EXPLICIT CONTINUED FRACTION EXPANSION OF A RATIONAL ROOT OF $1 + x^{-1}$ OVER $\mathbb{F}_p$

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ABSTRACT. Let p be a prime number, j and $d \geq 3$ positive integers coprime with p. We provide the explicit continued fraction expansion of the j/d-th root of $1 + x^{-1}$ in the power series field $\mathbb{F}_p((x^{-1}))$. We determine its irrationality exponent.

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1. Introduction

Throughout this paper, we let q be a power of a prime number p and $\mathbb{F}_q((x^{-1}))$ denote the field of power series in x^{-1} over $\mathbb{F}_q$. In parallel with the theory of continued fractions for real numbers, a theory of continued fractions has been developed in $\mathbb{F}_q((x^{-1}))$, where the partial quotients are (non-constant) polynomials in x; see the beginning of Section 3. Like in the real case, the continued fraction expansion of a power series ξ in $\mathbb{F}_q((x^{-1}))$ is ultimately periodic if and only if ξ is quadratic; see [20, Section 3]. While the continued fraction expansion of algebraic real numbers of degree at least three remains very mysterious, the continued fraction expansion of certain algebraic power series in $\mathbb{F}_q((x^{-1}))$ has been precisely determined. For instance, Baum and Sweet [2] have noticed that, for any positive integer s, the power series

$$\xi_{BS,s} = [x; x^{p^s}, x^{p^{2s}}, \ldots] = x + \frac{1}{x^{p^s} + \frac{1}{x^{p^{2s}} + \cdots}}$$

in $\mathbb{F}_q((x^{-1}))$, which satisfies $\xi_{BS,s} = x + 1/\xi_{BS,s}^{p^s}$, is a root of the polynomial

$$Z^{p^s+1} - xZ^{p^s} - 1,$$

hence is algebraic over $\mathbb{F}_q(x)$. Its degree is equal to $p^s + 1$ and its irrationality exponent (see Definition 3.1) is also equal to $p^s + 1$, by [10, p. 214]. This shows that a power series analogue of Roth's theorem does not hold and that the power series analogue of Liouville's theorem, which has been established by Mahler [13], is best possible; namely, an algebraic power series in $\mathbb{F}_q((x^{-1}))$ cannot be approximated by rational fractions at an order exceeding its degree. In the opposite direction, there are explicit examples of algebraic power series whose sequence of partial quotients are polynomials of bounded degree, that is, algebraic power series that are badly approximable by rational fractions, a first example in $\mathbb{F}_2((x^{-1}))$ being given in [2].

In the same paper [2], the authors derived the explicit continued fraction expansion of the power series

$$\left(\frac{P(x)}{P(x)+1}\right)^{1/(2^m-1)} \in \mathbb{F}_2((x^{-1})), \tag{1.1}$$

where $P(x)$ is in $\mathbb{F}_2[x] \setminus \mathbb{F}_2$. Since then, several explicit continued fractions of algebraic power series have been obtained; see [1, 11, 12, 15–17, 19, 20] for example. More recently, several families of continued fractions in $\mathbb{F}_2((x^{-1}))$ whose sequence of partial quotients can be generated by a finite automaton have been shown to be algebraic [4–8]. The main purpose of the present paper is to considerably extend the result of [2] on (1.1).

Let d, j be coprime integers with $d \geq 2$, $1 \leq j \leq d/2$ and such that p does not divide jd. Let s be the smallest positive integer such that d divides $p^s - 1$ and set $d' = (p^s - 1)/d$. We let $(1 + x^{-1})^{1/(p^s-1)}$ denote the unique power series $\xi = 1 + \dots$ in $\mathbb{F}_p((x^{-1}))$ such that $\xi^{p^s-1} = 1 + x^{-1}$ and we set

$$(1 + x^{-1})^{j/d} = \left((1 + x^{-1})^{1/(p^s-1)}\right)^{jd'}.$$

The case $d = 2$, $j = 1$, and p odd is easy. Namely, we have

$$(1 + x^{-1})^{1/2} = [1; 2x + 1/2, (-8x - 4, 2x + 1)^{\infty}] \tag{1.2}$$

in $\mathbb{Q}((x^{-1}))$, where the notation $(\ )^{\infty}$ means that the sequence of partial quotients of $(1 + x^{-1})^{1/2}$ is eventually periodic of period $-8x - 4, 2x + 1$; see Section 12 for a proof. Since the leading coefficient of each partial quotient is coprime with p, we immediately get that

$$(1 + x^{-1})^{1/2} = [1; 2x + 1/2, (-8x - 4, 2x + 1)^{\infty}]$$

in $\mathbb{F}_p((x^{-1}))$.

The main result of the present work is the following theorem.

Theorem 1.1. *Let p be a prime number, $d \geq 3$ an integer not divisible by p, and j a nonzero integer coprime with d. We give a full description of the continued fraction expansion of the power series $(1 + x^{-1})^{j/d}$ in $\mathbb{F}_p((x^{-1}))$.*

For the case $p = 2$, which is considerably simpler than the case p odd, precise statements are given in Theorems 2.2 and 4.2. The case p odd is much more complicated. To solve it, we have observed, experimentally, that in every case many convergents are of the form $(1 + x^{-1})^m$ for an integer m in $\mathbb{Z}$. Instead of giving a very technical, lengthy statement, we explain first how to find the infinite sequence $(m_k)_{k \geq 0}$ of integers such that $(|m_k|)_{k \geq 0}$ is increasing and $(1 + x^{-1})^m$ is a convergent to $(1 + x^{-1})^{j/d}$ if and only if m is an element of $(m_k)_{k \geq 0}$. Then, we give the continued fraction expansion of $(1 + x^{-1})^{m_0}$ and explain how to get the continued fraction expansion of $(1 + x^{-1})^{m_{k+1}}$ from the one of $(1 + x^{-1})^{m_k}$, for $k \geq 0$.

To our knowledge, Theorem 1.1 is the first general result on the continued fraction expansions of a 'natural', simple, infinite family of algebraic power series in arbitrary (finite) characteristic.

The precise knowledge of the continued fraction expansion of $(1 + x^{-1})^{j/d}$ in $\mathbb{F}_p((x^{-1}))$ allows us to determine its irrationality exponent and its approximation spectrum (see Definition 3.1). We gather several Diophantine results in Section 3.

2. Results

2.1. p-adic expansions of rational numbers. Let p be a prime number. Every p-adic integer λ in $\mathbb{Z}_p$ has a unique representation

$$\lambda = \lambda_0 + \lambda_1 p + \lambda_2 p^2 + \ldots, \quad \lambda_i \in \{0, 1, \ldots, p-1\},$$

called its Hensel expansion. We define the power series $(1 + x^{-1})^\lambda$ in $\mathbb{F}_p((x^{-1}))$ to be

$$(1 + x^{-1})^\lambda = (1 + x^{-1})^{\sum_{i \geq 0} \lambda_i p^i} = \prod_{i \geq 0} (1 + x^{-1})^{\lambda_i p^i} = \prod_{i \geq 0} (1 + x^{-p^i})^{\lambda_i}.$$

Mendès France and van der Poorten [14] proved that $(1 + x^{-1})^\lambda$ is algebraic over $\mathbb{F}_p(x)$ if and only if the p-adic integer λ is in $\mathbb{Q} \cap \mathbb{Z}_p$.

We check that if $\lambda = \lambda'_0 + \lambda'_1 p + \lambda'_2 p^2 + \ldots$, where $(\lambda'_i)_{i \geq 0}$ is a bounded sequence of integers, then

$$(1 + x^{-1})^\lambda = (1 + x^{-1})^{\sum_{i \geq 0} \lambda'_i p^i} = \prod_{i \geq 0} (1 + x^{-1})^{\lambda'_i p^i} = \prod_{i \geq 0} (1 + x^{-p^i})^{\lambda'_i},$$

as well. Furthermore, for λ in $\mathbb{Q} \cap \mathbb{Z}_p$, the expression $(1 + x^{-1})^\lambda$ gives the same result when λ is viewed as a rational number (as in Section 1) and when it is viewed as a p-adic number. These claims, as well as the other claims and the lemmas of this and the next sections, are proved in Section 5.

In the sequel, we sometimes assume that λ is a rational number j/d. Clearly, p and d are assumed to be coprime. We will often also assume that p and j are coprime. This is not at all restrictive, since the partial quotients of the power series ξ^p in $\mathbb{F}_p((x^{-1}))$ are the p-th powers of the partial quotients of the p-adic number ξ.

For the question investigated in this paper, the Hensel expansion is not the most appropriate way to represent the p-adic number λ. We make use in $\mathbb{F}_2$ of expansions with alternate signs over the set of digits $\{-1, 0, 1\}$ and, in $\mathbb{F}_p$ with p odd, of expansions over the set of digits $\{-(p-1)/2, \ldots, (p-1)/2\}$.

Our results give a precise description of the continued fraction expansion of any power series $(1 + x^{-1})^\lambda$ in $\mathbb{F}_p((x^{-1}))$ with λ in $\mathbb{Z}_p$.

2.2. Over the field $\mathbb{F}_2$. The continued fraction expansion of $(1 + x^{-1})^\lambda$ in $\mathbb{F}_2((x^{-1}))$ can be described in terms of the expansion with alternate signs of the 2-adic integer λ.

Lemma 2.1. *Every positive integer m has a unique binary representation with alternate signs*

$$m = 2^{v_\ell} - 2^{v_{\ell-1}} + \ldots - 2^{v_1} + 2^{v_0},$$

where $v_\ell > \ldots > v_0$ are non-negative integers and ℓ is even. Every negative integer m has a unique binary representation with alternate signs

$$m = -2^{v_\ell} + 2^{v_{\ell-1}} - \ldots - 2^{v_1} + 2^{v_0},$$

where $v_\ell > \ldots > v_0$ are non-negative integers and ℓ is odd. Every nonzero 2-adic integer λ which is not in $\mathbb{Z} \cap \mathbb{Z}_2$ has a unique expansion with alternate signs

$$\lambda = 2^{v_0} - 2^{v_1} + 2^{v_2} - \ldots,$$

where $(v_\ell)_{\ell \geq 0}$ is an increasing sequence of non-negative integers.

The following theorem asserts in particular that, for every λ in $\mathbb{Z}_2$, any convergent to the power series $(1+x^{-1})^\lambda$ is an integral power of $1+x^{-1}$. This holds in particular for any rational power $(1+x^{-1})^{j/d}$ with d odd.

To shorten the notation, we put

$$[k]_x = 1 + x + \ldots + x^{k-1}, \quad k \geq 1.$$

Theorem 2.2. *(i). Let m be a nonzero integer. Let*

$$m = (-1)^\ell 2^{v_\ell} + (-1)^{\ell-1} 2^{v_{\ell-1}} + \ldots + 2^{v_0}$$

denote its binary representation with alternate signs. Set

$$m_k = (-1)^k 2^{v_k} + (-1)^{k-1} 2^{v_{k-1}} + \ldots + 2^{v_0}, \quad k = 0, \ldots, \ell.$$

Then, the convergents to $(1+x^{-1})^m$ are

$$(1+x^{-1})^{m_0}, (1+x^{-1})^{m_1}, \ldots, (1+x^{-1})^{m_\ell} = (1+x^{-1})^m,$$

if $v_1 > v_0 + 1$, and

$$(1+x^{-1})^{m_1}, (1+x^{-1})^{m_2}, \ldots, (1+x^{-1})^{m_\ell} = (1+x^{-1})^m,$$

if $v_1 = v_0 + 1$.

(ii). Let λ be an element of $\mathbb{Z}_2 \setminus \mathbb{Z}$, whose expansion with alternate signs is given by

$$\lambda = 2^{v_0} - 2^{v_1} + 2^{v_2} - \ldots,$$

with $v_0 < v_1 < \ldots$ Set $m_{-1} = 0$ and

$$m_k = 2^{v_0} - 2^{v_1} + 2^{v_2} - \ldots + (-1)^k 2^{v_k}, \quad k \geq 0. \tag{2.1}$$

Then, the partial quotients a_n, $n \geq 1$, of

$$(1+x^{-1})^\lambda = [1; a_1, a_2, a_3, \ldots], \tag{2.2}$$

which are polynomials in $x^{2^{v_0}}$, and its convergents P_n/Q_n are given by

(ii.a) if $v_1 > v_0 + 1$:

$$a_1 = x^{2^{v_0}};$$

$$a_n = (x+x^2)^{|m_{n-3}|} [2^{v_{n-1}-v_{n-2}} - 1]_x^{2^{v_{n-2}}}, \quad n \geq 2;$$

$$\frac{P_n}{Q_n} = (1+x^{-1})^{m_{n-1}}, \quad n \geq 1;$$

(ii.b) if $v_1 = v_0 + 1$:

$$a_1 = 1 + x^{2^{v_0}};$$

$$a_2 = [2^{v_2-v_0} - 1]_x^{2^{v_0}};$$

$$a_n = (x+x^2)^{|m_{n-2}|} [2^{v_n-v_{n-1}} - 1]_x^{2^{v_{n-1}}}, \quad n \geq 3;$$

$$\frac{P_n}{Q_n} = (1 + x^{-1})^{m_n}, \quad n \geq 1.$$

With the notation of Theorem 2.2, we have $|m_{k+1}| \geq |m_k|$ for $k \geq 0$, with equality if and only if $k = 0$ and $v_1 = v_0 + 1$.

When $\lambda = j/d$ is rational and $1 \leq j < d/2$, we give in Theorem 4.2 below a closed formula for a_n depending on the value of n modulo a suitable integer at most equal to $(d-1)/4$. As an illustration of Theorem 4.2, we derive the continued fraction expansion of $(1 + x^{-1})^{1/13}$.

Corollary 2.3. *The continued fraction expansion of* $(1 + x^{-1})^{1/13}$ *is*

$$[1; x, [3]_x, (x + x^2)^1, (x + x^2)^3 \cdot [7]_x^8, (x + x^2)^5 \cdot [3]_x^{64}, (x + x^2)^{59}, \ldots],$$

where the partial quotients a_n *are given by*

$$\begin{aligned} a_1 &= x, \\ a_{3n} &= (x + x^2)^{(3 \cdot 2^{6n-4} - (-1)^n)/13}, \quad n \geq 1, \\ a_{3n+1} &= (x + x^2)^{(5 \cdot 2^{6n-3} + (-1)^n)/13} \cdot [7]_x^{2^{6n-3}}, \quad n \geq 1, \\ a_{3n+2} &= (x + x^2)^{(2^{6n} - (-1)^n)/13} \cdot [3]_x^{2^{6n}}, \quad n \geq 0. \end{aligned}$$

Further examples can be found on the webpage
`http://irma.math.unistra.fr/~guoniu/pthroot/` .

2.3. **Over the field $\mathbb{F}_p$ with p odd.** In this subsection, we let p denote an odd prime number. We use the signed p-adic expansion of an element of $\mathbb{Z}_p$ instead of its Hensel expansion.

Lemma 2.4. *Every nonzero integer* m *has a unique representation*

$$m = d_0 p^{v_0} + d_1 p^{v_1} + \ldots + d_\ell p^{v_\ell},$$

where $0 \leq v_0 < v_1 < \ldots < v_\ell$ *are integers and the digits* d_i *are in* $\{-(p-1)/2, \ldots, (p-1)/2\}$ *and are nonzero. We call it the signed* p*-adic representation of* m*. More generally, every* p*-adic integer* λ *which is not in* $\mathbb{Z} \cap \mathbb{Z}_p$ *has a unique expansion*

$$\lambda = d_0 p^{v_0} + d_1 p^{v_1} + \ldots, \tag{2.3}$$

where $0 \leq v_0 < v_1 < \ldots$ *are integers and all the digits* d_i *are in* $\{-(p-1)/2, \ldots, -1, 1, \ldots, (p-1)/2\}$*, and we call it its signed* p*-adic expansion.*

Let λ be a p-adic integer which is not in $\mathbb{Z} \cap \mathbb{Z}_p$ and let

$$\lambda = d_0 p^{v_0} + d_1 p^{v_1} + d_2 p^{v_2} + \ldots \tag{2.4}$$

denote its signed p-adic expansion. We want to compute the continued fraction expansion of $(1 + x^{-1})^\lambda$. We assume that $v_0 = 0$ in (2.4). This is not a restriction since every nonzero p-adic integer is equal to a power of p times a p-adic unit and, as already pointed out, the partial quotients of the p-th power of an element of $\mathbb{F}_p((x^{-1}))$ are the p-th powers of its partial quotients.

Define the integer sequence $(m_k)_{k\geq 0}$ by setting

$$m_k = d_0 + d_1 p^{v_1} + \ldots + d_k p^{v_k}, \quad k \geq 0.$$

Since

$$|m_k| \leq \frac{p-1}{2} \cdot \frac{p^{v_k+1}-1}{p-1} < \frac{p^{v_{k+1}}}{2}, \quad k \geq 0, \tag{2.5}$$

and $|m_{k+1} - m_k| \geq p^{v_{k+1}}$, the sequence $(|m_k|)_{k\geq 0}$ is increasing. Furthermore, m_k and d_k have the same sign.

Lemma 2.5. *Let λ be a nonzero p-adic integer in $\mathbb{Z}_p$ and set $\xi = (1+x^{-1})^{\lambda}$. Let $(m_k)_{k\geq 0}$ be the sequence defined above. For $k \geq 0$, the rational fraction $(1+x^{-1})^{m_k}$ is a convergent to ξ. Conversely, if m is a nonzero integer such that $(1+x^{-1})^m$ is a convergent to ξ, then there exists $k \geq 0$ such that $m = m_k$.*

We postpone its proof to Section 6. Theorem 2.2 shows that the analogue of Lemma 2.5 holds for $p = 2$.

By Lemma 2.5, to compute the continued fraction expansion of $(1+x^{-1})^{\lambda}$, it is sufficient to compute the continued fraction expansion of $(1+x^{-1})^{m_0}$ and to know how the continued fraction expansion of $(1+x^{-1})^{m_{k+1}}$ can be deduced from the one of $(1+x^{-1})^{m_k}$.

The next lemma answers the former question (recall that $|m_0| < p/2$ always holds). It follows from a classical result of Lagrange [9] and its proof is postponed to Section 12. Throughout this paper, by length of a finite continued fraction, we mean the number of its partial quotients (excluding the polynomial part). For example, the length of $1 + x^{-1} = [1, x]$ is 1 and the length of the ℓ-th convergent to the infinite continued fraction expansion $[a_0; a_1, \ldots]$ is equal to ℓ, for $\ell \geq 1$.

Lemma 2.6. *Let p be an odd prime number and m be a nonzero integer with $|m| < p/2$. For $j = 2, \ldots, |m|$, set*

$$f_j = \frac{(2j-1)(m-j+2)(m-j+4)\cdots(m+j-2)}{(m-j+1)(m-j+3)\cdots(m+j-1)}(2x+1).$$

Then, we have

$$(1+x^{-1})^m = \left[1, \frac{x}{m} + \frac{1-m}{2m}, 2f_2, \frac{f_3}{2}, \ldots, 2^{(-1)^m} f_{|m|}\right], \tag{2.6}$$

and the length of $(1+x^{-1})^m$ is equal to $|m|$.

Throughout this paper, $\mathbf{w}$ stands for a finite word over the alphabet composed of the non-constant polynomials in $\mathbb{F}_p[x]$. It may be the empty word. When it appears in a continued fraction expansion, we read the sequence of its letters as a sequence of partial quotients. Writing $\mathbf{w} = w_1, \ldots, w_k$, its reversal $\overleftarrow{\mathbf{w}}$ is the word

$$\overleftarrow{\mathbf{w}} = w_k, \ldots, w_1,$$

and, for a nonzero ρ in $\mathbb{Z}_p$, we set

$$\rho\mathbf{w} = \rho w_1, \rho^{-1} w_2, \dots, \rho^{(-1)^{k+1}} w_k,$$

and we call $\rho\mathbf{w}$ a twist of $\mathbf{w}$.

Let m be a nonzero integer coprime with p. The first partial quotient of $(1 + x^{-1})^m$ is 1 and the second one depends only on the value of m modulo p. By Lemma 2.6, it is equal to $\frac{x}{d_0} + \frac{1-d_0}{2d_0}$, where d_0 is the integer in $\{-(p-1)/2, \dots, (p-1)/2\}$ which is congruent to m modulo p. This justifies the first two partial quotients in (2.7) below.

The Key Lemma below is the main ingredient of the proof of Theorem 1.1 for odd primes.

Lemma 2.7 (Key Lemma). *Let $k \ge 0$ be an integer. Let $d_0, d_1, \dots, d_k, d_{k+1}$ be nonzero integers in $\{-(p-1)/2, \dots, (p-1)/2\}$. Let $v_1, \dots, v_k, v_{k+1}$ be integers with $0 < v_1 < \dots < v_k < v_{k+1}$ and set*

$$m_k := d_0 + d_1 p^{v_1} + \dots + d_k p^{v_k}, \quad m_{k+1} = m_k + d_{k+1} p^{v_{k+1}}.$$

Let

$$(1 + x^{-1})^{m_k} = \Big[1, \frac{x}{d_0} + \frac{1-d_0}{2d_0}, \mathbf{w}\Big] \tag{2.7}$$

denote the continued fraction expansion of $(1 + x^{-1})^{m_k}$, where $\mathbf{w}$ is a finite word, possibly empty. Set

$$\begin{aligned} \varepsilon &= -\operatorname{sgn}(m_k) d_{k+1} = -\operatorname{sgn}(m_k m_{k+1}) |d_{k+1}|, \\ y &= \frac{x}{m_k} + \frac{1-m_k}{2m_k}, \\ \mathbf{h} &= (\overleftarrow{\mathbf{w}/2}, 2y+1, -2y-1, -\mathbf{w}/2). \end{aligned}$$

For any non-negative integer j, put

$$\delta(4j) = 4, \;\; \delta(4j+1) = 16, \;\; \delta(4j+2) = \frac{1}{4}, \;\; \delta(4j+3) = 1,$$

and let $\eta_\varepsilon(j)$ be given by the Ultimate Triangle defined in Section 8. If m_k is odd, then the continued fraction expansion of $(1 + x^{-1})^{m_{k+1}}$ is given by

$$(1+x^{-1})^{m_{k+1}} = [1, y, \mathbf{w}, c_{k+1}, \eta_\varepsilon(0)\mathbf{h}, \eta_\varepsilon(1) c_{k+1}, \eta_\varepsilon(2)\mathbf{h}, \dots, \eta_\varepsilon(2|\varepsilon|-3) c_{k+1}],$$

if m_k and m_{k+1} have opposite signs, and by

$$(1 + x^{-1})^{m_{k+1}} = [1, y, \mathbf{w}, c_{k+1}, \eta_\varepsilon(0)\mathbf{h}, \eta_\varepsilon(1) c_{k+1}, \eta_\varepsilon(2)\mathbf{h}, \dots, \eta_\varepsilon(2|\varepsilon|-2)\mathbf{h}],$$

otherwise.

If m_k is even, then the continued fraction expansion of $(1 + x^{-1})^{m_{k+1}}$ is given by

$$\begin{aligned} (1 + x^{-1})^{m_{k+1}} = [1, y, \mathbf{w}, c_{k+1}, \delta(0)\eta_\varepsilon(0)\mathbf{h}, \delta(1)\eta_\varepsilon(1) c_{k+1}, \delta(2)\eta_\varepsilon(2)\mathbf{h}, \dots, \\ \delta(2|\varepsilon|-3)\eta_\varepsilon(2|\varepsilon|-3) c_{k+1}], \end{aligned}$$

if m_k and m_{k+1} have opposite signs, and by

$$(1+x^{-1})^{m_{k+1}} = [1, y, \mathbf{w}, c_{k+1}, \delta(0)\eta_\varepsilon(0)\mathbf{h}, \delta(1)\eta_\varepsilon(1)c_{k+1}, \delta(2)\eta_\varepsilon(2)\mathbf{h}, \ldots, \delta(2|\varepsilon|-2)\eta_\varepsilon(2|\varepsilon|-2)\mathbf{h}],$$

otherwise.

Here, c_{k+1} is the polynomial part of the rational fraction c'_{k+1} defined by

$$c'_{k+1} = (-1)^{m_k} 4^{\overline{m_k}-|m_k|} \frac{(1+x)^{p^{v_{k+1}}-|m_k|}}{d_{k+1}x^{|m_k|}},$$

where

$$\overline{m_k} = \begin{cases} 0, & \text{if } m_k \text{ is even,} \\ 1, & \text{if } m_k \text{ is odd.} \end{cases}$$

In particular, if ℓ_k and ℓ_{k+1} denote the lengths of the continued fraction expansions of $(1+x^{-1})^{m_k}$ and $(1+x^{-1})^{m_{k+1}}$, respectively, then

$$\ell_{k+1} = (2|d_{k+1}|-1)\ell_k + |d_{k+1}|, \quad \text{if } m_k m_{k+1} < 0,$$

and

$$\ell_{k+1} = (2|d_{k+1}|+1)\ell_k + |d_{k+1}|, \quad \text{if } m_k m_{k+1} > 0.$$

Moreover, ℓ_k and m_k have the same parity for every $k \geq 0$.

It should be pointed out that $\eta_\varepsilon(0), \ldots, \eta_\varepsilon(2|\varepsilon|-2)$ are rational numbers and do not depend on the prime p. Since their numerators and denominators are divisible only by primes less than p, they are nonzero modulo p. The shape of, say, the continued fraction expansion of $(1+x^{-1})^{1/4}$ in $\mathbb{F}_p((x^{-1}))$ depends on the signed p-adic expansion of $1/4$, but the coefficients $\eta_\varepsilon(h)$ are rational numbers independent of p.

To derive the continued fraction expansion of $(1+x^{-1})^{j/d}$ in $\mathbb{F}_p((x^{-1}))$, we first compute the signed p-adic expansion of j/d and then we apply Lemma 2.7 repeatedly. Two explicit examples are discussed in Section 13.

Lemma 2.7 allows us to describe precisely the degrees of the partial quotients of $(1+x^{-1})^\lambda$.

Corollary 2.8. *Let λ be in $\mathbb{Z} \setminus \mathbb{Z}_p$. Every partial quotient of $(1+x^{-1})^\lambda$ is a linear polynomial or a twist of a polynomial of degree $\delta_k := p^{v_{k+1}} - 2|m_k|$, for some $k \geq 0$. Furthermore, for every $\ell \geq 0$, the first occurrence of a partial quotient of degree δ_ℓ arises immediately after the convergent $(1+x^{-1})^{m_k}$, where k is the smallest integer with $\delta_k = \delta_\ell$.*

Note that the δ_k may not be all distinct. For example, if $m_k > 0$, $d_{k+1} = (p-1)/2$, and $v_{k+2} = v_{k+1} + 1$, then

$$p^{v_{k+2}} - 2|m_{k+1}| = p^{v_{k+1}+1} - 2m_k - (p-1)p^{v_{k+1}} = p^{v_{k+1}} - 2|m_k|.$$

For $p = 2$, the proof of Theorem 2.2 shows that every partial quotient of $(1+x^{-1})^\lambda$ is equal to the integer part of a rational fraction $(1+x)^{|m_{k+1}|}/x^{|m_k|}$, with m_k as in (2.1). Thus, the degree of the k-th (or $(k+1)$-th, depending whether or not v_1 is equal to v_0+1) partial quotient of $(1+x^{-1})^{\lambda}$ is equal to $|m_k|-|m_{k-1}|$, that is, to $2^{v_k}-2|m_{k-1}|$.

2.4. **Comments on the proofs.**

We had to compute many explicit examples to be able to guess the precise form of the continued fraction expansion of $(1+x^{-1})^{j/d}$ in $\mathbb{F}_p((x^{-1}))$. The case $p=2$ became simpler once we realized that all the convergents are integral powers of $(1+x^{-1})$. For the case p odd, once we found the expression of m_k and realized that one goes from the continued fraction expansion of $(1+x^{-1})^{m_k}$ to that of $(1+x^{-1})^{m_{k+1}}$ by adding partial quotients given, roughly speaking, by $\mathbf{w}, \overleftarrow{\mathbf{w}}$ (notations from Lemma 2.7), the most difficult point was to find the expression of the (nonzero) coefficients in $\mathbb{F}_p$ by which the partial quotients of $\mathbf{w}, \overleftarrow{\mathbf{w}}$ are multiplied. Subsequently, it was a great surprise when we noticed that these coefficients are given by the same array of rational numbers, that we simply have to take modulo p. To summarize, the most difficult part of the proof of Lemma 2.7 was to guess the conclusion of the lemma. It then only remained for us to check its correctness by a direct computation. That being said, this last step is quite lengthy and complicated.

In a forthcoming work, we will investigate how the constructions described in the present paper can be used to get new results on continued fraction expansions of real numbers. Namely, the formula obtained in Section 9 are established over an arbitrary field, hence are valid over the rationals.

3. Irrationality exponent and approximation spectrum

We define an absolute value $|\cdot|$ on $\mathbb{F}_q((x^{-1}))$ as follows. We set $|0|=0$ and, if $\xi = b_t x^t + b_{t-1}x^{t-1} + \dots$ with $b_t \neq 0$, we set $|\xi| = \mathrm{e}^t$. In particular, if ξ is a nonzero polynomial in $\mathbb{F}_q[x]$, then $|\xi| = \mathrm{e}^{\deg(\xi)}$.

Any element of $\mathbb{F}_q(x)$ is uniquely expressed as a finite continued fraction

$$[a_0; a_1, \dots, a_n] = \frac{P_n}{Q_n}, \tag{3.1}$$

where $a_0, a_1, \dots, a_n$ are in $\mathbb{F}_q[x]$ and have positive degree, except possibly for a_0. Every element ξ in $\mathbb{F}_q((x^{-1})) \setminus \mathbb{F}_q(x)$ can be uniquely represented as an infinite continued fraction

$$\xi = [a_0; a_1, \dots],$$

where $a_0, a_1, a_2, \dots$ are in $\mathbb{F}_q[x]$ and have positive degree, except possibly for a_0. The P_n/Q_n defined by (3.1) are the convergents to ξ and the a_i's are its partial quotients. We have

$$\Big|\xi - \frac{P_n}{Q_n}\Big| = \frac{1}{|Q_n Q_{n+1}|} = \frac{1}{|a_{n+1}| \cdot |Q_n|^2}, \quad n \geq 0, \tag{3.2}$$

and

$$P_{n+1} = a_{n+1}P_n + P_{n-1}, \quad Q_{n+1} = a_{n+1}Q_n + Q_{n-1}, \quad n \geq 1. \tag{3.3}$$

Observe that

$$\deg Q_n = \deg a_1 + \ldots + \deg a_n, \quad n \geq 1.$$

Standard references include [20, 21]. Unlike in the real case, the knowledge of the convergent P_n/Q_n (viewed as a rational fraction) does not determine P_n and Q_n, since it gives no information on their leading coefficient. For example,

$$\left(\frac{x+1}{x}\right)^m = [a_0; a_1, \ldots, a_n] = \frac{P_n}{Q_n},$$

with a positive integer m, does not imply that the polynomial P_n given by the recurrence (3.3) is equal to $(x+1)^m$; we only know that P_n is a constant multiple of $(x+1)^m$. A further illustration is given by

$$[1; -x-1] = 1 + \frac{1}{-x-1} = \frac{-x}{-x-1} = \frac{x}{x+1},$$

giving that $P_1 = -x$ and $Q_1 = -x - 1$.

