{133} + x^{132} + x^{130} + x^{129} + x^{128} + x^{127} + x^{126} \\
&\quad + x^{123} + x^{117} + x^{113} + x^{112} + x^{110} + x^{109} + x^{104} + x^{103} + x^{101} \\
&\quad + x^{98} + x^{96} + x^{95} + x^{93} + x^{91} + x^{90} + x^{88} + x^{87} + x^{85} + x^{83} + x^{81} \\
&\quad + x^{79} + x^{78} + x^{73} + x^{72} + x^{71} + x^{70} + x^{65} + x^{64} + x^{62} + x^{61} + x^{56} \\
&\quad + x^{51} + x^{50} + x^{49} + x^{46} + x^{45} + x^{44} + x^{42} + x^{40} + x^{36} + x^{35} + x^{32} \\
&\quad + x^{31} + x^{26} + x^{24} + x^{23} + x^{19} + x^{16}, \\
p_6(x) &= x^{141} + x^{138} + x^{137} + x^{136} + x^{134} + x^{132} + x^{131} + x^{125} + x^{123} + x^{119} \\
&\quad + x^{114} + x^{107} + x^{104} + x^{103} + x^{102} + x^{101} + x^{99} + x^{98} + x^{97} + x^{96} \\
&\quad + x^{95} + x^{92} + x^{81} + x^{80} + x^{79} + x^{77} + x^{76} + x^{74} + x^{73} + x^{71} + x^{70} \\
&\quad + x^{69} + x^{68} + x^{67} + x^{66} + x^{62} + x^{60} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53}
\end{aligned}$$

$$\begin{aligned}
& + x^{52} + x^{51} + x^{50} + x^{49} + x^{46} + x^{45} + x^{42} + x^{40} + x^{39} + x^{37} + x^{36} \\
& + x^{31} + x^{30} + x^{23} + x^{22} + x^{20} + x^{19} + x^{18} + x^{16} + x^{15} + x^{11} + x^8 \\
& + x^7 + x^4 + x^3 + 1, \\
p_9(x) = & x^{139} + x^{136} + x^{135} + x^{132} + x^{131} + x^{128} + x^{123} + x^{120} + x^{107} + x^{104} \\
& + x^{103} + x^{100} + x^{99} + x^{96} + x^{87} + x^{84} + x^{83} + x^{80} + x^{75} + x^{72} + x^{59} \\
& + x^{56} + x^{55} + x^{52} + x^{51} + x^{48} + x^{43} + x^{40} + x^{27} + x^{24} + x^{23} + x^{20} \\
& + x^{19} + x^{16} + x^{11} + x^8, \\
p_{12}(x) = & x^{147} + x^{144} + x^{143} + x^{140} + x^{139} + x^{136} + x^{131} + x^{128} + x^{127} + x^{124} \\
& + x^{123} + x^{120} + x^{119} + x^{116} + x^{111} + x^{108} + x^{107} + x^{104} + x^{87} + x^{84} \\
& + x^{79} + x^{76} + x^{75} + x^{72} + x^{71} + x^{68} + x^{63} + x^{60} + x^{59} + x^{56} + x^{51} \\
& + x^{48} + x^{47} + x^{44} + x^{43} + x^{40} + x^{39} + x^{36} + x^{31} + x^{28} + x^{27} + x^{24} \\
& + x^{19} + x^{16} + x^{15} + x^{12} + x^{11} + x^8 + x^7 + x^4 + x^3 + 1,
\end{aligned}$$

and the initial terms are $[1, 0, 0, 0, 1, 0, 1, 1, 0, 0, 1, 0, 0, 0, 0, 0]$.

For $\phi(W^o, 0, 1)$:

$$\begin{aligned}
p_0(x) = & x^{148} + x^{142} + x^{141} + x^{138} + x^{135} + x^{133} + x^{132} + x^{131} + x^{130} + x^{125} \\
& + x^{124} + x^{123} + x^{122} + x^{120} + x^{118} + x^{117} + x^{113} + x^{111} + x^{108} \\
& + x^{106} + x^{104} + x^{100} + x^{98} + x^{92} + x^{91} + x^{89} + x^{88} + x^{86} + x^{84} + x^{78} \\
& + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{67} + x^{66} + x^{65} + x^{63} + x^{62} \\
& + x^{59} + x^{58} + x^{57} + x^{51} + x^{50} + x^{48} + x^{47} + x^{46} + x^{43} + x^{40} + x^{37} \\
& + x^{34} + x^{33}, \\
p_3(x) = & x^{130} + x^{128} + x^{125} + x^{120} + x^{119} + x^{117} + x^{116} + x^{114} + x^{113} + x^{111} \\
& + x^{110} + x^{108} + x^{107} + x^{106} + x^{105} + x^{102} + x^{99} + x^{96} + x^{93} + x^{90} \\
& + x^{87} + x^{84} + x^{81} + x^{78} + x^{75} + x^{72} + x^{69} + x^{64} + x^{63} + x^{61} + x^{60} \\
& + x^{58} + x^{57} + x^{55} + x^{54} + x^{52} + x^{51} + x^{50} + x^{49} + x^{46} + x^{43} + x^{40} \\
& + x^{37} + x^{34} + x^{31} + x^{28} + x^{25} + x^{22} + x^{19} + x^{18}, \\
p_6(x) = & x^{136} + x^{128} + x^{126} + x^{122} + x^{120} + x^{114} + x^{112} + x^{110} + x^{106} + x^{102} \\
& + x^{100} + x^{96} + x^{94} + x^{90} + x^{88} + x^{82} + x^{78} + x^{74} + x^{72} + x^{66} + x^{62} \\
& + x^{60} + x^{58} + x^{56} + x^{48} + x^{44} + x^{38} + x^{36} + x^{24} + x^{20} + x^{14} + x^{12}, \\
p_9(x) = & x^{142} + x^{140} + x^{138} + x^{137} + x^{136} + x^{134} + x^{133} + x^{132} + x^{131} + x^{128} \\
& + x^{126} + x^{125} + x^{124} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{110} \\
& + x^{108} + x^{106} + x^{105} + x^{104} + x^{102} + x^{101} + x^{100} + x^{99} + x^{96} + x^{93} \\
& + x^{87} + x^{85} + x^{84} + x^{83} + x^{80} + x^{78} + x^{77} + x^{76} + x^{74} + x^{73} + x^{72} \\
& + x^{71} + x^{70} + x^{62} + x^{60} + x^{58} + x^{57} + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} \\
& + x^{48} + x^{46} + x^{45} + x^{44} + x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{30} + x^{28} \\
& + x^{26} + x^{25} + x^{24} + x^{22} + x^{21} + x^{20} + x^{19} + x^{16} + x^{14} + x^{13} + x^{12}
\end{aligned}$$

$$\begin{aligned}
& + x^{10} + x^9 + x^8 + x^7 + x^6, \\
p_{12}(x) &= x^{148} + x^{144} + x^{124} + x^{100} + x^{96} + x^{92} + x^{84} + x^{80} + x^{76} + x^{44} + x^{12} \\
& + x^4 + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 1, 1, 0, 1, 0, 1, 0, 1, 1, 0, 1, 1, 0, 1]$.

