# PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a, b) = (z^4 + z^3 + z + 1, z^3 + z^2 + z + 1)$

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Abstract. The continued fraction conjecture claims that the continued fraction defined by the $(a, b)$-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements $(a, b)$ in $\mathbb{F}_2[x] \setminus \mathbb{F}_2$. In this paper we prove the conjecture for $(a, b) = (z^4 + z^3 + z + 1, z^3 + z^2 + z + 1)$.

## 1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \ldots)$ of elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{\ddots + \cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let $a$ and $b$ be two distinct elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$. We consider the continued fraction (1.1) associated with the $(a, b)$-Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \ldots) = (a, b, b, a, b, a, a, b, \ldots)$$

that we shall denote

$$\mathrm{CF}(a, b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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**Theorem 1.1.** *When* $(a,b)=(z^4+z^3+z+1, z^3+z^2+z+1)$, *the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$, *with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

*where, for* $\mathrm{CF}(a,b)$,

$$\begin{aligned}
p_0(z)&=z^{13}+z^{10}+z^9+z^4+z^3+1,\\
p_1(z)&=z^{17}+z^{15}+z^{14}+z^{13}+z^{12}+z^{11}+z^{10}+z^8+z^7+z^5+z^4+z^2,\\
p_2(z)&=z^{20}+z^{14}+z^{12}+z^{10}+z^4+1,\\
p_3(z)&=z^{17}+z^{15}+z^{14}+z^{13}+z^{12}+z^{11}+z^{10}+z^8+z^7+z^5+z^4+z^2,\\
p_4(z)&=z^{14}+z^{13}+z^9+z^3+1,
\end{aligned}$$

*and for* $\mathrm{CF}(b,a)$,

$$\begin{aligned}
p_0(z)&=z^{13}+z^{10}+z^9+z^8+z^4+z^3+1,\\
p_1(z)&=z^{17}+z^{15}+z^{14}+z^{13}+z^{12}+z^{11}+z^{10}+z^8+z^7+z^5+z^4+z^2,\\
p_2(z)&=z^{20}+z^{14}+z^{12}+z^{10}+z^4+1,\\
p_3(z)&=z^{17}+z^{15}+z^{14}+z^{13}+z^{12}+z^{11}+z^{10}+z^8+z^7+z^5+z^4+z^2,\\
p_4(z)&=z^{14}+z^{13}+z^9+z^8+z^3+1.
\end{aligned}$$

## 2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x)={}&x^{\deg(t_{2^n-1})}\begin{pmatrix} t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\cdots\times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x)={}&x^{\deg(\bar t_{2^n-1})}\begin{pmatrix} \bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix} \bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\cdots\times x^{\deg(\bar t_0)}\begin{pmatrix} \bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the $(b,a)$-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\geq 0$

$$\begin{aligned}
M_{n+1}(x)&=W_n(x)\cdot M_n(x),\\
W_{n+1}(x)&=M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer $d_n$. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four $2\times 2$ matrices $M^e, M^o, W^e, W^o$ as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(175, 175, 175, 175, 175)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices $M^e, M^o, W^e, W^o$. The first one is about relations between them; the second, about the structure of the each matrix.

**Lemma 2.1.** *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

*Proof.* We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z \left(\phi(W^o,0,0)(x,z),\ z^{12} \cdot \phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity $m$ and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 630 and substitute it for $y$ in $Q(x,y)$. We get a series of valuation less than 630, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z \left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity $\mu$, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^{\mu}$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ to order 770 and substitute it for $y$ in $Q(x,y)$. We get a series of valuation less than 770, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. $\square$

Define

$$R^e = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

**Lemma 2.2.** *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned} M^e[:14u] = {} & M^e[:7u] + x^{3u}M^e[:4u] + x^{4u}M^e[:3u] + x^{6u}M^e[:u] + x^{3u}M^e[:7u] \\ & + x^{6u}M^e[:4u] + x^{7u}M^e[:3u] + x^{9u}M^e[:u] + x^{4u}M^e[:7u] \\ & + x^{7u}M^e[:4u] + x^{8u}M^e[:3u] + x^{10u}M^e[:u] + x^{7u}M^e[:7u] \\ & + x^{8u}M^e[:6u] + x^{12u}M^e[:2u] + x^{11u}M^e[:3u] + x^{13u}M^e[:u] \\ & + (x^{7u} + x^{10u} + x^{11u})R^e, \\ W^e[:14u] = {} & W^e[:7u] + x^{3u}W^e[:4u] + x^{4u}W^e[:3u] + x^{6u}W^e[:u] + x^{3u}W^e[:7u] \end{aligned}$$

$$
\begin{aligned}
&+ x^{6u}W^e[:4u] + x^{7u}W^e[:3u] + x^{9u}W^e[:u] + x^{4u}W^e[:7u] \\
&+ x^{7u}W^e[:4u] + x^{8u}W^e[:3u] + x^{10u}W^e[:u] + x^{7u}W^e[:7u] \\
&+ x^{8u}W^e[:6u] + x^{12u}W^e[:2u] + x^{11u}W^e[:3u] + x^{13u}W^e[:u] \\
&+ (x^{7u} + x^{10u} + x^{11u})R^e.
\end{aligned}
$$

*For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.*

$$
\begin{aligned}
M^o[:14u] = {} & M^o[:7u] + x^{3u}M^o[:4u] + x^{4u}M^o[:3u] + x^{6u}M^o[:u] + x^{3u}M^o[:7u] \\
&+ x^{6u}M^o[:4u] + x^{7u}M^o[:3u] + x^{9u}M^o[:u] + x^{4u}M^o[:7u] \\
&+ x^{7u}M^o[:4u] + x^{8u}M^o[:3u] + x^{10u}M^o[:u] + x^{7u}M^o[:7u] \\
&+ x^{8u}M^o[:6u] + x^{12u}M^o[:2u] + x^{11u}M^o[:3u] + x^{13u}M^o[:u] \\
&+ (x^{7u} + x^{10u} + x^{11u})R^o, \\
W^o[:14u] = {} & W^o[:7u] + x^{3u}W^o[:4u] + x^{4u}W^o[:3u] + x^{6u}W^o[:u] + x^{3u}W^o[:7u] \\
&+ x^{6u}W^o[:4u] + x^{7u}W^o[:3u] + x^{9u}W^o[:u] + x^{4u}W^o[:7u] \\
&+ x^{7u}W^o[:4u] + x^{8u}W^o[:3u] + x^{10u}W^o[:u] + x^{7u}W^o[:7u] \\
&+ x^{8u}W^o[:6u] + x^{12u}W^o[:2u] + x^{11u}W^o[:3u] + x^{13u}W^o[:u] \\
&+ (x^{7u} + x^{10u} + x^{11u})R^o.
\end{aligned}
$$

*Proof.* To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many $k$'s into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$
\begin{aligned}
T[:14u] = {} & T[:7u] + x^{3u}T[:4u] + x^{4u}T[:3u] + x^{6u}T[:u] + x^{3u}T[:7u] + x^{6u}T[:4u] \\
&+ x^{7u}T[:3u] + x^{9u}T[:u] + x^{4u}T[:7u] + x^{7u}T[:4u] + x^{8u}T[:3u] \\
&+ x^{10u}T[:u] + x^{7u}T[:7u] + x^{8u}T[:6u] + x^{12u}T[:2u] + x^{11u}T[:3u] \\
&+ x^{13u}T[:u] + (x^{7u} + x^{10u} + x^{11u}).
\end{aligned}
$$

The proof of the other 15 cases are similar. We break down the above identity into 7 parts:

$$
\begin{aligned}
x^7u &= x^{7u}T[:u] + x^{6u}T[u:2u] + x^{4u}T[3u:4u] + x^{3u}T[4u:5u] \\
&\quad + T[7u:8u], \\
0 &= x^{7u}T[u:2u] + x^{6u}T[2u:3u] + x^{4u}T[4u:5u] + x^{3u}T[5u:6u] \\
&\quad + T[8u:9u], \\
0 &= x^{9u}T[:u] + x^{7u}T[2u:3u] + x^{6u}T[3u:4u] + x^{4u}T[5u:6u] \\
&\quad + x^{3u}T[6u:7u] + T[9u:10u], \\
x^10u &= x^{10u}T[:u] + x^{4u}T[6u:7u] + T[10u:11u], \\
x^11u &= x^{11u}T[:u] + x^{8u}T[3u:4u] + x^{7u}T[4u:5u] + T[11u:12u], \\
0 &= x^{12u}T[:u] + x^{11u}T[u:2u] + x^{8u}T[4u:5u] + x^{7u}T[5u:6u] \\
&\quad + T[12u:13u], \\
0 &= x^{13u}T[:u] + x^{12u}T[u:2u] + x^{11u}T[2u:3u] + x^{8u}T[5u:6u]
\end{aligned}
$$

$$+ x^{7u}T[6u\!:\!7u] + T[13u\!:\!14u],$$

which can be rewritten as

$$0 = T[[w]_2] + T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[111w]_2], \tag{2.7}$$

$$0 = T[[1w]_2] + T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.8}$$

$$\begin{aligned} 0 = {} & T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2] \\ & + T[[1001w]_2], \end{aligned} \tag{2.9}$$

$$0 = T[[w]_2] + T[[110w]_2] + T[[1010w]_2], \tag{2.10}$$

$$0 = T[[w]_2] + T[[11w]_2] + T[[100w]_2] + T[[1011w]_2], \tag{2.11}$$

$$0 = T[[w]_2] + T[[1w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1100w]_2], \tag{2.12}$$

$$\begin{aligned} 0 = {} & T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[101w]_2] + T[[110w]_2] \\ & + T[[1101w]_2], \end{aligned} \tag{2.13}$$

for all binary word $w$ of length $2k$ and $w \neq 0^{2k}$, and

$$1 = T[[w]_2] + T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[111w]_2], \tag{2.14}$$

$$0 = T[[1w]_2] + T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.15}$$

$$\begin{aligned} 0 = {} & T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2] \\ & + T[[1001w]_2], \end{aligned} \tag{2.16}$$