The determination of the continued fraction expansion of ξ allows us to compute its irrationality exponent $\mu(\xi)$ and its approximation spectrum $\mathcal{S}(\xi)$, a more general notion introduced by Schmidt [20].

Definition 3.1. *Let $\xi = [a_0; a_1, a_2, \ldots]$ be in $\mathbb{F}_q((x^{-1})) \setminus \mathbb{F}_q(x)$ and denote by $(P_n/Q_n)_{n\geq 0}$ the sequence of its convergents. The irrationality exponent of ξ, denoted by $\mu(\xi)$, is defined by*

$$\mu(\xi) := \limsup_{n\to+\infty} -\frac{\log|\xi - P_n/Q_n|}{\log|Q_n|} = 2 + \limsup_{n\to+\infty} \frac{\deg a_{n+1}}{\deg Q_n}. \tag{3.4}$$

The approximation spectrum of ξ, denoted by $\mathcal{S}(\xi)$, is the set of limit points of the sequence

$$-\frac{\log|\xi Q_n - P_n|}{\log|Q_n|}, \quad n \geq 1.$$

We let

$$\nu(\xi) := \limsup_{n\to+\infty} -\frac{\log|\xi Q_n - P_n|}{\log|Q_n|} = \limsup_{n\to+\infty} \frac{\deg Q_{n+1}}{\deg Q_n} = \mu(\xi) - 1$$

and

$$\widehat{\nu}(\xi) := \liminf_{n\to+\infty} -\frac{\log|\xi Q_n - P_n|}{\log|Q_n|} = \liminf_{n\to+\infty} \frac{\deg Q_{n+1}}{\deg Q_n}$$

denote, respectively, the greatest and the smallest element of the approximation spectrum of ξ.

Since any element P/Q in $\mathbb{F}_q(x)$ such that $|\xi - P/Q| < 1/|Q|^2$ is a convergent to the irrational power series ξ (this statement is usually called Legendre's theorem for power series), the last equality in (3.4) follows straightforwardly from (3.2) and (3.3).

We derive from Theorems 2.2 and 2.7 the value of the irrationality exponent of $(1 + x^{-1})^{j/d}$ in $\mathbb{F}_p((x^{-1}))$. Since ξ, ξ^{-1}, and $(1 + x^{-1})\xi$ have the same irrationality exponent for every nonzero power series ξ, there is no restriction in assuming that $1 \le j < d/2$. As usual, an empty sum is equal to 0.

Theorem 3.2. *Let p be a prime number and j, d positive integers with $\gcd(p, jd) = 1$ and $1 \le j < d/2$. If $p = 2$, then write $j = j_0$ and*

$$d = j_0 + j_1\, 2^{u_1} = j_1 + j_2\, 2^{u_2} = \ldots = j_{s-1} + j_0\, 2^{u_{s-1}},$$

where $j_1, \ldots, j_{s-1}$ are distinct odd integers in $[1, d/2]$. Set $\mathcal{O}_{2,j,d} := \{j = j_0, j_1, \ldots, j_{s-1}\}$. Then,

$$\mu((1+x^{-1})^{j/d}) = \frac{d}{\min\{\bar{\jmath} : \bar{\jmath} \in \mathcal{O}_{2,j,d}\}} \ge \frac{d}{j},$$

$$\widehat{\nu}((1+x^{-1})^{j/d}) = \frac{d}{\max\{\bar{\jmath} : \bar{\jmath} \in \mathcal{O}_{2,j,d}\}} - 1 > 1,$$

and

$$\mathcal{S}((1+x^{-1})^{j/d}) = \Big\{ \frac{d-\bar{\jmath}}{\bar{\jmath}} : \bar{\jmath} \in \mathcal{O}_{2,j,d} \Big\}.$$

If p is odd, then write the signed p-adic expansion of j/d in $\mathbb{Z}_p$ as

$$\frac{j}{d} = (d_0 + d_1 p^{v_1} + \ldots + d_{s-1} p^{v_{s-1}})(1 + p^{v_s} + p^{2v_s} + \ldots),$$

with v_s minimal. Set $v_0 = 0$. For $h = 0, \ldots, s-1$, define

$$j_h = \frac{d(d_h + d_{h+1} p^{v_{h+1}-v_h} + \ldots + d_{s-1} p^{v_{s-1}-v_h} + d_0 p^{v_s - v_h} + \ldots + d_{h-1} p^{v_s + v_{h-1} - v_h})}{1 - p^{v_s}}$$

and $\mathcal{O}_{p,j,d} := \{j = j_0, j_1, \ldots, j_{s-1}\}$. Then,

$$\mu((1+x^{-1})^{j/d}) = \frac{d}{\min\{|\bar{\jmath}| : \bar{\jmath} \in \mathcal{O}_{p,j,d}\}} \ge \frac{d}{j}. \tag{3.5}$$

Theorem 3.2 extends a result of Osgood [18], who established in 1975 that the irrationality exponent of the power series $(1+x^{-1})^{1/d}$ is equal to d.

Note also that $j = j_0$ is always an element of the set $\mathcal{O}_{p,j,d}$. The appearence of the sets $\mathcal{O}_{p,j,d}$ in Theorem 3.2 is not surprising: it follows from the fact that $(1+x^{-1})^{j/d}$ and $(1+x^{-1})^{j/d-(d_0+d_1p^{v_1}+\ldots+d_ip^{v_i})}$ have the same irrationality exponent for every $i \ge 0$. Said differently, to compute the irrationality exponent of $(1+x^{-1})^{j/d}$, we have to consider the (finitely many) shifted sequences of the sequence of digits of j/d.

For an odd prime p, a simple algorithm allows us to compute the sets $\mathcal{O}_{p,j,d}$, where $1 \le j < d/2$ and $\gcd(p, jd) = 1$. There is a unique pair (a_0, ε_0) with $1 \le a_0 \le (p-1)/2$ and $\varepsilon_0 = \pm 1$ such that

$$p \mid (a_0 d - \varepsilon_0 j).$$

Then, define the integers j_1 and u_1 by

$$a_0 d = \varepsilon_0 j + j_1 p^{u_1}, \quad \gcd(p, j_1) = 1.$$

Observe that $1 \leq j_1 < d/2$. There is a unique pair (a_1, ε_1) with $1 \leq a_1 \leq (p-1)/2$ and $\varepsilon_1 = \pm 1$ such that

$$p \mid (a_1 d - \varepsilon_1 j_1).$$

Then, define the integers j_2 and u_2 by

$$a_1 d = \varepsilon_1 j_1 + j_2 p^{u_2}, \quad \gcd(p, j_2) = 1.$$

Observe that $1 \leq j_2 < d/2$. Continue like this to get $j_3, j_4, \ldots$ until one reaches $j_h = j$. This will always happen since the j_i's are in a finite set. We check that $h = s$, the values j_i are the same as in Theorem 3.2, and $u_i = v_i - v_{i-1}$. We also check that

$$\mathcal{O}_{p,j,d} = \{j, j_1, \ldots, j_{h-1}\}.$$

We can completely characterize the cases where $(1 + x^{-1})^{j/d}$ has a nontrivial uniform exponent of approximation and the cases where its approximation spectrum is finite.

Proposition 3.3. *Let p be a prime number, j, d coprime integers with $d \geq 3$, $1 \leq j < d/2$ and $\gcd(p, jd) = 1$. Set $\xi = (1 + x^{-1})^{j/d}$. If $p = 2$, then $\widehat{\nu}(\xi) > 1$ and the approximation spectrum $\mathcal{S}(\xi)$ is finite. If $p \geq 3$, write its signed p-adic expansion as*

$$\frac{j}{d} = (d_0 + d_1 p^{v_1} + \ldots + d_{s-1} p^{v_{s-1}})(1 + p^{v_s} + p^{2v_s} + \ldots),$$

with v_s minimal. Then, the three following properties are equivalent:

(i) $\widehat{\nu}(\xi) > 1$;

(ii) The approximation spectrum $\mathcal{S}(\xi)$ is finite;

(iii) s is even and $d_0, d_1, \ldots, d_{s-1}$ take alternatively the values 1 and -1.

To conclude this section, we briefly discuss simultaneous rational approximation of the first integral powers of $1 + x^{-1}$.

Observe that, for any integer $k \geq 1$ and any real number $\mu \geq 2$, if ξ and P/Q satisfy $|\xi - P/Q| = 1/|Q|^{\mu}$, then

$$|\xi^k - P^k/Q^k| \leq c(\xi, k)|Q^k|^{-\mu/k},$$

for some positive $c(\xi, k)$ independent of P/Q. Consequently, we have $\mu(\xi^k) \geq \mu(\xi)/k$ or, equivalently,

$$\nu(\xi^k) \geq \frac{\nu(\xi) - k + 1}{k}, \quad k \geq 1. \tag{3.6}$$

Definition 3.4. *Let ξ be in $\mathbb{F}_q((x^{-1})) \setminus \mathbb{F}_q(x)$. Let $k \geq 1$ be an integer. Let $\lambda_k(\xi)$ denote the supremum of the real numbers λ for which*

$$0 < \max\{|Q(T)\xi - P_1(T)|, \ldots, |Q(T)\xi^k - P_k(T)|\} < \mathrm{e}^{-\lambda \deg(Q)}$$

has infinitely many solutions in polynomials $Q(T), P_1(T), \ldots, P_k(T)$ in $\mathbb{F}_q[x]$.

The next result has been established in [3].

Theorem 3.5. *Let ξ be a power series in $\mathbb{F}_q((x^{-1}))$. For any positive integer k, we have*

$$(k+1)\big(1+\lambda_{k+1}(\xi)\big) \geq k\big(1+\lambda_k(\xi)\big),$$

with equality if $\lambda_{k+1}(\xi) > 1$. Consequently, for every integer n with $n \geq k$, we have

$$\lambda_n(\xi) \geq \frac{k\lambda_k(\xi) - n + k}{n}, \tag{3.7}$$

and equality holds if $\lambda_n(\xi) > 1$.

By combining Theorems 3.2 and 3.5, we easily get some partial results on the values of exponents of simultaneous approximation at the d-th root of $1 + x^{-1}$.

Theorem 3.6. *Let p be a prime number and $d \geq 3$ an integer. Let j be the greatest positive integer coprime with pd such that $j < d/2$ and j is the smallest element of the set $\mathcal{O}_{p,j,d}$. Then, we have*

$$\lambda_h\big((1+x^{-1})^{1/d}\big) = \frac{d-h}{h}, \quad h = 1, \ldots, j.$$

More generally, we can study the irrationality exponent of $\xi_\lambda = (1+x^{-1})^\lambda$ for an arbitrary λ in $\mathbb{Z}_p \setminus \mathbb{Z}$, since we have a precise description of its continued fraction expansion. We may return to this question in a subsequent work.

4. A simplified statement over $\mathbb{F}_2$

In the special case of $\mathbb{F}_2$, we can give a close formula for the partial quotients a_n defined in Theorem 2.2. First, we need to introduce several functions which play a crucial role in our approach. The correctness of the definition is not immediate and will be checked in the next half a page.

Definition 4.1. *Let $d \geq 3$ be an odd integer. Set*

$$\mathbb{A} = \mathbb{A}_d = \{j \mid 1 \leq j \leq (d-1)/2, \;\; j \text{ odd}, \gcd(j,d) = 1\}.$$

Let $\psi_d : \mathbb{A} \to \mathbb{A}$ be the map defined by the relation

$$d = j + \psi_d(j) 2^{\delta_d(j)}, \tag{4.1}$$

where $\delta(j) = \delta_d(j) \geq 1$. The map $\psi = \psi_d$ is a permutation on $\mathbb{A}$. Let j be in $\mathbb{A}$. Set

$$\sigma_j(n) = \sigma_{j,d}(n) = \delta(j) + \delta(\psi(j)) + \ldots + \delta(\psi^n(j)), \quad n \geq 0,$$

$\sigma_j(-1) = 0$. The function $\rho_j : \mathbb{N}_{\geq -1} \to \mathbb{N}_{\geq 0}$ is defined by

$$\rho_j(n) = \rho_{j,d}(n) = \big(\psi^{n+1}(j) \cdot 2^{\sigma_j(n)} + (-1)^n j\big)/d.$$

Furthermore, we let $\theta_j = \theta_{j,d}$ denote the length of the orbit of ψ containing j and we put $\beta_j = \beta_{j,d} = \sigma_j(\theta_j - 1)$. For $i = 0, 1, \ldots, \theta_j - 1$ and $m \geq -1$ such that $\theta_j m + i \geq -1$ we then have

$$\rho_j(\theta_j m + i) = \rho_{j,d}(\theta_j m + i) = \big(\psi^{i+1}(j) \cdot 2^{\beta_j m + \sigma_j(i)} + (-1)^{\theta_j m + i} j\big)/d.$$

When there is no risk of confusion, we write $\mathbb{A}, \psi, \delta, \sigma_j, \theta_j$, and β_j instead of $\mathbb{A}_d, \psi_d, \delta_d, \sigma_{j,d}, \theta_{j,d}$, and $\beta_{j,d}$.

For j in $\mathbb{A}$, observe that $\psi(j)$ is the odd part of $d-j$. Since any two distinct integers in $(d/2, d)$ have different odd parts, the map ψ is injective and is thus a permutation of the set $\mathbb{A}$.

Furthermore, the integer β_j is the smallest positive integer h such that 2^h is congruent to ± 1 modulo d. Therefore, β_j is independent of j. To see this, observe that, for j in $\mathbb{A}$, the integer $\psi^{-1}(j)$ is equal to $d-h$, where h is the unique integer of the form $2^a j$ in the interval $(d/2, d)$. Consider the multiplication by 2 map T_2 on $\mathbb{Z}/d\mathbb{Z}$, identified with $\{0, 1, \ldots, d-1\}$. Then, the orbit of j under ψ^{-1} can be derived from its orbit $\mathcal{O}$ under T_2 as follows: it is composed of the odd elements of $\mathcal{O}$ less than $d/2$ and of the integers $d-r$, where r runs through the even elements of $\mathcal{O}$ greater than $d/2$. Consequently, the cardinality of the orbit of j under ψ is equal to the smallest positive integer a such that $2^a j$ is congruent to $\pm j$ modulo d. Since j and d are coprime, this integer is independent of j: it is the smallest integer $h \geq 1$ such that 2^h is congruent to ± 1 modulo d.

However, for j and j' in $\mathbb{A}$, the integers θ_j and $\theta_{j'}$ may differ when j and j' do not belong to the same orbit of ψ.

The function ρ_j is fully determined by the permutation ψ and the values of the function δ at $j, \psi(j), \ldots, \psi^{\theta-1}(j)$. The fact that ρ_j takes integral values is not immediate from its definition and is established in the course of the proof of Theorem 4.2, where we obtain another expression for ρ_j.

Recall that

$$[k]_x = 1 + x + \ldots + x^{k-1}, \quad k \geq 1.$$

Theorem 4.2. *Let j, d be odd, coprime, positive integers with $d \geq 3$ and $j < d/2$. Let $\rho_{j,d}$ and $\sigma_{j,d}$ be as in Definition 4.1. Then, the continued fraction expansion of $(1+x^{-1})^{j/d}$ in $\mathbb{F}_2((x^{-1}))$ is given by*

$$(1+x^{-1})^{j/d} = [1; a_1, a_2, a_3, \ldots], \tag{4.2}$$

where the partial quotients a_n are polynomials in x, defined as follows:

(i) if $d - j \equiv 2 \pmod 4$, then

$$a_1 = 1 + x;$$

$$a_2 = [2^{\sigma_{j,d}(1)} - 1]_x;$$

$$a_n = (x+x^2)^{\rho_{j,d}(n-2)} [2^{\sigma_{j,d}(n-1) - \sigma_{j,d}(n-2)} - 1]_x^{2^{\sigma_{j,d}(n-2)}}, \quad n \geq 3;$$

$$\frac{P_n}{Q_n} = (1+x^{-1})^{(-1)^n \rho_{j,d}(n)}, \quad n \geq 1;$$

(ii) if $d \equiv j \pmod 4$, then

$$a_1 = x;$$

$$a_n = (x+x^2)^{\rho_{j,d}(n-3)} [2^{\sigma_{j,d}(n-2) - \sigma_{j,d}(n-3)} - 1]_x^{2^{\sigma_{j,d}(n-3)}}, \quad n \geq 2;$$

$$\frac{P_n}{Q_n} = (1+x^{-1})^{(-1)^{n-1}\rho_{j,d}(n-1)}, \quad n \geq 1.$$

Theorem 4.2 follows from Theorem 2.2 once we have checked that the expansion with alternate signs of the 2-adic number j/d is given by

$$1 - 2^{\sigma_{j,d}(0)} + 2^{\sigma_{j,d}(1)} - 2^{\sigma_{j,d}(2)} + \dots$$

To see this, keeping the notation from the beginning of this section and removing the subscript d, we get

$$\begin{aligned} \frac{j}{d} &= 1 - \frac{\psi(j)}{d} 2^{\delta(j)} \\ &= 1 - \Big(1 - \frac{\psi^2(j)}{d} 2^{\delta(\psi(j))}\Big) 2^{\delta(j)} \\ &= 1 - 2^{\delta(j)} + \frac{\psi^2(j)}{d} 2^{\delta(j)+\delta(\psi(j))} \end{aligned}$$

Also, we should add that $\sigma_{j,d}(0) = 1$ if $d - j \equiv 2 \pmod 4$, while $\sigma_{j,d}(0) > 1$ otherwise.

As an illustration of Theorem 4.2, we derive the continued fraction expansion of $(1+x^{-1})^{1/13}$. Since

$$13 = 1 + 3 \cdot 2^2 = 3 + 5 \cdot 2^1 = 5 + 1 \cdot 2^3,$$

we have

$$\sigma_1(3n) = 6n+2, \;\; \sigma_1(3n+1) = 6n+3, \;\; \sigma_1(3n+2) = 6n+6, \quad n \geq 0.$$

Furthermore,

$$\psi_1^{3n}(1) = 1, \;\; \psi_1^{3n+1}(1) = 3, \;\; \psi_1^{3n+2}(1) = 5.$$

By Theorem 4.2, this gives Corollary 2.3. Ignoring a_1 in Corollary 2.3, we see that the expression of a_n depends only on the value of n modulo 3, that is, modulo the length $\theta_{1,13}$ of the orbit of the permutation ψ_{13} containing 1. We point out that $\theta_{1,d}$ can be as large as $(d+1)/4$ (we have $\theta_{1,d} = (d+1)/4$ for $1/d = 1/11$ and for $1/d = 1/59$, for example).

More generally, up to the first one or two partial quotients, the expression of a_n depends only on the value of n modulo the length $\theta_{j,d}$ of the orbit of $\mathbb{A}_d$ containing j.

If $\theta_{1,d} = 1$, then $d = 1 + 2^u$ for some $u \geq 1$. Consequently, for an odd integer d, the partial quotients a_n in the continued fraction of $(1+x^{-1})^{1/d}$ have a unified formula for all n (that is, we do not need to distinguish congruence classes of n) if and only if there exists an integer $u \geq 1$ such that $d = 1 + 2^u$.

Since any element of $\mathbb{F}_2((x^{-1}))$ has a unique continued fraction expansion, Theorem 4.2 is still true if we replace x by any polynomial $P(x)$ in $\mathbb{F}_2[x] \setminus \mathbb{F}_2$. Baum–Sweet's result [2], that is, the continued fraction expansion of (1.1), is derived from Theorem 4.2 by replacing d by $2^m - 1$ and x by $1 + P(x)$. Notice that by replacing x by $1 + x$ in Theorem 4.2 we get that

$$(1 + x^{-1})^{-j/d} = [1; a_1 + 1, a_2, a_3, a_4, \dots], \tag{4.3}$$

where the a_i are the partial quotients of $(1 + x^{-1})^{j/d}$. To see this, observe that the terms $x + x^2$ and $[k]_x$ for k of the form $2^\ell - 1$ remain unchanged when we replace x by $1 + x$.

The case $j/d = 1/3$ in (4.3) was obtained by Mendès France and van der Poorten in [15].

Proof of Theorem 4.2. Let j, d be odd, coprime, positive integers with $d \geq 3$ and $j < d/2$. Theorem 2.2 gives the continued fraction expansion of $(1 + x^{-1})^\lambda$ for the 2-adic integer

$$\lambda = \lim_{k \to +\infty} \lambda_k,$$

where λ_k is the integer

$$\lambda_k = (-1)^k 2^{\sigma_j(k-1)} + (-1)^{k-1} 2^{\sigma_j(k-2)} + \cdots + 2^{\sigma_j(1)} - 2^{\sigma_j(0)} + 1, \quad k \geq 1.$$

Note that λ_k is positive when $k \geq 2$ is even and negative when $k \geq 1$ is odd. Let $i \geq 1$ be an integer. Recall that, by the definition of the map ψ, we have

$$d = \psi^i(j) + \psi^{i+1}(j) 2^{\delta(\psi^i(j))}.$$

By multiplying both sides by $(-1)^i 2^{\sigma_j(i-1)}$, we get

$$(-1)^i d 2^{\sigma_j(i-1)} = (-1)^i \psi^i(j) 2^{\sigma_j(i-1)} + (-1)^i \psi^{i+1}(j) 2^{\sigma_j(i)}.$$

For an integer $k \geq 1$, taking the summation of the above identity for $i = 1, 2, \dots, k$ yields

$$\begin{aligned}(-1)^1 d 2^{\sigma_j(0)} + (-1)^2 d 2^{\sigma_j(1)} + \cdots &+ (-1)^k d 2^{\sigma_j(k-1)} \\ &= (-1)^1 \psi(j) 2^{\sigma_j(0)} + (-1)^k \psi^{k+1}(j) 2^{\sigma_j(k)} \\ &= -(d - j) + (-1)^k \psi^{k+1}(j) 2^{\sigma_j(k)},\end{aligned}$$

since $d = j + \psi(j) 2^{\sigma_j(0)}$. Hence, we have

$$\begin{aligned}&(-1)^k \psi^{k+1}(j) 2^{\sigma_j(k)} \\ &\quad = (-1)^k d 2^{\sigma_j(k-1)} + (-1)^{k-1} d 2^{\sigma_j(k-2)} + \cdots + d 2^{\sigma_j(1)} - d 2^{\sigma_j(0)} + d - j \\ &\quad = d\lambda_k - j,\end{aligned}$$

This shows that the sequence of integers $(\lambda_k)_{k \geq 1}$ tends in $\mathbb{Z}_2$ to j/d, thus $\lambda = j/d$. Furthermore, since

$$(-1)^k \psi^{k+1}(j) 2^{\sigma_j(k)} = (-1)^k d \rho_j(k) - j,$$

we get that

$$(-1)^k \rho_j(k) = 1 - 2^{\sigma_j(0)} + 2^{\sigma_j(1)} - 2^{\sigma_j(2)} + \ldots + (-1)^k 2^{\sigma_j(k-1)},$$

thus $\rho_j(k)$ is an integer. Then, by applying Theorem 2.2 with $v_0 = 0$, $v_k = \sigma_j(k-1)$ for $k \geq 1$ and $m_k = (-1)^k \rho_j(k)$ for $k \geq 0$, we obtain Theorem 4.2. □

5. Proofs of the results from Subsections 2.1 and 2.2

Proofs of the claims in Subsection 2.1. Let λ be in $\mathbb{Z}_p$. Let M be an integer with $M \geq (p-1)/2$. Assume that

$$\lambda = \lambda_0 + \lambda_1 p + \ldots = \lambda'_0 + \lambda'_1 p + \ldots,$$

with $0 \leq \lambda_i \leq p-1$ and $|\lambda'_i| \leq M$ for $i \geq 0$. For an integer $k \geq 0$, set $p_k = \lambda_0 + \lambda_1 p + \ldots + \lambda_k p^k$ and observe that

$$|(1+x^{-1})^{\lambda} - (1+x^{-1})^{p_k}| = |(1+x^{-1})^{p_k}| \cdot |(1+x^{-1})^{\lambda_{k+1}p^{k+1}+\cdots} - 1| < \mathrm{e}^{-p^k}.$$

To see that

$$(1+x^{-1})^{\lambda} = (1+x^{-1})^{\sum_{i\geq 0}\lambda'_i p^i} = \prod_{i\geq 0}(1+x^{-1})^{\lambda'_i p^i} = \prod_{i\geq 0}(1+x^{-p^i})^{\lambda'_i},$$

it is then sufficient to note that, since $(|\lambda'_i|)_{i\geq 0}$ is bounded, we have

$$\mathrm{v}_p\big((\lambda_0+\lambda_1 p+\ldots+\lambda_k p^k) - (\lambda'_0+\lambda'_1 p+\ldots+\lambda'_k p^k)\big) \xrightarrow[k\to+\infty]{} +\infty,$$

where v_p denotes the p-adic valuation.

We check now that for λ in $\mathbb{Q}\cap\mathbb{Z}_p$ the expression $(1+x^{-1})^{\lambda}$ gives the same result when λ is viewed as a rational number (as in Section 1) and when it is viewed as a p-adic number.

Assume, without any loss of generality, that $0 < \lambda < 1$, $\lambda = j/d$, where $d \geq 3$ and $\gcd(jd, p) = 1$. Let s be the smallest positive integer such that d divides $p^s - 1$. The signed p-adic expansion of $1/(p^s-1)$ is $-1-p^s-p^{2s}-\ldots$. By definition of the p-adic power, we have

$$(1+x^{-1})^{\frac{1}{p^s-1}} = \prod_{i\geq 0}(1+x^{-1})^{-p^{si}} = \prod_{i\geq 0}(1+x^{-p^{si}})^{-1}. \tag{5.1}$$

Since

$$\Big((1+x^{-p^{si}})^{-1}\Big)^{p^s-1} = (1+x^{-p^{s(i+1)}})^{-1}\,(1+x^{-p^{si}}), \quad i \geq 0,$$

the last infinite product in (5.1), raised to the power $p^s - 1$, is equal to $1 + x^{-1}$. This proves our claim for $\lambda = 1/(p^s-1)$.

Now, write the representation in base p of the integer $j(p^s-1)/d$ in $[1, p^s-1]$ as

$$\frac{j(p^s-1)}{d} = j_0 + j_1 p + \ldots + j_{s-1}p^{s-1}, \quad \text{where } 0 \leq j_0, \ldots, j_{s-1} \leq p-1.$$

Then, the p-adic number $\lambda = j/d$ can be expressed as

$$\begin{aligned}\frac{j}{d} = \frac{j(p^s-1)/d}{p^s-1} &= -\frac{j(p^s-1)}{d}(1+p^s+p^{2s}+\ldots)\\ &= -(j_0+j_1p+\ldots+j_{s-1}p^{s-1})\cdot(1+p^s+p^{2s}+\ldots),\end{aligned}$$

and we have

$$(1+x^{-1})^{\lambda} = \prod_{i=0}^{s-1}\prod_{h=1}^{j_i}\left((1+x^{-p^i})^{-1}\right)^{1+p^s+\dots}$$
$$= \prod_{i=0}^{s-1}\prod_{h=1}^{j_i}\left((1+x^{-p^i})^{-1}\right)^{1/(1-p^s)},$$

where $1/(1-p^s)$ is viewed as a rational number. Then, we note that

$$\prod_{i=0}^{s-1}\prod_{h=1}^{j_i}\left((1+x^{-p^i})^{-1}\right)^{1/(1-p^s)} = \left((1+x^{-1})^{-(j_0+j_1p+\dots+j_{s-1}p^{s-1})}\right)^{1/(1-p^s)}$$
$$= (1+x^{-1})^{j/d}.$$

This shows that the expression $(1+x^{-1})^{\lambda}$ gives the same result when λ is viewed as a rational number and when it is viewed as a p-adic number. □

Proof of Lemma 2.1. We only justify the last assertion. Observe that, if λ is not in $\mathbb{Z}\cap\mathbb{Z}_2$, then

$$\lambda = \sum_{i\geq 1}(2^{a_i}+2^{a_i+1}+\dots+2^{a_i+c_i}),$$

where $(a_i)_{i\geq 1}$ is increasing and $a_i+c_i+1 < a_{i+1}$ for $i \geq 1$. Then,

$$\lambda = \sum_{i\geq 1}(-2^{a_i}+2^{a_i+c_i+1}) = 2^{a_1}-2^{a_1+1}+2^{a_1+c_1+1}+\sum_{i\geq 2}(-2^{a_i}+2^{a_i+c_i+1}),$$

which has the desired property (note that $-2^{a_1+1}+2^{a_1+c_1+1}$ vanishes if $c_1 = 0$). Furthermore, λ is the limit in $\mathbb{Z}_2$ of the sequence of positive integers

$$2^{a_1}-2^{a_1+1}+2^{a_1+c_1+1}+\sum_{i=2}^{m}(-2^{a_i}+2^{a_i+c_i+1}),\quad m\geq 2.$$

This proves the lemma. □

Until the end of this section, the last convergent of a finite continued fraction $[a_0; a_1, a_2, \dots, a_m]$ is the rational fraction $[a_0; a_1, a_2, \dots, a_{m-1}]$. The first assertion of the next lemma is proved at the end of [20, Section 1]. The second one is certainly well-known.

Lemma 5.1. *Let ξ be in $\mathbb{F}_q((x^{-1}))$. If the rational fraction P/Q satisfies $|\xi - P/Q| < 1/|Q|^2$, then P/Q is a convergent to ξ. If, moreover, ξ is a rational fraction R/S with $P/Q \neq R/S$ and if $|\xi - P/Q| = 1/|QS|$, then P/Q is the last convergent to ξ.*

Proof. If the last assertion does not hold, let U/V denote the last convergent to $\xi = R/S$. Then, $|Q| < |V| < |S|$ and

$$\frac{1}{|QS|} = \left|\xi - \frac{P}{Q}\right| \geq \frac{1}{|QV|} > \frac{1}{|QS|},$$

a contradiction. □

We are now in position to establish Theorem 2.2.