For $\phi(W^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{160} + x^{154} + x^{153} + x^{152} + x^{150} + x^{147} + x^{146} + x^{144} + x^{143} + x^{142} \\
& + x^{141} + x^{139} + x^{138} + x^{137} + x^{135} + x^{127} + x^{126} + x^{124} + x^{119} \\
& + x^{116} + x^{115} + x^{114} + x^{113} + x^{111} + x^{109} + x^{108} + x^{106} + x^{105} \\
& + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{98} + x^{94} + x^{92} + x^{90} + x^{89} \\
& + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{77} + x^{76} + x^{74} \\
& + x^{73} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{62} + x^{61} + x^{60} + x^{57} + x^{54} \\
& + x^{50} + x^{49} + x^{46} + x^{45} + x^{41} + x^{40} + x^{38} + x^{35} + x^{33}, \\
p_3(x) &= x^{142} + x^{140} + x^{137} + x^{134} + x^{131} + x^{130} + x^{102} + x^{100} + x^{97} + x^{92} \\
& + x^{91} + x^{90} + x^{89} + x^{86} + x^{83} + x^{82} + x^{46} + x^{44} + x^{41} + x^{36} + x^{35} \\
& + x^{34} + x^{33} + x^{28} + x^{27} + x^{26} + x^{25} + x^{22} + x^{19} + x^{18}, \\
p_6(x) &= x^{148} + x^{138} + x^{136} + x^{134} + x^{132} + x^{130} + x^{126} + x^{124} + x^{122} + x^{120} \\
& + x^{116} + x^{106} + x^{102} + x^{100} + x^{98} + x^{96} + x^{72} + x^{64} + x^{62} + x^{58} \\
& + x^{52} + x^{50} + x^{48} + x^{42} + x^{40} + x^{38} + x^{36} + x^{34} + x^{32} + x^{30} + x^{26} \\
& + x^{22} + x^{14} + x^{12}, \\
p_9(x) &= x^{154} + x^{152} + x^{150} + x^{149} + x^{148} + x^{145} + x^{143} + x^{141} + x^{140} + x^{137} \\
& + x^{135} + x^{133} + x^{132} + x^{129} + x^{127} + x^{125} + x^{124} + x^{121} + x^{119} \\
& + x^{117} + x^{116} + x^{113} + x^{111} + x^{109} + x^{108} + x^{106} + x^{105} + x^{104} \\
& + x^{103} + x^{102} + x^{90} + x^{88} + x^{86} + x^{85} + x^{84} + x^{81} + x^{79} + x^{77} + x^{76} \\
& + x^{73} + x^{71} + x^{69} + x^{68} + x^{65} + x^{63} + x^{61} + x^{60} + x^{57} + x^{55} + x^{53} \\
& + x^{52} + x^{49} + x^{47} + x^{45} + x^{44} + x^{41} + x^{39} + x^{37} + x^{36} + x^{33} + x^{31} \\
& + x^{29} + x^{28} + x^{25} + x^{23} + x^{21} + x^{20} + x^{17} + x^{15} + x^{13} + x^{12} + x^{10} \\
& + x^9 + x^8 + x^7 + x^6, \\
p_{12}(x) &= x^{160} + x^{156} + x^{152} + x^{132} + x^{128} + x^{116} + x^{108} + x^{104} + x^{100} + x^{92} \\
& + x^{88} + x^{68} + x^{64} + x^{52} + x^{48} + x^{36} + x^{32} + x^{20} + x^{12} + x^8 + x^4 + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 1]$.

For $\phi(W^o, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{147} + x^{144} + x^{143} + x^{141} + x^{140} + x^{138} + x^{137} + x^{136} + x^{135} + x^{134} \\
& + x^{133} + x^{131} + x^{130} + x^{128} + x^{127} + x^{125} + x^{124} + x^{114} + x^{113} \\
& + x^{112} + x^{111} + x^{110} + x^{109} + x^{106} + x^{105} + x^{103} + x^{101} + x^{100} \\
& + x^{99} + x^{97} + x^{92} + x^{88} + x^{85} + x^{84} + x^{80} + x^{79} + x^{77} + x^{75} + x^{74}
\end{aligned}$$

$$\begin{aligned}
& + x^{70} + x^{68} + x^{67} + x^{66} + x^{61} + x^{57} + x^{52} + x^{51} + x^{50} + x^{47} + x^{46} \\
& + x^{44} + x^{43} + x^{42}, \\
p_3(x) = & x^{139} + x^{136} + x^{135} + x^{133} + x^{132} + x^{131} + x^{130} + x^{129} + x^{127} + x^{126} \\
& + x^{123} + x^{119} + x^{116} + x^{115} + x^{114} + x^{111} + x^{108} + x^{107} + x^{106} \\
& + x^{105} + x^{103} + x^{102} + x^{101} + x^{98} + x^{96} + x^{95} + x^{93} + x^{90} + x^{84} \\
& + x^{83} + x^{82} + x^{79} + x^{73} + x^{72} + x^{71} + x^{70} + x^{65} + x^{64} + x^{63} + x^{62} \\
& + x^{60} + x^{59} + x^{58} + x^{57} + x^{55} + x^{54} + x^{53} + x^{52} + x^{48} + x^{45} + x^{44} \\
& + x^{42} + x^{40} + x^{36} + x^{29} + x^{28} + x^{25} + x^{24} + x^{23} + x^{22}, \\
p_6(x) = & x^{141} + x^{139} + x^{138} + x^{137} + x^{134} + x^{132} + x^{131} + x^{127} + x^{124} + x^{122} \\
& + x^{121} + x^{120} + x^{119} + x^{118} + x^{117} + x^{115} + x^{113} + x^{112} + x^{110} \\
& + x^{107} + x^{105} + x^{104} + x^{100} + x^{95} + x^{94} + x^{93} + x^{92} + x^{90} + x^{83} \\
& + x^{78} + x^{73} + x^{71} + x^{70} + x^{69} + x^{68} + x^{66} + x^{65} + x^{64} + x^{61} + x^{60} \\
& + x^{58} + x^{57} + x^{56} + x^{54} + x^{53} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} \\
& + x^{44} + x^{43} + x^{42} + x^{34} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} \\
& + x^{23} + x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{14} + x^{12} + x^7 + x^4 + x^3 + 1, \\
p_9(x) = & x^{139} + x^{136} + x^{135} + x^{132} + x^{131} + x^{128} + x^{123} + x^{120} + x^{107} + x^{104} \\
& + x^{103} + x^{100} + x^{99} + x^{96} + x^{87} + x^{84} + x^{83} + x^{80} + x^{75} + x^{72} + x^{59} \\
& + x^{56} + x^{55} + x^{52} + x^{51} + x^{48} + x^{43} + x^{40} + x^{27} + x^{24} + x^{23} + x^{20} \\
& + x^{19} + x^{16} + x^{11} + x^8, \\
p_{12}(x) = & x^{147} + x^{144} + x^{143} + x^{140} + x^{139} + x^{136} + x^{131} + x^{128} + x^{127} + x^{124} \\
& + x^{123} + x^{120} + x^{119} + x^{116} + x^{111} + x^{108} + x^{107} + x^{104} + x^{87} + x^{84} \\
& + x^{79} + x^{76} + x^{75} + x^{72} + x^{71} + x^{68} + x^{63} + x^{60} + x^{59} + x^{56} + x^{51} \\
& + x^{48} + x^{47} + x^{44} + x^{43} + x^{40} + x^{39} + x^{36} + x^{31} + x^{28} + x^{27} + x^{24} \\
& + x^{19} + x^{16} + x^{15} + x^{12} + x^{11} + x^8 + x^7 + x^4 + x^3 + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0]$.

Below is the transition function and output function of an 2-automaton that generates $T = M_{0,0}^o$:

Transition function $(n, j) \mapsto \delta(n, j)$ ($\Lambda(n) := [\delta(n, 0), \delta(n, 1)]$):

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	379	[587, 588]	758	[936, 999]	1137	[496, 935]
1	[3, 4]	380	[589, 590]	759	[416, 53]	1138	[994, 474]
2	[5, 6]	381	[591, 592]	760	[942, 928]	1139	[441, 1100]
3	[7, 8]	382	[593, 594]	761	[566, 570]	1140	[1051, 188]
4	[9, 10]	383	[595, 596]	762	[50, 1024]	1141	[1398, 1399]
5	[8, 11]	384	[597, 201]	763	[1025, 1026]	1142	[516, 1091]
6	[12, 13]	385	[598, 599]	764	[1027, 478]	1143	[390, 1017]
7	[14, 15]	386	[600, 601]	765	[412, 147]	1144	[380, 1400]