$$1 = T[[w]_2] + T[[110w]_2] + T[[1010w]_2], \tag{2.17}$$

$$1 = T[[w]_2] + T[[11w]_2] + T[[100w]_2] + T[[1011w]_2], \tag{2.18}$$

$$0 = T[[w]_2] + T[[1w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1100w]_2], \tag{2.19}$$

$$\begin{aligned} 0 = {} & T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[101w]_2] + T[[110w]_2] \\ & + T[[1101w]_2], \end{aligned} \tag{2.20}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates $T$ from its minimal polynomial and its first terms. This automaton has 509 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading $w$ from right to left starting from the state $s$, and $\tau$ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.20) can be written as

$$\begin{aligned} 0 = {} & \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,100)) \\ & + \tau(A(s,111)), \end{aligned} \tag{2.21}$$

$$\begin{aligned} 0 = {} & \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,101)) \\ & + \tau(A(s,1000)), \end{aligned} \tag{2.22}$$

$$\begin{aligned} 0 = {} & \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101)) \\ & + \tau(A(s,110)) + \tau(A(s,1001)), \end{aligned} \tag{2.23}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,110)) + \tau(A(s,1010)), \tag{2.24}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,11)) + \tau(A(s,100)) + \tau(A(s,1011)), \tag{2.25}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,100)) + \tau(A(s,101))$$

$$+\tau(A(s,1100)), \tag{2.26}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,101)) \\ &\quad + \tau(A(s,110)) + \tau(A(s,1101)), \end{aligned} \tag{2.27}$$

for all $s \in E_{2k}$, and

$$\begin{aligned} 1 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,100)) \\ &\quad + \tau(A(s,111)), \end{aligned} \tag{2.28}$$

$$\begin{aligned} 0 &= \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,101)) \\ &\quad + \tau(A(s,1000)), \end{aligned} \tag{2.29}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101)) \\ &\quad + \tau(A(s,110)) + \tau(A(s,1001)), \end{aligned} \tag{2.30}$$

$$1 = \tau(A(s,\epsilon)) + \tau(A(s,110)) + \tau(A(s,1010)), \tag{2.31}$$

$$1 = \tau(A(s,\epsilon)) + \tau(A(s,11)) + \tau(A(s,100)) + \tau(A(s,1011)), \tag{2.32}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,100)) + \tau(A(s,101)) \\ &\quad + \tau(A(s,1100)), \end{aligned} \tag{2.33}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,101)) \\ &\quad + \tau(A(s,110)) + \tau(A(s,1101)), \end{aligned} \tag{2.34}$$

for $s = A(i, 0^{2k})$. We find that $(A(i,0^{28}), E_{28}) = (A(i,0^{20}), E_{20})$, so that we only have to verify that identities (2.21) through (2.34) hold for $2 \le k \le 14$, which turns out to be true. □

In the following lemma, we express $M_{2k}$, $M_{2k+1}$, $W_{2k}$, and $W_{2k+1}$ in terms of $M^e$, $M^o$, $W^e$, and $W^o$.

**Lemma 2.3.** *For all integer $k \ge 2$, and $u = 2^{2k-1}$,*

$$\begin{aligned} M_{2k} &= M^e[:7u] + x^{3u} M^e[:4u] + x^{4u} M^e[:3u] + x^{6u} M^e[:u] + x^{7u} R^e, \\ W_{2k} &= W^e[:7u] + x^{3u} W^e[:4u] + x^{4u} W^e[:3u] + x^{6u} W^e[:u] + x^{7u} R^e. \end{aligned}$$

*For all integer $k \ge 2$, and $u = 2^{2k}$,*

$$\begin{aligned} M_{2k+1} &= M^o[:7u] + x^{3u} M^o[:4u] + x^{4u} M^o[:3u] + x^{6u} M^o[:u] + x^{7u} R^o, \\ W_{2k+1} &= W^o[:7u] + x^{3u} W^o[:4u] + x^{4u} W^o[:3u] + x^{6u} W^o[:u] + x^{7u} R^o. \end{aligned}$$

*Proof.* We call the four identities in Lemma 2.3 also by the name $M_{2k}$, $W_{2k}$, $M_{2k+1}$, and $W_{2k+1}$. For $n = 2$, the identities can be verified directly. For $n \ge 2$, we claim that

$$\begin{aligned} M_{2k} \wedge W_{2k} &\Rightarrow M_{2k+1} \wedge W_{2k+1}, \\ M_{2k+1} \wedge W_{2k+1} &\Rightarrow M_{2k+2} \wedge W_{2k+2}. \end{aligned}$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity $M_{2k+1}$ is equal to

$$\begin{aligned} W_{2k} M_{2k} &= \big(W^e[:7v] + x^{3v} W^e[:4v] + x^{4v} W^e[:3v] + x^{6v} W^e[:v] + x^{7u} R^e\big) \times \big( \\ &\quad M^e[:7v] + x^{3v} M^e[:4v] + x^{4v} M^e[:3v] + x^{6v} M^e[:v] + x^{7u} R^e\big); \end{aligned} \tag{2.35}$$

Call this expression $lhs$. Note that both sides of identity $M_{2k+1}$ have the same term of highest degree $x^{14v}R^o$. Therefore we only need to prove that their difference is $O(x^{14v})$. Using Lemma 2.1 it can be seen that the right side of identity $M_{2k+1}$ is congruent, modulo $x^{14v}$, to

$$W^e[:14v]M^e[:14v] + x^{6v}W^e[:8v]M^e[:8v] + x^{8v}W^e[:6v]M^e[:6v] \\ + x^{12v}W^e[:2v]M^e[:2v].$$

For all $n \le 14$, replace the occurrences of $W^e[:n\cdot v]$ and $M^e[:n\cdot v]$ in the above expression by the reduction modulo $x^{n\cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call $rhs$. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n\cdot v : (n+1)\cdot v]/X^n, \\ b_n &:= M^e[n\cdot v : (n+1)\cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions $lhs$ (2.35) and $rhs$ as polynomials in $\mathbb{F}_2[a_1, ..., a_{14}, b_1, ..., b_{14}, c][X]$. Note that it is not a problem that $a_j$ commutes with $b_k$ while $W^e$ does not commute with $M^e$, because in the expressions concerned, the products of $W^e$-terms and $M^e$-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{14})$, which completes the proof. □

*Proof of Theorem 1.1.* We prove the theorem for $\mathrm{CF}(a,b)$; for $\mathrm{CF}(b,a)$, the proof is similar. By Lemma 2.3, we have For all $0 \le j,k \le 1$,

$$\begin{aligned} \lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k}, \\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k}, \\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k}, \\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}. \end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a,b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e,0,1)$ and $\phi(M^e,0,0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x,y) = \mathrm{Res}_t\left(\phi(M^e,0,1)(x,t), y^{12}\phi(M^e,0,1)(x,t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x,y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned} q_0(x) &= x^{20} + x^{17} + x^{16} + x^{11} + x^{10} + x^7, \\ q_1(x) &= x^{18} + x^{16} + x^{15} + x^{13} + x^{12} + x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^3, \\ q_2(x) &= x^{20} + x^{16} + x^{10} + x^8 + x^6 + 1, \end{aligned}$$

$$q_3(x) = x^{18} + x^{16} + x^{15} + x^{13} + x^{12} + x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^3,$$
$$q_4(x) = x^{20} + x^{17} + x^{11} + x^7 + x^6.$$

The polynomial $Q(x, y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x, y)$ of multiplicity $m$ and $R(x, y) := P(x, y)/Q(x, y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 216, and substitute it for $y$ in $R(z, y)$, we get a series with valuation smaller than 216, which proves that $R(z, y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z, y)$ is the minimal polynomial of $\mathrm{CF}(a, b) = f(1/z)$. $\square$

## 3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e, 0, 0)$:

$$\begin{aligned}
p_0(x) = {} & x^{108} + x^{106} + x^{104} + x^{102} + x^{90} + x^{88} + x^{86} + x^{80} + x^{72} + x^{70} + x^{68} \\
& + x^{66} + x^{60} + x^{58} + x^{54} + x^{52} + x^{50} + x^{44} + x^{42} + x^{40} + x^{36}, \\
p_3(x) = {} & x^{104} + x^{102} + x^{98} + x^{94} + x^{92} + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{78} \\
& + x^{70} + x^{66} + x^{64} + x^{62} + x^{56} + x^{54} + x^{52} + x^{48} + x^{46} + x^{42} + x^{38} \\
& + x^{36} + x^{34} + x^{28} + x^{24} + x^{22} + x^{20}, \\
p_6(x) = {} & x^{102} + x^{100} + x^{92} + x^{88} + x^{86} + x^{80} + x^{76} + x^{66} + x^{60} + x^{58} + x^{56} \\
& + x^{54} + x^{52} + x^{50} + x^{48} + x^{40} + x^{38} + x^{34} + x^{28} + x^{24} + x^{22} + x^{20} \\
& + x^{16} + x^{10} + x^6 + x^4 + x^2 + 1, \\
p_9(x) = {} & x^{104} + x^{102} + x^{100} + x^{98} + x^{94} + x^{92} + x^{90} + x^{88} + x^{84} + x^{78} + x^{76} \\
& + x^{74} + x^{72} + x^{70} + x^{62} + x^{60} + x^{50} + x^{48} + x^{46} + x^{42} + x^{40} + x^{36} \\
& + x^{34} + x^{30} + x^{26} + x^{24} + x^{22} + x^{20} + x^{18} + x^{16} + x^{14} + x^{12} + x^8 \\
& + x^6, \\
p_{12}(x) = {} & x^{108} + x^{106} + x^{104} + x^{100} + x^{98} + x^{96} + x^{92} + x^{90} + x^{88} + x^{84} + x^{82} \\
& + x^{80} + x^{60} + x^{58} + x^{56} + x^{52} + x^{50} + x^{48} + x^{44} + x^{42} + x^{40} + x^{36} \\
& + x^{34} + x^{32} + x^{28} + x^{26} + x^{24} + x^{20} + x^{18} + x^{16} + x^{12} + x^{10} + x^8 \\
& + x^4 + x^2 + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0].

