On a family of automatic apwenian sequences

Ying-Jun GUO[*,a], Guo-Niu HAN[b]

[a] *Institute of Applied Mathematics, College of informatics, Huazhong Agricultural University, 430070, Wuhan, P. R. China*

[b] *Institute de Recherche Mathématique Avancée, Université de Strasbourg et CNRS, 7 rue René Descartes, 67084 Strasbourg, France*

Abstract

An integer sequence $\{a(n)\}_{n\geq 0}$ is called *apwenian* if $a(0) = 1$ and $a(n) \equiv a(2n+1)+a(2n+2) \pmod 2$ for all $n \geq 0$. The apwenian sequences are connected with the Hankel determinants, the continued fractions, the rational approximations and the measures of randomness for binary sequences. In this paper, we study the automatic apwenian sequences over different alphabets. On the alphabet $\{0, 1\}$, we give an extension of the generalized Rueppel sequences and characterize all the 2-automatic apwenian sequences in this class. On the alphabet $\{0, 1, 2\}$, we prove that the only apwenian sequence, among all fixed points of substitutions of constant length, is the period-doubling like sequence. On the other alphabets, we give a description of the 2-automatic apwenian sequences in terms of 2-uniform morphisms. Moreover, we find two 3-automatic apwenian sequences on the alphabet $\{1, 2, 3\}$.

Key words: Automatic sequences, apwenian sequences, Hankel determinants, Rueppel sequences, period-doubling sequence

1. Introduction

Apwenian sequences were introduced in the study of Hankel determinants of automatic sequences. Recall that the *Hankel determinant* of a sequence $\mathbf{a} = \{a(n)\}_{n\geq 0}$ of order n ($n \geq 1$) is defined by

$$H_n(\mathbf{a}) = \begin{vmatrix} a(0) & a(1) & \cdots & a(n-1) \\ a(1) & a(2) & \cdots & a(n) \\ \vdots & \vdots & \ddots & \vdots \\ a(n-1) & a(n) & \cdots & a(2n-2) \end{vmatrix} = \det(a(i+j))_{0\leq i,j\leq n-1}.$$

Hankel determinants play an important role in the study of rational approximation. In 1998, Allouche, Peyrière, Wen and Wen [2] studied the Hankel determinants of the Thue-Morse sequence $\mathbf{t}$ which is a famous 2-automatic sequence. They proved that the Hankel determinants of the ± 1 Thue-Morse sequence satisfy the congruence $H_n(\mathbf{t})/2^{n-1} \equiv 1 \pmod 2$ for all $n \geq 1$. Using this fact, Bugeaud [4] obtained the exact value of the irrationality exponent of the Thue-Morse number. Since then, several Hankel determinants of automatic sequences have been computed and several irrationality exponents of automatic numbers have been obtained. See [7, 10, 17, 5, 11] for example.

Apwenian sequences over $\{1, -1\}$ and $\{0, 1\}$ are studied in [8, 9, 1]. To study the non-purely automatic sequences over $\{0, 1\}$, let us slightly generalize the $\{0, 1\}$-apwenian sequences.

Definition 1. A nonnegative integer sequences $\mathbf{a} = \{a(n)\}_{n\geq 0}$ is called *apwenian*, if

$$a(0) = 1, \text{and } \forall\ n \geq 0, a(n) \equiv a(2n + 1) + a(2n + 2) \pmod 2. \tag{1}$$

As proved in [9], the sequence $\mathbf{a}$ is apwenian if and only if $H_n(\mathbf{a}) \equiv 1 \pmod 2$ for all $n \geq 1$. From Formula (1), Allouche et al. discovered that the apwenian sequences over $\{0, 1\}$ are the same as the sequences with perfect linear complexity profile (PLCP) up to indexing [1]. Here, the sequences with

*Ying-Jun GUO is the corresponding author.

Email addresses: guoyingjun2005@126.com (Ying-Jun GUO), guoniu.han@unistra.fr (Guo-Niu HAN)

perfect linear complexity profile were defined in the study of measures of randomness for binary sequences [15, 16, 13].