8	[16, 17]	387	[602, 603]	766	[1028, 465]	1145	[161, 159]
9	[18, 19]	388	[604, 605]	767	[1029, 1030]	1146	[479, 443]
10	[20, 21]	389	[606, 607]	768	[920, 501]	1147	[1097, 480]
11	[22, 23]	390	[608, 609]	769	[512, 166]	1148	[1401, 398]
12	[24, 25]	391	[610, 611]	770	[394, 950]	1149	[186, 411]
13	[26, 27]	392	[612, 613]	771	[1031, 994]	1150	[543, 1063]
14	[28, 29]	393	[614, 615]	772	[478, 1013]	1151	[31, 189]
15	[30, 31]	394	[616, 617]	773	[1012, 433]	1152	[1402, 1000]
16	[32, 33]	395	[210, 618]	774	[467, 1032]	1153	[374, 135]
17	[34, 35]	396	[74, 619]	775	[152, 1033]	1154	[137, 406]
18	[36, 37]	397	[620, 621]	776	[929, 158]	1155	[1403, 526]
19	[38, 39]	398	[622, 623]	777	[504, 918]	1156	[122, 975]
20	[40, 41]	399	[624, 625]	778	[463, 1034]	1157	[492, 136]
21	[42, 43]	400	[626, 627]	779	[1035, 1036]	1158	[198, 1056]
22	[44, 45]	401	[628, 629]	780	[1037, 117]	1159	[1391, 140]
23	[46, 47]	402	[630, 631]	781	[929, 402]	1160	[1404, 1017]
24	[48, 49]	403	[632, 633]	782	[403, 533]	1161	[524, 977]
25	[50, 51]	404	[634, 635]	783	[1038, 12]	1162	[432, 1405]
26	[46, 52]	405	[636, 637]	784	[953, 1039]	1163	[1406, 1399]
27	[53, 54]	406	[638, 639]	785	[502, 163]	1164	[935, 930]
28	[55, 56]	407	[640, 641]	786	[971, 115]	1165	[126, 47]
29	[57, 58]	408	[642, 367]	787	[1040, 378]	1166	[33, 154]
30	[59, 60]	409	[643, 269]	788	[1041, 32]	1167	[1407, 957]
31	[61, 62]	410	[644, 645]	789	[168, 985]	1168	[995, 160]
32	[63, 64]	411	[289, 646]	790	[1042, 522]	1169	[1068, 1032]
33	[65, 66]	412	[647, 648]	791	[541, 129]	1170	[192, 962]
34	[67, 68]	413	[649, 650]	792	[1043, 1044]	1171	[512, 1075]
35	[69, 70]	414	[651, 652]	793	[404, 196]	1172	[1408, 1044]
36	[71, 72]	415	[653, 654]	794	[987, 981]	1173	[1088, 944]
37	[73, 74]	416	[655, 65]	795	[1045, 1046]	1174	[38, 123]
38	[75, 76]	417	[656, 657]	796	[1011, 1007]	1175	[448, 1024]
39	[77, 78]	418	[658, 335]	797	[398, 406]	1176	[405, 560]
40	[79, 80]	419	[659, 660]	798	[1047, 1048]	1177	[402, 50]
41	[81, 82]	420	[661, 662]	799	[1049, 1050]	1178	[1061, 393]
42	[83, 84]	421	[663, 664]	800	[559, 399]	1179	[175, 573]
43	[85, 86]	422	[665, 666]	801	[126, 529]	1180	[32, 153]
44	[87, 88]	423	[667, 668]	802	[172, 517]	1181	[1409, 993]
45	[89, 90]	424	[578, 669]	803	[119, 492]	1182	[436, 546]
46	[91, 92]	425	[670, 671]	804	[1041, 171]	1183	[1035, 1022]
47	[93, 94]	426	[260, 672]	805	[1051, 447]	1184	[1410, 152]
48	[95, 96]	427	[673, 674]	806	[51, 534]	1185	[448, 533]
49	[97, 98]	428	[675, 676]	807	[380, 1048]	1186	[1411, 1412]
50	[99, 100]	429	[677, 678]	808	[508, 1052]	1187	[1413, 192]
51	[101, 102]	430	[678, 679]	809	[1053, 1021]	1188	[555, 950]

52	[103, 104]	431	[680, 681]	810	[496, 51]	1189	[1414, 919]
53	[105, 106]	432	[682, 683]	811	[465, 1054]	1190	[949, 1039]
54	[107, 108]	433	[684, 685]	812	[445, 484]	1191	[1041, 194]
55	[109, 110]	434	[686, 687]	813	[954, 468]	1192	[918, 945]
56	[111, 112]	435	[688, 638]	814	[538, 1055]	1193	[415, 1005]
57	[113, 114]	436	[689, 690]	815	[554, 158]	1194	[462, 561]
58	[111, 115]	437	[691, 692]	816	[168, 1056]	1195	[1051, 567]
59	[116, 117]	438	[693, 694]	817	[1057, 385]	1196	[386, 1415]
60	[118, 45]	439	[695, 696]	818	[955, 1058]	1197	[1004, 192]
61	[45, 119]	440	[697, 698]	819	[375, 136]	1198	[1416, 1072]
62	[34, 49]	441	[699, 219]	820	[553, 188]	1199	[430, 429]
63	[120, 121]	442	[700, 701]	821	[1059, 1048]	1200	[433, 1417]
64	[122, 123]	443	[702, 703]	822	[1060, 41]	1201	[1067, 482]
65	[124, 125]	444	[704, 705]	823	[1061, 192]	1202	[426, 160]
66	[126, 52]	445	[706, 707]	824	[177, 451]	1203	[445, 435]
67	[127, 128]	446	[708, 709]	825	[1062, 408]	1204	[993, 437]
68	[129, 130]	447	[56, 642]	826	[470, 145]	1205	[1418, 475]
69	[131, 132]	448	[710, 711]	827	[1063, 132]	1206	[190, 945]
70	[133, 134]	449	[712, 713]	828	[1064, 433]	1207	[1087, 1046]
71	[135, 136]	450	[714, 715]	829	[1008, 47]	1208	[1049, 1394]
72	[137, 138]	451	[716, 717]	830	[1065, 128]	1209	[996, 156]
73	[139, 140]	452	[718, 719]	831	[431, 471]	1210	[124, 120]
74	[141, 142]	453	[720, 721]	832	[134, 148]	1211	[923, 962]
75	[143, 144]	454	[722, 723]	833	[1066, 523]	1212	[428, 1419]
76	[145, 146]	455	[637, 724]	834	[435, 1054]	1213	[54, 528]
77	[147, 148]	456	[725, 726]	835	[178, 427]	1214	[125, 374]
78	[149, 150]	457	[727, 628]	836	[1067, 523]	1215	[163, 1397]
79	[151, 152]	458	[728, 729]	837	[493, 159]	1216	[503, 1397]
80	[153, 154]	459	[730, 731]	838	[485, 186]	1217	[1420, 1000]
81	[155, 156]	460	[732, 733]	839	[1068, 454]	1218	[33, 528]
82	[157, 158]	461	[734, 735]	840	[505, 561]	1219	[998, 937]
83	[159, 160]	462	[314, 736]	841	[1069, 140]	1220	[492, 1007]
84	[161, 162]	463	[737, 738]	842	[1070, 383]	1221	[1051, 1090]
85	[163, 164]	464	[739, 740]	843	[1071, 1072]	1222	[530, 932]
86	[165, 166]	465	[741, 587]	844	[146, 991]	1223	[1404, 391]
87	[167, 168]	466	[707, 742]	845	[484, 1054]	1224	[1053, 152]
88	[169, 170]	467	[743, 744]	846	[1068, 1073]	1225	[518, 190]
89	[171, 33]	468	[745, 231]	847	[1074, 459]	1226	[996, 1001]
90	[172, 173]	469	[746, 747]	848	[174, 1075]	1227	[943, 402]
91	[174, 166]	470	[681, 643]	849	[1076, 463]	1228	[428, 396]
92	[175, 176]	471	[96, 748]	850	[119, 1011]	1229	[513, 1421]
93	[177, 115]	472	[749, 750]	851	[570, 504]	1230	[548, 545]
94	[178, 13]	473	[751, 56]	852	[191, 921]	1231	[438, 966]
95	[179, 180]	474	[582, 752]	853	[1074, 453]	1232	[183, 1422]