Proof of Theorem 2.2. We only consider the case of λ in $\mathbb{Z}_2 \setminus \mathbb{Z}$, since the case of integers is similar (and easier). Let $(m_k)_{k \geq -1}$ be the sequence of integers defined in (2.1). If $v_1 > v_0 + 1$, then

$$|(1+x^{-1})^{\lambda} - (1+x^{-1})^{m_0}| < |x|^{-2 \cdot 2^{v_0}}.$$

Thus, $1 + x^{-2^{v_0}}$ is a convergent to $(1+x^{-1})^{\lambda}$ and $a_1 = x^{2^{v_0}}$. If $v_1 = v_0 + 1$, then $m_1 = -2^{v_0}$ and

$$|(1+x^{-1})^{\lambda} - (1+x^{-1})^{m_1}| < |x|^{-2 \cdot 2^{v_0}}.$$

Thus, $(1 + x^{-2^{v_0}})^{-1}$ is a convergent to $(1+x^{-1})^{\lambda}$ and $a_1 = 1 + x^{2^{v_0}}$.

Let $k \geq 2$ be an even integer. Thus, m_k is positive, m_{k-1} is negative, and $m_k - m_{k-1} = 2^{v_k}$. Since $(1+x)^u = 1 + x^u$ for any u which is a power of 2, we have

$$\begin{aligned}
\Big(\frac{1+x}{x}\Big)^{m_k} - \Big(\frac{x}{1+x}\Big)^{-m_{k-1}} &= \frac{(1+x)^{m_k - m_{k-1}} - x^{m_k - m_{k-1}}}{x^{m_k}(1+x)^{-m_{k-1}}} \\
&= \frac{(1+x)^{2^{v_k}} - x^{2^{v_k}}}{x^{m_k}(1+x)^{-m_{k-1}}} = \frac{1}{x^{m_k}(1+x)^{-m_{k-1}}}.
\end{aligned}$$

This shows that $(1+x^{-1})^{m_{k-1}}$ is a convergent to $(1+x^{-1})^{m_k}$ and, moreover, it is its last convergent, by Lemma 5.1. Assume that $k \geq 3$ is an odd integer, or $k = 1$ and $v_1 > v_0 + 1$. Then, a similar computation shows that $(1+x^{-1})^{m_{k-1}}$ is the last convergent to $(1+x^{-1})^{m_k}$. Consequently, if m_k is positive, then the last partial quotient A_k of $(1+x^{-1})^{m_k}$ satisfies

$$(1+x)^{m_k} = A_k x^{-m_{k-1}} + (1+x)^{m_{k-2}},$$

thus

$$\begin{aligned}
A_k x^{-m_{k-1}} &= (1+x)^{m_{k-2}} \big((1+x)^{m_k - m_{k-2}} - 1\big) \\
&= (1+x)^{m_{k-2}} \big((1+x)^{2^{v_{k-1}}(2^{v_k - v_{k-1}} - 1)} - 1\big) \\
&= (1+x)^{m_{k-2}} \big((1+x)^{2^{v_k - v_{k-1}} - 1} - 1\big)^{2^{v_{k-1}}} \\
&= (1+x)^{m_{k-2}} x^{2^{v_{k-1}}} [2^{v_k - v_{k-1}} - 1]_x^{2^{v_{k-1}}},
\end{aligned}$$

where we have used that $(1+x)^{u-1} = [u]_x$, when u is a power of 2. Since $m_{k-1} - m_{k-2} = -2^{v_{k-1}}$, we get that

$$A_k = (x + x^2)^{m_{k-2}} [2^{v_k - v_{k-1}} - 1]_x^{2^{v_{k-1}}}.$$

The same result holds if m_k is negative, by a similar computation (in this case the first factor is $(x+x^2)^{-m_{k-2}} = (x+x^2)^{|m_{k-2}|}$). This shows that $a_k = A_{k-1}$ for $k \geq 2$ in Case $(ii.a)$ and $a_k = A_k$ for $k \geq 3$ in Case $(ii.b)$. It only remains for us to compute a_2 when $v_1 = v_0 + 1$. Note that $-m_1 = m_0 = 2^{v_0}$. As $P_0 = Q_0 = 1$, we get

$$(1+x)^{m_2} = a_2 x^{-m_1} + 1.$$

A similar computation as above yields

$$a_2 x^{-m_1} = x^{2^{v_0}} [2^{v_2 - v_0} - 1]_x^{2^{v_0}},$$

giving $a_2 = [2^{v_2 - v_0} - 1]_x^{2^{v_0}}$, as asserted. □

6. Proofs of Lemmas 2.4 and 2.5

Proof of Lemma 2.4. We only justify the last assertion. Let λ be a p-adic integer and let

$$\lambda - \frac{1}{2} = \sum_{i \geq 0} \lambda_i p^i$$

be the Hensel expansion of $\lambda - 1/2$. Then,

$$\lambda = \frac{1}{2} + \sum_{i \geq 0} \lambda_i p^i = -\frac{p-1}{2}(1 + p + p^2 + \dots) + \sum_{i \geq 0} \lambda_i p^i$$
$$= \sum_{i \geq 0} \Big(\lambda_i - \frac{p-1}{2}\Big) p^i,$$

where all the digits $\lambda_i - (p-1)/2$ are in $\{-(p-1)/2, \dots, (p-1)/2\}$. □

Proof of Lemma 2.5. Observe that

$$\deg(\xi - (1 + x^{-1})^{m_k}) = \deg((1 + x^{-1})^{\lambda - m_k} - 1) = -p^{v_{k+1}},$$

while, by (2.5), we have $p^{v_{k+1}} > 2|m_k|$. It then follows from the first assertion of Lemma 5.1 that $(1 + x^{-1})^{m_k}$ is a convergent to ξ.

Let m be a nonzero integer not in the sequence $(m_k)_{k \geq 1}$ and let k be such that

$$|m_k| < |m| < |m_{k+1}|. \tag{6.1}$$

Assume that $(1 + x^{-1})^m$ is a convergent to ξ. Then, $(1 + x^{-1})^{m_k}$ is a convergent to $(1 + x^{-1})^m$ and $(1 + x^{-1})^m$ is a convergent to $(1 + x^{-1})^{m_{k+1}}$. Since

$$(1 + x^{-1})^m - (1 + x^{-1})^{m_k} = (1 + x^{-1})^{m_k} \big((1 + x^{-1})^{m - m_k} - 1\big),$$

we have

$$|(1 + x^{-1})^m - (1 + x^{-1})^{m_k}| = \mathrm{e}^{-p^u},$$

where p^u is the largest power of p dividing $m - m_k$. Since $(1 + x^{-1})^{m_k}$ is a convergent to $(1 + x^{-1})^m$, we get

$$|(1 + x^{-1})^m - (1 + x^{-1})^{m_k}| < \mathrm{e}^{-2|m_k|},$$

thus $p^u > 2|m_k|$. Likewise, if p^v denote the largest power of p dividing $m_{k+1} - m$, then $p^v > 2|m|$. It follows from (6.1) and the first inequality of (2.5) that $u, v \leq v_{k+1}$. Recall that $m_{k+1} - m_k = d_{k+1} p^{v_{k+1}}$. If $u \neq v$, then $|m - m_k|_p \neq |m - m_{k+1}|_p$ and

$$p^{-v_{k+1}} = |m_{k+1} - m_k|_p = \max\{|m - m_k|_p, |m - m_{k+1}|_p\} = p^{-\min\{u,v\}} > p^{-v_{k+1}},$$

which is absurd. Thus, we must have $u = v$. Then,

$$2|m| < p^v = p^u \leq |m - m_k| < 2|m|,$$

a contradiction. $\square$

7. Proofs of the results from Section 3

Proof of Theorem 3.2 for $p = 2$. Let j be in $\mathbb{A}$. Let n be a positive integer. By Definition 4.1, we have

$$\rho_j(n) = (\psi^{n+1}(j) \cdot 2^{\sigma_j(n)} + (-1)^n \cdot j)/d.$$

Since

$$d = \psi^{n+1}(j) + \psi^{n+2}(j) \cdot 2^{\delta(\psi^{n+1}(j))},$$

we get

$$\begin{aligned} d\rho_j(n+1) &= \psi^{n+2}(j) \cdot 2^{\sigma_j(n+1)} + (-1)^{n+1} \cdot j \\ &= 2^{\sigma_j(n+1) - \delta(\psi^{n+1}(j))}(d - \psi^{n+1}(j)) + (-1)^{n+1} \cdot j \\ &= 2^{\sigma_j(n)}(d - \psi^{n+1}(j)) + (-1)^{n+1} \cdot j, \end{aligned}$$

thus

$$\frac{\rho_j(n+1)}{\rho_j(n)} = \frac{d - \psi^{n+1}(j)}{\psi^{n+1}(j)} + o(1).$$

By Theorem 4.2, we get

$$\frac{\deg Q_{n+1}}{\deg Q_n} = \frac{\rho_j(n+1-\varepsilon)}{\rho_j(n-\varepsilon)},$$

with $\varepsilon = 1$ if 4 divides $d - j$ and $\varepsilon = 0$ otherwise.

Recall that the approximation spectrum $\mathcal{S}(\xi)$ of ξ is the set of limit points of the sequence $(\deg Q_{n+1}/\deg Q_n)_{n \geq 1}$. With θ_j as in Definition 4.1, we have established that

$$\mathcal{S}(\xi) = \Big\{\frac{d-j}{j}, \frac{d - \psi(j)}{\psi(j)}, \ldots, \frac{d - \psi^{\theta_j - 1}(j)}{\psi^{\theta_j - 1}(j)}\Big\}.$$

Since $\psi^i(j)$ is the integer j_i defined in the statement of the theorem, for $i = 0, \ldots, h-1$, the proof is complete. $\square$

Proof of Theorem 3.2 for p odd. It follows from (3.4) and Corollary 2.8 that

$$\mu\big((1 + x^{-1})^\lambda\big) = 2 + \limsup_{k \to +\infty} \frac{p^{v_{k+1}} - 2|m_k|}{|m_k|} = \limsup_{k \to +\infty} \frac{p^{v_{k+1}}}{|m_k|}.$$

Note that $p^{v_{k+1}} - 2|m_k| \geq 1$ for $k \geq 0$, by (2.5).

Let v_s denote the smallest positive integer such that d divides $p^{v_s} - 1$ and set $a = (p^{v_s} - 1)/d$. Then, the rational number j/d is equal to $aj/(p^{v_s} - 1)$. Let $d_0 + d_1 p^{v_1} + \ldots + d_{s-1} p^{v_{s-1}}$ denote the signed p-adic representation of $-aj$. Then, the signed p-adic expansion of j/d is purely periodic and equal to

$$j/d = (d_0 + d_1 p^{v_1} + \ldots + d_{s-1} p^{v_{s-1}}) \cdot (1 + p^{v_s} + p^{2v_s} + \ldots).$$

Since $v_{\ell s} = \ell v_s$ for $\ell \geq 1$, we get

$$m_{\ell s+h} = (d_0 + d_1 p^{v_1} + \ldots + d_{s-1}p^{v_{s-1}}) \cdot (1 + p^{v_s} + \ldots + p^{(\ell-1)v_s}) + d_0 p^{\ell v_s} + \ldots + d_h p^{v_h + \ell v_s},$$

for $\ell \geq 0$ and $h = 0, \ldots, s-1$. Then, we have (as usual, an empty sum is equal to 0)

$$\mu(\xi) = \max_{0 \leq h \leq s-1} \frac{p^{v_h}}{\left| \frac{d_0 + d_1 p^{v_1} + \ldots + d_{s-1}p^{v_{s-1}}}{p^{v_s}-1} + (d_0 + \ldots + d_{h-1}p^{v_{h-1}}) \right|},$$

Furthermore, since

$$\begin{aligned}&\big(d_0 + d_1 p^{v_1} + \ldots + d_{s-1}p^{v_{s-1}} + (p^{v_s}-1)(d_0 + \ldots + d_{h-1}p^{v_{h-1}})\big)p^{-v_h}\\ =&\big(d_h p^{v_h} + \ldots + d_{s-1}p^{v_{s-1}} + p^{v_s}(d_0 + \ldots + d_{h-1}p^{v_{h-1}})\big)p^{-v_h}\\ =&d_h + \ldots + d_{s-1}p^{v_{s-1}-v_h} + (d_0 p^{v_s - v_h} + \ldots + d_h p^{v_s + v_{h-1} - v_h})\\ =&\frac{j_h(1-p^{v_s})}{d},\end{aligned}$$

with j_h as in the statement of the theorem, we get

$$\mu(\xi) = \max_{0 \leq h \leq s-1} \frac{p^{v_s}-1}{|j_h(1-p^{v_s})/d|}.$$

This establishes (3.5). □

Proof of Proposition 3.3. The case $p = 2$ follows immediately from Theorem 4.2.

Assume that p is odd and keep the notation of the Proposition. Observe first that if $s = 1$, $v_1 = 1$, and $d_0 = \pm(p-1)/2$, then

$$\lambda = \pm\frac{p-1}{2} \cdot \frac{1}{1-p^{v_1}} = \mp\frac{1}{2}$$

and ξ is a quadratic number. We exclude this case. Set

$$m_k = d_0 + d_1 p^{v_1} + \ldots, \quad k \geq 0.$$

Since ξ and $(1+x^{-1})^h\xi$ have the same approximation spectrum for any integer h, we can replace ξ with $(1+x^{-1})^{\bar{j}/d}$, for any $\bar{j}$ in the set $\mathcal{O}_{p,j,d}$ defined in Theorem 3.2. This amounts to take any cyclic permutation of the digits $d_0, d_1, \ldots, d_{s-1}$ of j/d. The fact that some elements of $\mathcal{O}_{p,j,d}$ may be negative does not cause any trouble.

Assume that (iii) does not hold. By taking, if necessary, a cyclic permutation of $d_0, d_1, \ldots, d_{s-1}$, we can assume that d_{s-1} and d_s have the same sign (equivalently, that m_{s-1} and m_s have the same sign) or that $|d_s| \geq 2$. Let k be a positive integer. Since the sequences $(\mathrm{sgn}(m_i))_{i\geq 0}$ and $(d_i)_{i \geq 0}$ are periodic of period s, we get that m_{ks-1} and m_{ks} have the same sign or $|d_{ks}| \geq 2$. In particular, the ε defined in Lemma 2.7 to derive the continued fraction expansion of $(1+x^{-1})^{m_{ks}}$ from that of $(1+x^{-1})^{m_{ks-1}}$ is not equal to 1. Said differently, setting $(1+x^{-1})^{m_{ks-1}} = [1, y, \mathbf{w}_{ks-1}]$, we do not have $(1+x^{-1})^{m_{ks}} = [1, y, \mathbf{w}_{ks-1}, c_{ks}]$. Thus, writing

$$(1+x^{-1})^{m_{(k-h)s-1}} = [1, y, \mathbf{w}_{(k-h)s-1}], \quad 0 \le h \le k-1,$$

Lemma 2.7 asserts that there exist c_{ks} in $\mathbb{F}_p[x] \setminus \mathbb{F}_p$ and a nonzero η_k in $\mathbb{F}_p$ such that

$$(1+x^{-1})^{m_{ks}} = [1, y, \mathbf{w}_{ks-1}, c_{ks}, \eta_k(\overleftarrow{\mathbf{w}_{ks-1}/2}, 2y+1, -2y-1, -\mathbf{w}_{ks-1}/2), \ldots],$$

where $\deg c_{ks} = p^{v_{ks}} - 2|m_{ks-1}|$. Here and below, the notation $\eta(a_1, \ldots, a_m)$ for a nonzero η in $\mathbb{F}_p$ and $a_1, \ldots, a_m$ in $\mathbb{F}_p[x] \setminus \mathbb{F}_p$ stands for the sequence of m partial quotients $\eta a_1, \eta^{-1} a_2, \ldots, \eta^{(-1)^{m+1}} a_m$.

Let h be an integer with $0 < h \le k-1$. Then, there exist nonzero η'_k and $\rho_{h,k}$ in $\mathbb{F}_p$ such that

$$\begin{aligned}(1+x^{-1})^{m_{ks}} = [1, y, \mathbf{w}_{ks-1}, c_{ks}, \eta_k(\overleftarrow{\mathbf{w}_{ks-1}/2}, 2y+1, -2y-1),\\ \eta'_k(\mathbf{w}_{(k-h)s-1}/2), \rho_{h,k} c_{(k-h)s}, \ldots],\end{aligned}$$

Since

$$v_{\ell s} = \ell v_s, \quad \ell \ge 1,$$

the degree of the numerator (and of the denominator) of the rational fraction

$$\frac{r_{k,h}}{t_{k,h}} := [1, y, \mathbf{w}_{ks-1}, c_{ks}, \eta_k(\overleftarrow{\mathbf{w}_{ks-1}/2}, 2y+1, -2y-1), \eta'_k(\mathbf{w}_{(k-h)s-1}/2)]$$

is equal to

$$|m_{ks-1}| + (p^{v_{ks}} - 2|m_{ks-1}|) + |m_{ks-1}| + |m_{(k-h)s-1}| = p^{hv_s} p^{v_{(k-h)s}} + |m_{(k-h)s-1}|.$$

Furthermore,

$$\deg c_{(k-h)s} = p^{v_{(k-h)s}} - 2|m_{(k-h)s-1}|$$

and

$$m_{(k-h)s-1} = (d_0 + d_1 p^{v_1} + \ldots + d_{s-1} p^{v_{s-1}}) \cdot \frac{p^{(k-h)v_s} - 1}{p^{v_s} - 1}.$$

By the minimality of s, we cannot have $d_0 = \ldots = d_{s-1} = \pm(p-1)/2$ if $s \ge 2$. Thus,

$$|m_{(k-h)s-1}| < \frac{p-1}{2} \cdot \frac{p^{v_{s-1}+1} - 1}{p-1} \cdot \frac{p^{(k-h)v_s} - 1}{p^{v_s} - 1} \le \frac{p^{(k-h)v_s} - 1}{2}, \quad \text{if } s \ge 2.$$

This also holds if $s = 1$, since in that case we have excluded the values $d_0 = \pm(p-1)/2$. Consequently, setting

$$\begin{aligned}\sigma_h := \lim_{k \to +\infty} \frac{\deg c_{(k-h)s}}{\deg r_{k,h}} &= \lim_{k \to +\infty} \frac{p^{v_{(k-h)s}} - 2|m_{(k-h)s-1}|}{p^{hv_s} p^{v_{(k-h)s}} + |m_{(k-h)s-1}|}\\ &= \frac{p^{v_s} - 1 - 2|d_0 + d_1 p^{v_1} + \ldots + d_{s-1} p^{v_{s-1}}|}{p^{hv_s}(p^{v_s} - 1) + |d_0 + d_1 p^{v_1} + \ldots + d_{s-1} p^{v_{s-1}}|},\end{aligned}$$

the real number $1 + \sigma_h$ is greater than 1 and it belongs to the approximation spectrum of ξ. Since h is arbitrary, this shows that the approximation spectrum of ξ is infinite. Thus, (ii) does not hold. Furthermore, we have shown that there are infinitely many partial quotients of the form $\alpha y+\beta$, with α,β in $\mathbb{F}_p$ and α nonzero. This implies that $\widehat{\nu}(\xi)=1$.

Assume now that (iii) holds. Since $j\geq 1$, we have $d_0=1,\dots,d_{s-1}=-1$, thus

$$m_k=(-1)^k(p^{v_k}-p^{v_{k-1}}+\ldots+(-1)^k),\quad k\geq 0.$$

It then follows from Lemma 2.7 that there is a linear polynomial y and polynomials $c_1,c_2,\dots$ such that

$$\xi=[1;y,c_1,c_2,\dots,c_s,c_{s+1},\dots]$$

and

$$\deg c_i=p^{v_i}-2|m_{i-1}|=2|m_i|-p^{v_i},\quad i\geq 1.$$

Arguing then as in the proof of (3.5), we get that

$$\mathcal{S}(\xi)=\left\{\frac{p^{v_i}}{|-\frac{j}{d}+(1-p^{v_1}+\ldots+(-1)^{i-1}p^{v_{i-1}})|}:0\leq i\leq s-1\right\}.$$

This proves that the Diophantine spectrum is a finite set and that $\widehat{\nu}(\xi)>1$. The proof of the proposition is complete. □

Proof of Theorem 3.6. By Theorem 3.2, the condition on j implies that

$$\lambda_j\big((1+x^{-1})^{1/d}\big)\leq\nu\big((1+x^{-1})^{j/d}\big)=\frac{d-j}{j}.$$

Since $\nu((1+x^{-1})^{1/d})=d-1$, it follows from (3.7) than

$$\lambda_j\big((1+x^{-1})^{1/d}\big)\geq\frac{d-j}{j}.$$

Consequently, we have indeed equality, and we conclude by Theorem 3.5. □

8. Ultimate Triangle over $\mathbb{Q}$

Definition 8.1. *Let u and v be real numbers such that $u\leq v$ and $v-u$ is an even integer. We define their jump product $[\![u\bullet v]\!]$ by $[\![u\bullet v]\!]=1$ if $u=v$ and by*

$$[\![u\bullet v]\!]=u(u+2)(u+4)\cdots(v-2),\quad \textit{if } u<v.$$

We stress that the jump product $[\![\cdot\bullet\cdot]\!]$ has the lowest priority order among the other operations, like addition, subtraction and product. Consequently, for two integral valued expressions e_1 and e_2, we simply write $[\![e_1\bullet e_2]\!]$ instead of $[\![(e_1)\bullet(e_2)]\!]$.

For u,v two positive integers of the same parity with $u<v$, we check that

$$[\![u\bullet v]\!]=\begin{cases}2^{(v-u)/2}\dfrac{(v/2-1)!}{(u/2-1)!}, & \text{if } u \text{ is even};\\[2ex] 2^{(u-v)/2}\dfrac{((u-1)/2)!\,(v-1)!}{((v-1)/2)!\,(u-1)!}, & \text{if } u \text{ is odd}.\end{cases}\tag{8.1}$$

Definition 8.2. *Given a nonzero integer* $\varepsilon = 1, -1, 2, -2, 3, -3, \dots$ *Let* $m = 0, 1, \dots, 2|\varepsilon| - 3$ *if* $\varepsilon > 0$, *or* $m = 0, 1, \dots, 2|\varepsilon| - 2$ *if* $\varepsilon < 0$. *We define*

$$\eta_\varepsilon(m) = \begin{cases} -2^{m-4\lfloor m/4\rfloor -1} \dfrac{[\![|\varepsilon| - t \bullet |\varepsilon| + t + 2]\!]}{[\![|\varepsilon - 1| - t \bullet |\varepsilon - 1| + t + 2]\!]}, & \text{if } m = 2t \text{ is even;} \\ 2^{m-4\lfloor m/4\rfloor -3}(m+2)|\varepsilon| \dfrac{[\![|\varepsilon| - t \bullet |\varepsilon| + t + 2]\!]}{[\![|\varepsilon| - t - 1 \bullet |\varepsilon| + t + 3]\!]}, & \text{if } m = 2t+1 \text{ is odd.} \end{cases}$$

The rational numbers $\eta_\varepsilon(m)$ defined above form an infinite triangle, called the *Ultimate Triangle*, whose line numbered ε is given by

$$\begin{cases} [\eta_\varepsilon(0), \eta_\varepsilon(1), \dots, \eta_\varepsilon(2\varepsilon - 3)], & \text{if } \varepsilon > 0; \\ [\eta_\varepsilon(0), \eta_\varepsilon(1), \dots, \eta_\varepsilon(2|\varepsilon| - 3), \eta_\varepsilon(2|\varepsilon| - 2)], & \text{if } \varepsilon < 0. \end{cases}$$

The first few lines of the Ultimate Triangle are reproduced below.

ε	$\eta_\varepsilon(m)$						
1							
−1	−1/4						
2	−1	1					
−2	−1/3	1	−3/4				
3	−3/4	27/32	−16/3	8			
−3	−3/8	27/32	−16/15	8	−5/32		
4	−2/3	4/5	−15/4	25/4	−8/5	16/5	
−4	−2/5	4/5	−5/4	25/4	−8/35	16/5	−35/64
⋮			⋮				

Observe that $|\varepsilon|+t+3$ and $|\varepsilon-1|+t+2$ in Definition 8.2 are at most equal to $2|\varepsilon|+2$. Consequently, the largest prime number dividing the numerator or the denominator of the rational number $\eta_\varepsilon(m)$ is always less than $2|\varepsilon|+1$.

9. Preliminaries to the proof of the Key Lemma 2.7

We reformulate the lemma in terms of matrices, whose coefficients are in $\mathbb{F}_p[x]$ (actually, we can work over an arbitrary field). For each sequence $\mathbf{w} = w_1, w_2, \dots, w_k$ we define

$$\mathbf{M}(\mathbf{w}) = \begin{pmatrix} w_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} w_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} w_k & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} P_\mathbf{w} & P'_\mathbf{w} \\ Q_\mathbf{w} & Q'_\mathbf{w} \end{pmatrix}.$$

We stress that $'$ does not mean derivative. Then, we have

$$\frac{P'_\mathbf{w}}{Q'_\mathbf{w}} = [w_1, w_2, \dots, w_{k-1}], \quad \frac{P_\mathbf{w}}{Q_\mathbf{w}} = [w_1, w_2, \dots, w_k].$$

Recall that

$$\overleftarrow{\mathbf{w}} = w_k, \dots, w_2, w_1,$$

$$\rho\mathbf{w} = \rho w_1, \rho^{-1} w_2, \dots, \rho^{(-1)^{k+1}} w_k, \quad \rho \in \mathbb{F}_p \setminus \{0\},$$
$$-\mathbf{w} = (-1)\mathbf{w}.$$

Lemma 9.1. *Let ρ be a nonzero rational number. Keeping the above notation, we have*

$$\mathbf{M}(\overleftarrow{\mathbf{w}}) = \begin{pmatrix} P_{\mathbf{w}} & Q_{\mathbf{w}} \\ P'_{\mathbf{w}} & Q'_{\mathbf{w}} \end{pmatrix}, \quad \mathbf{M}(-\mathbf{w}) = (-1)^k \begin{pmatrix} P_{\mathbf{w}} & -P'_{\mathbf{w}} \\ -Q_{\mathbf{w}} & Q'_{\mathbf{w}} \end{pmatrix},$$

$$\mathbf{M}(\rho\mathbf{w}) = \begin{cases} \begin{pmatrix} P_{\mathbf{w}} & \rho P'_{\mathbf{w}} \\ \rho^{-1} Q_{\mathbf{w}} & Q'_{\mathbf{w}} \end{pmatrix}, & \text{if } k \text{ is even;} \\ \begin{pmatrix} \rho P_{\mathbf{w}} & P'_{\mathbf{w}} \\ Q_{\mathbf{w}} & \rho^{-1} Q'_{\mathbf{w}} \end{pmatrix}, & \text{if } k \text{ is odd.} \end{cases}$$

Moreover, we have

$$P_{\mathbf{w}} Q'_{\mathbf{w}} - P'_{\mathbf{w}} Q_{\mathbf{w}} = (-1)^k.$$

Proof. This is an easy computation. □

Lemma 9.2. *Let $\mathbf{w}$ be a finite or empty sequence of elements in $\mathbb{F}_p[x] \setminus \mathbb{F}_p$ such that*

$$\mathbf{M}(1, x, \mathbf{w}) = \begin{pmatrix} P & P' \\ Q & Q' \end{pmatrix}.$$

Let

$$\mathbf{v} = (2x+1, \frac{1}{2}\mathbf{w}) \quad \text{and} \quad \mathbf{h} = (\overleftarrow{\mathbf{v}}, -\mathbf{v}). \tag{9.1}$$

Then,

$$\mathbf{M}(\mathbf{h}) = \begin{cases} \begin{pmatrix} -4PQ & QP' + PQ' \\ -QP' - PQ' & P'Q' \end{pmatrix}, & \text{if the length of } \mathbf{w} \text{ is even;} \\ \begin{pmatrix} PQ & -(QP' + PQ') \\ QP' + PQ' & -4P'Q' \end{pmatrix}, & \text{if the length of } \mathbf{w} \text{ is odd.} \end{cases}$$

Proof. This follows from a direct matrix calculation. Note that when $\mathbf{w}$ is the empty sequence, we have $\mathbf{h} = (2x+1, -2x-1)$. □

Observe that the length of $\mathbf{h}$ is always even. Below, c_{k+1} is the polynomial defined in the Key Lemma. Throughout the end of this section, we assume that the length of $\mathbf{w}$ is even (we explain in Section 10 how to deduce the case of odd length from that of even length).

Definition 9.3. *For a nonzero rational number ρ, set*

$$\mathbf{M} = \begin{pmatrix} P & P' \\ Q & Q' \end{pmatrix}, \qquad \mathbf{C}(\rho) = \begin{pmatrix} \rho c_{k+1} & 1 \\ 1 & 0 \end{pmatrix},$$

$$\mathbf{H}(\rho) = \begin{pmatrix} -4PQ & \rho(QP' + PQ') \\ -\rho^{-1}(QP' + PQ') & P'Q' \end{pmatrix}.$$

In this and the next sections, we adopt the following notation: for an integer k, we define

$$\bar{k} = \begin{cases} 0, & \text{if } k \text{ is even,} \\ 1, & \text{if } k \text{ is odd.} \end{cases}$$

In view of Lemma 9.2, our goal is to establish the following

Theorem 9.4. *For every positive integer ε, set*

$$\mathbf{MC}(1)\mathbf{H}(\eta_\varepsilon(0))\mathbf{C}(\eta_\varepsilon(1))\cdots\mathbf{H}(\eta_\varepsilon(2\varepsilon-4))\mathbf{C}(\eta_\varepsilon(2\varepsilon-3)) = \begin{pmatrix} A_\varepsilon & A'_\varepsilon \\ B_\varepsilon & B'_\varepsilon \end{pmatrix}.$$

We have

$$A_\varepsilon = (-1)^{\varepsilon+1}2^{\varepsilon+\bar{\varepsilon}-2}(\varepsilon c_{k+1}PQ + P'Q)^\varepsilon/Q, \tag{9.2}$$

$$B_\varepsilon = (-1)^{\varepsilon+1}2^{\varepsilon+\bar{\varepsilon}-2}(\varepsilon c_{k+1}PQ + PQ')^\varepsilon/P. \tag{9.3}$$

For every negative integer ε, set

$$\mathbf{MC}(1)\mathbf{H}(\eta_\varepsilon(0))\mathbf{C}(\eta_\varepsilon(1))\cdots\mathbf{H}(\eta_\varepsilon(2|\varepsilon|-4))\times$$
$$\times\, \mathbf{C}(\eta_\varepsilon(2|\varepsilon|-3))\mathbf{H}(\eta_\varepsilon(2|\varepsilon|-2)) = \begin{pmatrix} A_\varepsilon & A'_\varepsilon \\ B_\varepsilon & B'_\varepsilon \end{pmatrix}.$$

We have

$$A_\varepsilon = 2^{-\varepsilon+\bar{\varepsilon}}(\varepsilon c_{k+1}PQ + PQ')^{-\varepsilon}P, \tag{9.4}$$

$$B_\varepsilon = 2^{-\varepsilon+\bar{\varepsilon}}(\varepsilon c_{k+1}PQ + P'Q)^{-\varepsilon}Q. \tag{9.5}$$

In both cases we have

$$\frac{A_\varepsilon}{B_\varepsilon} = \left(\frac{\varepsilon c_{k+1} + P'/P}{\varepsilon c_{k+1} + Q'/Q}\right)^\varepsilon \frac{P}{Q}. \tag{9.6}$$

Proof that Theorem 9.4 implies Lemma 2.7 when ℓ_k is odd. The assumption that ℓ_k is odd means that the word $\mathbf{w}$ defined in (2.7) has even length, thus Theorem 9.4 can be applied. We apply it with ε as in Lemma 2.7, that is, with $\varepsilon := -\operatorname{sgn}(m_k)d_{k+1}$. Assume that $(1+x^{-1})^{m_k} := P/Q$ and let P'/Q' denote the last convergent to P/Q. We have to show that

$$\left(\frac{\varepsilon c_{k+1} + P'/P}{\varepsilon c_{k+1} + Q'/Q}\right)^\varepsilon \frac{P}{Q} = (1+x^{-1})^{m_{k+1}}.$$

We first check that our choice of c_{k+1} implies (9.6).