Automatic apwenian sequences have also been studied. In [9], Guo et al. proved that the only apwenian sequence over $\{0,1\}$ which is purely automatic is the period-doubling sequence. In [1], Allouche et al. proved that for an apwenian sequence $\{u(n)\}_{n\geq 0}$, it is 2-automatic if and only if the subsequence $\{u(2n+1)\}_{n\geq 0}$ is 2-automatic. Moreover, they defined a map φ_3 from the set of $\{0,1\}$-sequences to the set of apwenian sequences, and showed that the so-called *generalized Rueppel sequences* $\varphi_3(\mathbf{b})$ is 2-automatic if and only if the $\{0,1\}$-sequence $\mathbf{b}$ is ultimately periodic. They also mentionned that the period-doubling sequence is not a generalized Rueppel sequence.

In this paper, we study the automatic apwenian sequences over different alphabets. In section 2, we recall some notation and definitions. In section 3, we construct a class of apwenian sequences on the alphabet $\{0,1\}$ which contains the generalized Rueppel sequences and the period-doubling sequence. All the automatic apwenian sequences in this class are determined (see Theorem 1). In section 4, we consider the apwenian sequences on the alphabet $\{0,1,2\}$, and prove that the only apwenian sequence, among all purely automatic sequences, is the period-doubling like sequence (Theorem 2). In the last section, we give a characterization of 2-automatic apwenian sequences in terms of 2-uniform morphisms (Theorem 3). Moreover, we find two 3-automatic apwenian sequences on the alphabet $\{1,2,3\}$ (Examples 13 and 14). This answers the following question by Allouche et al. [1]: "*Are there PLCP/apwenian sequences that are d-automatic for some d not a power of 2?* "

2. Preliminary

In this section, we recall some notation and definitions which can be found in [3].

Let Σ be a finite alphabet. We denote the set of finite words over the alphabet Σ by Σ^*. For a finite word $W \in \Sigma^*$, we denote the length of W by $|W|$. If $|W| = 0$, we call W the *empty word*. Together with the concatenation operation, the set Σ^* forms a free monoid. Let $\Sigma^{\mathbb{N}}$ be the set of infinite words over the alphabet Σ. There is a natural metric on $\Sigma^{\mathbb{N}}$: for $\mathbf{u} = \{u(n)\}_{n\geq 0}, \mathbf{v} = \{v(n)\}_{n\geq 0} \in \Sigma^{\mathbb{N}}$,

$$dist(\mathbf{u}, \mathbf{v}) = 2^{-\min\{n\geq 0: u(n)\neq v(n)\}}.$$

Let Σ and Δ be two finite alphabets. A morphism is a map σ from Σ^* to Δ^* satisfying $\sigma(UV) = \sigma(U)\sigma(V)$ for all $U, V \in \Sigma^*$. If there is a constant $k \geq 1$ such that $|\sigma(a)| = k$ for all $a \in \Sigma$, then we say that σ is *k-uniform* on Σ. A 1-uniform morphism is called a *coding*. For a morphism σ, if an infinite sequence $\mathbf{u}$ satisfies $\sigma(\mathbf{u}) = \mathbf{u}$, then the sequence $\mathbf{u}$ is called a *fixed point of* σ.

If $\Sigma = \Delta$, we can iterate the application of σ. We define $\sigma^0(a) = a$ and $\sigma^n(a) = \sigma(\sigma^{n-1}(a))$ for all $a \in \Sigma, n \geq 1$. If there is a letter $a \in \Sigma$ such that $\sigma(a) = aW$ for some $W \in \Sigma^*$, and $|\sigma^n(a)| \to \infty$ when $n \to \infty$, then we say that the morphism σ is *prolongable* on a. If σ is a prolongable morphism on a, then the limit of $\sigma^n(a)$, denoted by $\sigma^\infty(a)$, always exists under the natural metric, i.e., $\sigma^\infty(a) = \lim_{n\to\infty} \sigma^n(a)$. Clearly, the sequence $\sigma^\infty(a)$ is a fixed point of σ.