96	[181, 182]	475	[753, 285]	854	[511, 1074]	1233	[524, 1423]
97	[183, 184]	476	[754, 755]	855	[177, 972]	1234	[436, 1013]
98	[185, 186]	477	[756, 757]	856	[1077, 990]	1235	[442, 1098]
99	[187, 45]	478	[758, 759]	857	[1049, 391]	1236	[131, 1424]
100	[29, 188]	479	[760, 761]	858	[1078, 1079]	1237	[1420, 998]
101	[48, 35]	480	[762, 763]	859	[145, 471]	1238	[119, 375]
102	[189, 190]	481	[764, 339]	860	[1080, 441]	1239	[461, 1425]
103	[191, 176]	482	[765, 766]	861	[548, 439]	1240	[187, 125]
104	[192, 193]	483	[767, 768]	862	[1081, 1074]	1241	[1024, 930]
105	[194, 195]	484	[769, 770]	863	[403, 51]	1242	[400, 1056]
106	[48, 196]	485	[771, 772]	864	[1000, 1082]	1243	[120, 119]
107	[157, 197]	486	[773, 774]	865	[54, 154]	1244	[428, 940]
108	[198, 199]	487	[775, 776]	866	[998, 1083]	1245	[394, 1039]
109	[200, 201]	488	[777, 778]	867	[1084, 1000]	1246	[968, 1389]
110	[202, 203]	489	[696, 223]	868	[1025, 1033]	1247	[493, 995]
111	[204, 205]	490	[250, 779]	869	[118, 120]	1248	[1025, 1022]
112	[206, 207]	491	[780, 781]	870	[1085, 12]	1249	[554, 402]
113	[208, 101]	492	[222, 782]	871	[970, 922]	1250	[562, 375]
114	[209, 210]	493	[783, 784]	872	[450, 972]	1251	[1426, 475]
115	[211, 212]	494	[785, 786]	873	[1015, 1074]	1252	[964, 1427]
116	[213, 214]	495	[787, 788]	874	[513, 922]	1253	[416, 171]
117	[215, 216]	496	[292, 789]	875	[939, 396]	1254	[398, 560]
118	[217, 218]	497	[790, 791]	876	[191, 501]	1255	[1428, 152]
119	[219, 220]	498	[792, 793]	877	[1086, 415]	1256	[121, 375]
120	[221, 222]	499	[794, 795]	878	[30, 570]	1257	[152, 1022]
121	[223, 224]	500	[796, 797]	879	[154, 984]	1258	[457, 536]
122	[225, 226]	501	[798, 799]	880	[1087, 389]	1259	[961, 454]
123	[227, 228]	502	[800, 801]	881	[40, 933]	1260	[118, 125]
124	[229, 230]	503	[802, 803]	882	[1088, 520]	1261	[372, 1090]
125	[231, 232]	504	[804, 805]	883	[153, 528]	1262	[1097, 443]
126	[233, 234]	505	[62, 691]	884	[553, 447]	1263	[475, 545]
127	[235, 236]	506	[806, 26]	885	[1089, 378]	1264	[1096, 490]
128	[237, 238]	507	[807, 808]	886	[949, 556]	1265	[549, 439]
129	[239, 240]	508	[809, 810]	887	[1030, 938]	1266	[486, 1429]
130	[240, 241]	509	[811, 812]	888	[171, 195]	1267	[1028, 446]
131	[242, 243]	510	[813, 814]	889	[553, 1090]	1268	[374, 1011]
132	[244, 245]	511	[815, 816]	890	[125, 121]	1269	[478, 437]
133	[246, 247]	512	[817, 818]	891	[527, 1091]	1270	[478, 1427]
134	[248, 249]	513	[819, 820]	892	[1092, 144]	1271	[1405, 1417]
135	[90, 250]	514	[821, 822]	893	[130, 540]	1272	[1402, 998]
136	[251, 252]	515	[823, 824]	894	[1093, 1094]	1273	[11, 452]
137	[253, 254]	516	[825, 826]	895	[963, 1093]	1274	[574, 1421]
138	[255, 256]	517	[827, 828]	896	[1095, 126]	1275	[491, 171]
139	[257, 258]	518	[776, 829]	897	[157, 563]	1276	[1008, 1005]

140	[259, 260]	519	[830, 831]	898	[500, 921]	1277	[564, 517]
141	[261, 262]	520	[832, 833]	899	[1096, 474]	1278	[54, 983]
142	[263, 264]	521	[834, 739]	900	[1097, 1098]	1279	[1430, 926]
143	[265, 266]	522	[835, 653]	901	[1099, 994]	1280	[496, 1024]
144	[267, 268]	523	[836, 837]	902	[1096, 1100]	1281	[958, 396]
145	[269, 270]	524	[838, 839]	903	[197, 403]	1282	[456, 551]
146	[271, 272]	525	[840, 745]	904	[1088, 1058]	1283	[472, 454]
147	[273, 274]	526	[841, 842]	905	[413, 1094]	1284	[549, 545]
148	[275, 276]	527	[843, 844]	906	[1101, 507]	1285	[479, 488]
149	[277, 278]	528	[321, 762]	907	[519, 419]	1286	[1093, 1431]
150	[279, 280]	529	[652, 332]	908	[1102, 981]	1287	[1038, 1030]
151	[281, 282]	530	[845, 846]	909	[51, 930]	1288	[422, 1421]
152	[283, 284]	531	[847, 848]	910	[405, 138]	1289	[566, 31]
153	[285, 286]	532	[849, 850]	911	[955, 520]	1290	[117, 985]
154	[64, 287]	533	[631, 695]	912	[121, 135]	1291	[1078, 932]
155	[288, 289]	534	[715, 851]	913	[168, 199]	1292	[1070, 1432]
156	[290, 77]	535	[86, 369]	914	[1103, 587]	1293	[1413, 393]
157	[291, 292]	536	[852, 853]	915	[1104, 1105]	1294	[574, 514]
158	[293, 294]	537	[854, 855]	916	[1106, 1107]	1295	[1433, 372]
159	[295, 296]	538	[856, 351]	917	[60, 1108]	1296	[454, 1396]
160	[58, 259]	539	[857, 858]	918	[1109, 1110]	1297	[493, 162]
161	[297, 298]	540	[844, 859]	919	[1111, 1112]	1298	[1011, 136]
162	[299, 93]	541	[541, 541]	920	[1113, 1114]	1299	[415, 47]
163	[300, 301]	542	[68, 677]	921	[1115, 1116]	1300	[388, 1048]
164	[302, 105]	543	[860, 861]	922	[619, 1117]	1301	[1102, 988]
165	[303, 304]	544	[862, 852]	923	[1118, 1119]	1302	[1073, 1396]
166	[305, 306]	545	[863, 864]	924	[72, 313]	1303	[472, 1032]
167	[307, 308]	546	[865, 866]	925	[1120, 1121]	1304	[192, 924]
168	[309, 310]	547	[867, 863]	926	[1122, 1123]	1305	[948, 1390]
169	[311, 312]	548	[868, 869]	927	[1124, 1125]	1306	[1032, 455]
170	[313, 314]	549	[752, 107]	928	[1126, 827]	1307	[537, 426]
171	[315, 316]	550	[870, 871]	929	[1127, 1128]	1308	[12, 938]
172	[317, 239]	551	[607, 872]	930	[327, 1129]	1309	[1414, 539]
173	[318, 319]	552	[873, 874]	931	[1130, 1131]	1310	[1430, 155]
174	[320, 321]	553	[875, 876]	932	[1132, 741]	1311	[518, 918]
175	[322, 323]	554	[877, 878]	933	[1133, 1134]	1312	[1411, 1001]
176	[324, 325]	555	[879, 251]	934	[1135, 1136]	1313	[1414, 499]
177	[326, 327]	556	[880, 881]	935	[294, 1137]	1314	[970, 1421]
178	[328, 329]	557	[882, 883]	936	[1138, 1139]	1315	[1020, 999]
179	[330, 331]	558	[884, 63]	937	[641, 363]	1316	[1393, 1010]
180	[332, 333]	559	[885, 349]	938	[1140, 293]	1317	[393, 1434]
181	[76, 334]	560	[886, 887]	939	[1141, 1142]	1318	[462, 945]
182	[335, 336]	561	[805, 888]	940	[1143, 1144]	1319	[198, 985]
183	[337, 338]	562	[346, 889]	941	[98, 1145]	1320	[485, 410]