By the theory of continued fractions, we have

$$P'Q - PQ' = (-1)^{\ell_k}.$$

Assume first that $m_k > 0$. We claim that

$$f := (-1)^{\ell_k}\frac{(x+1)^{p^{v_{k+1}}}}{d_{k+1}PQ} + \frac{Q'}{d_{k+1}Q} = (-1)^{\ell_k}\,\frac{(x+1)^{p^{v_{k+1}}} + (-1)^{\ell_k}PQ'}{d_{k+1}PQ}$$

is a polynomial. Indeed, the numerator of f is clearly divisible by P, as P is a constant multiple of $(1+x)^{m_k}$ and $m_k < p^{v_{k+1}}$. Since this numerator is also equal to

$$x^{p^{v_{k+1}}} + 1 + (-1)^{\ell_k}(P'Q - (-1)^{\ell_k}) = x^{p^{v_{k+1}}} + (-1)^{\ell_k} P'Q,$$

it is also divisible by Q, as Q is a constant multiple of x^{m_k}.

Using that $\varepsilon = -d_{k+1}$, we get

$$\varepsilon f PQ = -(-1)^{\ell_k}(x+1)^{p^{v_{k+1}}} - PQ'$$

and we check that

$$\begin{aligned}\left(\frac{\varepsilon f + P'/P}{\varepsilon f + Q'/Q}\right)^{\varepsilon} \frac{P}{Q} &= \left(1 + \frac{(-1)^{\ell_k}}{\varepsilon f PQ + PQ'}\right)^{\varepsilon} \frac{P}{Q} \\ &= \left(1 - \frac{1}{(x+1)^{p^{v_{k+1}}}}\right)^{\varepsilon} \frac{P}{Q} \\ &= \left(\frac{x}{x+1}\right)^{\varepsilon p^{v_{k+1}}} \frac{P}{Q} \\ &= (1+x^{-1})^{d_{k+1}p^{v_{k+1}} + m_k} = (1+x^{-1})^{m_{k+1}}.\end{aligned}$$

Assume now that $m_k < 0$. Then, $\varepsilon = d_{k+1}$. Arguing as above, we check that

$$f' := (-1)^{\ell_k} \frac{(x+1)^{p^{v_{k+1}}}}{d_{k+1}PQ} - \frac{P'}{d_{k+1}P} = (-1)^{\ell_k} \frac{(x+1)^{p^{v_{k+1}}} - (-1)^{\ell_k} P'Q}{d_{k+1}PQ}$$

is a polynomial. Since

$$\begin{aligned}\varepsilon f' PQ &= (-1)^{\ell_k}(x+1)^{p^{v_{k+1}}} - P'Q \\ &= (-1)^{\ell_k} x^{p^{v_{k+1}}} + (-1)^{\ell_k} - P'Q = (-1)^{\ell_k} x^{p^{v_{k+1}}} - PQ',\end{aligned}$$

we get

$$\begin{aligned}\left(\frac{\varepsilon f' + P'/P}{\varepsilon f' + Q'/Q}\right)^{\varepsilon} \frac{P}{Q} &= \left(1 + \frac{(-1)^{\ell_k}}{\varepsilon f' PQ + PQ'}\right)^{\varepsilon} \frac{P}{Q} \\ &= \left(1 + \frac{1}{x^{p^{v_{k+1}}}}\right)^{\varepsilon} \frac{P}{Q} \\ &= (1+x^{-1})^{d_{k+1}p^{v_{k+1}} + m_k} = (1+x^{-1})^{m_{k+1}}.\end{aligned}$$

Recalling that the degree of P' (resp., of Q') is less than that of P (resp., of Q), this shows that in both cases c_{k+1} is the polynomial part of

$$(-1)^{\ell_k} \frac{(x+1)^{p^{v_{k+1}}}}{d_{k+1}PQ}. \tag{9.7}$$

One can check that $\ell_{k+1} = (2|d_{k+1}| - 1)\ell_k + |d_{k+1}|$ if ε is positive (or, equivalently, if m_k and m_{k+1} have opposite signs) and $\ell_{k+1} = (2|d_{k+1}| + 1)\ell_k + |d_{k+1}|$ if ε is negative (or, equivalently, if m_k and m_{k+1} have the same sign). The arguments given in Section 10 show that this remains true if ℓ_k is even. Recalling that $\ell_0 = |m_0|$ and noticing that

$$\ell_{k+1} - \ell_k \equiv d_{k+1} \equiv m_{k+1} - m_k \pmod 2,$$

and immediate induction shows that m_k and ℓ_k have the same parity for every $k \geq 0$.

By Lemma 9.5 below, the leading coefficient of P is equal to $\pm 2^{|m_k| - \overline{m_k}}$. Since Q has the same leading coefficient as P, we then derive from (9.7) that c_{k+1} is the polynomial part of

$$(-1)^{\ell_k} \frac{(x+1)^{p^{v_{k+1}} - |m_k|}}{d_{k+1} 4^{|m_k| - \overline{m_k}} x^{|m_k|}}.$$

This completes the proof, since ℓ_k and m_k have the same parity. □

Lemma 9.5. *Let m be an integer, and P be the numerator of the continued fraction of $(1 + x^{-1})^m$. Then, the leading coefficient of P is equal to*

$$G(P) = \kappa 2^{|m| - \overline{m}},$$

where $\kappa = 1$ if $m \geq 0$, and $\kappa = (-1)^m$ if $m < 0$.

Proof. A quick check shows that the lemma is true for $m = 0, \pm 1$. Assume that m satisfies $2 \leq |m| \leq (p-1)/2$. We apply Lemma 2.6 and keep its notation. Then, $f_2 \dots f_m$ is a polynomial in x whose leading coefficient is $m 2^{m-1}$ if $m > 0$ and $(-1)^m m 2^{|m|-1}$ if $m < 0$. The expression of $(1 + x^{-1})^m$ given in (2.6) then shows that the leading coefficient of P (which is equal to the product of the leading coefficients of the partial quotients) is equal to $2^{m-1+\delta}$ if $m > 0$ and to $(-1)^m 2^{|m|-1+\delta}$ if $m < 0$, where $\delta = 1$ if m is even and $\delta = 0$ if m is odd. The lemma is proved when $|m| \leq (p-1)/2$.

Let $(d_k)_{k \geq 0}$ be a sequence of nonzero integers in $\{-(p-1)/2, \dots, (p-1)/2\}$ and $(v_k)_{k \geq 1}$ an increasing sequence of positive integers. For $k \geq 0$, set

$$m_k = d_0 + d_1 p^{v_1} + \dots + d_k p^{v_k}.$$

We have checked above that the lemma holds for m_0. Let $k \geq 0$ be an integer such that the lemma holds for m_k. We show that it holds for m_{k+1}. We let $G(R)$ denote the leading coefficient of a non-constant polynomial R in $\mathbb{F}_p[x]$. We keep the notation of the preceding proof and also assume that m_k is odd. We know that $G(P) = G(Q)$ and recall that $\varepsilon = -\operatorname{sgn}(m_k) d_{k+1}$. It follows from (9.7) that

$$G(c_{k+1}) = (-1)^{m_k} \frac{1}{d_{k+1} G(P)^2},$$
$$G(c_{k+1} \varepsilon PQ) = (-1)^{m_k} \frac{\varepsilon}{d_{k+1}} = -(-1)^{m_k} \operatorname{sgn}(m_k).$$

Assume first that $\varepsilon > 0$. We then have

$$\operatorname{sgn}(m_{k+1}) = \operatorname{sgn}(d_{k+1}) = -\operatorname{sgn}(m_k),$$

and it follows from Theorem 9.4 that

$$\begin{aligned}
A_\varepsilon &= (-1)^{\varepsilon+1}2^{\varepsilon+\bar{\varepsilon}-2}(\varepsilon c_{k+1}PQ + P'Q)^\varepsilon/Q,\\
G(A_\varepsilon) &= (-1)^{\varepsilon+1}2^{\varepsilon+\bar{\varepsilon}-2}(G(\varepsilon c_{k+1}PQ + P'Q))^\varepsilon/G(Q)\\
&= (-1)^{\varepsilon+1}2^{\varepsilon+\bar{\varepsilon}-2}(G(\varepsilon c_{k+1}PQ))^\varepsilon/G(Q)\\
&= (-1)^{\varepsilon+1}2^{\varepsilon+\bar{\varepsilon}-2}(-(-1)^{m_k}\operatorname{sgn}(m_k))^\varepsilon/G(P)\\
&= (-1)^{\varepsilon+1}2^{\varepsilon+\bar{\varepsilon}-2}(\operatorname{sgn}(m_k))^\varepsilon/(\operatorname{sgn}(m_k)2^{|m_k|-1}),
\end{aligned}$$

since the lemma holds for m_k. Using Fermat's Little Theorem, we then get that, in $\mathbb{F}_p$,

$$\begin{aligned}
G(A_\varepsilon) &= (-\operatorname{sgn}(m_k))^{\varepsilon+1}2^{\varepsilon+\bar{\varepsilon}-|m_k|-1}\\
&= (-\operatorname{sgn}(m_k))^{\varepsilon+1}2^{|m|+\bar{\varepsilon}-1}\\
&= (\operatorname{sgn}(m))^{\varepsilon+1}2^{|m|-\overline{m}} = \kappa 2^{|m|-\overline{m}}.
\end{aligned}$$

Assume now that $\varepsilon < 0$. Then, we have,

$$\begin{aligned}
A_\varepsilon &= 2^{-\varepsilon+\bar{\varepsilon}}(\varepsilon c_{k+1}PQ + PQ')^{-\varepsilon}P\\
G(A_\varepsilon) &= 2^{-\varepsilon+\bar{\varepsilon}}(G(\varepsilon c_{k+1}PQ + PQ'))^{-\varepsilon}G(P)\\
&= 2^{-\varepsilon+\bar{\varepsilon}}(-(-1)^{m_k}\operatorname{sgn}(m_k))^{-\varepsilon}\times\operatorname{sgn}(m_k)2^{|m_k|-1},
\end{aligned}$$

since the lemma holds for m_k. Using Fermat's Little Theorem, we then get that, in $\mathbb{F}_p$,

$$\begin{aligned}
G(A_\varepsilon) &= (\operatorname{sgn}(m_k))^{\varepsilon+1}2^{-\varepsilon+\bar{\varepsilon}+|m_k|-1}\\
&= \kappa 2^{|m|+\bar{\varepsilon}-1} = \kappa 2^{|m|-\overline{m}}.
\end{aligned}$$

The case m_k even is similar in view of Theorem 9.4 and Section 10. We omit the details.

By Lemma 2.4, this shows that the lemma holds for every integer m which is not divisible by p. When m is a multiple of p, say $m = p^a m'$ with $\gcd(p, m') = 1$, then the partial quotients of $(1+x^{-1})^m$ are the p^a-th powers of the partial quotients of $(1+x^{-1})^{m'}$. In $\mathbb{F}_p$, their leading coefficients are the same. This completes the proof of the lemma. □

We will derive Theorem 9.4 from Theorem 9.9 below. First, we perform a change of variables to get an equivalent, but slightly simpler, statement. We define

$$Y := \varepsilon c_{k+1}PQ, \quad r := P'Q + Y, \quad q := PQ' + Y.$$

We use the letter r to avoid any confusion with the prime number p. Then (9.6) becomes

$$\frac{A_\varepsilon}{B_\varepsilon} = \left(\frac{r}{q}\right)^\varepsilon \frac{P}{Q}. \tag{9.8}$$

Now, we observe that $\mathbf{M}$, $\mathbf{MC}(1)$, and any finite product $\mathbf{MC}(1)\mathbf{H}(\eta_\varepsilon(0))\cdots$ take one of the two following forms

$$\begin{pmatrix} *P & */Q \\ *Q & */P \end{pmatrix}, \quad \begin{pmatrix} */Q & *P \\ */P & *Q \end{pmatrix},$$

where the factors $*$ are polynomials in Y, r, q. In particular, neither P, nor Q, occur in the factors $*$. Consequently, we can assume that $P = Q = 1$, thus

$$Y = \varepsilon c_{k+1}, \quad r = P' + Y, \quad q = Q' + Y.$$

We work in the polynomial ring over the rational number $\mathbb{Q}[Y, r, q]$ and view Y, r, q as indeterminates.

Definition 9.6. *For a nonzero rational number ρ, define*

$$\overline{\mathbf{M}} = \begin{pmatrix} 1 & r - Y \\ 1 & q - Y \end{pmatrix}, \qquad \overline{\mathbf{C}}(\rho) = \begin{pmatrix} \rho Y & 1 \\ 1 & 0 \end{pmatrix},$$

$$\overline{\mathbf{H}}(\rho) = \begin{pmatrix} -4 & \rho(r + q - 2Y) \\ -\rho^{-1}(r + q - 2Y) & (r - Y)(q - Y) \end{pmatrix}.$$

It is convenient to extend the definition of $\eta_\varepsilon(m)$ for $m = -2, -1, 0, 1, 2, \ldots$.

Definition 9.7. *Given a nonzero integer $\varepsilon = 1, -1, 2, -2, 3, -3, \ldots$, let $m = 0, 1, \ldots, 2|\varepsilon| - 1$ if $\varepsilon > 0$, or $m = 0, 1, \ldots, 2|\varepsilon|$ if $\varepsilon < 0$. We define*

$$\zeta_\varepsilon(m) = \begin{cases} \eta_\varepsilon(m-2), & \text{if } m = 2t \text{ is even;} \\ \eta_\varepsilon(m-2)/\varepsilon, & \text{if } m = 2t+1 \text{ is odd.} \end{cases}$$

For later use, we check that

$$\frac{\eta_\varepsilon(2t+1)}{\eta_\varepsilon(2t)} = -\frac{(2t+3)\varepsilon}{2(\varepsilon + t + 1)},$$

thus we get

$$\frac{\zeta_\varepsilon(2t+1)}{\zeta_\varepsilon(2t)} = \frac{\eta_\varepsilon(2t-1)/\varepsilon}{\eta_\varepsilon(2t-2)} = -\frac{2t+1}{2(\varepsilon + t)} \tag{9.9}$$

Furthermore, by combining

$$\eta_\varepsilon(2t) = \frac{-2^{2t-4\lfloor t/2 \rfloor - 1} [\![|\varepsilon| - t \bullet |\varepsilon| + t + 2]\!]}{[\![|\varepsilon - 1| - t \bullet |\varepsilon - 1| + t + 2]\!]},$$

with

$$\eta_\varepsilon(2t+2) = \frac{-2^{2t+2-4\lfloor (t+1)/2 \rfloor - 1} [\![|\varepsilon| - t - 1 \bullet |\varepsilon| + t + 1 + 2]\!]}{[\![|\varepsilon - 1| - t - 1 \bullet |\varepsilon - 1| + t + 1 + 2]\!]},$$

we get

$$\eta_\varepsilon(2t)\eta_\varepsilon(2t+2) = \frac{\varepsilon + t + 1}{\varepsilon - t - 2}. \tag{9.10}$$

This shows that Theorem 9.4 is equivalent to:

Theorem 9.8. *For every positive integer* ε*, let*

$$\overline{\mathbf{M}}\,\overline{\mathbf{C}}(1/\varepsilon)\overline{\mathbf{H}}(\zeta_\varepsilon(2))\overline{\mathbf{C}}(\zeta_\varepsilon(3))\cdots\overline{\mathbf{H}}(\zeta_\varepsilon(2\varepsilon-2))\overline{\mathbf{C}}(\zeta_\varepsilon(2\varepsilon-1)) = \begin{pmatrix} A_\varepsilon & A'_\varepsilon \\ B_\varepsilon & B'_\varepsilon \end{pmatrix}.$$

We have

$$A_\varepsilon = (-1)^{\varepsilon+1}2^{\varepsilon+\overline{\varepsilon}-2}r^\varepsilon, \tag{9.11}$$

$$B_\varepsilon = (-1)^{\varepsilon+1}2^{\varepsilon+\overline{\varepsilon}-2}q^\varepsilon. \tag{9.12}$$

For every negative integer ε*, let*

$$\overline{\mathbf{M}}\,\overline{\mathbf{C}}(1/\varepsilon)\overline{\mathbf{H}}(\zeta_\varepsilon(2))\overline{\mathbf{C}}(\zeta_\varepsilon(3))\cdots\overline{\mathbf{H}}(\zeta_\varepsilon(2|\varepsilon|-2))\times \overline{\mathbf{C}}(\zeta_\varepsilon(2|\varepsilon|-1))\overline{\mathbf{H}}(\zeta_\varepsilon(2|\varepsilon|)) = \begin{pmatrix} A_\varepsilon & A'_\varepsilon \\ B_\varepsilon & B'_\varepsilon \end{pmatrix}.$$

We have

$$A_\varepsilon = 2^{-\varepsilon+\overline{\varepsilon}}q^{-\varepsilon}, \tag{9.13}$$

$$B_\varepsilon = 2^{-\varepsilon+\overline{\varepsilon}}r^{-\varepsilon}. \tag{9.14}$$

In both cases we have

$$\frac{A_\varepsilon}{B_\varepsilon} = \left(\frac{r}{q}\right)^\varepsilon.$$

Since

$$\zeta_\varepsilon(0) = \eta_\varepsilon(-2) = -2, \quad \zeta_\varepsilon(1) = \eta_\varepsilon(-1)/\varepsilon = 1/\varepsilon,$$

we have

$$\begin{aligned}\overline{\mathbf{M}}\,\overline{\mathbf{C}}(1/\varepsilon)\overline{\mathbf{H}}(\zeta_\varepsilon(2))\overline{\mathbf{C}}(\zeta_\varepsilon(3))\cdots &= \overline{\mathbf{M}}\,(\overline{\mathbf{H}}(-2))^{-1}\times\overline{\mathbf{H}}(\zeta_\varepsilon(0))\overline{\mathbf{C}}(\zeta_\varepsilon(1))\times\overline{\mathbf{H}}(\zeta_\varepsilon(2))\overline{\mathbf{C}}(\zeta_\varepsilon(3))\cdots \\ &= \mathbf{U}\times\mathbf{D}_\varepsilon(0)\times\mathbf{D}_\varepsilon(1)\cdots,\end{aligned}$$

where

$$\mathbf{U} := \overline{\mathbf{M}}\,(\overline{\mathbf{H}}(-2))^{-1} = \frac{1}{r-q}\begin{pmatrix} (Y-r)/2 & -2 \\ -(Y-q)/2 & 2 \end{pmatrix}$$

and, for $t = 0, 1, \ldots, |\varepsilon|-1$,

$$\begin{aligned}\mathbf{D}_\varepsilon(t) &:= \overline{\mathbf{H}}(\zeta_\varepsilon(2t))\overline{\mathbf{C}}(\zeta_\varepsilon(2t+1)) \\ &= \begin{pmatrix} \zeta_\varepsilon(2t)\left(r+q-2Y\frac{\varepsilon-t-1}{\varepsilon+t}\right) & -4 \\ rq - \frac{(2\varepsilon-1)(r+q)}{2(\varepsilon+t)}Y + \frac{\varepsilon-t-1}{\varepsilon+t}Y^2 & -(\zeta_\varepsilon(2t))^{-1}(r+q-2Y) \end{pmatrix},\end{aligned}$$

by (9.9).

Consequently, it is sufficient to establish the following

Theorem 9.9. *For every positive integer* ε*, let*

$$\mathbf{U}\mathbf{D}_\varepsilon(0)\mathbf{D}_\varepsilon(1)\cdots\mathbf{D}_\varepsilon(\varepsilon-1) = \begin{pmatrix} A_\varepsilon & A'_\varepsilon \\ B_\varepsilon & B'_\varepsilon \end{pmatrix}.$$

Then, A_ε and B_ε are given by (9.11) and (9.12), respectively. For every negative integer ε, let

$$\mathbf{U}\mathbf{D}_\varepsilon(0)\mathbf{D}_\varepsilon(1)\cdots\mathbf{D}_\varepsilon(|\varepsilon|-1)\overline{\mathbf{H}}(\zeta_\varepsilon(2|\varepsilon|)) = \begin{pmatrix} A_\varepsilon & A'_\varepsilon \\ B_\varepsilon & B'_\varepsilon \end{pmatrix}.$$

Then, A_ε and B_ε are given by (9.13) and (9.14), respectively. In both cases we have

$$\frac{A_\varepsilon}{B_\varepsilon} = \left(\frac{r}{q}\right)^\varepsilon.$$

10. When the length of w is odd

We explain in this section how to deduce the results for $\mathbf{w}$ of odd length from those established for $\mathbf{w}$ of even length.

Observe first that in Lemma 9.2 in both cases the length of $\mathbf{h}$ is even.

Lemma 10.1. *Let $\mathbf{h}^e$ and $\mathbf{h}^o$ be two sequences of even length over $\mathbb{F}_p[x]\setminus\mathbb{F}_p$ such that*

$$\mathbf{M}(\mathbf{h}^e) = \begin{pmatrix} -4A & B \\ -B & C \end{pmatrix} \quad \text{and} \quad \mathbf{M}(\mathbf{h}^o) = \begin{pmatrix} A & -B \\ B & -4C \end{pmatrix}.$$

Let α, β, γ be nonzero elements of $\mathbb{F}_p$. Let c be in $\mathbb{F}_p[x]\setminus\mathbb{F}_p$ and $\mathbf{s}$ a finite sequence over $\mathbb{F}_p[x]$. Then, we have

$$\mathbf{M}(\alpha\mathbf{h}^e, \beta c, \gamma\mathbf{h}^e) = \mathbf{M}(4\alpha\mathbf{h}^o, 16\beta c, \frac{1}{4}\gamma\mathbf{h}^o),$$
$$[\mathbf{s}, \alpha\mathbf{h}^e] = [\mathbf{s}, 4\alpha\mathbf{h}^o],$$
$$[\mathbf{s}, \alpha\mathbf{h}^e, \beta c] = [\mathbf{s}, 4\alpha\mathbf{h}^o, 16\beta c],$$

The first equality is about matrices, while the last two equalities are about continued fractions and do not extend to matrices.

Proof. This follows from a direct computation. □

The form of the matrix $\mathbf{M}(\mathbf{h})$ in Lemma 9.2 allows us to apply Lemma 10.1 with

$$A := PQ, \quad B := QP' + PQ', \quad C := P'Q'$$

to deduce from Theorem 9.4 (which addresses the case where the length of $\mathbf{w}$ is even) its analogue for the case where the length of $\mathbf{w}$ is odd. Then, we derive the continued fraction expansion of $(1+x^{-1})^{m_{k+1}}$ when the length of $\mathbf{w}$ is odd (that is, when the length ℓ_k of $(1+x^{-1})^{m_k}$ is even). We omit the details.

11. Proof of Theorem 9.9

For i, j in $\{1, 2\}$, let $U[i, j]$ denote the $i \times j$ coefficient of the matrix U.

Lemma 11.1. *We have*

$$\begin{aligned} U[1,1](r,q) &= U[2,1](q,r), \\ U[1,2](r,q) &= U[2,2](q,r), \\ B_\varepsilon(r,q) &= A_\varepsilon(q,r), \\ B'_\varepsilon(r,q) &= A'_\varepsilon(q,r). \end{aligned}$$

Proof. The assertion about the coefficients of U is easy to check. It implies the last two assertions, since the matrices $\mathbf{D}$ and $\overline{\mathbf{H}}$ are symmetric in the indeterminates r and q. □

Definition 11.2. *For every positive integer k and $j = 0, 1, \ldots, k$, we use the following notations:*

$$\Phi(k,j;r,q) = \sum_{d=0}^{j} \frac{\binom{k+j+1}{d}\binom{j}{d}}{\binom{k}{d}} (-r)^d q^{j-d},$$

$$K_1(k,j) = \begin{cases} 2^{2-k-\bar{k}} \dfrac{[\![k+j \bullet 2k-2]\!]}{[\![1 \bullet k-j-1]\!]}, & \text{if } k-j \text{ is even,} \\ 2^{\bar{k}-1} \dfrac{[\![k-j-1 \bullet k-\bar{k}]\!]}{[\![k-1+\bar{k} \bullet k+j]\!]}, & \text{if } k-j \text{ is odd,} \end{cases}$$

$$K_2(k,j) = \begin{cases} -2^{-k-\bar{k}} \dfrac{[\![k+j+2 \bullet 2k+2]\!]}{k[\![1 \bullet k-j-1]\!]}, & \text{if } k-j \text{ is even,} \\ -2^{\bar{k}-1} \dfrac{[\![k-j-1 \bullet k+2-\bar{k}]\!]}{k[\![k+1+\bar{k} \bullet k+j+2]\!]}, & \text{if } k-j \text{ is odd,} \end{cases}$$

$$K_3(k,j) = \begin{cases} 2^{\bar{k}} \dfrac{[\![k-j \bullet k+\bar{k}]\!]}{[\![k+1-\bar{k} \bullet k+j+1]\!]}, & \text{if } k-j \text{ is even,} \\ 2^{2-k-\bar{k}} \dfrac{[\![k+j+1 \bullet 2k]\!]}{[\![1 \bullet k-j]\!]}, & \text{if } k-j \text{ is odd,} \end{cases}$$

$$K_4(k,j) = \begin{cases} (-1)^k 2^{-k-\bar{k}} \dfrac{[\![k+j+2 \bullet 2k+2]\!]}{[\![1 \bullet k-j+1]\!]}, & \text{if } k-j \text{ is even,} \\ (-1)^{k+1} 2^{\bar{k}-1} \dfrac{[\![k-j+1 \bullet k+2-\bar{k}]\!]}{[\![k+1+\bar{k} \bullet k+j+2]\!]}, & \text{if } k-j \text{ is odd,} \end{cases}$$

$$K_5(k,j) = \begin{cases} (-1)^{k+1} 2^{-k-\bar{k}} \dfrac{(k+j+1)[\![k+j+2 \bullet 2k+2]\!]}{k[\![1 \bullet k-j+1]\!]}, & \text{if } k-j \text{ is even,} \\ (-1)^k 2^{\bar{k}-1} \dfrac{(k+j+1)[\![k-j+1 \bullet k+2-\bar{k}]\!]}{k[\![k+1+\bar{k} \bullet k+j+2]\!]}, & \text{if } k-j \text{ is odd.} \end{cases}$$

It follows from (8.1) that

$$K_1(k,j)=\begin{cases}2^{-j-\bar k}\dfrac{(k-2)!((k-j)/2-1)!}{(k-j-2)!((k+j)/2-1)!}, & \text{if } k-j \text{ is even},\\ 1, & \text{if } k=1 \text{ and } j=0,\\ 0, & \text{if } k-j=1 \text{ and } k\ge 2,\\ 2^{j+\bar k}\dfrac{(k-2)!((k+j-1)/2)!}{((k-j-3)/2)!(k+j-1)!}, & \text{if } k-j \text{ is odd and } k-j\ge 3,\end{cases}$$

$$K_2(k,j)=\begin{cases}-2^{-j-\bar k-1}\dfrac{(k-1)!((k-j)/2-1)!}{((k+j)/2)!(k-j-2)!}, & \text{if } k-j \text{ is even},\\ 0, & \text{if } k-j=1,\\ -2^{j+\bar k+1}\dfrac{(k-1)!((k+j+1)/2)!}{((k-j-3)/2)!(k+j+1)!}, & \text{if } k-j \text{ is odd and } k-j\ge 3,\end{cases}$$

$$K_3(k,j)=\begin{cases}2^{j+\bar k+1}\dfrac{(k-1)!((k+j)/2)!}{((k-j)/2-1)!(k+j)!}, & \text{if } k-j \text{ is even},\\ 2^{1-j-\bar k}\dfrac{(k-1)!((k-j-1)/2)!}{((k+j-1)/2)!(k-j-1)!}, & \text{if } k-j \text{ is odd},\end{cases}$$

$$K_4(k,j)=\begin{cases}(-1)^k 2^{-j-\bar k}\dfrac{k!((k-j)/2)!}{((k+j)/2)!(k-j)!}, & \text{if } k-j \text{ is even},\\ (-1)^{k+1}2^{j+\bar k}\dfrac{k!((k+j+1)/2)!}{(k+j+1)!((k-j-1)/2)!}, & \text{if } k-j \text{ is odd},\end{cases}$$

$$K_5(k,j)=\begin{cases}(-1)^{k+1}2^{-j-\bar k}\dfrac{(k+j+1)(k-1)!((k-j)/2)!}{((k+j)/2)!(k-j)!}, & \text{if } k-j \text{ is even},\\ (-1)^k 2^{j+\bar k}\dfrac{(k-1)!((k+j+1)/2)!}{(k+j)!((k-j-1)/2)!}, & \text{if } k-j \text{ is odd}.\end{cases}$$

For each positive integer k and $j=0,1,\dots,k-2$ such that $k-j$ is odd, define

$$\begin{aligned}\Psi(k,j;r,q)&=K_1(k,j)\Phi(k-2,j;r,q)\\ &=2^{\bar k-1}\frac{[\![k-j-1\bullet k-\bar k]\!]}{[\![k-1+\bar k\bullet k+j]\!]}\sum_{d=0}^{j}\frac{\binom{k+j-1}{d}\binom{j}{d}}{\binom{k-2}{d}}(-r)^d q^{j-d}.\end{aligned}$$

Since $[\![k-j-1\bullet k-\bar k]\!]=(k-j-1)[\![k-j+1\bullet k-\bar k]\!]$ and, for $d\ge 1$,

$$\binom{k-2}{d}=\binom{k-2}{d-1}\cdot\frac{k-d-1}{d},$$

we have

$$\begin{aligned}\Psi(k,j;r,q)=2^{\bar k-1}&\frac{(k-j-1)[\![k-j+1\bullet k-\bar k]\!]}{[\![k-1+\bar k\bullet k+j]\!]}\\ &\times\left(q^j+\sum_{d=1}^{j}\frac{d\binom{k+j-1}{d}\binom{j}{d}}{(k-d-1)\binom{k-2}{d-1}}(-r)^d q^{j-d}\right)\end{aligned}$$

$$= 2^{\bar{k}-1} \frac{[\![k-j+1 \bullet k-\bar{k}]\!]}{[\![k-1+\bar{k} \bullet k+j]\!]} \times \left((k-j-1)q^j + \sum_{d=1}^{j} \frac{d(k-j-1)\binom{k+j-1}{d}\binom{j}{d}}{(k-d-1)\binom{k-2}{d-1}} (-r)^d q^{j-d} \right).$$

Hence we have

$$\Psi(k,j;r,q) = 2^{\bar{k}-1} \frac{[\![k-j+1 \bullet k-\bar{k}]\!]}{[\![k-1+\bar{k} \bullet k+j]\!]} \times \Big((k-j-1)q^j + \sum_{d=1}^{j-1} \frac{d(k-j-1)\binom{k+j-1}{d}\binom{j}{d}}{(k-d-1)\binom{k-2}{d-1}} (-r)^d q^{j-d} + \frac{j\binom{k+j-1}{j}}{\binom{k-2}{j-1}} (-r)^j \Big). \tag{11.1}$$

Formula (11.1) is defined for $j = 0, 1, \dots, k-2$. Its right-hand side is also valid for $j = k-1$. Thus, we use it to define $\Psi(k,j;r,q)$ when $j = k-1$.