Let $k \geq 2$ be an integer. We say a sequence $\{u(n)\}_{n\geq 0}$ is *k-automatic* if $u(n)$ is a finite-state function of the base-k expansion of n. Two equivalent definitions can be found in [6]. One is that a sequence is k-automatic if and only if it is the image, under a coding, of a fixed point of a k-uniform morphism. We also call the fixed point of a uniform morphism *a purely automatic* (or, more generally, *purely uniform morphic*) sequence. The other equivalent definition is that a sequence is k-automatic if and only if its k-kernel is finite, where the *k-kernel* of the sequence $\{u(n)\}_{n\geq 0}$, denoted by $\mathcal{K}_k(\mathbf{u})$, is defined as

$$\mathcal{K}_k(\mathbf{u}) = \{\{u(k^i n + j)\}_{n\geq 0} : i \geq 0, 0 \leq j \leq k^i - 1\}.$$

Throughout the paper, unless otherwise stated, we have the following assumptions and notation.

(1) Let $\mathbb{N}$ denote the set of all non-negative integers. All the alphabets we considered are finite subsets of $\mathbb{N}$.

(2) The symbol $\equiv$ means equality modulo 2.

(3) For any finite word $U = u(0)u(1)\cdots u(n), V = v(0)v(1)\cdots v(n)$, the symbol $U \equiv V$ means that $u(i) \equiv v(i)$ for all $i \in [0, n]$.

(4) For any k-uniform morphisms σ, τ over Σ, the symbol $\sigma \equiv \tau$ means that $\sigma(i) \equiv \tau(i)$ for all $i \in \Sigma$.

(5) Let N be an integer. The symbol $\{u(n)\}_{n=0}^{N}$ denote the finite word $u(0)u(1)\cdots u(N)$ if $N \geq 0$, and the empty word otherwise. Hence, we also write that $\{u(n)\}_{n\geq 0} = \{u(n)\}_{n=0}^{N} \cdot \{u(n)\}_{n\geq N+1}$.

(6) Let $n \geq 0$ and $b \geq 2$ be integers. The canonical base-b representation of n is denoted by the word $(n)_b$. For any $w_i w_{i-1} \cdots w_0 \in \{0, 1, \cdots, b-1\}^*$, the integer $\sum_{\ell=0}^{i} w_\ell b^\ell$ is denoted by $[w_i w_{i-1} \cdots w_0]_b$.

3. Extension of the generalized Rueppel sequences

Recall that a $\{0,1\}$-sequence $\{u(n)\}_{n\geq 0}$ is apwenian if $u(0) = 1$ and $u(n) \equiv u(2n+1) + u(2n+2)$ for all $n \geq 0$. Assume $\{u(n)\}_{n\geq 0}$ is apwenian and $u(2n+1)u(2n+2)$ is viewed as an image of $u(n)$ for some map ψ_n for every $n \geq 0$, i.e., $\psi_n(u(n)) = u(2n+1)u(2n+2)$, then ψ_n maps 1 to 10 or 01 and maps 0 to 00 or 11 for each n.

Consider the following four 2-uniform morphisms:

$$\tau_0 : 1 \mapsto 10, 0 \mapsto 00; \quad \tau_1 : 1 \mapsto 01, 0 \mapsto 00; \quad \tau_2 : 1 \mapsto 10, 0 \mapsto 11; \quad \tau_3 : 1 \mapsto 01, 0 \mapsto 11.$$

If $\psi_n \in \{\tau_0, \tau_1, \tau_2, \tau_3\}$ for all $n \geq 0$, then we can construct an apwenian sequence $\{u(n)\}_{n\geq 0}$ by taking $u(0) = 1$ and $u(2n+1)u(2n+2) = \psi_n(u(n))$ for all $n \geq 0$, i.e.,

$$u(0)u(1)u(2)\cdots = u(0)\psi_0(u(0))\psi_1(u(1))\psi_2(u(2))\cdots .$$

In particular, given a sequence $\sigma = \sigma_0\sigma_1\sigma_2\cdots \in \{\tau_0, \tau_1, \tau_2, \tau_3\}^{\mathbb{N}}$, taking $\psi_n = \sigma_{\lfloor \log_2(n+1) \rfloor}$ for all $n \geq 0$, then the generated sequence $\{u(n)\}_{n\geq 0}$, denoted by $\phi(\sigma)$, is called *the sequence generated by* σ. That is,

$$\phi(\sigma) := u(0)\sigma_0(u(0))\sigma_1(u(1))\sigma_1(u(2))\cdots = 1\sigma_0(1)\sigma_1(\sigma_0(1))\sigma_2(\sigma_1(\sigma_0(1)))\cdots .$$