For $\phi(M^e, 0, 1)$:

$$\begin{aligned}
p_0(x) = {} & x^{111} + x^{108} + x^{107} + x^{105} + x^{103} + x^{102} + x^{100} + x^{99} + x^{98} + x^{97} \\
& + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{88} + x^{87} + x^{86} + x^{81} + x^{79} + x^{78} \\
& + x^{76} + x^{75} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{65} + x^{64} + x^{62} + x^{60}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{59} + x^{58} + x^{56} + x^{55} + x^{52} + x^{50} + x^{46} + x^{45} + x^{43} + x^{42} + x^{41}\\
&+ x^{40} + x^{39},\\
p_3(x) &= x^{93} + x^{91} + x^{85} + x^{79} + x^{77} + x^{71} + x^{61} + x^{55} + x^{51} + x^{47} + x^{43}\\
&+ x^{39} + x^{37} + x^{31} + x^{29} + x^{27},\\
p_6(x) &= x^{99} + x^{96} + x^{95} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{86} + x^{85}\\
&+ x^{83} + x^{82} + x^{80} + x^{77} + x^{75} + x^{74} + x^{72} + x^{69} + x^{66} + x^{55} + x^{52}\\
&+ x^{51} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{42} + x^{41} + x^{38} + x^{31}\\
&+ x^{28} + x^{27} + x^{25} + x^{24} + x^{22} + x^{21} + x^{18},\\
p_9(x) &= x^{105} + x^{103} + x^{101} + x^{95} + x^{93} + x^{91} + x^{85} + x^{75} + x^{69} + x^{65} + x^{63}\\
&+ x^{51} + x^{49} + x^{47} + x^{45} + x^{43} + x^{41} + x^{39} + x^{37} + x^{35} + x^{33} + x^{31}\\
&+ x^{25} + x^{17} + x^{11} + x^{9},\\
p_{12}(x) &= x^{111} + x^{108} + x^{99} + x^{96} + x^{95} + x^{92} + x^{83} + x^{80} + x^{63} + x^{60} + x^{51}\\
&+ x^{48} + x^{47} + x^{44} + x^{35} + x^{32} + x^{31} + x^{28} + x^{19} + x^{16} + x^{15} + x^{12}\\
&+ x^{3} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0].

For $\phi(M^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{111} + x^{108} + x^{107} + x^{105} + x^{103} + x^{102} + x^{100} + x^{99} + x^{98} + x^{97}\\
&+ x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{88} + x^{87} + x^{86} + x^{81} + x^{79} + x^{78}\\
&+ x^{76} + x^{75} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{65} + x^{64} + x^{62} + x^{60}\\
&+ x^{59} + x^{58} + x^{56} + x^{55} + x^{52} + x^{50} + x^{46} + x^{45} + x^{43} + x^{42} + x^{41}\\
&+ x^{40} + x^{39},\\
p_3(x) &= x^{93} + x^{91} + x^{85} + x^{79} + x^{77} + x^{71} + x^{61} + x^{55} + x^{51} + x^{47} + x^{43}\\
&+ x^{39} + x^{37} + x^{31} + x^{29} + x^{27},\\
p_6(x) &= x^{99} + x^{96} + x^{95} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{86} + x^{85}\\
&+ x^{83} + x^{82} + x^{80} + x^{77} + x^{75} + x^{74} + x^{72} + x^{69} + x^{66} + x^{55} + x^{52}\\
&+ x^{51} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{42} + x^{41} + x^{38} + x^{31}\\
&+ x^{28} + x^{27} + x^{25} + x^{24} + x^{22} + x^{21} + x^{18},\\
p_9(x) &= x^{105} + x^{103} + x^{101} + x^{95} + x^{93} + x^{91} + x^{85} + x^{75} + x^{69} + x^{65} + x^{63}\\
&+ x^{51} + x^{49} + x^{47} + x^{45} + x^{43} + x^{41} + x^{39} + x^{37} + x^{35} + x^{33} + x^{31}\\
&+ x^{25} + x^{17} + x^{11} + x^{9},\\
p_{12}(x) &= x^{111} + x^{108} + x^{99} + x^{96} + x^{95} + x^{92} + x^{83} + x^{80} + x^{63} + x^{60} + x^{51}\\
&+ x^{48} + x^{47} + x^{44} + x^{35} + x^{32} + x^{31} + x^{28} + x^{19} + x^{16} + x^{15} + x^{12}\\
&+ x^{3} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0].

For $\phi(M^e, 1, 1)$:

$$\begin{aligned} p_0(x) &= x^{114} + x^{112} + x^{108} + x^{106} + x^{102} + x^{96} + x^{94} + x^{92} + x^{88} + x^{80} + x^{78} \\ &\quad + x^{76} + x^{68} + x^{64} + x^{62} + x^{60} + x^{54} + x^{50} + x^{48} + x^{44} + x^{42}, \\ p_3(x) &= x^{110} + x^{108} + x^{106} + x^{104} + x^{98} + x^{96} + x^{94} + x^{90} + x^{86} + x^{84} + x^{82} \\ &\quad + x^{80} + x^{78} + x^{74} + x^{72} + x^{70} + x^{64} + x^{56} + x^{50} + x^{46} + x^{44} + x^{38} \\ &\quad + x^{34} + x^{30} + x^{28} + x^{18}, \\ p_6(x) &= x^{108} + x^{102} + x^{96} + x^{94} + x^{90} + x^{88} + x^{84} + x^{78} + x^{74} + x^{72} + x^{70} \\ &\quad + x^{64} + x^{54} + x^{48} + x^{46} + x^{44} + x^{40} + x^{38} + x^{34} + x^{32} + x^{28} + x^{26} \\ &\quad + x^{18} + x^{6} + x^{2} + 1, \\ p_9(x) &= x^{110} + x^{108} + x^{104} + x^{94} + x^{88} + x^{86} + x^{82} + x^{80} + x^{76} + x^{74} + x^{72} \\ &\quad + x^{68} + x^{64} + x^{62} + x^{60} + x^{56} + x^{54} + x^{48} + x^{46} + x^{44} + x^{42} + x^{40} \\ &\quad + x^{38} + x^{36} + x^{28} + x^{26} + x^{18} + x^{16} + x^{12} + x^{10} + x^{8} + x^{6}, \\ p_{12}(x) &= x^{114} + x^{112} + x^{106} + x^{104} + x^{90} + x^{88} + x^{82} + x^{80} + x^{66} + x^{64} + x^{58} \\ &\quad + x^{56} + x^{42} + x^{40} + x^{26} + x^{24} + x^{10} + x^{8} + x^{2} + 1, \end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0].

For $\phi(W^e, 0, 0)$:

$$\begin{aligned} p_0(x) &= x^{114} + x^{112} + x^{108} + x^{104} + x^{102} + x^{98} + x^{94} + x^{90} + x^{88} + x^{84} + x^{82} \\ &\quad + x^{78} + x^{66} + x^{62} + x^{60} + x^{58} + x^{56} + x^{46} + x^{36}, \\ p_3(x) &= x^{110} + x^{108} + x^{106} + x^{104} + x^{102} + x^{100} + x^{96} + x^{94} + x^{90} + x^{80} \\ &\quad + x^{78} + x^{74} + x^{72} + x^{70} + x^{64} + x^{62} + x^{60} + x^{58} + x^{56} + x^{54} + x^{52} \\ &\quad + x^{42} + x^{38} + x^{34} + x^{30} + x^{28} + x^{22} + x^{20}, \\ p_6(x) &= x^{108} + x^{106} + x^{104} + x^{100} + x^{98} + x^{96} + x^{92} + x^{86} + x^{82} + x^{76} + x^{74} \\ &\quad + x^{72} + x^{70} + x^{66} + x^{62} + x^{60} + x^{56} + x^{54} + x^{46} + x^{44} + x^{42} + x^{40} \\ &\quad + x^{36} + x^{34} + x^{32} + x^{28} + x^{24} + x^{22} + x^{20} + x^{14} + x^{12} + x^{6} + x^{2} + 1, \\ p_9(x) &= x^{110} + x^{108} + x^{104} + x^{94} + x^{88} + x^{86} + x^{82} + x^{80} + x^{76} + x^{74} + x^{72} \\ &\quad + x^{68} + x^{64} + x^{62} + x^{60} + x^{56} + x^{54} + x^{48} + x^{46} + x^{44} + x^{42} + x^{40} \\ &\quad + x^{38} + x^{36} + x^{28} + x^{26} + x^{18} + x^{16} + x^{12} + x^{10} + x^{8} + x^{6}, \\ p_{12}(x) &= x^{114} + x^{112} + x^{106} + x^{104} + x^{90} + x^{88} + x^{82} + x^{80} + x^{66} + x^{64} + x^{58} \\ &\quad + x^{56} + x^{42} + x^{40} + x^{26} + x^{24} + x^{10} + x^{8} + x^{2} + 1, \end{aligned}$$

and the initial terms are [1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 1)$:

$$\begin{aligned} p_0(x) &= x^{111} + x^{108} + x^{107} + x^{105} + x^{103} + x^{102} + x^{100} + x^{98} + x^{97} + x^{96} \\ &\quad + x^{95} + x^{94} + x^{89} + x^{87} + x^{83} + x^{79} + x^{76} + x^{74} + x^{73} + x^{72} + x^{71} \\ &\quad + x^{69} + x^{67} + x^{66} + x^{65} + x^{63} + x^{61} + x^{57} + x^{55} + x^{54} + x^{51} + x^{50} \\ &\quad + x^{48} + x^{47} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39}, \end{aligned}$$

$$p_3(x) = x^{93} + x^{91} + x^{85} + x^{79} + x^{77} + x^{71} + x^{61} + x^{55} + x^{51} + x^{47} + x^{43} + x^{39} + x^{37} + x^{31} + x^{29} + x^{27},$$
$$p_6(x) = x^{99} + x^{96} + x^{95} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{86} + x^{85} + x^{83} + x^{82} + x^{80} + x^{77} + x^{75} + x^{74} + x^{72} + x^{69} + x^{66} + x^{55} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{42} + x^{41} + x^{38} + x^{31} + x^{28} + x^{27} + x^{25} + x^{24} + x^{22} + x^{21} + x^{18},$$
$$p_9(x) = x^{105} + x^{103} + x^{101} + x^{95} + x^{93} + x^{91} + x^{85} + x^{75} + x^{69} + x^{65} + x^{63} + x^{51} + x^{49} + x^{47} + x^{45} + x^{43} + x^{41} + x^{39} + x^{37} + x^{35} + x^{33} + x^{31} + x^{25} + x^{17} + x^{11} + x^{9},$$
$$p_{12}(x) = x^{111} + x^{108} + x^{99} + x^{96} + x^{95} + x^{92} + x^{83} + x^{80} + x^{63} + x^{60} + x^{51} + x^{48} + x^{47} + x^{44} + x^{35} + x^{32} + x^{31} + x^{28} + x^{19} + x^{16} + x^{15} + x^{12} + x^3 + 1,$$

and the initial terms are [0, 0, 0, 1, 1, 0, 0, 1, 1, 0, 1, 1, 0, 0, 0, 0].