Theorem 11.3. *For a nonzero integer ε and $j = 0, 1, \dots, |\varepsilon|-1$, set*

$$\Pi_\varepsilon(j) := \mathbf{U}\mathbf{D}_\varepsilon(0)\mathbf{D}_\varepsilon(1)\cdots\mathbf{D}_\varepsilon(j) = \begin{pmatrix} A_\varepsilon(j) & A'_\varepsilon(j) \\ B_\varepsilon(j) & B'_\varepsilon(j) \end{pmatrix}.$$

If $\varepsilon > 0$, then we have

$$\begin{aligned} A_\varepsilon(j) &= \Psi(|\varepsilon|, j; r, q)r + K_2(|\varepsilon|, j)\Phi(|\varepsilon|-1, j; r, q)Y \\ A'_\varepsilon(j) &= K_3(|\varepsilon|, j)\Phi(|\varepsilon|-1, j; r, q) \\ B_\varepsilon(j) &= \Psi(|\varepsilon|, j; q, r)q + K_2(|\varepsilon|, j)\Phi(|\varepsilon|-1, j; q, r)Y \\ B'_\varepsilon(j) &= K_3(|\varepsilon|, j)\Phi(|\varepsilon|-1, j; q, r) \end{aligned}$$

If $\varepsilon < 0$, then we have

$$\begin{aligned} A_\varepsilon(j) &= K_4(|\varepsilon|, j)\Phi(|\varepsilon|, j; q, r)r + K_5(|\varepsilon|, j)\Phi(|\varepsilon|-1, j; q, r)Y \\ A'_\varepsilon(j) &= (-1)^j K_3(|\varepsilon|, j)\Phi(|\varepsilon|-1, j; q, r) \\ B_\varepsilon(j) &= K_4(|\varepsilon|, j)\Phi(|\varepsilon|, j; r, q)q + K_5(|\varepsilon|, j)\Phi(|\varepsilon|-1, j; r, q)Y \\ B'_\varepsilon(j) &= (-1)^j K_3(|\varepsilon|, j)\Phi(|\varepsilon|-1, j; r, q) \end{aligned}$$

Sketch of the proof. We check by a direct computation that the theorem holds for $j = 0$. Then, we verify that $\Pi_\varepsilon(j+1) = \Pi_\varepsilon(j)\mathbf{D}_\varepsilon(j+1)$ for $j = 0, \dots, |\varepsilon|-2$. Consider only the special case where j is even and ε is positive and even. With $t = j+1$, we need to check that

$$\begin{aligned} A_\varepsilon(j+1) = A_\varepsilon(j)\zeta_\varepsilon(2t)&\left(r + q - 2Y\frac{\varepsilon-t-1}{\varepsilon+t}\right) \\ &+ A'_\varepsilon(j)\left(rq - \frac{(2\varepsilon-1)(r+q)}{2(\varepsilon+t)}Y + \frac{\varepsilon-t-1}{\varepsilon+t}Y^2\right), \end{aligned}$$

that is,

$$\begin{aligned} &K_1(\varepsilon, j+1)\Phi(\varepsilon-2, j+1; r, q)r + K_2(\varepsilon, j+1)\Phi(\varepsilon-1, j+1; r, q)Y \\ &= \big(K_1(\varepsilon, j)\Phi(\varepsilon-2, j; r, q)r \end{aligned}$$

$$+ K_2(\varepsilon, j)\Phi(\varepsilon-1, j; r, q)Y\big)\zeta_\varepsilon(2j+2)\left(r+q-2Y\frac{\varepsilon-j-2}{\varepsilon+j+1}\right)$$
$$+ K_3(\varepsilon, j)\Phi(\varepsilon-1, j; r, q)\left(rq-\frac{(2\varepsilon-1)(r+q)}{2(\varepsilon+j+1)}Y+\frac{\varepsilon-j-2}{\varepsilon+j+1}Y^2\right).$$

To check this, we view both expressions as polynomials in Y and verify that the coefficients of $1, Y$, and Y^2 coincide. Let us do this for the coefficients of Y. This amounts to check that

$$\begin{aligned}
&K_2(\varepsilon, j+1)\Phi(\varepsilon-1, j+1; r, q)\\
&= (K_1(\varepsilon, j)\Phi(\varepsilon-2, j; r, q)r)\,\zeta_\varepsilon(2j+2)\left(-2\frac{\varepsilon-j-2}{\varepsilon+j+1}\right)\\
&\quad + (K_2(\varepsilon, j)\Phi(\varepsilon-1, j; r, q))\,\zeta_\varepsilon(2j+2)\,(r+q)\\
&\quad + K_3(\varepsilon, j)\Phi(\varepsilon-1, j; r, q)\left(-\frac{(2\varepsilon-1)(r+q)}{2(\varepsilon+j+1)}\right).
\end{aligned}$$

By Definition 9.7, this is equivalent to

$$\begin{aligned}
&2K_2(\varepsilon, j+1)\Phi(\varepsilon-1, j+1; r, q)(\varepsilon+j+1)\\
&= -4K_1(\varepsilon, j)\Phi(\varepsilon-2, j; r, q)r\eta_\varepsilon(2j)(\varepsilon-j-2)\\
&\quad + 2K_2(\varepsilon, j)\Phi(\varepsilon-1, j; r, q)\eta_\varepsilon(2j)\,(r+q)\,(\varepsilon+j+1)\\
&\quad - K_3(\varepsilon, j)\Phi(\varepsilon-1, j; r, q)(2\varepsilon-1)(r+q).
\end{aligned}$$

Recalling that that ε and j are assumed to be even, we get

$$\begin{aligned}
\eta_\varepsilon(2j) &= -2^{2j-2j-1}\,\frac{[\![\varepsilon-j\bullet\varepsilon+j+2]\!]}{[\![\varepsilon-1-j\bullet\varepsilon+j+1]\!]},\\
K_2(\varepsilon, j+1) &= -2^{-1}\,\frac{[\![\varepsilon-j-2\bullet\varepsilon+2]\!]}{\varepsilon[\![\varepsilon+1\bullet\varepsilon+j+3]\!]},\\
K_1(\varepsilon, j) &= 2^{2-\varepsilon}\,\frac{[\![\varepsilon+j\bullet 2\varepsilon-2]\!]}{[\![1\bullet\varepsilon-j-1]\!]},\\
K_2(\varepsilon, j) &= -2^{-\varepsilon}\,\frac{[\![\varepsilon+j+2\bullet 2\varepsilon+2]\!]}{\varepsilon[\![1\bullet\varepsilon-j-1]\!]},\\
K_3(\varepsilon, j) &= 2^{0}\,\frac{[\![\varepsilon-j\bullet\varepsilon]\!]}{[\![\varepsilon+1\bullet\varepsilon+j+1]\!]}.
\end{aligned}$$

Inserting this in the equality to be checked and noticing that $[\![u\bullet v]\!]\cdot[\![v\bullet w]\!] = [\![u\bullet w]\!]$ for every integers u, v, w, we use (8.1) to show that $2^{\varepsilon-2}[\![1\bullet\varepsilon-1]\!] = [\![\varepsilon\bullet 2\varepsilon-2]\!]$ and we see after some simplification that we need to verify that

$$\begin{aligned}
&-(\varepsilon-1)\Phi(\varepsilon-1, j+1; r, q)\\
&= 2r(\varepsilon+j)\Phi(\varepsilon-2, j; r, q)\\
&\quad - (\varepsilon-1)(r+q)\Phi(\varepsilon-1, j; r, q).
\end{aligned}$$

We use the definition of Φ to conclude. We omit the details. We check analogously that the values of $A'_\varepsilon(j+1), B_\varepsilon(j+1)$, and $B'_\varepsilon(j+1)$ are those given by the theorem. We also have to consider the remaining cases (ε negative, ε or j odd). A full proof is given in Appendix A. □

Deduction of Theorem 9.9 from Theorem 11.3. If $\varepsilon > 0$ and $j = \varepsilon - 1$, we have

$$\begin{aligned} A_\varepsilon = A_\varepsilon(\varepsilon-1) &= \Psi(\varepsilon, \varepsilon-1; r, q) r \\ &= 2^{\bar\varepsilon-1} \frac{[\![\varepsilon - j + 1 \bullet \varepsilon - \bar\varepsilon]\!]}{[\![\varepsilon - 1 + \bar\varepsilon \bullet \varepsilon + j]\!]} \frac{j\binom{\varepsilon+j-1}{j}}{\binom{\varepsilon-2}{j-1}} (-r)^j \\ &= \frac{2^{\bar\varepsilon-1}(\varepsilon-1)[\![2 \bullet \varepsilon - \bar\varepsilon]\!] r}{[\![\varepsilon - 1 + \bar\varepsilon \bullet 2\varepsilon - 1]\!]} \binom{2\varepsilon-2}{\varepsilon-1} (-r)^{\varepsilon-1} \\ &= (-1)^{\varepsilon+1} 2^{\varepsilon+\bar\varepsilon-2} r^\varepsilon. \end{aligned}$$

Hence, by Lemma 11.1, we get

$$B_\varepsilon = (-1)^{\varepsilon+1} 2^{\varepsilon+\bar\varepsilon-2} q^\varepsilon,$$

and conclude that

$$\frac{A_\varepsilon}{B_\varepsilon} = \left(\frac{r}{q}\right)^\varepsilon.$$

If $\varepsilon < 0$, $j = |\varepsilon| - 1$, then

$$\begin{pmatrix} A_\varepsilon(j) & A'_\varepsilon(j) \\ B_\varepsilon(j) & B'_\varepsilon(j) \end{pmatrix} \overline{\mathbf{H}}(\zeta_\varepsilon(2|\varepsilon|)) = \begin{pmatrix} A_\varepsilon & A'_\varepsilon \\ B_\varepsilon & B'_\varepsilon \end{pmatrix}.$$

Hence,

$$\begin{aligned} A_\varepsilon &= -4A_\varepsilon(j) - A'_\varepsilon(j) \frac{r + q - 2Y}{\zeta_\varepsilon(2|\varepsilon|)} \\ &= -4K_4(|\varepsilon|, j)\Phi(|\varepsilon|, j; q, r) r - 4K_5(|\varepsilon|, j)\Phi(|\varepsilon| - 1, j; q, r) Y \\ &\quad - (-1)^j K_3(|\varepsilon|, j)\Phi(|\varepsilon| - 1, j; q, r) \frac{r + q - 2Y}{\zeta_\varepsilon(2|\varepsilon|)}. \end{aligned}$$

With $k = |\varepsilon|$ and $j = k - 1$,

$$\begin{aligned} K_3(k, j) &= 2^{2-k-\bar k}, \\ K_4(k, j) &= (-1)^{k+1} 2^{\bar k - 1} \frac{[\![2 \bullet k + 2 - \bar k]\!]}{[\![k + 1 + \bar k \bullet 2k + 1]\!]}, \\ K_5(k, j) &= (-1)^k 2^{\bar k} \frac{[\![2 \bullet k + 2 - \bar k]\!]}{[\![k + 1 + \bar k \bullet 2k + 1]\!]}, \\ \zeta_\varepsilon(2k) &= -2^{(-1)^k} \frac{[\![1 \bullet 2k + 1]\!]}{[\![2 \bullet 2k + 2]\!]}, \\ \Phi(|\varepsilon|, j; q, r) &= \sum_{d=0}^{k-1} \frac{\binom{2k}{d}\binom{k-1}{d}}{\binom{k}{d}} (-q)^d r^{k-1-d} \end{aligned}$$

$$= \sum_{d=0}^{k-1} \frac{\binom{2k}{d}(k-d)}{k}(-q)^d r^{k-1-d},$$

$$\Phi(|\varepsilon|-1, j; q, r) = \sum_{d=0}^{k-1} \binom{2k-1}{d}(-q)^d r^{k-1-d}.$$

We check that

$$\begin{aligned}
\alpha &= -4K_5(|\varepsilon|, j)Y - (-1)^j K_3(|\varepsilon|, j)\frac{r+q-2Y}{\zeta_\varepsilon(2|\varepsilon|)} \\
&= -4\frac{(-1)^k 2^{\bar{k}}[\![2 \bullet k+2-\bar{k}]\!]}{[\![k+1+\bar{k} \bullet 2k+1]\!]}Y + (-1)^j 2^{2-k-\bar{k}}(r+q-2Y)\frac{[\![2 \bullet 2k+2]\!]}{2^{(-1)^k}[\![1 \bullet 2k+1]\!]} \\
&= (-1)^j 2^{1-k+\bar{k}}(r+q)\frac{[\![2 \bullet 2k+2]\!]}{[\![1 \bullet 2k+1]\!]},
\end{aligned}$$

thus

$$\begin{aligned}
A_\varepsilon &= -4K_4(|\varepsilon|, j)\Phi(|\varepsilon|, j; q, r)r + \alpha\Phi(|\varepsilon|-1, j; q, r) \\
&= -4r\frac{(-1)^{k+1}2^{\bar{k}-1}[\![2 \bullet k+2-\bar{k}]\!]}{[\![k+1+\bar{k} \bullet 2k+1]\!]}\sum_{d=0}^{k-1}\frac{\binom{2k}{d}(k-d)}{k}(-q)^d r^{k-1-d} \\
&\quad + (-1)^j 2^{1-k+\bar{k}}(r+q)\frac{[\![2 \bullet 2k+2]\!]}{[\![1 \bullet 2k+1]\!]}\sum_{d=0}^{k-1}\binom{2k-1}{d}(-q)^d r^{k-1-d} \\
&= (-1)^{k+1}2^{\bar{k}+1}\frac{k!}{[\![1 \bullet 2k+1]\!]}\sum_{d=0}^{k-1}\left[(-r)\frac{\binom{2k}{d}(k-d)}{k} + (r+q)\binom{2k-1}{d}\right](-q)^d r^{k-1-d}.
\end{aligned}$$

Since

$$\begin{aligned}
T &:= \sum_{d=0}^{k-1}\left[(-r)\frac{\binom{2k}{d}(k-d)}{k} + (r+q)\binom{2k-1}{d}\right](-q)^d r^{k-1-d} \\
&= \sum_{d=0}^{k-1}\left[r\binom{2k-1}{d-1} + q\binom{2k-1}{d}\right](-q)^d r^{k-1-d} \\
&= \sum_{d=0}^{k-1}\binom{2k-1}{d-1}(-q)^d r^{k-d} - \sum_{d=0}^{k-1}\binom{2k-1}{d}(-q)^{d+1} r^{k-d-1} \\
&= -\binom{2k-1}{k-1}(-q)^k,
\end{aligned}$$

we derive that

$$\begin{aligned}
A_\varepsilon &= (-1)^{k+1}2^{\bar{k}+1}\frac{k!}{[\![1 \bullet 2k+1]\!]}T \\
&= 2^{\bar{k}+1}\frac{k!}{[\![1 \bullet 2k+1]\!]}\binom{2k-1}{k-1}q^k = 2^{k+\bar{k}}q^k.
\end{aligned}$$

By Lemma 11.1 we get $B_\varepsilon = 2^{k+\bar{k}} r^k$, and we conclude that
$$\frac{A_\varepsilon}{B_\varepsilon} = \left(\frac{r}{q}\right)^\varepsilon.$$
This proves the theorem. □

12. Proof of Lemma 2.6

We start with an easy lemma.

Lemma 12.1. *If the generalized continued fraction*
$$f(z) = c_0 + \frac{a_0 z}{1} + \frac{a_1 z}{1} + \frac{a_2 z}{1} + \cdots$$
is equal to the continued fraction
$$f(z) = c_0 + \cfrac{1}{d_1(u_1 + z^{-1}) + \cfrac{1}{d_2(u_2 + z^{-1}) + \ddots}}$$
then
$$\begin{aligned}
u_1 &= a_1,\\
u_k &= a_{2k-2} + a_{2k-1}, \quad k \geq 2,\\
d_1 &= a_0^{-1}\\
d_{2k+1} &= \frac{(a_1a_2)(a_5a_6)\cdots(a_{4k-3}a_{4k-2})}{a_0(a_3a_4)\cdots(a_{4k-1}a_{4k})}, \quad k \geq 1,\\
d_{2k} &= -\frac{a_0(a_3a_4)\cdots(a_{4k-5}a_{4k-4})}{(a_1a_2)(a_5a_6)\cdots(a_{4k-3}a_{4k-2})}, \quad k \geq 1.
\end{aligned}$$

Proof. This follows from an elementary continued fraction manipulation. □

Lagrange [9] showed that, for every real number r, we have
$$(1+z)^r = 1 + \frac{rz}{1} - \frac{\frac{(r-1)z}{2}}{1} + \frac{\frac{(r+1)z}{2\cdot 3}}{1} - \frac{\frac{(r-2)z}{2\cdot 3}}{1} + \frac{\frac{(r+2)z}{2\cdot 5}}{1} - \frac{\frac{(r-3)z}{2\cdot 5}}{1} + \frac{\frac{(r+3)z}{2\cdot 7}}{1} + \cdots$$
By setting $x = 1/z$, taking $r = 1/2$ and applying Lemma 12.1 with $c_0 = 1, a_0 = 1/2$, and $a_k = 1/4$ for $k \geq 1$, we derive the continued fraction expansion of $(1 + x^{-1})^{1/2}$ given in (1.2).

By Lemma 12.1 applied with
$$\begin{aligned}
c_0 &= 1,\\
a_0 &= r,\\
a_{2k} &= (r+k)/(2(2k+1)), \quad k \geq 1,\\
a_{2k-1} &= -(r-k)/(2(2k-1)), \quad k \geq 1,
\end{aligned}$$

and with the notation of Lemma 12.1, we get

$$\begin{aligned} u_1 &= -(r-1)/2, \\ u_k &= 1/2, \quad k \geq 2, \\ d_1 &= r^{-1}, \\ d_{2k+1} &= \frac{(4k+1)[\![r-2k+1 \bullet r+2k+1]\!]}{[\![r-2k \bullet r+2k+2]\!]}, \quad k \geq 1, \\ d_{2k} &= \frac{4(4k-1)[\![r-2k+2 \bullet r+2k]\!]}{[\![r-2k+1 \bullet r+2k+1]\!]}, \quad k \geq 1. \end{aligned}$$

This implies the following result when r is an integer.

Lemma 12.2. *For any nonzero integer r, we have*

$$(1+x^{-1})^r = 1 + \cfrac{1}{r^{-1}(x-(r-1)/2) + \cfrac{1}{d_2(x+1/2) + \cfrac{1}{d_3(x+1/2) + \ddots}}}$$

where

$$\begin{aligned} d_{2k+1} &= \frac{(4k+1)[\![r-2k+1 \bullet r+2k+1]\!]}{[\![r-2k \bullet r+2k+2]\!]}, \quad 1 \leq k \leq (|r|-1)/2, \\ d_{2k} &= \frac{4(4k-1)[\![r-2k+2 \bullet r+2k]\!]}{[\![r-2k+1 \bullet r+2k+1]\!]}, \quad 1 \leq k \leq |r|/2. \end{aligned}$$

Lemma 2.6 directly follows from Lemma 12.2. If p is an odd prime number with $p > 2|r|$, then the prime divisors of the denominators of $d_2, \ldots, d_r$ obtained in Lemma 12.2 are all less than p, thus they are not divisible by p. In this case, the continued fraction expansion of $(1+x^{-1})^r$ in $\mathbb{F}_p((x^{-1}))$ follows from its continued fraction expansion in $\mathbb{Q}((x^{-1}))$ by simply taking all the partial quotients modulo p. This does not remain the case when $|r|$ exceeds $2p$. To see this, take for instance $p = 3$ and $r = 11$ (see Subsection 13.1 below). Then, we get

$$d_3 = 5 \cdot \frac{10 \cdot 12}{9 \cdot 11 \cdot 13},$$

which is not an element of $\mathbb{Z}_3$.

13. Two examples

We keep the notation of Lemma 2.7 and write

$$\xi = (1+x^{-1})^{j/d} = [1; a_1, a_2, \ldots], \quad \frac{p_k}{q_k} := [1; a_1, a_2, \ldots, a_k], \quad k \geq 1.$$

13.1. **Continued fraction expansion of $(1+x^{-1})^{1/5}$ in $\mathbb{F}_3((x^{-1}))$.** We explain how Lemmata 2.5 to 2.7 allow us to compute by hand the continued fraction expansion of $(1+x^{-1})^{1/5}$ in $\mathbb{F}_3((x^{-1}))$. Since

$$\frac{1}{5} = \frac{16}{80} = (-1+3+9-27)(1+81+81^2+\dots)$$

in $\mathbb{F}_3$, we have

$$d_{4h} = d_{4h+3} = -1, \quad d_{4h+1} = d_{4h+2} = 1, \quad h \geq 0,$$

and the sequence $(m_k)_{k\geq 0}$ starts with

$$-1, 2, 11, -16, -97, 146, 875, \dots$$

By Lemma 2.5, the convergents to $\xi := (1+x^{-1})^{1/5}$ of the form $(1+x^{-1})^m$ are then

$$\frac{x}{x+1}, \Big(\frac{x+1}{x}\Big)^2, \Big(\frac{x+1}{x}\Big)^{11}, \Big(\frac{x}{x+1}\Big)^{16}, \Big(\frac{x}{x+1}\Big)^{97} \dots$$

For $k \geq 0$, let ℓ_k denote the length of the continued fraction expansion of $(1+x^{-1})^{m_k}$. This means that $p_{\ell_k}/q_{\ell_k} = (1+x^{-1})^{m_k}$. We apply Lemma 2.7 to compute ℓ_k. We get $\ell_0 = 1, \ell_1 = 2, \ell_2 = 3 \cdot 2 + 1, \dots$ and an easy induction shows that

$$\ell_{2h-2} = 3^h - 2, \quad \ell_{2h-1} = 3^h - 1, \quad h \geq 1.$$

Consequently, we have

$$\frac{p_{3^{2h+1}-2}}{q_{3^{2h+1}-2}} = \Big(\frac{x}{x+1}\Big)^{|m_{4h}|}, \quad \frac{p_{3^{2h+1}-1}}{q_{3^{2h+1}-1}} = \Big(\frac{x+1}{x}\Big)^{|m_{4h+1}|}, \quad h \geq 0,$$

and

$$\frac{p_{3^{2h+2}-2}}{q_{3^{2h+2}-2}} = \Big(\frac{x+1}{x}\Big)^{|m_{4h+2}|}, \quad \frac{p_{3^{2h+2}-1}}{q_{3^{2h+2}-1}} = \Big(\frac{x}{x+1}\Big)^{|m_{4h+3}|}, \quad h \geq 0.$$

By Lemma 2.6 and Lemma 2.7 applied with $\mathbf{w}$ being the empty (alternatively, this can be easily done by hand), we get

$$\frac{p_1}{q_1} = \frac{x}{x+1} = [1; -x-1], \quad \frac{p_2}{q_2} = \Big(\frac{x+1}{x}\Big)^2 = [1; -x-1, -x+1].$$

Also, observe that

$$\eta_{-1}(0) = -1, \quad \delta(0) = 1.$$

Furthermore, m_{2h} is odd and m_{2h+1} is even for $h \geq 0$. We deduce from Lemma 2.7 that there are only two cases. Let $h \geq 1$ be an integer.

- If $(1+x^{-1})^{m_{2h}} = [1; -x-1, \mathbf{w}]$, then

$$(1+x^{-1})^{m_{2h+1}} = [1; -x-1, \mathbf{w}, c_{2h+1}].$$

- If $(1+x^{-1})^{m_{2h+1}} = [1; -x-1, \mathbf{w}]$, then $(1+x^{-1})^{m_{2h+2}} = [1; -x-1, \mathbf{w}, c_{2h+2}, -\mathbf{h}]$, where $\mathbf{h} = (-\overleftarrow{\mathbf{w}}, x-1, -x+1, \mathbf{w})$, thus

$$(1+x^{-1})^{m_{2h+2}} = [1; -x-1, \mathbf{w}, c_{2h+2}, \overleftarrow{\mathbf{w}}, -x+1, x-1, -\mathbf{w}].$$

Let us now compute the polynomials c_k. Let $h \ge 0$ be an integer. First, note that

$$|m_{2h}| + |m_{2h+1}| = 3^{2h+1}.$$

Since

$$x^{3^{2h+1}}(x-1)^{3^{2h+1}} = (x+1)^{2\cdot 3^{2h+1}} - 1,$$

we get

$$\begin{aligned}(x+1)^{3^{2h+2}-|m_{2h+1}|} &= (x+1)^{|m_{2h}|}(x+1)^{3^{2h+2}-|m_{2h+1}|-|m_{2h}|}\\ &= (x+1)^{|m_{2h}|}(x+1)^{2\cdot 3^{2h+1}}\\ &= (x+1)^{|m_{2h}|}\big(x^{3^{2h+1}}(x-1)^{3^{2h+1}}+1\big).\end{aligned}$$

Since $|m_{2h+1}| > |m_{2h}|$, this shows that

$$\begin{aligned}\mathrm{Polpart}\Big(\frac{(x+1)^{3^{2h+2}-|m_{2h+1}|}}{x^{|m_{2h+1}|}}\Big) &= (x+1)^{|m_{2h}|}x^{3^{2h+1}-|m_{2h+1}|}(x-1)^{3^{2h+1}}\\ &= (x+1)^{|m_{2h}|}x^{|m_{2h}|}(x-1)^{3^{2h+1}},\end{aligned}$$

where Polpart means the polynomial part. Likewise, by using that

$$|m_{2h+2}| + |m_{2h}| = 4\cdot 3^{2h+1},$$

we obtain

$$\begin{aligned}(x+1)^{3^{2h+3}-|m_{2h+2}|} &= (x+1)^{|m_{2h}|}(x+1)^{3^{2h+3}-|m_{2h+2}|-|m_{2h}|}\\ &= (x+1)^{|m_{2h}|}(x+1)^{5\cdot 3^{2h+1}}\\ &= (x+1)^{|m_{2h}|}\big((x+1)^3x(x-1)+x^3+1\big)^{3^{2h+1}}\\ &= (x+1)^{|m_{2h}|}\big(x^{4\cdot 3^{2h+1}}(x-1)^{3^{2h+1}}+x^{3^{2h+1}}(x-1)^{3^{2h+1}}+x^{3\cdot 3^{2h+1}}+1\big),\end{aligned}$$

thus, since $3\cdot 3^{2h+1} + |m_{2h}| < 3\cdot 3^{2h+1} + |m_{2h+1}| = |m_{2h+2}|$,

$$\begin{aligned}\mathrm{Polpart}\Big(\frac{(x+1)^{3^{2h+3}-|m_{2h+2}|}}{x^{|m_{2h+2}|}}\Big) &= (x+1)^{|m_{2h}|}x^{4\cdot 3^{2h+1}-|m_{2h+2}|}(x-1)^{3^{2h+1}}\\ &= (x+1)^{|m_{2h}|}x^{|m_{2h}|}(x-1)^{3^{2h+1}}.\end{aligned}$$

Since

$$\mathrm{sgn}\big((-1)^{m_{4h}}d_{4h+1}\big) = \mathrm{sgn}\big((-1)^{m_{4h+3}}d_{4h+4}\big) = -1, \quad h \ge 0,$$

and

$$\mathrm{sgn}\big((-1)^{m_{4h+1}}d_{4h+2}\big) = \mathrm{sgn}\big((-1)^{m_{4h+2}}d_{4h+3}\big) = 1, \quad h \ge 0,$$

we have proved that

$$c_{2h+2} = c_{2h+3} = (x+1)^{|m_{2h}|}x^{|m_{2h}|}(x-1)^{3^{2h+1}}, \quad \text{for } h \ge 0 \text{ even},$$

and

$$c_{2h+2} = c_{2h+3} = -(x+1)^{|m_{2h}|}x^{|m_{2h}|}(x-1)^{3^{2h+1}}, \quad \text{for } h \ge 1 \text{ odd}.$$

Let us apply Lemma 2.7 to derive the continued fraction expansion of $(1+x^{-1})^{11}$ from that of $(1+x^{-1})^2$. Since $c_2 = x(x+1)(x-1)^3$ and $\mathbf{h} = (x-1, x-1, -x+1, -x+1)$, we get

$$\begin{aligned}\frac{p_7}{q_7} = \Big(\frac{x+1}{x}\Big)^{11} &= [1; -x-1, -x+1, c_2, -\mathbf{h}]\\ &= [1; -x-1, -x+1, x(x+1)(x-1)^3, -x+1, -x+1, x-1, x-1]\\ &= [1; -v+1, -v, c_2, -v, -v, v, v],\end{aligned}$$

where v denotes the polynomial $x-1$ (we use this to shorten the notation). Let us apply Lemma 2.7 to derive the continued fraction expansion of $(1+x^{-1})^{-16}$ from that of $(1+x^{-1})^{11}$. There is only one partial quotient to add, which is the polynomial part of $(1+x)^{16}/x^{11}$, that is, $c_2 = c_3 = x(x+1)(x-1)^3$. Consequently, we obtain

$$\frac{p_8}{q_8} = \Big(\frac{x}{x+1}\Big)^{16} = [1; -v+1, -v, c_2, -v, -v, v, v, c_2].$$

Then, we find

$$\begin{aligned}\frac{p_{25}}{q_{25}} = \Big(\frac{x}{x+1}\Big)^{97} &= [1; -x-1, -v, c_2, -v, -v, v, v, c_2, -c_4, -\mathbf{h}]\\ &= [1; -v+1, -v, c_2, -v, -v, v, v, c_2, -c_4, c_2, v, v, \ldots, -v, -c_2],\end{aligned}$$

with $c_4 = x^{11}(x+1)^{11}(x-1)^{27}$, and

$$\frac{p_{26}}{q_{26}} = \Big(\frac{x+1}{x}\Big)^{146} = [1; -v+1, -v, c_2, -v, -v, v, v, c_2, -c_4, c_2, v, v, \ldots, -v, -c_2, -c_4].$$

Furthermore,

$$\frac{p_{79}}{q_{79}} = \Big(\frac{x+1}{x}\Big)^{875} = [1; -x-1, -x+1, \ldots, -c_2, -c_4, c_6, -c_4, -c_2, \ldots, c_2, c_4],$$

with $c_6 = x^{97}(x+1)^{97}(x-1)^{243}$.