Let $X_0 = 1$ and $X_{n+1} = \sigma_n(X_n)$ for all $n \geq 0$. Then, we have

$$\phi(\sigma) = \lim_{n\to\infty} X_0 X_1 \cdots X_n = \prod_{n=0}^{\infty} X_n = X_0 X_1 X_2 \cdots . \tag{2}$$

The sequences $\phi(\sigma)$ can be viewed as generated by a map ϕ from the set $\{\tau_0, \tau_1, \tau_2, \tau_3\}^{\mathbb{N}}$ to the set of the apwenian sequences. We illustrate the construction of $\phi(\sigma)$ with three basic cases.

Example 1. *[**The characteristic sequence of the powers of** 2] If $\sigma_n = \tau_0$ for all $n \geq 0$, then the sequence $\phi(\sigma)$ is the* characteristic sequence of the powers of 2. *Give a shift of indices by $\phi(\sigma) = \{v(n)\}_{n\geq 1}$. Then, for all $n \geq 1$, we have $v(2n)v(2n+1) = \tau_0(v(n)) = v(n)0$. Hence, the sequence $\phi(\sigma)$ satisfies that $v(n) = 1$ if $n = 2^k$ for some $k \geq 0$, and $v(n) = 0$ otherwise.*

Example 2. *[**The period-doubling sequence**] If $\sigma_n = \tau_3$ for all $n \geq 0$, then the sequence $\phi(\sigma)$ is the* period-doubling sequence. *Let $\phi(\sigma) = \{p(n)\}_{n\geq 0}$. Note that $p(0) = 1$ and $p(2n+1)p(2n+2) = \tau_3(p(n)) = (1-p(n))1$ for all $n \geq 0$. Hence, the sequence $\phi(\sigma)$ satisfies that $p(2n) = 1$ and $p(2n+1) = 1 - p(n)$ for all $n \geq 0$. This implies that $\phi(\sigma)$ is the period-doubling sequence.*

Example 3. *[**The generalized Rueppel sequences**] If $\sigma_n \in \{\tau_0, \tau_1\}$ for all $n \geq 0$, then the sequence $\phi(\sigma)$ is a* generalized Rueppel sequence *[15]. In fact, let $\sigma_i = \tau_{b_i}$ with $b_i \in \{0, 1\}$ for all $i \geq 0$ and let $\phi(\sigma) = \{v(n)\}_{n\geq 1}$. Since $v(2n)v(2n+1) = \sigma_{i_n}(v(n))$ for all $n \geq 1$, the sequence $\{v(n)\}_{n\geq 1}$ can be defined as follows:*

$$v(n) = \begin{cases} 1 & \text{if } n = n_i \text{ for some } i \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

where the integers n_i are defined by $n_0 = 1$ and $n_{i+1} = 2n_i + b_i$ for all $i \geq 0$.

Allouche et al. proved that the generalized Rueppel sequence $\phi((\tau_{b_i})_i)$ is 2-automatic if and only if $(b_i)_i$ is ultimately periodic [1]. We establish a similar result for the sequence generated by σ, as stated in the following Theorem.

Theorem 1. *Let $\sigma = \sigma_0\sigma_1\sigma_2\cdots \in \{\tau_0, \tau_1, \tau_2, \tau_3\}^{\mathbb{N}}$. Then the sequence $\phi(\sigma)$ is 2-automatic apwenian if and only if σ is ultimately periodic.*

To prove Theorem 1, we need some lemmas. Let us recall some properties of automatic sequences (see, e.g. [3]) firstly.