184	[339, 340]	563	[890, 891]	942	[1146, 1147]	1321	[446, 466]
185	[341, 342]	564	[892, 893]	943	[1148, 632]	1322	[550, 968]
186	[343, 344]	565	[894, 895]	944	[1149, 1150]	1323	[1084, 998]
187	[345, 346]	566	[896, 897]	945	[350, 1151]	1324	[504, 190]
188	[347, 348]	567	[898, 237]	946	[1152, 1153]	1325	[495, 194]
189	[349, 350]	568	[899, 900]	947	[596, 1154]	1326	[553, 567]
190	[351, 352]	569	[901, 902]	948	[1155, 1156]	1327	[484, 466]
191	[353, 354]	570	[903, 904]	949	[1157, 1158]	1328	[434, 446]
192	[355, 356]	571	[905, 732]	950	[1159, 1160]	1329	[1066, 482]
193	[357, 358]	572	[906, 907]	951	[1161, 1162]	1330	[961, 1073]
194	[359, 360]	573	[795, 908]	952	[1163, 1109]	1331	[1435, 487]
195	[361, 362]	574	[909, 910]	953	[1164, 1165]	1332	[1097, 488]
196	[363, 364]	575	[911, 912]	954	[17, 1166]	1333	[968, 536]
197	[365, 366]	576	[913, 365]	955	[1167, 680]	1334	[456, 968]
198	[367, 368]	577	[914, 137]	956	[1168, 1169]	1335	[1436, 548]
199	[369, 370]	578	[555, 395]	957	[1170, 1171]	1336	[487, 1098]
200	[371, 32]	579	[915, 916]	958	[1172, 1173]	1337	[1437, 479]
201	[372, 188]	580	[182, 542]	959	[1174, 1175]	1338	[1438, 1030]
202	[373, 35]	581	[415, 529]	960	[1176, 1177]	1339	[971, 112]
203	[374, 375]	582	[917, 918]	961	[1178, 1179]	1340	[435, 466]
204	[376, 377]	583	[518, 505]	962	[201, 1180]	1341	[425, 159]
205	[378, 379]	584	[197, 496]	963	[1181, 1182]	1342	[1410, 1021]
206	[380, 381]	585	[538, 919]	964	[1183, 710]	1343	[917, 505]
207	[382, 383]	586	[920, 921]	965	[1184, 1185]	1344	[1032, 1396]
208	[384, 385]	587	[422, 922]	966	[102, 1186]	1345	[967, 535]
209	[386, 387]	588	[923, 924]	967	[1187, 1188]	1346	[557, 164]
210	[388, 381]	589	[426, 424]	968	[1189, 1190]	1347	[191, 573]
211	[42, 389]	590	[467, 473]	969	[1191, 302]	1348	[418, 944]
212	[390, 391]	591	[110, 924]	970	[1192, 1193]	1349	[917, 190]
213	[392, 393]	592	[449, 469]	971	[1194, 1195]	1350	[532, 41]
214	[394, 395]	593	[925, 926]	972	[747, 1196]	1351	[963, 147]
215	[9, 396]	594	[562, 492]	973	[1197, 103]	1352	[967, 457]
216	[177, 112]	595	[927, 928]	974	[1198, 1199]	1353	[154, 377]
217	[397, 398]	596	[929, 197]	975	[1200, 1201]	1354	[162, 160]
218	[187, 120]	597	[533, 930]	976	[1202, 1203]	1355	[544, 535]
219	[30, 399]	598	[931, 932]	977	[1204, 1146]	1356	[1030, 427]
220	[400, 401]	599	[386, 933]	978	[1205, 275]	1357	[1428, 1021]
221	[402, 403]	600	[934, 155]	979	[1206, 884]	1358	[195, 528]
222	[404, 35]	601	[448, 935]	980	[1207, 1208]	1359	[483, 435]
223	[44, 125]	602	[936, 937]	981	[1209, 1210]	1360	[550, 535]
224	[405, 406]	603	[559, 570]	982	[1211, 1212]	1361	[433, 1439]
225	[407, 408]	604	[551, 536]	983	[304, 1213]	1362	[133, 413]
226	[409, 181]	605	[483, 446]	984	[66, 1214]	1363	[11, 1074]
227	[410, 411]	606	[178, 938]	985	[786, 1215]	1364	[555, 1039]

228	[412, 413]	607	[939, 940]	986	[1216, 4]	1365	[535, 1389]
229	[414, 415]	608	[941, 463]	987	[1217, 1218]	1366	[967, 551]
230	[416, 194]	609	[54, 376]	988	[1219, 1191]	1367	[1440, 180]
231	[417, 119]	610	[942, 464]	989	[1220, 1221]	1368	[471, 430]
232	[418, 419]	611	[495, 53]	990	[1222, 1223]	1369	[1085, 1030]
233	[420, 421]	612	[943, 197]	991	[226, 271]	1370	[1074, 1441]
234	[422, 423]	613	[415, 52]	992	[1224, 1225]	1371	[486, 568]
235	[162, 424]	614	[519, 944]	993	[703, 99]	1372	[444, 486]
236	[425, 426]	615	[496, 533]	994	[1226, 1227]	1373	[1073, 455]
237	[12, 427]	616	[505, 945]	995	[1228, 1229]	1374	[483, 465]
238	[428, 10]	617	[137, 560]	996	[1230, 1231]	1375	[1064, 131]
239	[429, 409]	618	[946, 142]	997	[1203, 1232]	1376	[161, 426]
240	[430, 431]	619	[931, 531]	998	[276, 244]	1377	[441, 1003]
241	[409, 430]	620	[947, 163]	999	[236, 1233]	1378	[1029, 12]
242	[432, 433]	621	[920, 573]	1000	[1234, 1235]	1379	[111, 972]
243	[434, 435]	622	[948, 421]	1001	[1121, 1236]	1380	[495, 32]
244	[436, 437]	623	[949, 950]	1002	[1237, 1238]	1381	[1411, 156]
245	[438, 439]	624	[53, 33]	1003	[698, 1239]	1382	[1442, 1000]
246	[440, 441]	625	[38, 951]	1004	[1240, 357]	1383	[189, 505]
247	[442, 443]	626	[952, 499]	1005	[672, 267]	1384	[918, 561]
248	[444, 186]	627	[953, 556]	1006	[1241, 1242]	1385	[398, 379]
249	[445, 446]	628	[920, 176]	1007	[617, 1243]	1386	[1443, 461]
250	[33, 376]	629	[954, 459]	1008	[1244, 1245]	1387	[403, 1024]
251	[372, 447]	630	[32, 195]	1009	[1246, 1247]	1388	[1444, 72]
252	[402, 448]	631	[955, 944]	1010	[1248, 1249]	1389	[871, 1308]
253	[449, 166]	632	[943, 158]	1011	[1173, 1250]	1390	[1223, 880]
254	[450, 451]	633	[398, 138]	1012	[1251, 1252]	1391	[1445, 1446]
255	[450, 112]	634	[956, 957]	1013	[352, 760]	1392	[1447, 295]
256	[452, 453]	635	[542, 470]	1014	[1125, 1234]	1393	[1448, 1449]
257	[454, 455]	636	[958, 940]	1015	[1253, 1254]	1394	[1450, 1451]
258	[456, 457]	637	[574, 922]	1016	[1255, 1256]	1395	[1452, 1453]
259	[458, 459]	638	[175, 921]	1017	[1257, 22]	1396	[586, 1211]
260	[420, 114]	639	[959, 960]	1018	[1258, 1259]	1397	[801, 1454]
261	[460, 461]	640	[473, 455]	1019	[1260, 1261]	1398	[1455, 1456]
262	[189, 462]	641	[961, 473]	1020	[1262, 1263]	1399	[1457, 1111]
263	[463, 464]	642	[110, 962]	1021	[1264, 1265]	1400	[1458, 1281]
264	[169, 399]	643	[431, 146]	1022	[282, 1266]	1401	[1459, 255]
265	[465, 466]	644	[963, 413]	1023	[833, 1267]	1402	[828, 1360]
266	[467, 454]	645	[537, 162]	1024	[662, 1268]	1403	[1460, 1461]
267	[458, 468]	646	[964, 437]	1025	[1269, 1270]	1404	[1462, 1463]
268	[165, 469]	647	[965, 487]	1026	[331, 1271]	1405	[1464, 1465]
269	[182, 409]	648	[475, 966]	1027	[1272, 865]	1406	[1466, 913]
270	[146, 470]	649	[569, 523]	1028	[1273, 1274]	1407	[1467, 1126]
271	[471, 181]	650	[967, 968]	1029	[1275, 1276]	1408	[1468, 1176]