The partial quotients of degree greater than one are

$$\pm g_h, \text{ with } g_h := (x^2+x)^{|m_{2h}|}(x-1)^{3^{2h+1}}, \quad h \geq 0.$$

We show by induction on h that

$$(1+x^{-1})^{m_{4h+2}} = [1; -x-1, \ldots, g_{2h+1}, \ldots, g_0, g_1, g_2, \ldots, g_{2h}],$$

$$(1+x^{-1})^{m_{4h+3}} = [1; -x-1, \ldots, g_{2h+1}, \ldots, g_0, g_1, g_2, \ldots, g_{2h}, g_{2h+1}],$$

$$\begin{aligned}(1+x^{-1})^{m_{4h+4}} = [1; -x-1, \ldots, g_{2h+1}, \ldots, g_0, g_1, g_2, \ldots, g_{2h}, g_{2h+1}, -g_{2h+2}, g_{2h+1}, \ldots,\\ g_2, g_1, g_0, \ldots, -g_0, -g_1, -g_2, \ldots, -g_{2h+1}],\end{aligned}$$

$$(1+x^{-1})^{m_{4h+5}} = [1; -x-1, \ldots, -g_{2h+2}, \ldots, -g_0, -g_1, -g_2, \ldots, -g_{2h+1}, -g_{2h+2}],$$

and

$$\begin{aligned}(1+x^{-1})^{m_{4h+6}} = [1; -x-1, \ldots, g_{2h+1}, \ldots, -g_0, -g_1, -g_2, \ldots, -g_{2h+1}, -g_{2h+2},\\ g_{2h+3}, g_{2h+2}, \ldots, -g_0, -g_1, \ldots, g_0, g_1, g_2, \ldots, g_{2h+2}].\end{aligned}$$

Indeed, assuming that the continued fraction expansions of $(1+x^{-1})^{m_{4h+2}}$ is as above, we use our preceding observations to derive the continued fraction expansions of $(1+x^{-1})^{m_{4h+3}}, \ldots, (1+x^{-1})^{m_{4h+6}}$, and we observe that the latter one has the same form as the continued fraction expansion of $(1+x^{-1})^{m_{4h+2}}$. Furthermore, we have shown that the continued fraction expansion of $(1+x^{-1})^{m_6} = (1+x^{-1})^{875}$ has the requested form.

Since the lengths of the continued fraction expansions of $(1+x^{-1})^{m_{4h+2}}, \ldots, (1+x^{-1})^{m_{4h+5}}$ are $3^{2h+2}-2, 3^{2h+2}-1, 3^{2h+3}-2, 3^{2h+3}-1$, respectively, we derive that

$$a_{3^h} = (-1)^{h-1} g_{h-1}, \quad a_{3^h \pm i} = (-1)^h g_{h-1-i}, \quad i = 1, \ldots, h-2,$$

and

$$\deg q_{3^h-1} = |m_{2h-1}|, \quad h \geq 1.$$

This allows us to determine infinitely many elements of the approximation spectrum $\mathcal{S}(\xi)$ of ξ. Namely, an easy computation shows that

$$\frac{\deg q_{3^h}}{\deg q_{3^h-1}} = 1 + \frac{\deg g_{h-1}}{\deg q_{3^h-1}} = 1 + \frac{2|m_{2h-2}| + 3^{2h-1}}{|m_{2h-1}|} = 2 + 3\frac{|m_{2h-2}|}{|m_{2h-1}|}.$$

Likewise,

$$\frac{\deg q_{3^h-i}}{\deg q_{3^h-i-1}} = 1 + \frac{\deg g_{h-i-1}}{\deg q_{3^h-1} - \deg g_{h-2} - \ldots - \deg g_{h-i-1}}.$$

An easy induction shows that

$$|m_{2h}| = \frac{6 \cdot 3^{2h} - (-1)^h}{5}, \quad |m_{2h+1}| = \frac{3 \cdot 3^{2h+1} + (-1)^h}{5}, \quad h \geq 0.$$

Using that

$$\deg g_{h-1-i} = 2|m_{2h-2-2i}| + 3^{2h-1-2i}, \quad i = 0, 1, \ldots, h-1,$$

we get that

$$\lim_{h \to +\infty} \frac{\deg q_{3^h-i}}{\deg q_{3^h-i-1}} = 1 + \frac{8}{5 \cdot 3^{2i-1} + 1}, \quad i \geq 0,$$

and, similarly,

$$\lim_{h \to +\infty} \frac{\deg q_{3^h+i}}{\deg q_{3^h+i-1}} = 1 + \frac{8}{35 \cdot 3^{2i-1} - 9}, \quad i \geq 1.$$

This shows that

$$\mathcal{S}(\xi) \supset \Big\{1 + \frac{8}{5 \cdot 3^{2i-1} + 1} : i \geq 0\Big\} \cup \Big\{1 + \frac{8}{35 \cdot 3^{2i-1} - 9} : i \geq 1\Big\},$$

thus the set $\mathcal{S}(\xi)$ is infinite. The largest elements of these two infinite families are

$$4, \ \frac{3}{2}, \ \frac{13}{12}, \ \frac{18}{17}, \ \frac{118}{117}, \ldots$$

A further look at the continued fraction expansion of $(1+x^{-1})^{1/5}$ shows that there are other elements in the approximation spectrum, an example being given by 44/41. We leave to the interested reader the precise determination of the set $\mathcal{S}(\xi)$.

13.2. **Continued fraction expansion of** $(1+x^{-1})^{1/4}$ **in** $\mathbb{F}_7((x^{-1}))$**.** Since

$$\frac{1}{4} = \frac{-12}{-49+1} = -12(1+7^2+7^4+\ldots) = (2-2\cdot 7)(1+7^2+7^4+\ldots)$$

in $\mathbb{F}_7$, the sequence $(m_k)_{k\geq 0}$ starts with

$$2, -12, 86, -600, \ldots,$$

and we have

$$m_k = (-1)^k \, \frac{7^{k+1}+(-1)^k}{4}, \quad k \geq 0.$$

In particular, m_k and m_{k+1} have opposite signs and it follows from Lemmas 2.6 and 2.7 that the length ℓ_k of the continued fraction expansion of $(1+x^{-1})^{m_k}$ satisfies $\ell_k = 3^{k+1}-1$ for $k \geq 0$. By Lemma 2.7, all the partial quotients of $(1+x^{-1})^{1/4}$ in $\mathbb{F}_7((x^{-1}))$ are either linear or constant multiples of the polynomial part of

$$\Pi_k := \frac{(1+x)^{7^k-|m_{k-1}|}}{x^{|m_{k-1}|}},$$

for some $k \geq 1$. Setting $m_{-1}=0$, we claim that

$$\mathrm{Polpart}(\Pi_k) = x^{|m_{k-2}|}(1+x)^{|m_{k-2}|}(x+4)^{7^{k-1}}(x^2+x+6)^{7^{k-1}} =: P_k, \quad k \geq 1,$$

where, as above, Polpart means the polynomial part. This is true for $k=1$ since

$$P_1 = \mathrm{Polpart}\Big(\frac{(1+x)^5}{x^2}\Big) = x^3+5x^2+10x+10 = (x+4)(x^2+x+6).$$

Let $k \geq 3$ be an odd integer. Then, $m_{k-1} = m_{k-2} + 2\cdot 7^{k-1}$ and

$$7^k - |m_{k-1}| = 5\cdot 7^{k-1} - m_{k-2} = 5\cdot 7^{k-1} + |m_{k-2}|, \quad |m_{k-1}| = 2\cdot 7^{k-1} - |m_{k-2}|,$$

thus, since $7^{k-1} > 2|m_{k-2}|$, we get

$$\begin{aligned}\mathrm{Polpart}(\Pi_k) &= x^{|m_{k-2}|}(1+x)^{|m_{k-2}|}\mathrm{Polpart}\Big(\frac{(1+x)^{5\cdot 7^{k-1}}}{x^{2\cdot 7^{k-1}}}\Big)\\ &= x^{|m_{k-2}|}(1+x)^{|m_{k-2}|}\mathrm{Polpart}\Big(\frac{(1+x)^5}{x^2}\Big)^{7^{k-1}},\end{aligned}$$

as claimed. The case k even is analogous and we omit it. By Lemma 2.7, the first partial quotient of $(1+x^{-1})^{1/4}$ in $\mathbb{F}_7((x^{-1}))$ which is a constant multiple of P_k is a_{3^k}.

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Appendix A. Full proof of Theorem 11.3

We provide a full proof of Theorem 11.3. It contains several steps. For the idea of the proof, see the sketch of proof below the statement of Theorem 11.3.

A.1. **Rewriting $\zeta_k(2t)$.** The definition of $\zeta_\varepsilon(m)$ is given in Definition 9.7. We rewrite it according to the parity of ε.

Lemma A.1. *For a positive integer k and $t = 0, 1, \dots, k-1$, we have*

$$\zeta_k(2t) = -2^{2t-3-4\lfloor (t-1)/2 \rfloor} \frac{[\![k-t+1 \bullet k+t+1]\!]}{[\![k-t \bullet k+t]\!]}$$

and

$$\zeta_{-k}(2t) = -2^{2t-3-4\lfloor (t-1)/2 \rfloor} \frac{[\![k-t+1 \bullet k+t+1]\!]}{[\![k-t+2 \bullet k+t+2]\!]}.$$

Lemma A.2. *For a positive integer k and $t = 0, 1, \dots, k-1$, we have*

$$\begin{aligned}
\zeta_{2k}(4t) &= -\frac{2^{-4t+1}\,(2k+2t)!\,(k-t)!\,(k-t-1)!}{(2k-2t)!\,(k+t)!\,(k+t-1)!},\\
\zeta_{-2k}(4t) &= -\frac{2^{-4t+1}\,(2k+2t)!\,(k-t)!^2}{(2k-2t)!\,(k+t)!^2},\\
\zeta_{2k}(4t+2) &= -\frac{2^{4t+1}\,(2k-2t-2)!\,(k+t)!^2}{(2k+2t)!\,(k-t-1)!^2},\\
\zeta_{-2k}(4t+2) &= -\frac{2^{4t+1}\,(2k-2t)!\,(k+t+1)!\,(k+t)!}{(2k+2t+2)!\,(k-t)!\,(k-t-1)!}.
\end{aligned}$$

For a non-negative integer k and $t = 0, 1, \dots, k-1$, we have

$$\begin{aligned}
\zeta_{2k+1}(4t) &= -\frac{2^{4t+1}\,(2k-2t)!\,(k+t)!^2}{(2k+2t)!\,(k-t)!^2},\\
\zeta_{-2k-1}(4t) &= -\frac{2^{4t+1}\,(2k-2t+2)!\,(k+t+1)!\,(k+t)!}{(2k+2t+2)!\,(k-t+1)!\,(k-t)!},\\
\zeta_{2k+1}(4t+2) &= -\frac{2^{-4t-3}\,(2k+2t+2)!\,(k-t)!\,(k-t-1)!}{(2k-2t)!\,(k+t+1)!\,(k+t)!},\\
\zeta_{-2k-1}(4t+2) &= -\frac{2^{-4t-3}\,(2k+2t+2)!\,(k-t)!^2}{(2k-2t)!\,(k+t+1)!^2}.
\end{aligned}$$

A.2. **Rewriting $K_i(k, j)$.** We rewrite $K_1(k,j), K_2(k,j), K_3(k,j), K_4(k,j), K_5(k,j)$ given in Definition 11.2 according to the parities of k and j.

Lemma A.3. *We have*

$$\begin{aligned}
K_1(1,0) &= 1,\\
K_1(j+1,j) &= 0, \quad j \ge 1,\\
K_2(j+1,j) &= 0, \quad j \ge 0,
\end{aligned}$$

and

$$
\begin{aligned}
K_1(2k,2j) &= \frac{(-j+k-1)!\,(2\,k-2)!}{2^{2\,j}\,(j+k-1)!\,(-2\,j+2\,k-2)!},\\
K_1(2k,2j+1) &= \frac{2^{2\,j+1}\,(j+k)!\,(2\,k-2)!}{(2\,j+2\,k)!\,(-j+k-2)!},\\
K_1(2k+1,2j) &= \frac{2^{2\,j+1}\,(j+k)!\,(2\,k-1)!}{(2\,j+2\,k)!\,(-j+k-1)!},\\
K_1(2k+1,2j+1) &= \frac{2^{-2\,j-2}\,(-j+k-1)!\,(2\,k-1)!}{(j+k)!\,(-2\,j+2\,k-2)!},\\
K_2(2k,2j) &= -\frac{2^{-2\,j-1}\,(-j+k-1)!\,(2\,k-1)!}{(j+k)!\,(-2\,j+2\,k-2)!},\\
K_2(2k,2j+1) &= -\frac{2^{2\,j+2}\,(j+k+1)!\,(2\,k-1)!}{(2\,j+2\,k+2)!\,(-j+k-2)!},\\
K_2(2k+1,2j) &= -\frac{2^{2\,j+2}\,(j+k+1)!\,(2\,k)!}{(2\,j+2\,k+2)!\,(-j+k-1)!},\\
K_2(2k+1,2j+1) &= -\frac{2^{-2\,j-3}\,(-j+k-1)!\,(2\,k)!}{(j+k+1)!\,(-2\,j+2\,k-2)!},\\
K_3(2k,2j) &= \frac{2^{2\,j+1}\,(j+k)!\,(2\,k-1)!}{(2\,j+2\,k)!\,(-j+k-1)!},\\
K_3(2k,2j+1) &= \frac{(-j+k-1)!\,(2\,k-1)!}{2^{2\,j}\,(j+k)!\,(-2\,j+2\,k-2)!},\\
K_3(2k+1,2j) &= \frac{(-j+k)!\,(2\,k)!}{2^{2\,j}\,(j+k)!\,(-2\,j+2\,k)!},\\
K_3(2k+1,2j+1) &= \frac{2^{2\,j+3}\,(j+k+1)!\,(2\,k)!}{(2\,j+2\,k+2)!\,(-j+k-1)!},\\
K_4(2k,2j) &= \frac{(-j+k)!\,(2\,k)!}{2^{2\,j}\,(j+k)!\,(-2\,j+2\,k)!},\\
K_4(2k,2j+1) &= -\frac{2^{2\,j+1}\,(j+k+1)!\,(2\,k)!}{(2\,j+2\,k+2)!\,(-j+k-1)!},\\
K_4(2k+1,2j) &= \frac{2^{2\,j+1}\,(j+k+1)!\,(2\,k+1)!}{(2\,j+2\,k+2)!\,(-j+k)!},\\
K_4(2k+1,2j+1) &= -\frac{2^{-2\,j-2}\,(-j+k)!\,(2\,k+1)!}{(j+k+1)!\,(-2\,j+2\,k)!},\\
K_5(2k,2j) &= -\frac{(2\,j+2\,k+1)\,(-j+k)!\,(2\,k-1)!}{2^{2\,j}\,(j+k)!\,(-2\,j+2\,k)!},\\
K_5(2k,2j+1) &= \frac{2^{2\,j+1}\,(j+k+1)!\,(2\,k-1)!}{(2\,j+2\,k+1)!\,(-j+k-1)!},
\end{aligned}
$$

$$K_5(2k+1,2j) = -\frac{2^{2\,j+1}\,(j+k+1)!\,(2\,k)!}{(2\,j+2\,k+1)!\,(-j+k)!},$$
$$K_5(2k+1,2j+1) = \frac{2^{-2\,j-2}(2\,j+2\,k+3)\,(-j+k)!\,(2\,k)!}{(j+k+1)!\,(-2\,j+2\,k)!}.$$

A.3. **Rewriting $\mathbf{D}_\varepsilon(t)$.** The definition of $\mathbf{D}_\varepsilon(t)$ is given just above Theorem 9.9. Since $\mathbf{D}_\varepsilon(t)$ involves $\zeta_\varepsilon(2t)$, we need to rewrite $\mathbf{D}_\varepsilon(t)$ according to the parity of ε. Notice that all the entries of $\mathbf{D}_\varepsilon(t)$ are polynomials in Y of degree at most 2.

Below, we let $D_\varepsilon(t)[i,j]$ denote the $i \times j$ entry of the matrix $\mathbf{D}_\varepsilon(t)$. We assume implicitly that ε is nonzero and $t = 0, \ldots, |\varepsilon| - 1$.

Lemma A.4. *We have*

$$D_{2k}(2t)[1,1] = \frac{2^{-4\,t+1}\left(\frac{Y(2\,k-2\,t-1)}{k+t} - r - q\right)(2\,k+2\,t)!\,(k-t)!\,(k-t-1)!}{(2\,k-2\,t)!\,(k+t)!\,(k+t-1)!},$$
$$D_{2k}(2t)[1,2] = -4,$$
$$D_{2k}(2t)[2,1] = \frac{Y^2(2\,k-2\,t-1)}{2\,(k+t)} - \frac{Y(4\,k-1)(r+q)}{4\,(k+t)} + rq,$$
$$D_{2k}(2t)[2,2] = -\frac{(2\,Y-r-q)\,(2\,k-2\,t)!\,(k+t)!\,(k+t-1)!}{2^{-4\,t+1}\,(2\,k+2\,t)!\,(k-t)!\,(k-t-1)!},$$
$$D_{-2k}(2t)[1,1] = \frac{2^{-4\,t+1}\left(\frac{Y(2\,k+2\,t+1)}{k-t} - r - q\right)(2\,k+2\,t)!\,(k-t)!^2}{(2\,k-2\,t)!\,(k+t)!^2},$$
$$D_{-2k}(2t)[1,2] = -4,$$
$$D_{-2k}(2t)[2,1] = \frac{Y^2(2\,k+2\,t+1)}{2\,(k-t)} - \frac{Y(4\,k+1)(r+q)}{4\,(k-t)} + rq,$$
$$D_{-2k}(2t)[2,2] = -\frac{(2\,Y-r-q)\,(2\,k-2\,t)!\,(k+t)!^2}{2^{-4\,t+1}\,(2\,k+2\,t)!\,(k-t)!^2},$$
$$D_{2k+1}(2t)[1,1] = \frac{2^{4\,t+1}\left(\frac{4\,Y(k-t)}{2\,k+2\,t+1} - r - q\right)(2\,k-2\,t)!\,(k+t)!^2}{(2\,k+2\,t)!\,(k-t)!^2},$$
$$D_{2k+1}(2t)[1,2] = -4,$$
$$D_{2k+1}(2t)[2,1] = \frac{2\,Y^2(k-t)}{2\,k+2\,t+1} - \frac{Y(4\,k+1)(r+q)}{2\,(2\,k+2\,t+1)} + rq,$$
$$D_{2k+1}(2t)[2,2] = -\frac{(2\,Y-r-q)\,(2\,k+2\,t)!\,(k-t)!^2}{2^{4\,t+1}\,(2\,k-2\,t)!\,(k+t)!^2},$$
$$D_{-2k-1}(2t)[1,1] = \frac{2^{4\,t+1}\left(\frac{4\,Y(k+t+1)}{2\,k-2\,t+1} - r - q\right)(2\,k-2\,t+2)!\,(k+t+1)!\,(k+t)!}{(2\,k+2\,t+2)!\,(k-t+1)!\,(k-t)!},$$
$$D_{-2k-1}(2t)[1,2] = -4,$$

$$D_{-2k-1}(2t)[2,1] = \frac{2\,Y^2(k+t+1)}{2\,k-2\,t+1} - \frac{Y(4\,k+3)(r+q)}{2\,(2\,k-2\,t+1)} + rq,$$

$$D_{-2k-1}(2t)[2,2] = -\frac{(2\,Y-r-q)\,(2\,k+2\,t+2)!\,(k-t+1)!\,(k-t)!}{2^{4\,t+1}\,(2\,k-2\,t+2)!\,(k+t+1)!\,(k+t)!},$$

$$D_{2k}(2t+1)[1,1] = \frac{2^{4\,t+1}\left(\frac{4\,Y(k-t-1)}{2\,k+2\,t+1} - r - q\right)(2\,k-2\,t-2)!\,(k+t)!^2}{(2\,k+2\,t)!\,(k-t-1)!^2},$$

$$D_{2k}(2t+1)[1,2] = -4,$$

$$D_{2k}(2t+1)[2,1] = \frac{2\,Y^2(k-t-1)}{2\,k+2\,t+1} - \frac{Y(4\,k-1)(r+q)}{2\,(2\,k+2\,t+1)} + rq,$$

$$D_{2k}(2t+1)[2,2] = -\frac{(2\,Y-r-q)\,(2\,k+2\,t)!\,(k-t-1)!^2}{2^{4\,t+1}\,(2\,k-2\,t-2)!\,(k+t)!^2},$$

$$D_{-2k}(2t+1)[1,1] = \frac{2^{4\,t+1}\left(\frac{4\,Y(k+t+1)}{2\,k-2\,t-1} - r - q\right)(2\,k-2\,t)!\,(k+t+1)!\,(k+t)!}{(2\,k+2\,t+2)!\,(k-t)!\,(k-t-1)!},$$

$$D_{-2k}(2t+1)[1,2] = -4,$$

$$D_{-2k}(2t+1)[2,1] = \frac{2\,Y^2(k+t+1)}{2\,k-2\,t-1} - \frac{Y(4\,k+1)(r+q)}{2\,(2\,k-2\,t-1)} + rq,$$

$$D_{-2k}(2t+1)[2,2] = -\frac{(2\,Y-r-q)\,(2\,k+2\,t+2)!\,(k-t)!\,(k-t-1)!}{2^{4\,t+1}\,(2\,k-2\,t)!\,(k+t+1)!\,(k+t)!},$$

$$D_{2k+1}(2t+1)[1,1] = \frac{2^{-4\,t-3}\left(\frac{Y(2\,k-2\,t-1)}{k+t+1} - r - q\right)(2\,k+2\,t+2)!\,(k-t)!\,(k-t-1)!}{(2\,k-2\,t)!\,(k+t+1)!\,(k+t)!},$$

$$D_{2k+1}(2t+1)[1,2] = -4,$$

$$D_{2k+1}(2t+1)[2,1] = \frac{Y^2(2\,k-2\,t-1)}{2\,(k+t+1)} - \frac{Y(4\,k+1)(r+q)}{4\,(k+t+1)} + rq,$$

$$D_{2k+1}(2t+1)[2,2] = -\frac{(2\,Y-r-q)\,(2\,k-2\,t)!\,(k+t+1)!\,(k+t)!}{2^{-4\,t-3}\,(2\,k+2\,t+2)!\,(k-t)!\,(k-t-1)!},$$

$$D_{-2k-1}(2t+1)[1,1] = \frac{2^{-4\,t-3}\left(\frac{Y(2\,k+2\,t+3)}{k-t} - r - q\right)(2\,k+2\,t+2)!\,(k-t)!^2}{(2\,k-2\,t)!\,(k+t+1)!^2},$$

$$D_{-2k-1}(2t+1)[1,2] = -4,$$

$$D_{-2k-1}(2t+1)[2,1] = \frac{Y^2(2\,k+2\,t+3)}{2\,(k-t)} - \frac{Y(4\,k+3)(r+q)}{4\,(k-t)} + rq,$$

$$D_{-2k-1}(2t+1)[2,2] = -\frac{(2\,Y-r-q)\,(2\,k-2\,t)!\,(k+t+1)!^2}{2^{-4\,t-3}\,(2\,k+2\,t+2)!\,(k-t)!^2}.$$

A.4. **Rewriting $A_\varepsilon(j), A'_\varepsilon(j), B_\varepsilon(j), B'_\varepsilon(j)$.** We rewrite $A_\varepsilon(j), A'_\varepsilon(j), B_\varepsilon(j), B'_\varepsilon(j)$ given in Theorem 11.3 according to the sign of ε and the parities of ε and j. Below, we have replaced $\Psi(k,j;r,q)$ by the product $K_1(k,j)\Phi(k-2,j;r,q)$.

$$
\begin{aligned}
A_{2k}(2j) &= YK_2\,(2\,k,2\,j)\,\Phi\,(2\,k-1,2\,j;r,q)\\
&\quad + rK_1\,(2\,k,2\,j)\,\Phi\,(2\,k-2,2\,j;r,q)\,,\\
A_{2k}(2j+1) &= YK_2\,(2\,k,2\,j+1)\,\Phi\,(2\,k-1,2\,j+1;r,q)\\
&\quad + rK_1\,(2\,k,2\,j+1)\,\Phi\,(2\,k-2,2\,j+1;r,q)\,,\\
A_{-2k}(2j) &= rK_4\,(2\,k,2\,j)\,\Phi\,(2\,k,2\,j;q,r)\\
&\quad + YK_5\,(2\,k,2\,j)\,\Phi\,(2\,k-1,2\,j;q,r)\,,\\
A_{-2k}(2j+1) &= rK_4\,(2\,k,2\,j+1)\,\Phi\,(2\,k,2\,j+1;q,r)\\
&\quad + YK_5\,(2\,k,2\,j+1)\,\Phi\,(2\,k-1,2\,j+1;q,r)\,,\\
A_{2k+1}(2j) &= YK_2\,(2\,k+1,2\,j)\,\Phi\,(2\,k,2\,j;r,q)\\
&\quad + rK_1\,(2\,k+1,2\,j)\,\Phi\,(2\,k-1,2\,j;r,q)\,,\\
A_{2k+1}(2j+1) &= YK_2\,(2\,k+1,2\,j+1)\,\Phi\,(2\,k,2\,j+1;r,q)\\
&\quad + rK_1\,(2\,k+1,2\,j+1)\,\Phi\,(2\,k-1,2\,j+1;r,q)\,,\\
A_{-2k-1}(2j) &= YK_5\,(2\,k+1,2\,j)\,\Phi\,(2\,k,2\,j;q,r)\\
&\quad + rK_4\,(2\,k+1,2\,j)\,\Phi\,(2\,k+1,2\,j;q,r)\,,\\
A_{-2k-1}(2j+1) &= YK_5\,(2\,k+1,2\,j+1)\,\Phi\,(2\,k,2\,j+1;q,r)\\
&\quad + rK_4\,(2\,k+1,2\,j+1)\,\Phi\,(2\,k+1,2\,j+1;q,r)\,,\\
A'_{2k}(2j) &= K_3\,(2\,k,2\,j)\,\Phi\,(2\,k-1,2\,j;r,q)\,,\\
A'_{2k}(2j+1) &= K_3\,(2\,k,2\,j+1)\,\Phi\,(2\,k-1,2\,j+1;r,q)\,,\\
A'_{-2k}(2j) &= K_3\,(2\,k,2\,j)\,\Phi\,(2\,k-1,2\,j;q,r)\,,\\
A'_{-2k}(2j+1) &= -K_3\,(2\,k,2\,j+1)\,\Phi\,(2\,k-1,2\,j+1;q,r)\,,\\
A'_{2k+1}(2j) &= K_3\,(2\,k+1,2\,j)\,\Phi\,(2\,k,2\,j;r,q)\,,\\
A'_{2k+1}(2j+1) &= K_3\,(2\,k+1,2\,j+1)\,\Phi\,(2\,k,2\,j+1;r,q)\,,\\
A'_{-2k-1}(2j) &= K_3\,(2\,k+1,2\,j)\,\Phi\,(2\,k,2\,j;q,r)\,,\\
A'_{-2k-1}(2j+1) &= -K_3\,(2\,k+1,2\,j+1)\,\Phi\,(2\,k,2\,j+1;q,r)\,,\\
B_{2k}(2j) &= YK_2\,(2\,k,2\,j)\,\Phi\,(2\,k-1,2\,j;q,r)\\
&\quad + qK_1\,(2\,k,2\,j)\,\Phi\,(2\,k-2,2\,j;q,r)\,,\\
B_{2k}(2j+1) &= YK_2\,(2\,k,2\,j+1)\,\Phi\,(2\,k-1,2\,j+1;q,r)\\
&\quad + qK_1\,(2\,k,2\,j+1)\,\Phi\,(2\,k-2,2\,j+1;q,r)\,,\\
B_{-2k}(2j) &= qK_4\,(2\,k,2\,j)\,\Phi\,(2\,k,2\,j;r,q)\\
&\quad + YK_5\,(2\,k,2\,j)\,\Phi\,(2\,k-1,2\,j;r,q)\,,\\
B_{-2k}(2j+1) &= qK_4\,(2\,k,2\,j+1)\,\Phi\,(2\,k,2\,j+1;r,q)\\
&\quad + YK_5\,(2\,k,2\,j+1)\,\Phi\,(2\,k-1,2\,j+1;r,q)\,,\\
B_{2k+1}(2j) &= YK_2\,(2\,k+1,2\,j)\,\Phi\,(2\,k,2\,j;q,r)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + qK_1\left(2\,k+1,2\,j\right)\Phi\left(2\,k-1,2\,j;q,r\right),\\
B_{2k+1}(2j+1) &= YK_2\left(2\,k+1,2\,j+1\right)\Phi\left(2\,k,2\,j+1;q,r\right)\\
&\quad + qK_1\left(2\,k+1,2\,j+1\right)\Phi\left(2\,k-1,2\,j+1;q,r\right),\\
B_{-2k-1}(2j) &= YK_5\left(2\,k+1,2\,j\right)\Phi\left(2\,k,2\,j;r,q\right)\\
&\quad + qK_4\left(2\,k+1,2\,j\right)\Phi\left(2\,k+1,2\,j;r,q\right),\\
B_{-2k-1}(2j+1) &= YK_5\left(2\,k+1,2\,j+1\right)\Phi\left(2\,k,2\,j+1;r,q\right)\\
&\quad + qK_4\left(2\,k+1,2\,j+1\right)\Phi\left(2\,k+1,2\,j+1;r,q\right),\\
B'_{2k}(2j) &= K_3\left(2\,k,2\,j\right)\Phi\left(2\,k-1,2\,j;q,r\right),\\
B'_{2k}(2j+1) &= K_3\left(2\,k,2\,j+1\right)\Phi\left(2\,k-1,2\,j+1;q,r\right),\\
B'_{-2k}(2j) &= K_3\left(2\,k,2\,j\right)\Phi\left(2\,k-1,2\,j;r,q\right),\\
B'_{-2k}(2j+1) &= -K_3\left(2\,k,2\,j+1\right)\Phi\left(2\,k-1,2\,j+1;r,q\right),\\
B'_{2k+1}(2j) &= K_3\left(2\,k+1,2\,j\right)\Phi\left(2\,k,2\,j;q,r\right),\\
B'_{2k+1}(2j+1) &= K_3\left(2\,k+1,2\,j+1\right)\Phi\left(2\,k,2\,j+1;q,r\right),\\
B'_{-2k-1}(2j) &= K_3\left(2\,k+1,2\,j\right)\Phi\left(2\,k,2\,j;r,q\right),\\
B'_{-2k-1}(2j+1) &= -K_3\left(2\,k+1,2\,j+1\right)\Phi\left(2\,k,2\,j+1;r,q\right).
\end{aligned}
$$

By replacing the $K_i(k,t)$ by their expressions given in Lemma A.3, we obtain the following identities.