Lemma 1. *(1) If a sequence differs only in finitely many terms from a k-automatic sequence, then it is k-automatic.*

(2) Let $\mathbf{u}$ be a k-automatic sequence, and let ρ be a ℓ-uniform morphism for some $\ell \geq 1$. Then the sequence $\rho(\mathbf{u})$ is also k-automatic.

(3) Let $\{u(n)\}_{n\geq 0}$ and $\{v(n)\}_{n\geq 0}$ be k-automatic sequences. Then $\{u(n)+v(n)\}_{n\geq 0}$ and $\{u(n)v(n)\}_{n\geq 0}$ are k-automatic.

(4) Let $m \geq 1$ be an integer. Then a sequence is k-automatic if and only if it is k^m-automatic.

Lemma 2. *Let $\{u(n)\}_{n\geq 0}$ be a k-automatic sequence and j be an integer, then the subsequence $\{u(k^n + j)\}_{n\geq 0}$ is ultimately periodic. Moreover, the subsequences $\{u(k^n + j)\}_{n\geq 0}$ have the same period for all $j \geq 0$.*

Proof. The periodicity of $\{u(k^n + j)\}_{n\geq 0}$ follows from Theorem 5.5.2 in [3]. In fact, for any $j \geq 0$, let $(j)_k = W$, then $(k^n + j)_k = 10^{n-|W|}W$ for any $n \geq |W|$. Hence, for any $j \geq 0$, $(k^n + j)_k$ have the same periodic prefix when n is large enough. This implies that the subsequences $\{u(k^n + j)\}_{n\geq 0}$ have the same period. □

Lemma 3. *Let $S : \Sigma^* \to \Sigma^*$ be a k-uniform morphism and $W \in \Sigma^*$ be a finite non-empty word. Then the sequence $\prod_{n=0}^{\infty} S^n(W)$ is k-automatic.*

Proof. For each $i \in \{0, 1, \cdots, k-1\}$ and $a \in \Sigma$, let $S_i(a)$ be the $(i+1)$-th element of $S(a)$, then S_i are codings from Σ to Σ, and $S(a) = S_0(a)S_1(a)\cdots S_{k-1}(a)$ for all $a \in \Sigma$. Let $|W| = M$ with $M \geq 1$, and $\{v(n)\}_{n\geq 0} = \prod_{n=0}^{\infty} S^n(W)$, then $v(kn + M + i) = S_i(v(n))$ for all $n \geq 0$ and $i \in \{0, 1, \cdots, k-1\}$. Let $i \geq 1$ and $j \in [0, k^i - 1]$, then, for any $n \geq 0$, we have

$$v\left(k^i n + j + \frac{k^i - 1}{k - 1}M\right) = S_{j_0} \circ \cdots \circ S_{j_{i-2}} \circ S_{j_{i-1}}(v(n)),$$

where $[j_{i-1}j_{i-2}\cdots j_0]_k = j$.

Note that $S_{j_0} \circ \cdots \circ S_{j_{i-2}} \circ S_{j_{i-1}}$ are also codings from Σ to Σ for each $j \geq 0$, and there are finitely many codings from Σ to Σ. Hence, the set $\left\{\left\{v\left(k^i n + j + \frac{k^i-1}{k-1}M\right)\right\}_{n\geq 0} : i \geq 1, 0 \leq j \leq k^i - 1\right\}$ is finite.

For any $i \geq 1, j \in [0, k^i - 1]$, there exist integers $N \geq 0$ and $j' \in [0, k^i - 1]$ such that $\frac{k^i-1}{k-1}M - j = k^i N - j'$. So, we have

$$\begin{aligned}
\{v(k^i n + j)\}_{n\geq 0} &= \{v(k^i n + j)\}_{0\leq n\leq N-1} \cdot \{v(k^i n + j)\}_{n\geq N} \\
&= \{v(k^i n + j)\}_{0\leq n\leq N-1} \cdot \{v(k^i (n + N) + j)\}_{n\geq 0} \\
&= \{v(k^i n + j)\}_{0\leq n\leq N-1} \cdot \left\{v(k^i n + j' + \frac{k^i - 1}{k - 1}M)\right\}_{n\geq 0}.
\end{aligned}$$

Since $N = \frac{j'-j+\frac{k^i-1}{k-1}M}{k^i} \leq \frac{k^i+\frac{k^i}{k-1}M}{k^i} \leq 1 + \frac{M}{k-1}$, there are finitely many terms $\{v(k^i n + j)\}_{0\leq n\leq N-1}$. Hence, the set $\{\{v(k^i n + j)\}_{n\geq 0} : i \geq 0, 0 \leq j \leq k^i - 1\}$ is finite. This completes the proof. □

The proof of Lemma 3 also can be found in [12]. Now, let us prove Theorem 1.