272	[181, 145]	651	[969, 503]	1030	[1277, 1278]	1409	[1469, 1470]
273	[133, 147]	652	[970, 423]	1031	[1279, 1280]	1410	[1471, 1472]
274	[472, 473]	653	[498, 919]	1032	[1281, 347]	1411	[646, 899]
275	[441, 474]	654	[971, 972]	1033	[1282, 659]	1412	[1345, 227]
276	[475, 476]	655	[973, 126]	1034	[1283, 318]	1413	[264, 1473]
277	[477, 478]	656	[399, 504]	1035	[1284, 1285]	1414	[1474, 1475]
278	[479, 480]	657	[974, 975]	1036	[266, 1286]	1415	[1476, 1275]
279	[481, 482]	658	[976, 916]	1037	[1287, 1288]	1416	[1477, 1478]
280	[483, 484]	659	[482, 977]	1038	[1289, 1290]	1417	[1263, 1264]
281	[485, 486]	660	[978, 147]	1039	[1291, 1292]	1418	[1479, 754]
282	[434, 484]	661	[979, 980]	1040	[1293, 1294]	1419	[1480, 667]
283	[487, 488]	662	[564, 173]	1041	[1295, 361]	1420	[1481, 1482]
284	[489, 490]	663	[382, 981]	1042	[1296, 1297]	1421	[1483, 1484]
285	[491, 194]	664	[530, 982]	1043	[1298, 1299]	1422	[1285, 1485]
286	[137, 379]	665	[983, 984]	1044	[1300, 1301]	1423	[284, 1124]
287	[121, 492]	666	[400, 985]	1045	[1302, 1303]	1424	[683, 1486]
288	[441, 490]	667	[139, 986]	1046	[1304, 1305]	1425	[1145, 1487]
289	[487, 443]	668	[987, 988]	1047	[1306, 1307]	1426	[1488, 1489]
290	[493, 426]	669	[178, 576]	1048	[1308, 1309]	1427	[1225, 775]
291	[494, 168]	670	[392, 192]	1049	[1310, 1311]	1428	[228, 1490]
292	[495, 171]	671	[513, 423]	1050	[1312, 15]	1429	[772, 1491]
293	[158, 496]	672	[953, 395]	1051	[1313, 1314]	1430	[660, 290]
294	[497, 39]	673	[378, 138]	1052	[1315, 1240]	1431	[738, 1492]
295	[498, 499]	674	[45, 374]	1053	[1201, 1282]	1432	[1493, 1494]
296	[500, 501]	675	[989, 990]	1054	[687, 85]	1433	[1495, 898]
297	[502, 503]	676	[34, 571]	1055	[1316, 798]	1434	[613, 903]
298	[500, 176]	677	[542, 991]	1056	[214, 1317]	1435	[1496, 777]
299	[174, 469]	678	[130, 130]	1057	[1318, 1319]	1436	[1497, 1498]
300	[404, 49]	679	[991, 429]	1058	[1271, 1320]	1437	[1499, 702]
301	[504, 505]	680	[991, 431]	1059	[1321, 1322]	1438	[1500, 673]
302	[198, 401]	681	[470, 182]	1060	[1323, 1324]	1439	[1332, 283]
303	[506, 507]	682	[992, 993]	1061	[1325, 1326]	1440	[1501, 1502]
304	[373, 49]	683	[994, 490]	1062	[1327, 1328]	1441	[1254, 89]
305	[508, 142]	684	[149, 486]	1063	[1329, 1330]	1442	[1372, 1503]
306	[509, 510]	685	[425, 995]	1064	[1331, 1332]	1443	[1504, 1283]
307	[511, 452]	686	[969, 163]	1065	[1333, 1334]	1444	[1505, 170]
308	[175, 501]	687	[949, 395]	1066	[1335, 1336]	1445	[995, 424]
309	[512, 469]	688	[9, 940]	1067	[1337, 1204]	1446	[434, 465]
310	[513, 514]	689	[461, 464]	1068	[1338, 1339]	1447	[393, 962]
311	[515, 372]	690	[554, 563]	1069	[1340, 1341]	1448	[1023, 926]
312	[491, 53]	691	[119, 135]	1070	[1342, 1343]	1449	[195, 983]
313	[170, 504]	692	[996, 997]	1071	[1344, 1345]	1450	[1000, 999]
314	[516, 517]	693	[926, 156]	1072	[1346, 636]	1451	[943, 563]
315	[31, 518]	694	[559, 170]	1073	[592, 1347]	1452	[1019, 503]

316	[519, 520]	695	[195, 154]	1074	[1348, 1349]	1453	[953, 950]
317	[521, 522]	696	[927, 464]	1075	[1350, 716]	1454	[399, 189]
318	[523, 184]	697	[941, 461]	1076	[1351, 1352]	1455	[983, 377]
319	[481, 524]	698	[374, 492]	1077	[1353, 730]	1456	[29, 567]
320	[525, 526]	699	[998, 999]	1078	[1354, 1355]	1457	[42, 1046]
321	[527, 517]	700	[1000, 937]	1079	[1356, 708]	1458	[452, 468]
322	[528, 377]	701	[124, 417]	1080	[1357, 1358]	1459	[1081, 452]
323	[46, 529]	702	[153, 376]	1081	[15, 720]	1460	[935, 534]
324	[530, 531]	703	[155, 1001]	1082	[1359, 905]	1461	[378, 406]
325	[532, 387]	704	[1002, 479]	1083	[1360, 1361]	1462	[1442, 998]
326	[533, 534]	705	[489, 1003]	1084	[1362, 1359]	1463	[50, 533]
327	[117, 199]	706	[1004, 393]	1085	[82, 1140]	1464	[978, 413]
328	[497, 123]	707	[450, 115]	1086	[1363, 1364]	1465	[456, 535]
329	[518, 462]	708	[572, 421]	1087	[1365, 1366]	1466	[375, 1007]
330	[535, 536]	709	[971, 451]	1088	[1367, 1368]	1467	[457, 1389]
331	[537, 159]	710	[169, 31]	1089	[1369, 886]	1468	[528, 984]
332	[503, 164]	711	[46, 1005]	1090	[874, 1370]	1469	[460, 463]
333	[538, 539]	712	[1006, 980]	1091	[1371, 1372]	1470	[50, 935]
334	[540, 129]	713	[497, 951]	1092	[1373, 1374]	1471	[543, 433]
335	[540, 541]	714	[135, 1007]	1093	[1375, 1376]	1472	[425, 162]
336	[429, 542]	715	[1008, 52]	1094	[247, 1377]	1473	[405, 379]
337	[543, 131]	716	[1009, 932]	1095	[1378, 1379]	1474	[1506, 37]
338	[544, 457]	717	[141, 1010]	1096	[755, 1380]	1475	[172, 565]
339	[438, 545]	718	[974, 39]	1097	[1381, 804]	1476	[1035, 1033]
340	[478, 546]	719	[562, 1011]	1098	[287, 758]	1477	[446, 1054]
341	[547, 548]	720	[29, 447]	1099	[1382, 1383]	1478	[161, 995]
342	[549, 476]	721	[31, 917]	1100	[1280, 693]	1479	[151, 1021]
343	[185, 486]	722	[948, 114]	1101	[1384, 1385]	1480	[1070, 988]
344	[550, 551]	723	[970, 514]	1102	[1386, 1387]	1481	[569, 482]
345	[552, 553]	724	[959, 558]	1103	[1388, 923]	1482	[537, 995]
346	[554, 197]	725	[947, 503]	1104	[551, 1389]	1483	[509, 1046]
347	[555, 556]	726	[111, 451]	1105	[472, 1073]	1484	[508, 1507]
348	[557, 558]	727	[113, 421]	1106	[1074, 468]	1485	[549, 966]
349	[559, 31]	728	[516, 173]	1107	[113, 1390]	1486	[993, 1427]
350	[378, 560]	729	[403, 935]	1108	[117, 1056]	1487	[186, 1508]
351	[418, 520]	730	[117, 401]	1109	[974, 123]	1488	[1076, 461]
352	[448, 51]	731	[563, 448]	1110	[153, 983]	1489	[33, 983]
353	[190, 561]	732	[1012, 131]	1111	[1060, 387]	1490	[961, 1032]
354	[168, 401]	733	[445, 465]	1112	[1391, 1392]	1491	[994, 1100]
355	[122, 39]	734	[438, 476]	1113	[1024, 534]	1492	[442, 480]
356	[195, 376]	735	[994, 1003]	1114	[29, 1090]	1493	[1021, 1033]
357	[400, 199]	736	[504, 462]	1115	[1069, 986]	1494	[566, 170]
358	[125, 562]	737	[993, 1013]	1116	[1393, 1052]	1495	[1438, 12]
359	[563, 496]	738	[548, 476]	1117	[1016, 1394]	1496	[1443, 463]