Lemma A.5. *We have*

$$
\begin{aligned}
A_{2k}(2j) &= -\frac{2^{-2\,j-1}Y\Phi\left(2\,k-1,2\,j;r,q\right)\left(-j+k-1\right)!\left(2\,k-1\right)!}{\left(j+k\right)!\left(-2\,j+2\,k-2\right)!}\\
&\quad+\frac{r\Phi\left(2\,k-2,2\,j;r,q\right)\left(-j+k-1\right)!\left(2\,k-2\right)!}{2^{2\,j}\left(j+k-1\right)!\left(-2\,j+2\,k-2\right)!},\\
A_{2k}(2j+1) &= -\frac{2^{2\,j+2}Y\Phi\left(2\,k-1,2\,j+1;r,q\right)\left(j+k+1\right)!\left(2\,k-1\right)!}{\left(2\,j+2\,k+2\right)!\left(-j+k-2\right)!}\\
&\quad+\frac{2^{2\,j+1}r\Phi\left(2\,k-2,2\,j+1;r,q\right)\left(j+k\right)!\left(2\,k-2\right)!}{\left(2\,j+2\,k\right)!\left(-j+k-2\right)!},\\
A_{-2k}(2j) &= -\frac{Y\left(2\,j+2\,k+1\right)\Phi\left(2\,k-1,2\,j;q,r\right)\left(-j+k\right)!\left(2\,k-1\right)!}{2^{2\,j}\left(j+k\right)!\left(-2\,j+2\,k\right)!}\\
&\quad+\frac{r\Phi\left(2\,k,2\,j;q,r\right)\left(-j+k\right)!\left(2\,k\right)!}{2^{2\,j}\left(j+k\right)!\left(-2\,j+2\,k\right)!},\\
A_{-2k}(2j+1) &= -\frac{2^{2\,j+1}r\Phi\left(2\,k,2\,j+1;q,r\right)\left(j+k+1\right)!\left(2\,k\right)!}{\left(2\,j+2\,k+2\right)!\left(-j+k-1\right)!}\\
&\quad+\frac{2^{2\,j+1}Y\Phi\left(2\,k-1,2\,j+1;q,r\right)\left(j+k+1\right)!\left(2\,k-1\right)!}{\left(2\,j+2\,k+1\right)!\left(-j+k-1\right)!},\\
A_{2k+1}(2j) &= -\frac{2^{2\,j+2}Y\Phi\left(2\,k,2\,j;r,q\right)\left(j+k+1\right)!\left(2\,k\right)!}{\left(2\,j+2\,k+2\right)!\left(-j+k-1\right)!}
\end{aligned}
$$

$$+\frac{2^{2\,j+1}r\Phi\left(2\,k-1,2\,j;r,q\right)\left(j+k\right)!\left(2\,k-1\right)!}{\left(2\,j+2\,k\right)!\left(-j+k-1\right)!},$$

$$A_{2k+1}(2j+1)=-\frac{2^{-2\,j-3}Y\Phi\left(2\,k,2\,j+1;r,q\right)\left(-j+k-1\right)!\left(2\,k\right)!}{\left(j+k+1\right)!\left(-2\,j+2\,k-2\right)!}+\frac{2^{-2\,j-2}r\Phi\left(2\,k-1,2\,j+1;r,q\right)\left(-j+k-1\right)!\left(2\,k-1\right)!}{\left(j+k\right)!\left(-2\,j+2\,k-2\right)!},$$

$$A_{-2k-1}(2j)=-\frac{2^{2\,j+1}Y\Phi\left(2\,k,2\,j;q,r\right)\left(j+k+1\right)!\left(2\,k\right)!}{\left(2\,j+2\,k+1\right)!\left(-j+k\right)!}+\frac{2^{2\,j+1}r\Phi\left(2\,k+1,2\,j;q,r\right)\left(j+k+1\right)!\left(2\,k+1\right)!}{\left(2\,j+2\,k+2\right)!\left(-j+k\right)!},$$

$$A_{-2k-1}(2j+1)=\frac{2^{-2\,j-2}Y(2\,j+2\,k+3)\Phi\left(2\,k,2\,j+1;q,r\right)\left(-j+k\right)!\left(2\,k\right)!}{\left(j+k+1\right)!\left(-2\,j+2\,k\right)!}-\frac{2^{-2\,j-2}r\Phi\left(2\,k+1,2\,j+1;q,r\right)\left(-j+k\right)!\left(2\,k+1\right)!}{\left(j+k+1\right)!\left(-2\,j+2\,k\right)!},$$

$$A'_{2k}(2j)=\frac{2^{2\,j+1}\Phi\left(2\,k-1,2\,j;r,q\right)\left(j+k\right)!\left(2\,k-1\right)!}{\left(2\,j+2\,k\right)!\left(-j+k-1\right)!},$$

$$A'_{2k}(2j+1)=\frac{\Phi\left(2\,k-1,2\,j+1;r,q\right)\left(-j+k-1\right)!\left(2\,k-1\right)!}{2^{2\,j}\left(j+k\right)!\left(-2\,j+2\,k-2\right)!},$$

$$A'_{-2k}(2j)=\frac{2^{2\,j+1}\Phi\left(2\,k-1,2\,j;q,r\right)\left(j+k\right)!\left(2\,k-1\right)!}{\left(2\,j+2\,k\right)!\left(-j+k-1\right)!},$$

$$A'_{-2k}(2j+1)=-\frac{\Phi\left(2\,k-1,2\,j+1;q,r\right)\left(-j+k-1\right)!\left(2\,k-1\right)!}{2^{2\,j}\left(j+k\right)!\left(-2\,j+2\,k-2\right)!},$$

$$A'_{2k+1}(2j)=\frac{\Phi\left(2\,k,2\,j;r,q\right)\left(-j+k\right)!\left(2\,k\right)!}{2^{2\,j}\left(j+k\right)!\left(-2\,j+2\,k\right)!},$$

$$A'_{2k+1}(2j+1)=\frac{2^{2\,j+3}\Phi\left(2\,k,2\,j+1;r,q\right)\left(j+k+1\right)!\left(2\,k\right)!}{\left(2\,j+2\,k+2\right)!\left(-j+k-1\right)!},$$

$$A'_{-2k-1}(2j)=\frac{\Phi\left(2\,k,2\,j;q,r\right)\left(-j+k\right)!\left(2\,k\right)!}{2^{2\,j}\left(j+k\right)!\left(-2\,j+2\,k\right)!},$$

$$A'_{-2k-1}(2j+1)=-\frac{2^{2\,j+3}\Phi\left(2\,k,2\,j+1;q,r\right)\left(j+k+1\right)!\left(2\,k\right)!}{\left(2\,j+2\,k+2\right)!\left(-j+k-1\right)!},$$

$$B_{2k}(2j)=-\frac{2^{-2\,j-1}Y\Phi\left(2\,k-1,2\,j;q,r\right)\left(-j+k-1\right)!\left(2\,k-1\right)!}{\left(j+k\right)!\left(-2\,j+2\,k-2\right)!}+\frac{q\Phi\left(2\,k-2,2\,j;q,r\right)\left(-j+k-1\right)!\left(2\,k-2\right)!}{2^{2\,j}\left(j+k-1\right)!\left(-2\,j+2\,k-2\right)!},$$

$$B_{2k}(2j+1)=-\frac{2^{2\,j+2}Y\Phi\left(2\,k-1,2\,j+1;q,r\right)\left(j+k+1\right)!\left(2\,k-1\right)!}{\left(2\,j+2\,k+2\right)!\left(-j+k-2\right)!}+\frac{2^{2\,j+1}q\Phi\left(2\,k-2,2\,j+1;q,r\right)\left(j+k\right)!\left(2\,k-2\right)!}{\left(2\,j+2\,k\right)!\left(-j+k-2\right)!},$$

$$\begin{aligned}
B_{-2k}(2j) &= -\frac{Y(2\,j+2\,k+1)\Phi\,(2\,k-1,2\,j;r,q)\,(-j+k)!\,(2\,k-1)!}{2^{2\,j}\,(j+k)!\,(-2\,j+2\,k)!} \\
&\quad + \frac{q\Phi\,(2\,k,2\,j;r,q)\,(-j+k)!\,(2\,k)!}{2^{2\,j}\,(j+k)!\,(-2\,j+2\,k)!}, \\
B_{-2k}(2j+1) &= -\frac{2^{2\,j+1}q\Phi\,(2\,k,2\,j+1;r,q)\,(j+k+1)!\,(2\,k)!}{(2\,j+2\,k+2)!\,(-j+k-1)!} \\
&\quad + \frac{2^{2\,j+1}Y\Phi\,(2\,k-1,2\,j+1;r,q)\,(j+k+1)!\,(2\,k-1)!}{(2\,j+2\,k+1)!\,(-j+k-1)!}, \\
B_{2k+1}(2j) &= -\frac{2^{2\,j+2}Y\Phi\,(2\,k,2\,j;q,r)\,(j+k+1)!\,(2\,k)!}{(2\,j+2\,k+2)!\,(-j+k-1)!} \\
&\quad + \frac{2^{2\,j+1}q\Phi\,(2\,k-1,2\,j;q,r)\,(j+k)!\,(2\,k-1)!}{(2\,j+2\,k)!\,(-j+k-1)!}, \\
B_{2k+1}(2j+1) &= -\frac{2^{-2\,j-3}Y\Phi\,(2\,k,2\,j+1;q,r)\,(-j+k-1)!\,(2\,k)!}{(j+k+1)!\,(-2\,j+2\,k-2)!} \\
&\quad + \frac{2^{-2\,j-2}q\Phi\,(2\,k-1,2\,j+1;q,r)\,(-j+k-1)!\,(2\,k-1)!}{(j+k)!\,(-2\,j+2\,k-2)!}, \\
B_{-2k-1}(2j) &= -\frac{2^{2\,j+1}Y\Phi\,(2\,k,2\,j;r,q)\,(j+k+1)!\,(2\,k)!}{(2\,j+2\,k+1)!\,(-j+k)!} \\
&\quad + \frac{2^{2\,j+1}q\Phi\,(2\,k+1,2\,j;r,q)\,(j+k+1)!\,(2\,k+1)!}{(2\,j+2\,k+2)!\,(-j+k)!}, \\
B_{-2k-1}(2j+1) &= \frac{2^{-2\,j-2}Y(2\,j+2\,k+3)\Phi\,(2\,k,2\,j+1;r,q)\,(-j+k)!\,(2\,k)!}{(j+k+1)!\,(-2\,j+2\,k)!} \\
&\quad - \frac{2^{-2\,j-2}q\Phi\,(2\,k+1,2\,j+1;r,q)\,(-j+k)!\,(2\,k+1)!}{(j+k+1)!\,(-2\,j+2\,k)!}, \\
B'_{2k}(2j) &= \frac{2^{2\,j+1}\Phi\,(2\,k-1,2\,j;q,r)\,(j+k)!\,(2\,k-1)!}{(2\,j+2\,k)!\,(-j+k-1)!}, \\
B'_{2k}(2j+1) &= \frac{\Phi\,(2\,k-1,2\,j+1;q,r)\,(-j+k-1)!\,(2\,k-1)!}{2^{2\,j}\,(j+k)!\,(-2\,j+2\,k-2)!}, \\
B'_{-2k}(2j) &= \frac{2^{2\,j+1}\Phi\,(2\,k-1,2\,j;r,q)\,(j+k)!\,(2\,k-1)!}{(2\,j+2\,k)!\,(-j+k-1)!}, \\
B'_{-2k}(2j+1) &= -\frac{\Phi\,(2\,k-1,2\,j+1;r,q)\,(-j+k-1)!\,(2\,k-1)!}{2^{2\,j}\,(j+k)!\,(-2\,j+2\,k-2)!}, \\
B'_{2k+1}(2j) &= \frac{\Phi\,(2\,k,2\,j;q,r)\,(-j+k)!\,(2\,k)!}{2^{2\,j}\,(j+k)!\,(-2\,j+2\,k)!}, \\
B'_{2k+1}(2j+1) &= \frac{2^{2\,j+3}\Phi\,(2\,k,2\,j+1;q,r)\,(j+k+1)!\,(2\,k)!}{(2\,j+2\,k+2)!\,(-j+k-1)!}, \\
B'_{-2k-1}(2j) &= \frac{\Phi\,(2\,k,2\,j;r,q)\,(-j+k)!\,(2\,k)!}{2^{2\,j}\,(j+k)!\,(-2\,j+2\,k)!},
\end{aligned}$$

$$B'_{-2k-1}(2j+1) = -\frac{2^{2\,j+3}\Phi\,(2\,k, 2\,j+1; r, q)\,(j+k+1)!\,(2\,k)!}{(2\,j+2\,k+2)!\,(-j+k-1)!}.$$

A.5. **Proof of Theorem 11.3 for $j = 0$.** For a nonzero integer ε and $j = 0, 1, \ldots, |\varepsilon| - 1$, recall that

$$\Pi_\varepsilon(j) := \mathbf{U}\mathbf{D}_\varepsilon(0)\mathbf{D}_\varepsilon(1)\cdots\mathbf{D}_\varepsilon(j) = \begin{pmatrix} A_\varepsilon(j) & A'_\varepsilon(j) \\ B_\varepsilon(j) & B'_\varepsilon(j) \end{pmatrix}.$$

Idea of the proof: We check by a direct computation that the theorem holds for $j = 0$. Then, we verify that $\Pi_\varepsilon(j+1) = \Pi_\varepsilon(j)\mathbf{D}_\varepsilon(j+1)$ for $j = 0, \ldots, |\varepsilon| - 2$.

In this section, we check by a direct computation that the theorem holds for $j = 0$. We have:

$$\mathbf{U} = \frac{1}{r-q}\begin{pmatrix} (Y-r)/2 & -2 \\ -(Y-q)/2 & 2 \end{pmatrix},$$
$$K_1(k,0) = 1,$$
$$K_2(k,0) = -\frac{k-1}{k},$$
$$K_3(k,0) = 1,$$
$$K_4(k,0) = 1,$$
$$K_5(k,0) = -\frac{k+1}{k},$$

and

$$\Phi(k,0;r,q) = 1.$$

This gives:

- $\varepsilon = k > 0$:

$$A_k(0) = r + K_2(k,0)Y = r - \frac{k-1}{k}Y,$$
$$A'_k(0) = 1,$$
$$B_k(0) = q + K_2(k,0)Y = q - \frac{k-1}{k}Y,$$
$$B'_k(0) = 1.$$

- $\varepsilon = -k < 0$:

$$A_{-k}(0) = r + K_5(k,0)Y = r - \frac{k+1}{k}Y,$$
$$A'_{-k}(0) = 1,$$
$$B_{-k}(0) = q + K_5(k,0)Y = q - \frac{k+1}{k}Y,$$
$$B'_{-k}(0) = 1.$$

In both cases, we have

$$\zeta_\varepsilon(0) = -2.$$

We distinguish four cases to check that $\mathbf{U}\times\mathbf{D}_\varepsilon(0)=\Pi_\varepsilon(0)$:

• $\varepsilon=2k$:

$$\mathbf{U}\times\begin{pmatrix}\frac{2Y(2k-1)}{k}-2q-2r & -4\\ \frac{Y^2(2k-1)}{2k}-\frac{Y(4k-1)(q+r)}{4k}+qr & -Y+\frac{1}{2}q+\frac{1}{2}r\end{pmatrix}=\begin{pmatrix}-\frac{2Yk-2kr-Y}{2k} & 1\\ -\frac{2Yk-2kq-Y}{2k} & 1\end{pmatrix}.$$

• $\varepsilon=-2k$:

$$\mathbf{U}\times\begin{pmatrix}\frac{2Y(2k+1)}{k}-2q-2r & -4\\ \frac{Y^2(2k+1)}{2k}-\frac{Y(4k+1)(q+r)}{4k}+qr & -Y+\frac{1}{2}q+\frac{1}{2}r\end{pmatrix}=\begin{pmatrix}-\frac{2Yk-2kr+Y}{2k} & 1\\ -\frac{2Yk-2kq+Y}{2k} & 1\end{pmatrix}.$$

• $\varepsilon=2k+1$:

$$\mathbf{U}\times\begin{pmatrix}\frac{8Yk}{2k+1}-2q-2r & -4\\ \frac{2Y^2k}{2k+1}-\frac{Y(4k+1)(q+r)}{2(2k+1)}+qr & -Y+\frac{1}{2}q+\frac{1}{2}r\end{pmatrix}=\begin{pmatrix}-\frac{2Yk-(2k+1)r}{2k+1} & 1\\ -\frac{2Yk-(2k+1)q}{2k+1} & 1\end{pmatrix}.$$

• $\varepsilon=-2k-1$:

$$\mathbf{U}\times\begin{pmatrix}\frac{8Y(k+1)}{2k+1}-2q-2r & -4\\ \frac{2Y^2(k+1)}{2k+1}-\frac{Y(4k+3)(q+r)}{2(2k+1)}+qr & -Y+\frac{1}{2}q+\frac{1}{2}r\end{pmatrix}=\begin{pmatrix}-\frac{2Yk-(2k+1)r+2Y}{2k+1} & 1\\ -\frac{2Yk-(2k+1)q+2Y}{2k+1} & 1\end{pmatrix}.$$

This completes the case $j=0$.

A.6. **The key identities for the inductive step.** We need to verify that, for $j=0,\dots,|\varepsilon|-2$, we have

$$\Pi_\varepsilon(j+1)=\Pi_\varepsilon(j)\mathbf{D}_\varepsilon(j+1),$$

that is,

$$\text{(A.1)}\quad \begin{pmatrix}A_\varepsilon(j+1) & A'_\varepsilon(j+1)\\ B_\varepsilon(j+1) & B'_\varepsilon(j+1)\end{pmatrix}=\begin{pmatrix}A_\varepsilon(j) & A'_\varepsilon(j)\\ B_\varepsilon(j) & B'_\varepsilon(j)\end{pmatrix}$$
$$\times\begin{pmatrix}\zeta_\varepsilon(2j+2)\left(r+q-2Y\frac{\varepsilon-(j+1)-1}{\varepsilon+j+1}\right) & -4\\ rq-\frac{(2\varepsilon-1)(r+q)}{2(\varepsilon+j+1)}Y+\frac{\varepsilon-(j+1)-1}{\varepsilon+j+1}Y^2 & -(\zeta_\varepsilon(2j+2))^{-1}(r+q-2Y)\end{pmatrix}.$$

The entries of these matrices are polynomials in Y of degree at most 2. The above identity is equivalent to *four* identities for quadratic polynomials in Y. One of these four identities is (recall that $t=j+1$)

$$A_\varepsilon(j+1)=A_\varepsilon(j)\zeta_\varepsilon(2t)\left(r+q-2Y\frac{\varepsilon-t-1}{\varepsilon+t}\right)$$
$$+A'_\varepsilon(j)\left(rq-\frac{(2\varepsilon-1)(r+q)}{2(\varepsilon+t)}Y+\frac{\varepsilon-t-1}{\varepsilon+t}Y^2\right).$$

For example, when $\varepsilon=2k$ and j is replaced by $2j$, the above identity becomes

$$-\frac{2^{4j+1}\left(\frac{4Y(j-k+1)}{2j+2k+1}+r+q\right)(j+k)!^2}{(2j+2k)!\,(-j+k-1)!}$$
$$\times\left(\frac{2^{-2j-1}Y\Phi\,(2k-1,2j;r,q)\,(2k-1)!}{(j+k)!}-\frac{r\Phi\,(2k-2,2j;r,q)\,(2k-2)!}{2^{2j}\,(j+k-1)!}\right)$$

$$+\frac{\left(\frac{4\,Y^2(j-k+1)}{2\,j+2\,k+1}+\frac{Y\,(4\,k-1)(r+q)}{2\,j+2\,k+1}-2\,rq\right)2^{2\,j+1}\Phi\left(2\,k-1,2\,j;r,q\right)(j+k)!\,(2\,k-1)!}{2\;(2\,j+2\,k)!\,(-j+k-1)!}$$
$$-\frac{2^{2\,j+2}Y\Phi\left(2\,k-1,2\,j+1;r,q\right)(j+k+1)!\,(2\,k-1)!}{(2\,j+2\,k+2)!\,(-j+k-2)!}$$
$$+\frac{2^{2\,j+1}r\Phi\left(2\,k-2,2\,j+1;r,q\right)(j+k)!\,(2\,k-2)!}{(2\,j+2\,k)!\,(-j+k-2)!}=0.$$

To prove each such identity, we equate each coefficient of Y^d, for $d = 0, 1, 2$. We need to prove that (take the numerator, then simplify):

$$\begin{aligned}
[Y^0]\ 0 &= (j+k)(r+q)\Phi\left(2\,k-2,2\,j;r,q\right)-(2k-1)q\Phi\left(2\,k-1,2\,j;r,q\right)\\
&\quad+(k-j-1)\Phi\left(2\,k-2,2\,j+1;r,q\right),\\
[Y^1]\ 0 &= (2\,k-1)\Phi\left(2\,k-1,2\,j+1;r,q\right)\\
&\quad-(2\,k-1)(r+q)\Phi\left(2\,k-1,2\,j;r,q\right)\\
&\quad+4\,r\,(j+k)\,\Phi\left(2\,k-2,2\,j;r,q\right),\\
[Y^2]\ 0 &= 0.
\end{aligned}$$

The notation $[Y^d]$ means that we consider the coefficient of Y^d. Doing the same calculations for all other cases, we get a full list of $4 \times 3 \times 8 = 96$ identities to prove (the 4 entries of the matrices are quadratic polynomials, and we distinguish according to the sign of k and to the parities of j and k).

The notation $(\varepsilon, j)_{ij}[Y^d]$ used below means that the identity is obtained by considering the coefficient of Y^d in the $i \times j$ entry of (A.1).

In all the cases, the identities obtained by considering the coefficients of Y^2 are tautologies. This is also the case for the identities obtained by considering the coefficients of Y in the entries of the second column of each matrix. Consequently, we are left with 48 identities to be checked, namely:

$$\begin{aligned}
(2k,2j)_{11}[Y^0]\ 0 &= (2\,k-1)q\Phi\left(2\,k-1,2\,j;r,q\right)\\
&\quad-(j+k)(q+r)\Phi\left(2\,k-2,2\,j;r,q\right)\\
&\quad+(j-k+1)\Phi\left(2\,k-2,2\,j+1;r,q\right),\\
(2k,2j)_{11}[Y^1]\ 0 &= (2\,k-1)(q+r)\Phi\left(2\,k-1,2\,j;r,q\right)\\
&\quad-4\,(j+k)r\Phi\left(2\,k-2,2\,j;r,q\right)\\
&\quad-(2\,k-1)\Phi\left(2\,k-1,2\,j+1;r,q\right),\\
(2k,2j)_{12}[Y^0]\ 0 &= (2\,k-1)(q+r)\Phi\left(2\,k-1,2\,j;r,q\right)\\
&\quad-4\,(j+k)r\Phi\left(2\,k-2,2\,j;r,q\right)\\
&\quad-(2\,k-1)\Phi\left(2\,k-1,2\,j+1;r,q\right),\\
(2k,2j)_{21}[Y^0]\ 0 &= (2\,k-1)r\Phi\left(2\,k-1,2\,j;q,r\right)\\
&\quad-(j+k)(q+r)\Phi\left(2\,k-2,2\,j;q,r\right)
\end{aligned}$$

$$
\begin{aligned}
&+(j-k+1)\Phi\,(2\,k-2,2\,j+1;q,r)\,,\\
(2k,2j)_{21}[Y^1]\;\;0 &= (2\,k-1)(q+r)\Phi\,(2\,k-1,2\,j;q,r)\\
&-4\,(j+k)q\Phi\,(2\,k-2,2\,j;q,r)\\
&-(2\,k-1)\Phi\,(2\,k-1,2\,j+1;q,r)\,,\\
(2k,2j)_{22}[Y^0]\;\;0 &= (2\,k-1)(q+r)\Phi\,(2\,k-1,2\,j;q,r)\\
&-4\,(j+k)q\Phi\,(2\,k-2,2\,j;q,r)\\
&-(2\,k-1)\Phi\,(2\,k-1,2\,j+1;q,r)\,,\\
(2k,2j+1)_{11}[Y^0]\;\;0 &= 2\,(2\,k-1)q\Phi\,(2\,k-1,2\,j+1;r,q)\\
&-(2\,j+2\,k+1)(q+r)\Phi\,(2\,k-2,2\,j+1;r,q)\\
&+(2\,j-2\,k+3)\Phi\,(2\,k-2,2\,j+2;r,q)\,,\\
(2k,2j+1)_{11}[Y^1]\;\;0 &= (2\,k-1)(q+r)\Phi\,(2\,k-1,2\,j+1;r,q)\\
&-2\,(2\,j+2\,k+1)r\Phi\,(2\,k-2,2\,j+1;r,q)\\
&-(2\,k-1)\Phi\,(2\,k-1,2\,j+2;r,q)\,,\\
(2k,2j+1)_{12}[Y^0]\;\;0 &= (2\,k-1)(q+r)\Phi\,(2\,k-1,2\,j+1;r,q)\\
&-2\,(2\,j+2\,k+1)r\Phi\,(2\,k-2,2\,j+1;r,q)\\
&-(2\,k-1)\Phi\,(2\,k-1,2\,j+2;r,q)\,,\\
(2k,2j+1)_{21}[Y^0]\;\;0 &= 2\,(2\,k-1)r\Phi\,(2\,k-1,2\,j+1;q,r)\\
&-(2\,j+2\,k+1)(q+r)\Phi\,(2\,k-2,2\,j+1;q,r)\\
&+(2\,j-2\,k+3)\Phi\,(2\,k-2,2\,j+2;q,r)\,,\\
(2k,2j+1)_{21}[Y^1]\;\;0 &= (2\,k-1)(q+r)\Phi\,(2\,k-1,2\,j+1;q,r)\\
&-2\,(2\,j+2\,k+1)q\Phi\,(2\,k-2,2\,j+1;q,r)\\
&-(2\,k-1)\Phi\,(2\,k-1,2\,j+2;q,r)\,,\\
(2k,2j+1)_{22}[Y^0]\;\;0 &= (2\,k-1)(q+r)\Phi\,(2\,k-1,2\,j+1;q,r)\\
&-2\,(2\,j+2\,k+1)q\Phi\,(2\,k-2,2\,j+1;q,r)\\
&-(2\,k-1)\Phi\,(2\,k-1,2\,j+2;q,r)\,,\\
(-2k,2j)_{11}[Y^0]\;\;0 &= k(q+r)\Phi\,(2\,k,2\,j;q,r)\\
&-(2\,j+2\,k+1)q\Phi\,(2\,k-1,2\,j;q,r)\\
&-k\Phi\,(2\,k,2\,j+1;q,r)\,,\\
(-2k,2j)_{11}[Y^1]\;\;0 &= 4\,kr\Phi\,(2\,k,2\,j;q,r)\\
&-(2\,j+2\,k+1)(q+r)\Phi\,(2\,k-1,2\,j;q,r)\\
&+(2\,j-2\,k+1)\Phi\,(2\,k-1,2\,j+1;q,r)\,,\\
(-2k,2j)_{12}[Y^0]\;\;0 &= 4\,kr\Phi\,(2\,k,2\,j;q,r)\\
&-(2\,j+2\,k+1)(q+r)\Phi\,(2\,k-1,2\,j;q,r)\\
&+(2\,j-2\,k+1)\Phi\,(2\,k-1,2\,j+1;q,r)\,,
\end{aligned}
$$