For $\phi(W^e, 1, 0)$:

$$p_0(x) = x^{111} + x^{108} + x^{107} + x^{105} + x^{103} + x^{102} + x^{100} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{89} + x^{87} + x^{83} + x^{79} + x^{76} + x^{74} + x^{73} + x^{72} + x^{71} + x^{69} + x^{67} + x^{66} + x^{65} + x^{63} + x^{61} + x^{57} + x^{55} + x^{54} + x^{51} + x^{50} + x^{48} + x^{47} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39},$$
$$p_3(x) = x^{93} + x^{91} + x^{85} + x^{79} + x^{77} + x^{71} + x^{61} + x^{55} + x^{51} + x^{47} + x^{43} + x^{39} + x^{37} + x^{31} + x^{29} + x^{27},$$
$$p_6(x) = x^{99} + x^{96} + x^{95} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{86} + x^{85} + x^{83} + x^{82} + x^{80} + x^{77} + x^{75} + x^{74} + x^{72} + x^{69} + x^{66} + x^{55} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{42} + x^{41} + x^{38} + x^{31} + x^{28} + x^{27} + x^{25} + x^{24} + x^{22} + x^{21} + x^{18},$$
$$p_9(x) = x^{105} + x^{103} + x^{101} + x^{95} + x^{93} + x^{91} + x^{85} + x^{75} + x^{69} + x^{65} + x^{63} + x^{51} + x^{49} + x^{47} + x^{45} + x^{43} + x^{41} + x^{39} + x^{37} + x^{35} + x^{33} + x^{31} + x^{25} + x^{17} + x^{11} + x^{9},$$
$$p_{12}(x) = x^{111} + x^{108} + x^{99} + x^{96} + x^{95} + x^{92} + x^{83} + x^{80} + x^{63} + x^{60} + x^{51} + x^{48} + x^{47} + x^{44} + x^{35} + x^{32} + x^{31} + x^{28} + x^{19} + x^{16} + x^{15} + x^{12} + x^3 + 1,$$

and the initial terms are [0, 0, 0, 1, 1, 0, 0, 1, 1, 0, 1, 1, 0, 0, 0, 0].

For $\phi(W^e, 1, 1)$:

$$p_0(x) = x^{108} + x^{106} + x^{104} + x^{102} + x^{100} + x^{98} + x^{96} + x^{92} + x^{78} + x^{74} + x^{66} + x^{64} + x^{62} + x^{56} + x^{50} + x^{48} + x^{46} + x^{44} + x^{42},$$
$$p_3(x) = x^{104} + x^{102} + x^{98} + x^{96} + x^{92} + x^{90} + x^{80} + x^{76} + x^{74} + x^{70} + x^{68}$$

$$
\begin{aligned}
&+ x^{56} + x^{54} + x^{50} + x^{48} + x^{46} + x^{42} + x^{40} + x^{36} + x^{34} + x^{32} + x^{30} \\
&+ x^{26} + x^{24} + x^{22} + x^{18}, \\
p_6(x) &= x^{102} + x^{98} + x^{94} + x^{92} + x^{88} + x^{86} + x^{82} + x^{78} + x^{76} + x^{72} + x^{66} \\
&+ x^{64} + x^{60} + x^{56} + x^{52} + x^{46} + x^{42} + x^{40} + x^{34} + x^{22} + x^{18} + x^{14} \\
&+ x^{12} + x^{10} + x^{6} + x^{4} + x^{2} + 1, \\
p_9(x) &= x^{104} + x^{102} + x^{100} + x^{98} + x^{94} + x^{92} + x^{90} + x^{88} + x^{84} + x^{78} + x^{76} \\
&+ x^{74} + x^{72} + x^{70} + x^{62} + x^{60} + x^{50} + x^{48} + x^{46} + x^{42} + x^{40} + x^{36} \\
&+ x^{34} + x^{30} + x^{26} + x^{24} + x^{22} + x^{20} + x^{18} + x^{16} + x^{14} + x^{12} + x^{8} \\
&+ x^{6}, \\
p_{12}(x) &= x^{108} + x^{106} + x^{104} + x^{100} + x^{98} + x^{96} + x^{92} + x^{90} + x^{88} + x^{84} + x^{82} \\
&+ x^{80} + x^{60} + x^{58} + x^{56} + x^{52} + x^{50} + x^{48} + x^{44} + x^{42} + x^{40} + x^{36} \\
&+ x^{34} + x^{32} + x^{28} + x^{26} + x^{24} + x^{20} + x^{18} + x^{16} + x^{12} + x^{10} + x^{8} \\
&+ x^{4} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0].

For $\phi(M^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{111} + x^{108} + x^{105} + x^{103} + x^{102} + x^{97} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} \\
&+ x^{89} + x^{86} + x^{85} + x^{83} + x^{82} + x^{80} + x^{78} + x^{74} + x^{73} + x^{71} + x^{70} \\
&+ x^{69} + x^{68} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{57} + x^{53} + x^{51} \\
&+ x^{46} + x^{41} + x^{40} + x^{39} + x^{38} + x^{36}, \\
p_3(x) &= x^{103} + x^{100} + x^{97} + x^{94} + x^{92} + x^{85} + x^{84} + x^{83} + x^{79} + x^{74} + x^{72} \\
&+ x^{71} + x^{69} + x^{68} + x^{67} + x^{66} + x^{64} + x^{63} + x^{61} + x^{59} + x^{57} + x^{54} \\
&+ x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{44} + x^{42} + x^{40} + x^{36} + x^{34} + x^{33} \\
&+ x^{32} + x^{31} + x^{30} + x^{29} + x^{27} + x^{26} + x^{24}, \\
p_6(x) &= x^{105} + x^{102} + x^{100} + x^{95} + x^{90} + x^{87} + x^{85} + x^{83} + x^{81} + x^{77} + x^{76} \\
&+ x^{74} + x^{73} + x^{70} + x^{69} + x^{68} + x^{67} + x^{64} + x^{62} + x^{60} + x^{58} + x^{57} \\
&+ x^{55} + x^{53} + x^{49} + x^{46} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39} + x^{38} + x^{35} \\
&+ x^{31} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{21} + x^{18} + x^{17} \\
&+ x^{16} + x^{14} + x^{3} + 1, \\
p_9(x) &= x^{103} + x^{100} + x^{95} + x^{92} + x^{55} + x^{52} + x^{47} + x^{44} + x^{23} + x^{20} + x^{15} \\
&+ x^{12}, \\
p_{12}(x) &= x^{111} + x^{108} + x^{103} + x^{100} + x^{99} + x^{96} + x^{83} + x^{80} + x^{63} + x^{60} + x^{55} \\
&+ x^{52} + x^{51} + x^{48} + x^{35} + x^{32} + x^{31} + x^{28} + x^{23} + x^{20} + x^{19} + x^{16} \\
&+ x^{3} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 1, 1, 0, 1, 1].

For $\phi(M^o,0,1)$:

$$\begin{aligned}p_0(x) &= x^{114}+x^{112}+x^{110}+x^{107}+x^{105}+x^{103}+x^{102}+x^{101}+x^{97}+x^{96}\\ &+x^{95}+x^{90}+x^{89}+x^{87}+x^{86}+x^{85}+x^{84}+x^{83}+x^{81}+x^{80}+x^{78}\\ &+x^{76}+x^{75}+x^{74}+x^{71}+x^{66}+x^{65}+x^{64}+x^{63}+x^{61}+x^{60}+x^{56}\\ &+x^{55}+x^{47}+x^{44}+x^{43}+x^{42}+x^{40}+x^{39},\\ p_3(x) &= x^{96}+x^{93}+x^{90}+x^{88}+x^{87}+x^{85}+x^{84}+x^{82}+x^{81}+x^{79}+x^{78}\\ &+x^{76}+x^{75}+x^{73}+x^{72}+x^{70}+x^{69}+x^{67}+x^{66}+x^{64}+x^{63}+x^{61}\\ &+x^{60}+x^{58}+x^{57}+x^{56}+x^{55}+x^{54}+x^{53}+x^{52}+x^{51}+x^{50}+x^{49}\\ &+x^{48}+x^{47}+x^{46}+x^{45}+x^{44}+x^{43}+x^{42}+x^{41}+x^{39}+x^{38}+x^{36}\\ &+x^{35}+x^{33}+x^{30}+x^{27},\\ p_6(x) &= x^{102}+x^{100}+x^{98}+x^{94}+x^{92}+x^{90}+x^{82}+x^{80}+x^{78}+x^{74}+x^{72}\\ &+x^{68}+x^{66}+x^{58}+x^{56}+x^{54}+x^{50}+x^{48}+x^{46}+x^{42}+x^{40}+x^{38}\\ &+x^{34}+x^{32}+x^{30}+x^{22}+x^{20}+x^{18},\\ p_9(x) &= x^{108}+x^{105}+x^{104}+x^{101}+x^{96}+x^{93}+x^{92}+x^{89}+x^{60}+x^{57}+x^{56}\\ &+x^{53}+x^{48}+x^{45}+x^{44}+x^{41}+x^{28}+x^{25}+x^{24}+x^{21}+x^{16}+x^{13}\\ &+x^{12}+x^{9},\\ p_{12}(x) &= x^{114}+x^{112}+x^{94}+x^{92}+x^{90}+x^{88}+x^{82}+x^{80}+x^{66}+x^{64}+x^{46}\\ &+x^{44}+x^{42}+x^{40}+x^{14}+x^{12}+x^{10}+x^{8}+x^{2}+1,\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1].