360	[564, 565]	739	[544, 551]	1118	[527, 173]	1497	[934, 926]
361	[566, 399]	740	[134, 1014]	1119	[189, 918]	1498	[917, 462]
362	[372, 567]	741	[952, 919]	1120	[432, 131]	1499	[925, 155]
363	[150, 568]	742	[393, 193]	1121	[1068, 473]	1500	[124, 45]
364	[569, 183]	743	[1015, 452]	1122	[442, 488]	1501	[159, 424]
365	[570, 518]	744	[394, 556]	1123	[993, 546]	1502	[550, 457]
366	[373, 571]	745	[1008, 529]	1124	[489, 474]	1503	[467, 1073]
367	[572, 114]	746	[1016, 1017]	1125	[964, 1013]	1504	[481, 523]
368	[500, 573]	747	[1018, 140]	1126	[1028, 484]	1505	[1509, 804]
369	[574, 423]	748	[129, 540]	1127	[1395, 553]	1506	[1510, 1511]
370	[575, 576]	749	[1019, 163]	1128	[44, 417]	1507	[1512, 1513]
371	[577, 285]	750	[422, 514]	1129	[53, 153]	1508	[342, 1230]
372	[268, 578]	751	[538, 499]	1130	[473, 1396]	1509	[28, 1051]
373	[579, 580]	752	[1020, 937]	1131	[1028, 435]	1510	[376, 984]
374	[230, 581]	753	[463, 928]	1132	[557, 1397]	1511	[126, 1005]
375	[19, 582]	754	[562, 135]	1133	[461, 928]	1512	[926, 1001]
376	[316, 583]	755	[1021, 1022]	1134	[118, 417]	1513	[416, 32]
377	[108, 584]	756	[1023, 155]	1135	[149, 186]		
378	[585, 586]	757	[121, 1011]	1136	[544, 968]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	253	0	506	1	759	0	1012	1	1265	1
1	0	254	1	507	1	760	0	1013	1	1266	1
2	0	255	1	508	0	761	1	1014	1	1267	1
3	0	256	0	509	1	762	0	1015	0	1268	0
4	1	257	0	510	1	763	0	1016	1	1269	0
5	0	258	1	511	1	764	0	1017	1	1270	0
6	1	259	0	512	0	765	1	1018	1	1271	0
7	0	260	1	513	1	766	1	1019	1	1272	0
8	0	261	1	514	0	767	1	1020	1	1273	1
9	1	262	0	515	1	768	0	1021	0	1274	1
10	0	263	1	516	0	769	0	1022	0	1275	1
11	1	264	1	517	1	770	1	1023	1	1276	0
12	1	265	1	518	1	771	0	1024	0	1277	0
13	1	266	0	519	0	772	0	1025	0	1278	0
14	0	267	0	520	1	773	1	1026	1	1279	0
15	0	268	1	521	1	774	0	1027	0	1280	1
16	0	269	1	522	0	775	1	1028	1	1281	1
17	1	270	1	523	0	776	1	1029	1	1282	1
18	1	271	1	524	0	777	0	1030	0	1283	1
19	1	272	1	525	1	778	1	1031	0	1284	1
20	0	273	0	526	1	779	1	1032	1	1285	0
21	0	274	1	527	1	780	1	1033	1	1286	0

22	1	275	1	528	1	781	1	1034	1	1287	1
23	1	276	1	529	0	782	0	1035	1	1288	0
24	1	277	1	530	0	783	1	1036	0	1289	1
25	0	278	0	531	1	784	0	1037	1	1290	1
26	1	279	0	532	1	785	0	1038	1	1291	1
27	1	280	1	533	1	786	0	1039	1	1292	1
28	0	281	0	534	0	787	1	1040	1	1293	1
29	1	282	0	535	1	788	0	1041	0	1294	1
30	0	283	1	536	1	789	0	1042	0	1295	0
31	0	284	1	537	1	790	0	1043	0	1296	0
32	0	285	1	538	1	791	0	1044	0	1297	1
33	0	286	0	539	0	792	0	1045	0	1298	0
34	1	287	1	540	1	793	0	1046	1	1299	0
35	0	288	1	541	0	794	1	1047	1	1300	0
36	1	289	1	542	0	795	0	1048	1	1301	0
37	0	290	1	543	1	796	0	1049	0	1302	0
38	1	291	0	544	0	797	0	1050	1	1303	1
39	0	292	1	545	0	798	1	1051	0	1304	1
40	0	293	0	546	0	799	0	1052	1	1305	0
41	1	294	0	547	0	800	0	1053	0	1306	1
42	0	295	0	548	0	801	1	1054	0	1307	1
43	0	296	0	549	1	802	1	1055	1	1308	1
44	1	297	0	550	0	803	0	1056	1	1309	0
45	0	298	0	551	0	804	0	1057	0	1310	0
46	1	299	1	552	0	805	0	1058	0	1311	1
47	1	300	0	553	0	806	1	1059	0	1312	1
48	1	301	0	554	1	807	1	1060	0	1313	0
49	1	302	0	555	1	808	0	1061	1	1314	0
50	0	303	1	556	1	809	0	1062	0	1315	1
51	1	304	0	557	0	810	1	1063	1	1316	1
52	1	305	0	558	0	811	1	1064	0	1317	0
53	1	306	1	559	0	812	0	1065	0	1318	0
54	0	307	1	560	0	813	1	1066	1	1319	0
55	0	308	1	561	0	814	1	1067	0	1320	0
56	0	309	0	562	1	815	1	1068	0	1321	0
57	1	310	1	563	1	816	0	1069	1	1322	0
58	0	311	1	564	0	817	0	1070	1	1323	0
59	0	312	1	565	0	818	1	1071	1	1324	0
60	1	313	0	566	1	819	1	1072	0	1325	1
61	0	314	0	567	0	820	0	1073	0	1326	0
62	1	315	0	568	0	821	0	1074	1	1327	0
63	0	316	0	569	1	822	0	1075	1	1328	0
64	1	317	1	570	1	823	1	1076	1	1329	1
65	0	318	0	571	1	824	1	1077	1	1330	1

66	1	319	0	572	0	825	0	1078	1	1331	0
67	1	320	1	573	0	826	0	1079	0	1332	1
68	0	321	1	574	1	827	1	1080	1	1333	0
69	0	322	1	575	1	828	0	1081	0	1334	1
70	0	323	1	576	0	829	0	1082	1	1335	1
71	1	324	0	577	0	830	0	1083	0	1336	1
72	0	325	1	578	1	831	0	1084	0	1337	0
73	0	326	1	579	0	832	1	1085	0	1338	0
74	1	327	1	580	1	833	1	1086	1	1339	0
75	1	328	0	581	0	834	1	1087	1	1340	1
76	1	329	1	582	1	835	0	1088	0	1341	0
77	0	330	1	583	1	836	0	1089	0	1342	1
78	1	331	1	584	1	837	1	1090	1	1343	1
79	0	332	1	585	1	838	0	1091	1	1344	1
80	1	333	1	586	0	839	0	1092	0	1345	1
81	1	334	1	587	0	840	1	1093	0	1346	0
82	0	335	1	588	1	841	1	1094	0	1347	1
83	0	336	0	589	1	842	1	1095	1	1348	1
84	0	337	1	590	0	843	1	1096	0	1349	1
85	0	338	0	591	0	844	1	1097	1	1350	1
86	1	339	0	592	0	845	0	1098	1	1351	1
87	1	340	0	593	0	846	0	1099	1	1352	1
88	1	341	0	594	1	847	1	1100	1	1353	1
89	0	342	1	595	1	848	1	1101	0	1354	1
90	1	343	0	596	1	849	1	1102	0	1355	0
91	1	344	0	597	1	850	0	1103	0	1356	0
92	1	345	0	598	1	851	1	1104	0	1357	1
93	1	346	1	599	1	852	1	1105	1	1358	1
94	0	347	1	600	1	853	1	1106	1	1359	1
95	1	348	0	601	1	854	1	1107	1	1360	0
96	1	349	0	602	0	855	1	1108	1	1361	1
97	1	350	1	603	0	856	1	1109	0	1362	0
98	0	351	1	604	0	857	0	1110	1	1363	1
99	0	352	1	605	1	858	1	1111	0	1364	1
100	1	353	1	606	0	859	1	1112	0	1365	1
101	1	354	0	607	0	860	1	1113	0	1366	1
102	0	355	1	608	0	861	0	1114	1	1367	0
103	1	356	1	609	0	862	0	1115	1	1368	1
104	1	357	1	610	0	863	0	1116	1	1369	0
105	1	358	1	611	1	864	0	1117	1	1370	1
106	1	359	1	612	0	865	0	1118	1	1371	1
107	0	360	0	613	0	866	1	1119	0	1372	1
108	0	361	1	614	0	867	0	1120	0	1373	0
109	0	362	1	615	1	868	0	1121	0	1374	1