Proof of Theorem 1. (i) The "if" part. Assume σ is ultimately periodic, i.e., there exist integers $N \geq 0, p \geq 1$ such that $\sigma_n = \sigma_{n+p}$ for all $n \geq N$. Then, by Formula (2), we have

$$\phi(\sigma) = \prod_{i=0}^{N-1} X_i \prod_{n=0}^{\infty} (\mu_0^n(X_N)\mu_1^n(X_{N+1})\cdots\mu_{p-1}^n(X_{N+p-1})),$$

where $\mu_i = \sigma_{N+p-1+i} \circ \sigma_{N+p-2+i} \circ \cdots \circ \sigma_{N+i}$ for $i \in [0, p-1]$.

For any $i \in [0, p-1], n \geq 0$, let $r_{i,n}, s_{i,n}$ be the length of the word $\mu_0^n(X_N)\cdots\mu_{i-1}^n(X_{N+i-1})$ and the word $\mu_{i+1}^n(X_{N+i+1})\cdots\mu_{p-1}^n(X_{N+p-1})$ respectively. That is, for all $n \geq 0$,

$$\begin{aligned} r_{0,n} = 0 \quad &\text{and} \quad r_{i,n} = |\mu_0^n(X_N)\cdots\mu_{i-1}^n(X_{N+i-1})|(1 \leq i \leq p-1), \\ s_{p-1,n} = 0 \quad &\text{and} \quad s_{i,n} = |\mu_{i+1}^n(X_{N+i+1})\cdots\mu_{p-1}^n(X_{N+p-1})|(0 \leq i \leq p-2). \end{aligned}$$

Define a $\{0,1\}$-sequence $\mathbf{u}^{(i)} = \prod_{n=0}^{\infty}(0^{r_{i,n}}\mu_i^n(X_{N+i})0^{s_{i,n}})$, then

$$\mathbf{u}^{(0)} + \mathbf{u}^{(1)} + \cdots + \mathbf{u}^{(p-1)} = \prod_{n=0}^{\infty}(\mu_0^n(X_N)\mu_1^n(X_{N+1})\cdots\mu_{p-1}^n(X_{N+p-1})).$$

Hence, we have

$$\phi(\sigma) = X_0X_1\cdots X_{N-1}(\mathbf{u}^{(0)} + \mathbf{u}^{(1)} + \cdots + \mathbf{u}^{(p-1)}).$$

Note that the morphisms μ_i are 2^p-uniform for all $i \in [0, p-1]$. Hence, $\frac{r_{i,n+1}}{r_{i,n}} = \frac{s_{i,n+1}}{s_{i,n}} = 2^p$ for all $n \geq 0$. Let $W_i = x^{r_{i,0}}X_{N+i}x^{s_{i,0}}$ be a finite word over $\{0,1,x\}$ and S be a 2^p-uniform morphism over $\{0,1,x\}$ defined by $S(x) = x^{2^p}$ and $S(y) = \mu_i(y)$ for $y \in \{0,1\}$. Then, we have $\mathbf{u}^{(i)} = \rho(\prod_{n=0}^{\infty} S^n(W_i))$, where ρ is a coding defined by $\rho(x) = 0, \rho(y) = y$ with $y \in \{0,1\}$. Hence, by Lemma 3 and Lemma 1, the sequences $\mathbf{u}^{(i)}$ are 2-automatic for all $i \in [0, p-1]$. This implies that $\phi(\sigma)$ is 2-automatic.

(ii) The "only if" part. Consider the following two cases.