$$
\begin{aligned}
(-2k,2j)_{21}[Y^0]\ 0 &= k(q+r)\Phi\left(2\,k,2\,j;r,q\right)\\
&-\left(2\,j+2\,k+1\right)r\Phi\left(2\,k-1,2\,j;r,q\right)\\
&-k\Phi\left(2\,k,2\,j+1;r,q\right),\\
(-2k,2j)_{21}[Y^1]\ 0 &= 4\,kq\Phi\left(2\,k,2\,j;r,q\right)\\
&-\left(2\,j+2\,k+1\right)(q+r)\Phi\left(2\,k-1,2\,j;r,q\right)\\
&+\left(2\,j-2\,k+1\right)\Phi\left(2\,k-1,2\,j+1;r,q\right),\\
(-2k,2j)_{22}[Y^0]\ 0 &= 4\,kq\Phi\left(2\,k,2\,j;r,q\right)\\
&-\left(2\,j+2\,k+1\right)(q+r)\Phi\left(2\,k-1,2\,j;r,q\right)\\
&+\left(2\,j-2\,k+1\right)\Phi\left(2\,k-1,2\,j+1;r,q\right),\\
(-2k,2j+1)_{11}[Y^0]\ 0 &= k(q+r)\Phi\left(2\,k,2\,j+1;q,r\right)\\
&-2\left(j+k+1\right)q\Phi\left(2\,k-1,2\,j+1;q,r\right)\\
&-k\Phi\left(2\,k,2\,j+2;q,r\right),\\
(-2k,2j+1)_{11}[Y^1]\ 0 &= 2\,kr\Phi\left(2\,k,2\,j+1;q,r\right)\\
&-(j+k+1)(q+r)\Phi\left(2\,k-1,2\,j+1;q,r\right)\\
&+(j-k+1)\Phi\left(2\,k-1,2\,j+2;q,r\right),\\
(-2k,2j+1)_{12}[Y^0]\ 0 &= 2\,kr\Phi\left(2\,k,2\,j+1;q,r\right)\\
&-(j+k+1)(q+r)\Phi\left(2\,k-1,2\,j+1;q,r\right)\\
&+(j-k+1)\Phi\left(2\,k-1,2\,j+2;q,r\right),\\
(-2k,2j+1)_{21}[Y^0]\ 0 &= k(q+r)\Phi\left(2\,k,2\,j+1;r,q\right)\\
&-2\left(j+k+1\right)r\Phi\left(2\,k-1,2\,j+1;r,q\right)\\
&-k\Phi\left(2\,k,2\,j+2;r,q\right),\\
(-2k,2j+1)_{21}[Y^1]\ 0 &= 2\,kq\Phi\left(2\,k,2\,j+1;r,q\right)\\
&-(j+k+1)(q+r)\Phi\left(2\,k-1,2\,j+1;r,q\right)\\
&+(j-k+1)\Phi\left(2\,k-1,2\,j+2;r,q\right),\\
(-2k,2j+1)_{22}[Y^0]\ 0 &= 2\,kq\Phi\left(2\,k,2\,j+1;r,q\right)\\
&-(j+k+1)(q+r)\Phi\left(2\,k-1,2\,j+1;r,q\right)\\
&+(j-k+1)\Phi\left(2\,k-1,2\,j+2;r,q\right),\\
(2k+1,2j)_{11}[Y^0]\ 0 &= 4\,kq\Phi\left(2\,k,2\,j;r,q\right)\\
&-\left(2\,j+2\,k+1\right)(q+r)\Phi\left(2\,k-1,2\,j;r,q\right)\\
&+\left(2\,j-2\,k+1\right)\Phi\left(2\,k-1,2\,j+1;r,q\right),\\
(2k+1,2j)_{11}[Y^1]\ 0 &= k(q+r)\Phi\left(2\,k,2\,j;r,q\right)\\
&-\left(2\,j+2\,k+1\right)r\Phi\left(2\,k-1,2\,j;r,q\right)\\
&-k\Phi\left(2\,k,2\,j+1;r,q\right),\\
(2k+1,2j)_{12}[Y^0]\ 0 &= k(q+r)\Phi\left(2\,k,2\,j;r,q\right)
\end{aligned}
$$

$$
\begin{aligned}
&\quad - (2\,j + 2\,k + 1)r\Phi\,(2\,k - 1, 2\,j; r, q) \\
&\quad - k\Phi\,(2\,k, 2\,j + 1; r, q)\,, \\
(2k+1, 2j)_{21}[Y^0]\ 0 &= 4\,kr\Phi\,(2\,k, 2\,j; q, r) \\
&\quad - (2\,j + 2\,k + 1)(q + r)\Phi\,(2\,k - 1, 2\,j; q, r) \\
&\quad + (2\,j - 2\,k + 1)\Phi\,(2\,k - 1, 2\,j + 1; q, r)\,, \\
(2k+1, 2j)_{21}[Y^1]\ 0 &= k(q + r)\Phi\,(2\,k, 2\,j; q, r) \\
&\quad - (2\,j + 2\,k + 1)q\Phi\,(2\,k - 1, 2\,j; q, r) \\
&\quad - k\Phi\,(2\,k, 2\,j + 1; q, r)\,, \\
(2k+1, 2j)_{22}[Y^0]\ 0 &= k(q + r)\Phi\,(2\,k, 2\,j; q, r) \\
&\quad - (2\,j + 2\,k + 1)q\Phi\,(2\,k - 1, 2\,j; q, r) \\
&\quad - k\Phi\,(2\,k, 2\,j + 1; q, r)\,, \\
(2k+1, 2j+1)_{11}[Y^0]\ 0 &= 2\,kq\Phi\,(2\,k, 2\,j + 1; r, q) \\
&\quad - (j + k + 1)(q + r)\Phi\,(2\,k - 1, 2\,j + 1; r, q) \\
&\quad + (j - k + 1)\Phi\,(2\,k - 1, 2\,j + 2; r, q)\,, \\
(2k+1, 2j+1)_{11}[Y^1]\ 0 &= k(q + r)\Phi\,(2\,k, 2\,j + 1; r, q) \\
&\quad - 2\,(j + k + 1)r\Phi\,(2\,k - 1, 2\,j + 1; r, q) \\
&\quad - k\Phi\,(2\,k, 2\,j + 2; r, q)\,, \\
(2k+1, 2j+1)_{12}[Y^0]\ 0 &= k(q + r)\Phi\,(2\,k, 2\,j + 1; r, q) \\
&\quad - 2\,(j + k + 1)r\Phi\,(2\,k - 1, 2\,j + 1; r, q) \\
&\quad - k\Phi\,(2\,k, 2\,j + 2; r, q)\,, \\
(2k+1, 2j+1)_{21}[Y^0]\ 0 &= 2\,kr\Phi\,(2\,k, 2\,j + 1; q, r) \\
&\quad - (j + k + 1)(q + r)\Phi\,(2\,k - 1, 2\,j + 1; q, r) \\
&\quad + (j - k + 1)\Phi\,(2\,k - 1, 2\,j + 2; q, r)\,, \\
(2k+1, 2j+1)_{21}[Y^1]\ 0 &= k(q + r)\Phi\,(2\,k, 2\,j + 1; q, r) \\
&\quad - 2\,(j + k + 1)q\Phi\,(2\,k - 1, 2\,j + 1; q, r) \\
&\quad - k\Phi\,(2\,k, 2\,j + 2; q, r)\,, \\
(2k+1, 2j+1)_{22}[Y^0]\ 0 &= k(q + r)\Phi\,(2\,k, 2\,j + 1; q, r) \\
&\quad - 2\,(j + k + 1)q\Phi\,(2\,k - 1, 2\,j + 1; q, r) \\
&\quad - k\Phi\,(2\,k, 2\,j + 2; q, r)\,, \\
(-2k-1, 2j)_{11}[Y^0]\ 0 &= 4\,(j + k + 1)q\Phi\,(2\,k, 2\,j; q, r) \\
&\quad - (2\,k + 1)(q + r)\Phi\,(2\,k + 1, 2\,j; q, r) \\
&\quad + (2\,k + 1)\Phi\,(2\,k + 1, 2\,j + 1; q, r)\,, \\
(-2k-1, 2j)_{11}[Y^1]\ 0 &= (j + k + 1)(q + r)\Phi\,(2\,k, 2\,j; q, r) \\
&\quad - (2\,k + 1)r\Phi\,(2\,k + 1, 2\,j; q, r)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad - (j-k)\Phi\,(2\,k, 2\,j+1; q, r)\,,\\
(-2k-1, 2j)_{12}[Y^0]\ 0 &= (j+k+1)(q+r)\Phi\,(2\,k, 2\,j; q, r)\\
&\quad - (2\,k+1)r\Phi\,(2\,k+1, 2\,j; q, r)\\
&\quad - (j-k)\Phi\,(2\,k, 2\,j+1; q, r)\,,\\
(-2k-1, 2j)_{21}[Y^0]\ 0 &= 4\,(j+k+1)r\Phi\,(2\,k, 2\,j; r, q)\\
&\quad - (2\,k+1)(q+r)\Phi\,(2\,k+1, 2\,j; r, q)\\
&\quad + (2\,k+1)\Phi\,(2\,k+1, 2\,j+1; r, q)\,,\\
(-2k-1, 2j)_{21}[Y^1]\ 0 &= (j+k+1)(q+r)\Phi\,(2\,k, 2\,j; r, q)\\
&\quad - (2\,k+1)q\Phi\,(2\,k+1, 2\,j; r, q)\\
&\quad - (j-k)\Phi\,(2\,k, 2\,j+1; r, q)\,,\\
(-2k-1, 2j)_{22}[Y^0]\ 0 &= (j+k+1)(q+r)\Phi\,(2\,k, 2\,j; r, q)\\
&\quad - (2\,k+1)q\Phi\,(2\,k+1, 2\,j; r, q)\\
&\quad - (j-k)\Phi\,(2\,k, 2\,j+1; r, q)\,,\\
(-2k-1, 2j+1)_{11}[Y^0]\ 0 &= 2\,(2\,j+2\,k+3)q\Phi\,(2\,k, 2\,j+1; q, r)\\
&\quad - (2\,k+1)(q+r)\Phi\,(2\,k+1, 2\,j+1; q, r)\\
&\quad + (2\,k+1)\Phi\,(2\,k+1, 2\,j+2; q, r)\,,\\
(-2k-1, 2j+1)_{11}[Y^1]\ 0 &= (2\,j+2\,k+3)(q+r)\Phi\,(2\,k, 2\,j+1; q, r)\\
&\quad - 2\,(2\,k+1)r\Phi\,(2\,k+1, 2\,j+1; q, r)\\
&\quad - (2\,j-2\,k+1)\Phi\,(2\,k, 2\,j+2; q, r)\,,\\
(-2k-1, 2j+1)_{12}[Y^0]\ 0 &= (2\,j+2\,k+3)(q+r)\Phi\,(2\,k, 2\,j+1; q, r)\\
&\quad - 2\,(2\,k+1)r\Phi\,(2\,k+1, 2\,j+1; q, r)\\
&\quad - (2\,j-2\,k+1)\Phi\,(2\,k, 2\,j+2; q, r)\,,\\
(-2k-1, 2j+1)_{21}[Y^0]\ 0 &= 2\,(2\,j+2\,k+3)r\Phi\,(2\,k, 2\,j+1; r, q)\\
&\quad - (2\,k+1)(q+r)\Phi\,(2\,k+1, 2\,j+1; r, q)\\
&\quad + (2\,k+1)\Phi\,(2\,k+1, 2\,j+2; r, q)\,,\\
(-2k-1, 2j+1)_{21}[Y^1]\ 0 &= (2\,j+2\,k+3)(q+r)\Phi\,(2\,k, 2\,j+1; r, q)\\
&\quad - 2\,(2\,k+1)q\Phi\,(2\,k+1, 2\,j+1; r, q)\\
&\quad - (2\,j-2\,k+1)\Phi\,(2\,k, 2\,j+2; r, q)\,,\\
(-2k-1, 2j+1)_{22}[Y^0]\ 0 &= (2\,j+2\,k+3)(q+r)\Phi\,(2\,k, 2\,j+1; r, q)\\
&\quad - 2\,(2\,k+1)q\Phi\,(2\,k+1, 2\,j+1; r, q)\\
&\quad - (2\,j-2\,k+1)\Phi\,(2\,k, 2\,j+2; r, q)\,.
\end{aligned}
$$

We delete duplicates and use the symmetry in r and q. We are left with the following 12 identities:

$$\begin{aligned}
0 &= (2\,k-1)r\Phi\,(2\,k-1,2\,j;q,r) - (j+k)(r+q)\Phi\,(2\,k-2,2\,j;q,r)\\
&\quad + (j-k+1)\Phi\,(2\,k-2,2\,j+1;q,r)\,,\\
0 &= (2\,k-1)(r+q)\Phi\,(2\,k-1,2\,j;q,r) - 4\,(j+k)q\Phi\,(2\,k-2,2\,j;q,r)\\
&\quad - (2\,k-1)\Phi\,(2\,k-1,2\,j+1;q,r)\,,\\
0 &= 2\,(2\,k-1)r\Phi\,(2\,k-1,2\,j+1;q,r)\\
&\quad - (2\,j+2\,k+1)(r+q)\Phi\,(2\,k-2,2\,j+1;q,r)\\
&\quad + (2\,j-2\,k+3)\Phi\,(2\,k-2,2\,j+2;q,r)\,,\\
0 &= (2\,k-1)(r+q)\Phi\,(2\,k-1,2\,j+1;q,r)\\
&\quad - 2\,(2\,j+2\,k+1)q\Phi\,(2\,k-2,2\,j+1;q,r)\\
&\quad - (2\,k-1)\Phi\,(2\,k-1,2\,j+2;q,r)\,,\\
0 &= k(r+q)\Phi\,(2\,k,2\,j;r,q) - (2\,j+2\,k+1)r\Phi\,(2\,k-1,2\,j;r,q)\\
&\quad - k\Phi\,(2\,k,2\,j+1;r,q)\,,\\
0 &= 4\,kq\Phi\,(2\,k,2\,j;r,q) - (2\,j+2\,k+1)(r+q)\Phi\,(2\,k-1,2\,j;r,q)\\
&\quad + (2\,j-2\,k+1)\Phi\,(2\,k-1,2\,j+1;r,q)\,,\\
0 &= k(r+q)\Phi\,(2\,k,2\,j+1;r,q) - 2\,(j+k+1)r\Phi\,(2\,k-1,2\,j+1;r,q)\\
&\quad - k\Phi\,(2\,k,2\,j+2;r,q)\,,\\
0 &= 2\,kq\Phi\,(2\,k,2\,j+1;r,q) - (j+k+1)(r+q)\Phi\,(2\,k-1,2\,j+1;r,q)\\
&\quad + (j-k+1)\Phi\,(2\,k-1,2\,j+2;r,q)\,,\\
0 &= 4\,(j+k+1)r\Phi\,(2\,k,2\,j;r,q) - (2\,k+1)(r+q)\Phi\,(2\,k+1,2\,j;r,q)\\
&\quad + (2\,k+1)\Phi\,(2\,k+1,2\,j+1;r,q)\,,\\
0 &= (j+k+1)(r+q)\Phi\,(2\,k,2\,j;r,q) - (2\,k+1)q\Phi\,(2\,k+1,2\,j;r,q)\\
&\quad - (j-k)\Phi\,(2\,k,2\,j+1;r,q)\,,\\
0 &= 2\,(2\,j+2\,k+3)r\Phi\,(2\,k,2\,j+1;r,q)\\
&\quad - (2\,k+1)(r+q)\Phi\,(2\,k+1,2\,j+1;r,q)\\
&\quad + (2\,k+1)\Phi\,(2\,k+1,2\,j+2;r,q)\,,\\
0 &= (2\,j+2\,k+3)(r+q)\Phi\,(2\,k,2\,j+1;r,q)\\
&\quad - 2\,(2\,k+1)q\Phi\,(2\,k+1,2\,j+1;r,q)\\
&\quad - (2\,j-2\,k+1)\Phi\,(2\,k,2\,j+2;r,q)\,.
\end{aligned}$$

Four of these identities are obtained by shifting by 1 another identity, so we are left with the following 8 identities, where, as in the sequel, we use the shortened notation

$$\Phi\,(k,j) = \Phi\,(k,j;r,q)\,.$$

$$0 = (2\,k+1)q\Phi\,(2\,k+1,2\,j) - (j+k+1)(q+r)\Phi\,(2\,k,2\,j)$$

$$+ (j-k)\Phi(2k, 2j+1),$$
$$0 = 2(2k+1)q\Phi(2k+1, 2j+1) - (2j+2k+3)(q+r)\Phi(2k, 2j+1)$$
$$+ (2j-2k+1)\Phi(2k, 2j+2),$$
$$0 = (2k+1)(q+r)\Phi(2k+1, 2j) - 4(j+k+1)r\Phi(2k, 2j)$$
$$- (2k+1)\Phi(2k+1, 2j+1),$$
$$0 = (2k+1)(q+r)\Phi(2k+1, 2j+1) - 2(2j+2k+3)r\Phi(2k, 2j+1)$$
$$- (2k+1)\Phi(2k+1, 2j+2),$$
$$0 = k(r+q)\Phi(2k, 2j) - (2j+2k+1)r\Phi(2k-1, 2j) - k\Phi(2k, 2j+1),$$
$$0 = k(r+q)\Phi(2k, 2j+1) - 2(j+k+1)r\Phi(2k-1, 2j+1)$$
$$- k\Phi(2k, 2j+2),$$
$$0 = 4kq\Phi(2k, 2j) - (2j+2k+1)(r+q)\Phi(2k-1, 2j)$$
$$+ (2j-2k+1)\Phi(2k-1, 2j+1),$$
$$0 = 2kq\Phi(2k, 2j+1) - (j+k+1)(r+q)\Phi(2k-1, 2j+1)$$
$$+ (j-k+1)\Phi(2k-1, 2j+2).$$

The cases $2j$ and $2j+1$ can be merged and we are left with the following 4 identities:

$$0 = 2(2k+1)q\Phi(2k+1, j) - (j+2k+2)(q+r)\Phi(2k, j)$$
$$+ (j-2k)\Phi(2k, j+1),$$
$$0 = (2k+1)(q+r)\Phi(2k+1, j) - 2(j+2k+2)r\Phi(2k, j)$$
$$- (2k+1)\Phi(2k+1, j+1),$$
$$0 = k(r+q)\Phi(2k, j) - (j+2k+1)r\Phi(2k-1, j) - k\Phi(2k, j+1),$$
$$0 = 4kq\Phi(2k, j) - (j+2k+1)(r+q)\Phi(2k-1, j)$$
$$+ (j-2k+1)\Phi(2k-1, j+1).$$

We merge the cases $2k$ and $2k-1$, thus we are eventually left with only two identities to check.

Lemma A.6. *For $k \ge 1$ and $j = 0, 1, \ldots, k-2$, we have*

$$0 = (k-1)(r+q)\Phi(k-1, j) - 2(j+k)r\Phi(k-2, j) - (k-1)\Phi(k-1, j+1)$$

and

$$0 = 2kq\Phi(k, j) - (j+k+1)(r+q)\Phi(k-1, j) + (j-k+1)\Phi(k-1, j+1).$$

A.7. **Proof of Lemma A.6 (the inductive step).** We prove the first identity of Lemma A.6.

Define

$$\delta = (k-1)\Phi(k-1, j+1) + 2(j+k)r\Phi(k-2, j) - (k-1)(r+q)\Phi(k-1, j)$$

We need to prove that $\delta = 0$.

We use the definition of Φ and put all the terms r and q after the signs $\sum$ to get

$$\begin{aligned}
\delta &= (k-1)\sum_{d=0}^{j+1}\frac{\binom{k+j+1}{d}\binom{j+1}{d}}{\binom{k-1}{d}}(-r)^d q^{j+1-d} \\
&\quad + 2r(k+j)\sum_{d=0}^{j}\frac{\binom{k+j-1}{d}\binom{j}{d}}{\binom{k-2}{d}}(-r)^d q^{j-d} \\
&\quad - (k-1)(r+q)\sum_{d=0}^{j}\frac{\binom{k+j}{d}\binom{j}{d}}{\binom{k-1}{d}}(-r)^d q^{j-d} \\
&= (k-1)\sum_{d=0}^{j+1}\frac{\binom{k+j+1}{d}\binom{j+1}{d}}{\binom{k-1}{d}}(-r)^d q^{j+1-d} \\
&\quad - 2(k+j)\sum_{d=0}^{j}\frac{\binom{k+j-1}{d}\binom{j}{d}}{\binom{k-2}{d}}(-r)^{d+1} q^{j-d} \\
&\quad + (k-1)\sum_{d=0}^{j}\frac{\binom{k+j}{d}\binom{j}{d}}{\binom{k-1}{d}}(-r)^{d+1} q^{j-d} \\
&\quad - (k-1)\sum_{d=0}^{j}\frac{\binom{k+j}{d}\binom{j}{d}}{\binom{k-1}{d}}(-r)^d q^{j+1-d}.
\end{aligned}$$

By shifting the indices of the second and third sums, we obtain

$$\begin{aligned}
\delta &= (k-1)\sum_{d=0}^{j+1}\frac{\binom{k+j+1}{d}\binom{j+1}{d}}{\binom{k-1}{d}}(-r)^d q^{j+1-d} \\
&\quad - 2(k+j)\sum_{d=1}^{j+1}\frac{\binom{k+j-1}{d-1}\binom{j}{d-1}}{\binom{k-2}{d-1}}(-r)^d q^{j-d+1} \\
&\quad + (k-1)\sum_{d=1}^{j+1}\frac{\binom{k+j}{d-1}\binom{j}{d-1}}{\binom{k-1}{d-1}}(-r)^d q^{j-d+1} \\
&\quad - (k-1)\sum_{d=0}^{j}\frac{\binom{k+j}{d}\binom{j}{d}}{\binom{k-1}{d}}(-r)^d q^{j+1-d} \\
&= \sum_{d=1}^{j}\delta_1(d)(-r)^d q^{j-d+1} + \delta_2,
\end{aligned}$$

where

$$\delta_1(d) = (k-1)\frac{\binom{k+j+1}{d}\binom{j+1}{d}}{\binom{k-1}{d}} - 2(k+j)\frac{\binom{k+j-1}{d-1}\binom{j}{d-1}}{\binom{k-2}{d-1}} + (k-1)\frac{\binom{k+j}{d-1}\binom{j}{d-1}}{\binom{k-1}{d-1}}$$

$$
\begin{aligned}
&- (k-1)\frac{\binom{k+j}{d}\binom{j}{d}}{\binom{k-1}{d}} \\
&= \frac{j!(k-1-d)!(k+j)!}{d!(k-2)!(j-d+1)!(k+j-d+1)!}\Big[(k+j+1)(j+1) - 2d(k+j-d+1) \\
&\quad + d(k-d) - (j-d+1)(k+j-d+1)\Big] \\
&= 0
\end{aligned}
$$

and

$$
\begin{aligned}
\delta_2 &= +(k-1)q^{j+1} + (k-1)\frac{\binom{k+j+1}{j+1}\binom{j+1}{j+1}}{\binom{k-1}{j+1}}(-r)^{j+1} \\
&\quad - 2(k+j)\frac{\binom{k+j-1}{j}\binom{j}{j}}{\binom{k-2}{j}}(-r)^{j+1} + (k-1)\frac{\binom{k+j}{j}\binom{j}{j}}{\binom{k-1}{j}}(-r)^{j+1} - (k-1)q^{j+1} \\
&= (k-1)\frac{\binom{k+j+1}{j+1}}{\binom{k-1}{j+1}}(-r)^{j+1} - 2(k+j)\frac{\binom{k+j-1}{j}}{\binom{k-2}{j}}(-r)^{j+1} \\
&\quad + (k-1)\frac{\binom{k+j}{j}}{\binom{k-1}{j}}(-r)^{j+1} \\
&= (-r)^{j+1}\frac{(k+j)!(k-j-2)!}{k!(k-2)!}\Big[(k+j+1) - 2k + (k-1-j)\Big] \\
&= 0.
\end{aligned}
$$

We conclude that

$$
\delta = \sum_{d=1}^{j} \delta_1(d)(-r)^d q^{j-d+1} + \delta_2 = 0,
$$

and the first identity holds. .

We prove the second identity of Lemma A.6. Define

$$
\delta' = 2\,kq\Phi\,(k,j) - (j+k+1)(r+q)\Phi\,(k-1,j) + (j-k+1)\Phi\,(k-1,j+1)\,.
$$

We need to prove that $\delta' = 0$.

We use the definition of Φ and put all the terms r and q after the signs $\sum$ to get

$$
\begin{aligned}
\delta &= 2kq\sum_{d=0}^{j}\frac{\binom{k+j+1}{d}\binom{j}{d}}{\binom{k}{d}}(-r)^d q^{j-d} \\
&\quad - (j+k+1)(r+q)\sum_{d=0}^{j}\frac{\binom{k+j}{d}\binom{j}{d}}{\binom{k-1}{d}}(-r)^d q^{j-d} \\
&\quad + (j-k+1)\sum_{d=0}^{j+1}\frac{\binom{k+j+1}{d}\binom{j+1}{d}}{\binom{k-1}{d}}(-r)^d q^{j+1-d}
\end{aligned}
$$

$$= 2k\sum_{d=0}^{j}\frac{\binom{k+j+1}{d}\binom{j}{d}}{\binom{k}{d}}(-r)^d q^{j+1-d}$$
$$+ (j+k+1)\sum_{d=0}^{j}\frac{\binom{k+j}{d}\binom{j}{d}}{\binom{k-1}{d}}(-r)^{d+1} q^{j-d}$$
$$- (j+k+1)\sum_{d=0}^{j}\frac{\binom{k+j}{d}\binom{j}{d}}{\binom{k-1}{d}}(-r)^{d} q^{j+1-d}$$
$$+ (j-k+1)\sum_{d=0}^{j+1}\frac{\binom{k+j+1}{d}\binom{j+1}{d}}{\binom{k-1}{d}}(-r)^{d} q^{j+1-d}.$$

By shifting the indices of the second sum, we obtain

$$\delta = 2k\sum_{d=0}^{j}\frac{\binom{k+j+1}{d}\binom{j}{d}}{\binom{k}{d}}(-r)^d q^{j+1-d}$$
$$+ (j+k+1)\sum_{d=1}^{j+1}\frac{\binom{k+j}{d-1}\binom{j}{d-1}}{\binom{k-1}{d-1}}(-r)^{d} q^{j-d+1}$$
$$- (j+k+1)\sum_{d=0}^{j}\frac{\binom{k+j}{d}\binom{j}{d}}{\binom{k-1}{d}}(-r)^{d} q^{j+1-d}$$
$$+ (j-k+1)\sum_{d=0}^{j+1}\frac{\binom{k+j+1}{d}\binom{j+1}{d}}{\binom{k-1}{d}}(-r)^{d} q^{j+1-d}$$
$$= \sum_{d=1}^{j}\delta_1'(d)(-r)^d q^{j-d+1} + \delta_2',$$

where

$$\delta_1'(d) = 2k\frac{\binom{k+j+1}{d}\binom{j}{d}}{\binom{k}{d}} + (j+k+1)\frac{\binom{k+j}{d-1}\binom{j}{d-1}}{\binom{k-1}{d-1}} - (j+k+1)\frac{\binom{k+j}{d}\binom{j}{d}}{\binom{k-1}{d}}$$
$$+ (j-k+1)\frac{\binom{k+j+1}{d}\binom{j+1}{d}}{\binom{k-1}{d}}$$
$$= \frac{(j+k+1)!(k-d-1)!j!}{d!(k+j-d+1)!(k-1)!(j-d+1)!}\Big[2(k-d)(j-d+1) + d(k-d)$$
$$- (k+j-d+1)(j-d+1) + (j-k+1)(j+1)\Big]$$
$$= 0$$

and

$$\delta_2' = 2kq^{j+1} + (j+k+1)\frac{\binom{k+j}{j}\binom{j}{j}}{\binom{k-1}{j}}(-r)^{j+1} - (j+k+1)q^{j+1}$$

$$
\begin{aligned}
&+(j-k+1)q^{j+1}+(j-k+1)\frac{\binom{k+j+1}{j+1}\binom{j+1}{j+1}}{\binom{k-1}{j+1}}(-r)^{j+1}\\
&=(j+k+1)\frac{\binom{k+j}{j}}{\binom{k-1}{j}}(-r)^{j+1}+(j-k+1)\frac{\binom{k+j+1}{j+1}}{\binom{k-1}{j+1}}(-r)^{j+1}\\
&=\frac{(k+j+1)!(k-j-2)!}{k!(k-1)!}(-r)^{j+1}\Big[(k-1-j)+(j-k+1)\Big]\\
&=0.
\end{aligned}
$$

We conclude that

$$\delta' = \sum_{d=1}^{j} \delta_1'(d)(-r)^d q^{j-d+1} + \delta_2' = 0,$$

and the proof is complete.

References

[1] J.-P. Allouche, M. Mendès France, and A. J. van der Poorten, *An infinite product with bounded partial quotients*, Acta Arith. 59 (1991), 171–182. 3

[2] L. E. Baum and M. M. Sweet, *Continued fractions of algebraic power series in characteristic* 2, Ann. of Math. 103 (1976), 593–610. 2, 3, 16

[3] Y. Bugeaud, *On simultaneous rational approximation in fields of power series*, Bull. Math. Soc. Sci. Math. Roumanie 65 (2022), 191–202. 13

[4] Y. Bugeaud and G.-N. Han, *The Thue–Morse continued fractions in characteristic* 2 *are algebraic*, Acta Arith. To appear. 3

[5] Y. Hu, *On a family of* 2*-automatic sequences generating algebraic continued fractions in characteristic* 2, Finite Fields Appl. 88 (2023), Paper No. 102191.

[6] Y. Hu, *Algebraic automatic continued fractions in characteristic* 2, Int. Math. Res. Not. IMRN 2024, no. 9, 7255–7280.

[7] Y. Hu and G.-N. Han, *On the algebraicity of Thue-Morse and period-doubling continued fractions*, Acta Arith. 203 (2022), 353–381.

[8] Y. Hu and A. Lasjaunias. *Period-doubling continued fractions are algebraic in characteristic* 2, Ann. Inst. Fourier (Grenoble). To appear. 3

[9] J.-L. Lagrange, *Sur l'usage des fractions continues dans le calcul intégral*, Gauthier-Villars, (1869), 301–332. 7, 41

[10] A. Lasjaunias, *A survey of diophantine approximation in fields of power series*, Monatsh. Math. 130 (2000), 211–229. 2

[11] A. Lasjaunias, *Algebraic continued fractions in* $\mathbb{F}_q((T^{-1}))$ *and recurrent sequences in* $\mathbb{F}_q$, Acta Arith. 133 (2008), 251–265. 3

[12] A. Lasjaunias and J.-Y. Yao, *Hyperquadratic continued fractions in odd characteristic with partial quotients of degree one*, J. Number Theory 149 (2015), 259–284. 3

[13] K. Mahler, *On a theorem of Liouville in fields of positive characteristic*, Canad. J. Math. 1 (1949), 397–400. 2

[14] M. Mendès France and A. J. van der Poorten. *Automata and the arithmetic of formal power series*, Acta Arith. 46 (1986), 211–214. 4

[15] M. Mendès France and A. J. van der Poorten. *Some explicit continued fraction expansions*, Mathematika 38 (1991), 1–9. 3, 17

[16] W. H. Mills and D. P. Robbins, *Continued fractions for certain algebraic power series*, J. Number Theory 23 (1986), 388–404.

[17] M. Mkaouar, *Sur le développement en fraction continue de la série de Baum et Sweet*, Bull. Soc. Math. France 123 (1995), 361–374. 3

[18] C. Osgood, *Effective bounds on the "Diophantine approximation" of algebraic functions over fields of arbitrary characteristic and applications to differential equations*, Nederl. Akad. Wetensch. Proc. Ser. A 78 (1975), 105–119. 12

[19] A. J. van der Poorten and J. Shallit, *A specialised continued fraction*, Canad. J. Math. 45 (1993), 1067–1079. 3

[20] W. M. Schmidt, *On continued fractions and Diophantine approximation in power series fields*, Acta Arith. 95 (2000), 139–166. 2, 3, 11, 19

[21] D. S. Thakur, Function field arithmetic. World Scientific Publishing Co., Inc., River Edge, NJ, 2004. 11

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