110	0	363	0	616	1	869	1	1122	0	1375	0
111	0	364	1	617	0	870	0	1123	1	1376	0
112	1	365	1	618	0	871	0	1124	1	1377	1
113	1	366	0	619	1	872	1	1125	1	1378	1
114	1	367	0	620	1	873	0	1126	1	1379	0
115	0	368	0	621	0	874	1	1127	1	1380	1
116	0	369	1	622	0	875	0	1128	1	1381	1
117	1	370	1	623	0	876	1	1129	1	1382	1
118	1	371	0	624	1	877	1	1130	1	1383	0
119	0	372	1	625	1	878	0	1131	1	1384	0
120	0	373	0	626	1	879	1	1132	0	1385	0
121	1	374	0	627	0	880	1	1133	0	1386	0
122	1	375	1	628	0	881	0	1134	1	1387	0
123	1	376	0	629	1	882	0	1135	1	1388	0
124	0	377	0	630	0	883	1	1136	0	1389	0
125	1	378	1	631	1	884	0	1137	1	1390	1
126	1	379	0	632	0	885	0	1138	0	1391	0
127	1	380	1	633	0	886	0	1139	1	1392	0
128	1	381	0	634	0	887	0	1140	0	1393	1
129	0	382	0	635	0	888	0	1141	0	1394	0
130	1	383	1	636	1	889	0	1142	0	1395	1
131	0	384	1	637	1	890	1	1143	0	1396	0
132	0	385	1	638	1	891	1	1144	1	1397	1
133	0	386	1	639	1	892	0	1145	0	1398	0
134	1	387	0	640	1	893	1	1146	0	1399	0
135	1	388	0	641	1	894	0	1147	1	1400	0
136	1	389	0	642	0	895	1	1148	0	1401	0
137	0	390	0	643	0	896	1	1149	0	1402	0
138	1	391	0	644	1	897	0	1150	1	1403	0
139	0	392	0	645	1	898	0	1151	0	1404	1
140	0	393	0	646	1	899	0	1152	0	1405	0
141	1	394	1	647	1	900	1	1153	0	1406	1
142	1	395	0	648	1	901	1	1154	0	1407	1
143	1	396	1	649	1	902	0	1155	0	1408	1
144	0	397	1	650	1	903	1	1156	1	1409	1
145	1	398	0	651	0	904	0	1157	0	1410	1
146	1	399	1	652	0	905	1	1158	0	1411	1
147	0	400	1	653	0	906	0	1159	0	1412	1
148	1	401	0	654	0	907	0	1160	1	1413	1
149	1	402	0	655	0	908	0	1161	0	1414	0
150	0	403	0	656	1	909	1	1162	0	1415	1
151	0	404	0	657	0	910	1	1163	1	1416	0
152	1	405	1	658	1	911	1	1164	0	1417	1
153	1	406	1	659	1	912	1	1165	1	1418	0

154	1	407	1	660	0	913	0	1166	0	1419	1
155	1	408	0	661	1	914	0	1167	1	1420	1
156	1	409	0	662	0	915	0	1168	0	1421	1
157	0	410	1	663	0	916	1	1169	0	1422	0
158	0	411	1	664	0	917	1	1170	1	1423	1
159	0	412	1	665	0	918	0	1171	0	1424	0
160	0	413	1	666	1	919	0	1172	1	1425	0
161	0	414	0	667	0	920	0	1173	0	1426	1
162	1	415	0	668	1	921	1	1174	1	1427	1
163	0	416	0	669	0	922	1	1175	1	1428	1
164	0	417	1	670	0	923	1	1176	1	1429	0
165	1	418	1	671	1	924	0	1177	0	1430	0
166	0	419	1	672	0	925	0	1178	1	1431	0
167	1	420	1	673	1	926	0	1179	1	1432	0
168	0	421	0	674	0	927	1	1180	0	1433	0
169	1	422	0	675	0	928	1	1181	1	1434	0
170	0	423	0	676	1	929	1	1182	0	1435	0
171	0	424	1	677	0	930	1	1183	1	1436	1
172	1	425	0	678	1	931	1	1184	1	1437	0
173	0	426	1	679	0	932	0	1185	1	1438	0
174	1	427	1	680	0	933	0	1186	1	1439	1
175	1	428	0	681	0	934	1	1187	1	1440	0
176	0	429	0	682	0	935	0	1188	1	1441	0
177	1	430	1	683	0	936	0	1189	0	1442	1
178	0	431	0	684	1	937	1	1190	0	1443	0
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181	1	434	0	687	0	940	0	1193	0	1446	0
182	1	435	1	688	1	941	0	1194	0	1447	0
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184	0	437	0	690	1	943	0	1196	1	1449	1
185	0	438	0	691	0	944	0	1197	0	1450	0
186	0	439	1	692	0	945	1	1198	0	1451	0
187	0	440	0	693	0	946	0	1199	1	1452	1
188	1	441	1	694	0	947	1	1200	1	1453	0
189	0	442	0	695	1	948	0	1201	0	1454	1
190	1	443	1	696	1	949	0	1202	1	1455	0
191	1	444	1	697	0	950	0	1203	0	1456	1
192	1	445	0	698	0	951	0	1204	1	1457	0
193	1	446	0	699	1	952	1	1205	0	1458	0
194	1	447	0	700	0	953	0	1206	1	1459	0
195	1	448	1	701	0	954	1	1207	1	1460	0
196	0	449	0	702	1	955	1	1208	0	1461	1
197	1	450	1	703	1	956	0	1209	0	1462	1

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199	1	452	0	705	1	958	1	1211	1	1464	0
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201	1	454	0	707	1	960	1	1213	0	1466	1
202	0	455	1	708	0	961	1	1214	1	1467	1
203	0	456	1	709	0	962	1	1215	0	1468	1
204	0	457	1	710	1	963	1	1216	1	1469	1
205	1	458	0	711	1	964	1	1217	1	1470	0
206	1	459	1	712	0	965	1	1218	0	1471	1
207	0	460	1	713	0	966	0	1219	1	1472	0
208	1	461	0	714	1	967	1	1220	0	1473	1
209	1	462	0	715	0	968	0	1221	0	1474	0
210	0	463	1	716	0	969	0	1222	0	1475	1
211	0	464	0	717	1	970	0	1223	1	1476	1
212	0	465	1	718	0	971	0	1224	0	1477	0
213	0	466	1	719	1	972	1	1225	1	1478	0
214	1	467	0	720	1	973	0	1226	0	1479	0
215	1	468	0	721	0	974	0	1227	0	1480	1
216	1	469	1	722	0	975	1	1228	0	1481	1
217	1	470	0	723	0	976	1	1229	1	1482	1
218	0	471	1	724	1	977	1	1230	0	1483	1
219	0	472	1	725	1	978	0	1231	0	1484	0
220	1	473	1	726	0	979	1	1232	1	1485	1
221	0	474	1	727	1	980	1	1233	0	1486	1
222	0	475	1	728	0	981	0	1234	0	1487	0
223	1	476	1	729	0	982	1	1235	0	1488	1
224	1	477	1	730	1	983	0	1236	0	1489	0
225	1	478	0	731	1	984	1	1237	1	1490	1
226	0	479	0	732	1	985	0	1238	0	1491	0
227	1	480	0	733	0	986	1	1239	0	1492	0
228	1	481	0	734	0	987	1	1240	0	1493	0
229	0	482	1	735	0	988	1	1241	0	1494	1
230	0	483	1	736	0	989	0	1242	1	1495	0
231	1	484	0	737	1	990	0	1243	0	1496	0
232	1	485	0	738	0	991	0	1244	0	1497	1
233	1	486	1	739	0	992	0	1245	1	1498	1
234	0	487	1	740	1	993	1	1246	0	1499	0
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236	0	489	1	742	0	995	0	1248	0	1501	0
237	1	490	0	743	0	996	0	1249	1	1502	0
238	0	491	1	744	1	997	0	1250	1	1503	0
239	0	492	0	745	0	998	1	1251	1	1504	0
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241	0	494	0	747	1	1000	0	1253	0	1506	0

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243	0	496	1	749	1	1002	1	1255	1	1508	1
244	0	497	0	750	0	1003	0	1256	1	1509	0
245	0	498	0	751	1	1004	0	1257	1	1510	0
246	0	499	1	752	1	1005	0	1258	1	1511	1
247	0	500	0	753	1	1006	0	1259	1	1512	0
248	1	501	1	754	1	1007	0	1260	1	1513	0
249	0	502	0	755	0	1008	0	1261	1		
250	0	503	1	756	1	1009	0	1262	1		
251	1	504	0	757	1	1010	0	1263	1		
252	0	505	1	758	0	1011	0	1264	0		

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