For $\phi(M^o,1,0)$:

$$\begin{aligned}p_0(x) &= x^{112}+x^{106}+x^{105}+x^{104}+x^{100}+x^{96}+x^{95}+x^{94}+x^{93}+x^{91}+x^{90}\\ &+x^{89}+x^{88}+x^{86}+x^{84}+x^{80}+x^{68}+x^{67}+x^{64}+x^{63}+x^{62}+x^{58}\\ &+x^{57}+x^{56}+x^{54}+x^{53}+x^{52}+x^{49}+x^{48}+x^{45}+x^{44}+x^{43}+x^{41}\\ &+x^{40}+x^{39},\\ p_3(x) &= x^{94}+x^{92}+x^{91}+x^{89}+x^{88}+x^{86}+x^{85}+x^{83}+x^{78}+x^{76}+x^{75}\\ &+x^{73}+x^{72}+x^{69}+x^{68}+x^{65}+x^{64}+x^{62}+x^{61}+x^{59}+x^{38}+x^{36}\\ &+x^{35}+x^{33}+x^{32}+x^{30}+x^{29}+x^{27},\\ p_6(x) &= x^{100}+x^{94}+x^{88}+x^{84}+x^{82}+x^{80}+x^{78}+x^{76}+x^{74}+x^{72}+x^{70}\\ &+x^{66}+x^{56}+x^{50}+x^{44}+x^{38}+x^{32}+x^{26}+x^{24}+x^{18},\\ p_9(x) &= x^{106}+x^{104}+x^{103}+x^{102}+x^{101}+x^{100}+x^{99}+x^{98}+x^{97}+x^{96}\\ &+x^{95}+x^{94}+x^{93}+x^{92}+x^{91}+x^{89}+x^{58}+x^{56}+x^{55}+x^{54}+x^{53}\\ &+x^{52}+x^{51}+x^{50}+x^{49}+x^{48}+x^{47}+x^{46}+x^{45}+x^{44}+x^{43}+x^{41}\\ &+x^{26}+x^{24}+x^{23}+x^{22}+x^{21}+x^{20}+x^{19}+x^{18}+x^{17}+x^{16}+x^{15}\\ &+x^{14}+x^{13}+x^{12}+x^{11}+x^{9},\\ p_{12}(x) &= x^{112}+x^{108}+x^{104}+x^{96}+x^{84}+x^{80}+x^{64}+x^{60}+x^{56}+x^{48}+x^{36}\end{aligned}$$

$$+ x^{28} + x^{24} + x^{16} + x^4 + 1,$$

and the initial terms are [0, 0, 0, 1, 1, 0, 0, 1, 1, 1, 1, 1, 1, 0, 0, 0].

For $\phi(M^o, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{111} + x^{108} + x^{105} + x^{102} + x^{100} + x^{99} + x^{97} + x^{96} + x^{94} + x^{90} + x^{89} \\
&\quad + x^{88} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{77} + x^{76} + x^{75} \\
&\quad + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{63} + x^{62} + x^{60} + x^{59} + x^{58} + x^{56} \\
&\quad + x^{55} + x^{54} + x^{52} + x^{50} + x^{48} + x^{47} + x^{45} + x^{42}, \\
p_3(x) &= x^{103} + x^{100} + x^{97} + x^{95} + x^{94} + x^{91} + x^{88} + x^{87} + x^{83} + x^{82} + x^{81} \\
&\quad + x^{79} + x^{78} + x^{77} + x^{73} + x^{72} + x^{70} + x^{69} + x^{67} + x^{66} + x^{64} + x^{60} \\
&\quad + x^{59} + x^{58} + x^{55} + x^{53} + x^{52} + x^{51} + x^{45} + x^{43} + x^{41} + x^{38} + x^{37} \\
&\quad + x^{36} + x^{32} + x^{31} + x^{29} + x^{26}, \\
p_6(x) &= x^{105} + x^{103} + x^{102} + x^{99} + x^{96} + x^{93} + x^{92} + x^{91} + x^{89} + x^{88} + x^{86} \\
&\quad + x^{84} + x^{83} + x^{82} + x^{81} + x^{76} + x^{75} + x^{73} + x^{72} + x^{71} + x^{70} + x^{66} \\
&\quad + x^{63} + x^{62} + x^{59} + x^{58} + x^{57} + x^{56} + x^{53} + x^{52} + x^{51} + x^{48} + x^{47} \\
&\quad + x^{45} + x^{43} + x^{39} + x^{35} + x^{29} + x^{27} + x^{26} + x^{24} + x^{22} + x^{20} + x^{17} \\
&\quad + x^{16} + x^{15} + x^{14} + x^{12} + x^{3} + 1, \\
p_9(x) &= x^{103} + x^{100} + x^{95} + x^{92} + x^{55} + x^{52} + x^{47} + x^{44} + x^{23} + x^{20} + x^{15} \\
&\quad + x^{12}, \\
p_{12}(x) &= x^{111} + x^{108} + x^{103} + x^{100} + x^{99} + x^{96} + x^{83} + x^{80} + x^{63} + x^{60} + x^{55} \\
&\quad + x^{52} + x^{51} + x^{48} + x^{35} + x^{32} + x^{31} + x^{28} + x^{23} + x^{20} + x^{19} + x^{16} \\
&\quad + x^{3} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 0, 0, 0, 1].

For $\phi(W^o, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{111} + x^{108} + x^{105} + x^{103} + x^{102} + x^{97} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} \\
&\quad + x^{89} + x^{86} + x^{85} + x^{83} + x^{82} + x^{80} + x^{78} + x^{74} + x^{73} + x^{71} + x^{70} \\
&\quad + x^{69} + x^{68} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{57} + x^{53} + x^{51} \\
&\quad + x^{46} + x^{41} + x^{40} + x^{39} + x^{38} + x^{36}, \\
p_3(x) &= x^{103} + x^{100} + x^{97} + x^{94} + x^{92} + x^{85} + x^{84} + x^{83} + x^{79} + x^{74} + x^{72} \\
&\quad + x^{71} + x^{69} + x^{68} + x^{67} + x^{66} + x^{64} + x^{63} + x^{61} + x^{59} + x^{57} + x^{54} \\
&\quad + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{44} + x^{42} + x^{40} + x^{36} + x^{34} + x^{33} \\
&\quad + x^{32} + x^{31} + x^{30} + x^{29} + x^{27} + x^{26} + x^{24}, \\
p_6(x) &= x^{105} + x^{102} + x^{100} + x^{95} + x^{90} + x^{87} + x^{85} + x^{83} + x^{81} + x^{77} + x^{76} \\
&\quad + x^{74} + x^{73} + x^{70} + x^{69} + x^{68} + x^{67} + x^{64} + x^{62} + x^{60} + x^{58} + x^{57} \\
&\quad + x^{55} + x^{53} + x^{49} + x^{46} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39} + x^{38} + x^{35} \\
&\quad + x^{31} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{21} + x^{18} + x^{17}
\end{aligned}$$

$$
\begin{aligned}
&+x^{16}+x^{14}+x^{3}+1,\\
p_9(x) &= x^{103}+x^{100}+x^{95}+x^{92}+x^{55}+x^{52}+x^{47}+x^{44}+x^{23}+x^{20}+x^{15}\\
&\quad +x^{12},\\
p_{12}(x) &= x^{111}+x^{108}+x^{103}+x^{100}+x^{99}+x^{96}+x^{83}+x^{80}+x^{63}+x^{60}+x^{55}\\
&\quad +x^{52}+x^{51}+x^{48}+x^{35}+x^{32}+x^{31}+x^{28}+x^{23}+x^{20}+x^{19}+x^{16}\\
&\quad +x^{3}+1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 1, 1, 0, 1, 1].

For $\phi(W^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{112}+x^{106}+x^{105}+x^{104}+x^{100}+x^{96}+x^{95}+x^{94}+x^{93}+x^{91}+x^{90}\\
&\quad +x^{89}+x^{88}+x^{86}+x^{84}+x^{80}+x^{68}+x^{67}+x^{64}+x^{63}+x^{62}+x^{58}\\
&\quad +x^{57}+x^{56}+x^{54}+x^{53}+x^{52}+x^{49}+x^{48}+x^{45}+x^{44}+x^{43}+x^{41}\\
&\quad +x^{40}+x^{39},\\
p_3(x) &= x^{94}+x^{92}+x^{91}+x^{89}+x^{88}+x^{86}+x^{85}+x^{83}+x^{78}+x^{76}+x^{75}\\
&\quad +x^{73}+x^{72}+x^{69}+x^{68}+x^{65}+x^{64}+x^{62}+x^{61}+x^{59}+x^{38}+x^{36}\\
&\quad +x^{35}+x^{33}+x^{32}+x^{30}+x^{29}+x^{27},\\
p_6(x) &= x^{100}+x^{94}+x^{88}+x^{84}+x^{82}+x^{80}+x^{78}+x^{76}+x^{74}+x^{72}+x^{70}\\
&\quad +x^{66}+x^{56}+x^{50}+x^{44}+x^{38}+x^{32}+x^{26}+x^{24}+x^{18},\\
p_9(x) &= x^{106}+x^{104}+x^{103}+x^{102}+x^{101}+x^{100}+x^{99}+x^{98}+x^{97}+x^{96}\\
&\quad +x^{95}+x^{94}+x^{93}+x^{92}+x^{91}+x^{89}+x^{58}+x^{56}+x^{55}+x^{54}+x^{53}\\
&\quad +x^{52}+x^{51}+x^{50}+x^{49}+x^{48}+x^{47}+x^{46}+x^{45}+x^{44}+x^{43}+x^{41}\\
&\quad +x^{26}+x^{24}+x^{23}+x^{22}+x^{21}+x^{20}+x^{19}+x^{18}+x^{17}+x^{16}+x^{15}\\
&\quad +x^{14}+x^{13}+x^{12}+x^{11}+x^{9},\\
p_{12}(x) &= x^{112}+x^{108}+x^{104}+x^{96}+x^{84}+x^{80}+x^{64}+x^{60}+x^{56}+x^{48}+x^{36}\\
&\quad +x^{28}+x^{24}+x^{16}+x^{4}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 1, 0, 0, 1, 1, 1, 1, 1, 1, 0, 0, 0].

For $\phi(W^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{114}+x^{112}+x^{110}+x^{107}+x^{105}+x^{103}+x^{102}+x^{101}+x^{97}+x^{96}\\
&\quad +x^{95}+x^{90}+x^{89}+x^{87}+x^{86}+x^{85}+x^{84}+x^{83}+x^{81}+x^{80}+x^{78}\\
&\quad +x^{76}+x^{75}+x^{74}+x^{71}+x^{66}+x^{65}+x^{64}+x^{63}+x^{61}+x^{60}+x^{56}\\
&\quad +x^{55}+x^{47}+x^{44}+x^{43}+x^{42}+x^{40}+x^{39},\\
p_3(x) &= x^{96}+x^{93}+x^{90}+x^{88}+x^{87}+x^{85}+x^{84}+x^{82}+x^{81}+x^{79}+x^{78}\\
&\quad +x^{76}+x^{75}+x^{73}+x^{72}+x^{70}+x^{69}+x^{67}+x^{66}+x^{64}+x^{63}+x^{61}\\
&\quad +x^{60}+x^{58}+x^{57}+x^{56}+x^{55}+x^{54}+x^{53}+x^{52}+x^{51}+x^{50}+x^{49}\\
&\quad +x^{48}+x^{47}+x^{46}+x^{45}+x^{44}+x^{43}+x^{42}+x^{41}+x^{39}+x^{38}+x^{36}\\
&\quad +x^{35}+x^{33}+x^{30}+x^{27},
\end{aligned}
$$

$$
\begin{aligned}
p_6(x) &= x^{102} + x^{100} + x^{98} + x^{94} + x^{92} + x^{90} + x^{82} + x^{80} + x^{78} + x^{74} + x^{72} \\
&\quad + x^{68} + x^{66} + x^{58} + x^{56} + x^{54} + x^{50} + x^{48} + x^{46} + x^{42} + x^{40} + x^{38} \\
&\quad + x^{34} + x^{32} + x^{30} + x^{22} + x^{20} + x^{18}, \\
p_9(x) &= x^{108} + x^{105} + x^{104} + x^{101} + x^{96} + x^{93} + x^{92} + x^{89} + x^{60} + x^{57} + x^{56} \\
&\quad + x^{53} + x^{48} + x^{45} + x^{44} + x^{41} + x^{28} + x^{25} + x^{24} + x^{21} + x^{16} + x^{13} \\
&\quad + x^{12} + x^{9}, \\
p_{12}(x) &= x^{114} + x^{112} + x^{94} + x^{92} + x^{90} + x^{88} + x^{82} + x^{80} + x^{66} + x^{64} + x^{46} \\
&\quad + x^{44} + x^{42} + x^{40} + x^{14} + x^{12} + x^{10} + x^{8} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1].

For $\phi(W^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{111} + x^{108} + x^{105} + x^{102} + x^{100} + x^{99} + x^{97} + x^{96} + x^{94} + x^{90} + x^{89} \\
&\quad + x^{88} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{77} + x^{76} + x^{75} \\
&\quad + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{63} + x^{62} + x^{60} + x^{59} + x^{58} + x^{56} \\
&\quad + x^{55} + x^{54} + x^{52} + x^{50} + x^{48} + x^{47} + x^{45} + x^{42}, \\
p_3(x) &= x^{103} + x^{100} + x^{97} + x^{95} + x^{94} + x^{91} + x^{88} + x^{87} + x^{83} + x^{82} + x^{81} \\
&\quad + x^{79} + x^{78} + x^{77} + x^{73} + x^{72} + x^{70} + x^{69} + x^{67} + x^{66} + x^{64} + x^{60} \\
&\quad + x^{59} + x^{58} + x^{55} + x^{53} + x^{52} + x^{51} + x^{45} + x^{43} + x^{41} + x^{38} + x^{37} \\
&\quad + x^{36} + x^{32} + x^{31} + x^{29} + x^{26}, \\
p_6(x) &= x^{105} + x^{103} + x^{102} + x^{99} + x^{96} + x^{93} + x^{92} + x^{91} + x^{89} + x^{88} + x^{86} \\
&\quad + x^{84} + x^{83} + x^{82} + x^{81} + x^{76} + x^{75} + x^{73} + x^{72} + x^{71} + x^{70} + x^{66} \\
&\quad + x^{63} + x^{62} + x^{59} + x^{58} + x^{57} + x^{56} + x^{53} + x^{52} + x^{51} + x^{48} + x^{47} \\
&\quad + x^{45} + x^{43} + x^{39} + x^{35} + x^{29} + x^{27} + x^{26} + x^{24} + x^{22} + x^{20} + x^{17} \\
&\quad + x^{16} + x^{15} + x^{14} + x^{12} + x^{3} + 1, \\
p_9(x) &= x^{103} + x^{100} + x^{95} + x^{92} + x^{55} + x^{52} + x^{47} + x^{44} + x^{23} + x^{20} + x^{15} \\
&\quad + x^{12}, \\
p_{12}(x) &= x^{111} + x^{108} + x^{103} + x^{100} + x^{99} + x^{96} + x^{83} + x^{80} + x^{63} + x^{60} + x^{55} \\
&\quad + x^{52} + x^{51} + x^{48} + x^{35} + x^{32} + x^{31} + x^{28} + x^{23} + x^{20} + x^{19} + x^{16} \\
&\quad + x^{3} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 0, 0, 0, 1].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

| $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | [1, 2] | 102 | [162, 14] | 204 | [41, 286] | 306 | [377, 378] | 408 | [448, 208] |
| 1 | [3, 4] | 103 | [47, 113] | 205 | [263, 287] | 307 | [379, 380] | 409 | [449, 450] |
| 2 | [5, 6] | 104 | [163, 164] | 206 | [139, 118] | 308 | [381, 382] | 410 | [451, 452] |
| 3 | [7, 8] | 105 | [38, 126] | 207 | [43, 288] | 309 | [383, 384] | 411 | [201, 453] |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 4 | [9, 10] | 106 | [165, 166] | 208 | [289, 290] | 310 | [385, 366] | 412 | [454, 455] |
| 5 | [11, 12] | 107 | [120, 37] | 209 | [291, 274] | 311 | [386, 387] | 413 | [456, 457] |
| 6 | [13, 14] | 108 | [167, 168] | 210 | [165, 164] | 312 | [388, 389] | 414 | [458, 459] |
| 7 | [15, 16] | 109 | [28, 169] | 211 | [292, 149] | 313 | [4, 390] | 415 | [460, 461] |
| 8 | [17, 18] | 110 | [87, 170] | 212 | [135, 286] | 314 | [391, 392] | 416 | [226, 372] |
| 9 | [19, 20] | 111 | [58, 171] | 213 | [281, 147] | 315 | [382, 248] | 417 | [462, 216] |
| 10 | [21, 22] | 112 | [172, 173] | 214 | [293, 294] | 316 | [7, 141] | 418 | [463, 464] |
| 11 | [23, 24] | 113 | [174, 175] | 215 | [134, 14] | 317 | [38, 274] | 419 | [465, 375] |
| 12 | [25, 26] | 114 | [176, 177] | 216 | [123, 279] | 318 | [393, 297] | 420 | [466, 467] |
| 13 | [27, 28] | 115 | [178, 179] | 217 | [295, 114] | 319 | [130, 294] | 421 | [468, 469] |
| 14 | [29, 30] | 116 | [99, 180] | 218 | [296, 297] | 320 | [394, 46] | 422 | [470, 220] |
| 15 | [31, 11] | 117 | [181, 182] | 219 | [56, 126] | 321 | [8, 160] | 423 | [471, 472] |
| 16 | [32, 33] | 118 | [85, 183] | 220 | [283, 161] | 322 | [12, 154] | 424 | [473, 474] |
| 17 | [34, 11] | 119 | [184, 185] | 221 | [153, 155] | 323 | [395, 279] | 425 | [59, 227] |
| 18 | [35, 36] | 120 | [186, 187] | 222 | [298, 113] | 324 | [112, 48] | 426 | [367, 210] |
| 19 | [37, 38] | 121 | [188, 189] | 223 | [299, 13] | 325 | [143, 46] | 427 | [32, 284] |
| 20 | [39, 40] | 122 | [190, 191] | 224 | [300, 155] | 326 | [126, 273] | 428 | [293, 425] |
| 21 | [41, 42] | 123 | [192, 193] | 225 | [55, 125] | 327 | [396, 397] | 429 | [414, 305] |
| 22 | [43, 44] | 124 | [183, 194] | 226 | [301, 266] | 328 | [47, 398] | 430 | [135, 475] |
| 23 | [45, 46] | 125 | [195, 196] | 227 | [146, 114] | 329 | [117, 264] | 431 | [314, 157] |
| 24 | [47, 48] | 126 | [105, 197] | 228 | [158, 8] | 330 | [143, 271] | 432 | [155, 140] |
| 25 | [12, 49] | 127 | [198, 199] | 229 | [302, 136] | 331 | [399, 134] | 433 | [303, 36] |
| 26 | [46, 50] | 128 | [200, 201] | 230 | [303, 304] | 332 | [156, 400] | 434 | [476, 114] |
| 27 | [51, 52] | 129 | [202, 203] | 231 | [159, 111] | 333 | [166, 165] | 435 | [477, 418] |
| 28 | [53, 44] | 130 | [204, 205] | 232 | [295, 147] | 334 | [271, 50] | 436 | [159, 12] |
| 29 | [39, 54] | 131 | [206, 207] | 233 | [155, 305] | 335 | [123, 315] | 437 | [283, 145] |
| 30 | [55, 56] | 132 | [208, 209] | 234 | [146, 300] | 336 | [401, 402] | 438 | [162, 478] |
| 31 | [57, 58] | 133 | [210, 211] | 235 | [128, 305] | 337 | [126, 37] | 439 | [404, 152] |
| 32 | [16, 59] | 134 | [212, 213] | 236 | [278, 306] | 338 | [51, 400] | 440 | [300, 115] |
| 33 | [60, 61] | 135 | [214, 215] | 237 | [152, 307] | 339 | [147, 128] | 441 | [165, 163] |
| 34 | [62, 63] | 136 | [216, 217] | 238 | [302, 152] | 340 | [311, 397] | 442 | [405, 275] |
| 35 | [64, 65] | 137 | [218, 219] | 239 | [144, 308] | 341 | [166, 164] | 443 | [288, 160] |
| 36 | [66, 67] | 138 | [220, 221] | 240 | [268, 109] | 342 | [53, 8] | 444 | [158, 141] |
| 37 | [68, 69] | 139 | [67, 222] | 241 | [309, 288] | 343 | [403, 145] | 445 | [308, 270] |
| 38 | [70, 71] | 140 | [223, 224] | 242 | [303, 263] | 344 | [163, 163] | 446 | [293, 131] |
| 39 | [72, 73] | 141 | [217, 60] | 243 | [46, 118] | 345 | [55, 38] | 447 | [272, 310] |
| 40 | [74, 75] | 142 | [225, 226] | 244 | [123, 306] | 346 | [404, 136] | 448 | [37, 291] |
| 41 | [76, 77] | 143 | [227, 94] | 245 | [152, 310] | 347 | [144, 118] | 449 | [399, 162] |
| 42 | [78, 79] | 144 | [228, 229] | 246 | [311, 312] | 348 | [158, 288] | 450 | [109, 111] |
| 43 | [80, 81] | 145 | [230, 231] | 247 | [274, 127] | 349 | [139, 405] | 451 | [41, 267] |
| 44 | [82, 83] | 146 | [232, 233] | 248 | [53, 288] | 350 | [406, 46] | 452 | [285, 277] |
| 45 | [84, 85] | 147 | [234, 223] | 249 | [264, 49] | 351 | [45, 144] | 453 | [476, 147] |
| 46 | [86, 87] | 148 | [235, 236] | 250 | [295, 153] | 352 | [50, 119] | 454 | [8, 283] |
| 47 | [88, 89] | 149 | [237, 238] | 251 | [111, 313] | 353 | [407, 306] | 455 | [394, 139] |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 48 | [90, 91] | 150 | [239, 240] | 252 | [141, 32] | 354 | [408, 113] | 456 | [296, 479] |
| 49 | [92, 93] | 151 | [30, 218] | 253 | [299, 314] | 355 | [111, 409] | 457 | [273, 38] |
| 50 | [94, 95] | 152 | [73, 241] | 254 | [288, 32] | 356 | [410, 411] | 458 | [12, 409] |
| 51 | [96, 97] | 153 | [91, 242] | 255 | [278, 315] | 357 | [130, 412] | 459 | [117, 111] |
| 52 | [98, 99] | 154 | [24, 243] | 256 | [55, 291] | 358 | [48, 413] | 460 | [278, 124] |
| 53 | [100, 101] | 155 | [233, 98] | 257 | [126, 127] | 359 | [153, 414] | 461 | [398, 413] |
| 54 | [102, 103] | 156 | [244, 245] | 258 | [291, 126] | 360 | [116, 11] | 462 | [32, 161] |
| 55 | [104, 105] | 157 | [246, 247] | 259 | [125, 120] | 361 | [45, 271] | 463 | [36, 480] |
| 56 | [106, 107] | 158 | [248, 249] | 260 | [273, 56] | 362 | [405, 270] | 464 | [151, 136] |
| 57 | [108, 109] | 159 | [250, 92] | 261 | [316, 251] | 363 | [415, 42] | 465 | [283, 33] |
| 58 | [50, 110] | 160 | [251, 252] | 262 | [317, 318] | 364 | [13, 416] | 466 | [404, 481] |
| 59 | [11, 111] | 161 | [253, 254] | 263 | [319, 320] | 365 | [399, 13] | 467 | [11, 264] |
| 60 | [112, 113] | 162 | [255, 23] | 264 | [168, 321] | 366 | [163, 165] | 468 | [111, 49] |
| 61 | [114, 115] | 163 | [256, 257] | 265 | [322, 323] | 367 | [417, 418] | 469 | [482, 286] |
| 62 | [116, 117] | 164 | [257, 258] | 266 | [193, 324] | 368 | [125, 282] | 470 | [146, 153] |
| 63 | [118, 119] | 165 | [165, 165] | 267 | [221, 232] | 369 | [419, 290] | 471 | [304, 483] |
| 64 | [120, 55] | 166 | [259, 260] | 268 | [325, 25] | 370 | [408, 48] | 472 | [299, 134] |
| 65 | [121, 122] | 167 | [261, 8] | 269 | [326, 327] | 371 | [155, 420] | 473 | [156, 52] |
| 66 | [123, 124] | 168 | [155, 129] | 270 | [328, 329] | 372 | [125, 126] | 474 | [481, 137] |
| 67 | [116, 109] | 169 | [262, 263] | 271 | [330, 331] | 373 | [421, 40] | 475 | [321, 484] |
| 68 | [37, 125] | 170 | [147, 155] | 272 | [332, 17] | 374 | [163, 166] | 476 | [79, 485] |
| 69 | [37, 56] | 171 | [109, 264] | 273 | [107, 333] | 375 | [156, 411] | 477 | [486, 487] |
| 70 | [126, 55] | 172 | [38, 120] | 274 | [69, 195] | 376 | [139, 50] | 478 | [211, 488] |
| 71 | [127, 38] | 173 | [265, 266] | 275 | [331, 334] | 377 | [44, 403] | 479 | [461, 489] |
| 72 | [128, 129] | 174 | [135, 267] | 276 | [335, 102] | 378 | [422, 11] | 480 | [490, 19] |
| 73 | [130, 131] | 175 | [268, 11] | 277 | [336, 337] | 379 | [289, 423] | 481 | [491, 492] |
| 74 | [113, 132] | 176 | [116, 159] | 278 | [338, 190] | 380 | [127, 56] | 482 | [493, 494] |
| 75 | [133, 134] | 177 | [269, 152] | 279 | [339, 80] | 381 | [128, 420] | 483 | [495, 496] |
| 76 | [135, 42] | 178 | [50, 270] | 280 | [340, 341] | 382 | [159, 264] | 484 | [309, 44] |
| 77 | [136, 137] | 179 | [271, 118] | 281 | [342, 343] | 383 | [295, 300] | 485 | [138, 154] |
| 78 | [11, 138] | 180 | [115, 140] | 282 | [344, 345] | 384 | [405, 119] | 486 | [405, 110] |
| 79 | [34, 109] | 181 | [34, 117] | 283 | [81, 206] | 385 | [401, 122] | 487 | [497, 52] |
| 80 | [45, 139] | 182 | [133, 13] | 284 | [346, 347] | 386 | [12, 313] | 488 | [164, 165] |
| 81 | [128, 140] | 183 | [144, 50] | 285 | [236, 348] | 387 | [424, 425] | 489 | [298, 48] |
| 82 | [53, 141] | 184 | [151, 272] | 286 | [349, 350] | 388 | [398, 426] | 490 | [301, 498] |
| 83 | [142, 134] | 185 | [44, 160] | 287 | [20, 317] | 389 | [269, 136] | 491 | [41, 475] |
| 84 | [143, 144] | 186 | [120, 273] | 288 | [351, 346] | 390 | [8, 32] | 492 | [406, 139] |
| 85 | [32, 145] | 187 | [274, 37] | 289 | [352, 353] | 391 | [293, 412] | 493 | [51, 411] |
| 86 | [146, 147] | 188 | [50, 275] | 290 | [245, 354] | 392 | [422, 109] | 494 | [304, 287] |
| 87 | [9, 36] | 189 | [276, 131] | 291 | [345, 104] | 393 | [427, 428] | 495 | [292, 499] |
| 88 | [120, 127] | 190 | [113, 277] | 292 | [355, 356] | 394 | [360, 429] | 496 | [282, 37] |
| 89 | [148, 149] | 191 | [35, 263] | 293 | [357, 358] | 395 | [430, 431] | 497 | [500, 501] |
| 90 | [51, 150] | 192 | [278, 279] | 294 | [359, 360] | 396 | [432, 21] | 498 | [502, 503] |
| 91 | [143, 139] | 193 | [36, 280] | 295 | [361, 178] | 397 | [97, 433] | 499 | [494, 504] |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 92 | [151, 152] | 194 | [281, 114] | 296 | [362, 363] | 398 | [428, 434] | 500 | [130, 425] |
| 93 | [153, 115] | 195 | [125, 274] | 297 | [364, 365] | 399 | [337, 435] | 501 | [10, 280] |
| 94 | [111, 154] | 196 | [56, 274] | 298 | [366, 367] | 400 | [436, 325] | 502 | [481, 505] |
| 95 | [114, 155] | 197 | [164, 164] | 299 | [368, 369] | 401 | [437, 212] | 503 | [142, 13] |
| 96 | [156, 150] | 198 | [38, 282] | 300 | [348, 370] | 402 | [438, 439] | 504 | [9, 263] |
| 97 | [13, 157] | 199 | [282, 127] | 301 | [371, 27] | 403 | [384, 440] | 505 | [506, 88] |
| 98 | [147, 115] | 200 | [283, 284] | 302 | [372, 373] | 404 | [441, 385] | 506 | [417, 507] |
| 99 | [158, 44] | 201 | [44, 32] | 303 | [374, 340] | 405 | [442, 443] | 507 | [474, 508] |
| 100 | [34, 159] | 202 | [112, 285] | 304 | [375, 86] | 406 | [444, 445] | 508 | [262, 36] |
| 101 | [160, 161] | 203 | [141, 160] | 305 | [242, 376] | 407 | [446, 447] | | |

Output function $n \mapsto \tau(n)$:

| $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 73 | 0 | 146 | 0 | 219 | 0 | 292 | 0 | 365 | 1 | 438 | 1 |
| 1 | 0 | 74 | 0 | 147 | 0 | 220 | 1 | 293 | 0 | 366 | 0 | 439 | 0 |
| 2 | 0 | 75 | 0 | 148 | 1 | 221 | 0 | 294 | 0 | 367 | 1 | 440 | 1 |
| 3 | 0 | 76 | 0 | 149 | 0 | 222 | 0 | 295 | 0 | 368 | 1 | 441 | 0 |
| 4 | 0 | 77 | 1 | 150 | 1 | 223 | 1 | 296 | 0 | 369 | 1 | 442 | 0 |
| 5 | 0 | 78 | 0 | 151 | 0 | 224 | 1 | 297 | 1 | 370 | 0 | 443 | 0 |
| 6 | 1 | 79 | 1 | 152 | 0 | 225 | 0 | 298 | 0 | 371 | 1 | 444 | 1 |
| 7 | 0 | 80 | 0 | 153 | 0 | 226 | 1 | 299 | 1 | 372 | 1 | 445 | 1 |
| 8 | 1 | 81 | 1 | 154 | 1 | 227 | 0 | 300 | 1 | 373 | 0 | 446 | 0 |
| 9 | 0 | 82 | 1 | 155 | 1 | 228 | 1 | 301 | 1 | 374 | 0 | 447 | 1 |
| 10 | 0 | 83 | 0 | 156 | 1 | 229 | 1 | 302 | 1 | 375 | 1 | 448 | 0 |
| 11 | 0 | 84 | 0 | 157 | 0 | 230 | 0 | 303 | 0 | 376 | 1 | 449 | 1 |
| 12 | 0 | 85 | 1 | 158 | 1 | 231 | 0 | 304 | 1 | 377 | 1 | 450 | 1 |
| 13 | 1 | 86 | 0 | 159 | 0 | 232 | 0 | 305 | 0 | 378 | 0 | 451 | 0 |
| 14 | 1 | 87 | 0 | 160 | 0 | 233 | 1 | 306 | 1 | 379 | 0 | 452 | 1 |
| 15 | 0 | 88 | 1 | 161 | 1 | 234 | 0 | 307 | 0 | 380 | 1 | 453 | 1 |
| 16 | 1 | 89 | 1 | 162 | 1 | 235 | 1 | 308 | 1 | 381 | 1 | 454 | 1 |
| 17 | 1 | 90 | 1 | 163 | 0 | 236 | 1 | 309 | 0 | 382 | 0 | 455 | 1 |
| 18 | 1 | 91 | 0 | 164 | 1 | 237 | 0 | 310 | 1 | 383 | 0 | 456 | 0 |
| 19 | 0 | 92 | 0 | 165 | 0 | 238 | 1 | 311 | 0 | 384 | 0 | 457 | 1 |
| 20 | 1 | 93 | 0 | 166 | 1 | 239 | 1 | 312 | 0 | 385 | 1 | 458 | 0 |
| 21 | 0 | 94 | 0 | 167 | 0 | 240 | 0 | 313 | 0 | 386 | 0 | 459 | 1 |
| 22 | 0 | 95 | 1 | 168 | 1 | 241 | 0 | 314 | 0 | 387 | 1 | 460 | 1 |
| 23 | 0 | 96 | 1 | 169 | 1 | 242 | 0 | 315 | 0 | 388 | 0 | 461 | 0 |
| 24 | 1 | 97 | 1 | 170 | 0 | 243 | 0 | 316 | 0 | 389 | 1 | 462 | 1 |
| 25 | 0 | 98 | 0 | 171 | 1 | 244 | 1 | 317 | 1 | 390 | 1 | 463 | 1 |
| 26 | 0 | 99 | 1 | 172 | 1 | 245 | 0 | 318 | 1 | 391 | 0 | 464 | 0 |
| 27 | 1 | 100 | 1 | 173 | 0 | 246 | 0 | 319 | 0 | 392 | 0 | 465 | 1 |
| 28 | 1 | 101 | 0 | 174 | 0 | 247 | 0 | 320 | 1 | 393 | 1 | 466 | 0 |
| 29 | 1 | 102 | 1 | 175 | 0 | 248 | 1 | 321 | 1 | 394 | 1 | 467 | 0 |
| 30 | 0 | 103 | 1 | 176 | 1 | 249 | 1 | 322 | 0 | 395 | 0 | 468 | 0 |

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 31 | 0 | 104 | 0 | 177 | 1 | 250 | 0 | 323 | 0 | 396 | 1 | 469 | 1 |
| 32 | 1 | 105 | 1 | 178 | 0 | 251 | 0 | 324 | 1 | 397 | 1 | 470 | 0 |
| 33 | 1 | 106 | 0 | 179 | 0 | 252 | 0 | 325 | 0 | 398 | 0 | 471 | 1 |
| 34 | 1 | 107 | 1 | 180 | 0 | 253 | 1 | 326 | 1 | 399 | 1 | 472 | 1 |
| 35 | 1 | 108 | 0 | 181 | 1 | 254 | 0 | 327 | 1 | 400 | 0 | 473 | 1 |
| 36 | 1 | 109 | 1 | 182 | 0 | 255 | 1 | 328 | 1 | 401 | 1 | 474 | 0 |
| 37 | 0 | 110 | 0 | 183 | 1 | 256 | 0 | 329 | 1 | 402 | 1 | 475 | 1 |
| 38 | 1 | 111 | 0 | 184 | 0 | 257 | 1 | 330 | 0 | 403 | 0 | 476 | 1 |
| 39 | 1 | 112 | 1 | 185 | 1 | 258 | 0 | 331 | 1 | 404 | 0 | 477 | 0 |
| 40 | 0 | 113 | 0 | 186 | 1 | 259 | 1 | 332 | 1 | 405 | 0 | 478 | 0 |
| 41 | 0 | 114 | 1 | 187 | 0 | 260 | 1 | 333 | 1 | 406 | 1 | 479 | 0 |
| 42 | 0 | 115 | 0 | 188 | 0 | 261 | 0 | 334 | 0 | 407 | 0 | 480 | 1 |
| 43 | 0 | 116 | 1 | 189 | 1 | 262 | 1 | 335 | 1 | 408 | 0 | 481 | 0 |
| 44 | 1 | 117 | 1 | 190 | 0 | 263 | 0 | 336 | 1 | 409 | 1 | 482 | 1 |
| 45 | 0 | 118 | 1 | 191 | 1 | 264 | 1 | 337 | 1 | 410 | 0 | 483 | 0 |
| 46 | 0 | 119 | 0 | 192 | 1 | 265 | 0 | 338 | 1 | 411 | 1 | 484 | 0 |
| 47 | 1 | 120 | 1 | 193 | 1 | 266 | 1 | 339 | 0 | 412 | 1 | 485 | 1 |
| 48 | 1 | 121 | 0 | 194 | 1 | 267 | 0 | 340 | 0 | 413 | 0 | 486 | 0 |
| 49 | 0 | 122 | 0 | 195 | 1 | 268 | 0 | 341 | 1 | 414 | 0 | 487 | 0 |
| 50 | 0 | 123 | 1 | 196 | 0 | 269 | 1 | 342 | 1 | 415 | 1 | 488 | 1 |
| 51 | 1 | 124 | 1 | 197 | 1 | 270 | 1 | 343 | 0 | 416 | 1 | 489 | 0 |
| 52 | 0 | 125 | 1 | 198 | 1 | 271 | 0 | 344 | 0 | 417 | 1 | 490 | 1 |
| 53 | 1 | 126 | 1 | 199 | 0 | 272 | 1 | 345 | 0 | 418 | 1 | 491 | 0 |
| 54 | 1 | 127 | 1 | 200 | 1 | 273 | 1 | 346 | 0 | 419 | 1 | 492 | 1 |
| 55 | 0 | 128 | 1 | 201 | 1 | 274 | 0 | 347 | 1 | 420 | 0 | 493 | 1 |
| 56 | 0 | 129 | 1 | 202 | 1 | 275 | 1 | 348 | 1 | 421 | 0 | 494 | 1 |
| 57 | 0 | 130 | 0 | 203 | 0 | 276 | 1 | 349 | 1 | 422 | 0 | 495 | 0 |
| 58 | 0 | 131 | 1 | 204 | 0 | 277 | 1 | 350 | 1 | 423 | 1 | 496 | 0 |
| 59 | 0 | 132 | 0 | 205 | 0 | 278 | 1 | 351 | 0 | 424 | 1 | 497 | 0 |
| 60 | 1 | 133 | 0 | 206 | 1 | 279 | 0 | 352 | 0 | 425 | 0 | 498 | 0 |
| 61 | 1 | 134 | 0 | 207 | 0 | 280 | 0 | 353 | 0 | 426 | 1 | 499 | 1 |
| 62 | 1 | 135 | 0 | 208 | 0 | 281 | 1 | 354 | 0 | 427 | 1 | 500 | 0 |
| 63 | 1 | 136 | 1 | 209 | 0 | 282 | 0 | 355 | 0 | 428 | 0 | 501 | 0 |
| 64 | 1 | 137 | 0 | 210 | 0 | 283 | 1 | 356 | 0 | 429 | 0 | 502 | 0 |
| 65 | 0 | 138 | 1 | 211 | 0 | 284 | 0 | 357 | 0 | 430 | 0 | 503 | 0 |
| 66 | 1 | 139 | 1 | 212 | 0 | 285 | 1 | 358 | 1 | 431 | 0 | 504 | 0 |
| 67 | 1 | 140 | 1 | 213 | 1 | 286 | 1 | 359 | 0 | 432 | 1 | 505 | 1 |
| 68 | 0 | 141 | 0 | 214 | 0 | 287 | 1 | 360 | 1 | 433 | 0 | 506 | 1 |
| 69 | 0 | 142 | 0 | 215 | 0 | 288 | 0 | 361 | 0 | 434 | 1 | 507 | 0 |
| 70 | 1 | 143 | 0 | 216 | 1 | 289 | 0 | 362 | 0 | 435 | 0 | 508 | 1 |
| 71 | 1 | 144 | 1 | 217 | 0 | 290 | 0 | 363 | 1 | 436 | 0 | | |
| 72 | 1 | 145 | 0 | 218 | 0 | 291 | 0 | 364 | 1 | 437 | 1 | | |

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*Email address*: huyining@hust.edu.cn