Case 1: $\exists\ M \geq 0$ such that $\sigma_k \in \{\tau_0, \tau_1\}$ for all $k \geq M$. Let $\phi(\sigma) = \{v(n)\}_{n\geq 1}$, we see that $v(2n)v(2n+1) = \sigma_{\lfloor \log_2 n \rfloor}(v(n))$ for all $n \geq 1$. Note that $\tau_0(a) = a0$ and $\tau_1(a) = 0a$ for all $a \in \{0,1\}$. Hence, when $n \geq 2^M$, we have $v(2n)v(2n+1) = v(n)0$ or $v(2n)v(2n+1) = 0v(n)$. Let $\{b_i\}_i$ be a $\{0,1\}$-sequence and $\sigma_i = \tau_{b_i}$ for all $i \geq M$. Define $L_i = \{(n)_2 : 2^{M+i} \leq n < 2^{M+1+i}, v(n) = 1\}$ for all $i \geq 0$. Then $L_{i+1} = L_ib_{M+i} = \{wb_{M+i} : w \in L_i\}$ for all $i \geq 0$. Hence,

$$\begin{aligned} \{(n)_2 : v(n) = 1\} &= \{(n)_2 : 0 \leq n < 2^{M+1}, v(n) = 1\} \cup \bigcup_{i\geq 1}^{\infty} L_i \\ &= \{(n)_2 : 0 \leq n < 2^{M+1}, v(n) = 1\} \cup L_0\{b_Mb_{M+1}\cdots b_{M+i} : i \geq 0\}. \end{aligned}$$

Note that a binary sequence $\{v(n)\}_{n\geq 1}$ is 2-automatic if and only if $\{(n)_2 : v(n) = 1\}$ forms a regular set. Since the sets $\{(n)_2 : 0 \leq n < 2^{M+1}, v(n) = 1\}$ and L_0 are finite, the automaticity of $\{v(n)\}_{n\geq 1}$ implies the regularity of the set $\{b_Mb_{M+1}\cdots b_{M+i} : i \geq 0\}$. An easy classical result in [14, 18] asserts that the set of all prefixes of an infinite word is regular if and only if that word is ultimately periodic. Hence, if $\phi(\sigma)$ is 2-automatic, we conclude that the sequence $\{b_{M+i}\}_{i\geq 0}$ is ultimately periodic. This implies that the sequence $\sigma = \{\sigma_n\}_{n\geq 0}$ is ultimately periodic.

Case 2: There exist infinitely many k such that $\sigma_k \in \{\tau_2, \tau_3\}$. For any finite word W and integer i with $0 \leq i < |W|$, let $f_i(W)$ denote the $(i+1)$-th letter of W. Let $\phi(\sigma) = \{v(n)\}_{n\geq 0}$. Then, by Formula (2), we have $f_i(X_n) = v(2^n - 1 + i)$ for any $n \geq 0, i \in [0, 2^n - 1]$. Since $\{v(n)\}_{n\geq 0}$ is 2-automatic, by Lemma 2, we see that the sequences $\{f_i(X_n)\}_n$ are ultimately periodic with the same period P for all $i \geq 0$. Hence, for any fixed i and sufficiently large n, we have

$$\begin{aligned} \sigma_n(f_i(X_n)) &= f_{2i}(\sigma_n(X_n))f_{2i+1}(\sigma_n(X_n)) \\ &= f_{2i}(X_{n+1})f_{2i+1}(X_{n+1}) \\ &= f_{2i}(X_{n+1+P})f_{2i+1}(X_{n+1+P}) \\ &= f_{2i}(\sigma_{n+P}(X_{n+P}))f_{2i+1}(\sigma_{n+P}(X_{n+P})) \\ &= \sigma_{n+P}(f_i(X_{n+P})) \\ &= \sigma_{n+P}(f_i(X_n)). \end{aligned} \tag{3}$$

Now, we claim that $\{f_i(X_n) : 0 \leq i \leq 2^{P+2}\} = \{0,1\}$ for all large enough n. Note that if there exists an integer N such that $f_i(X_N) = 1$ for all $i \in [0, 2^{P+2}]$, then $f_i(X_{N-1}) = 0$ for all $i \in [0, 2^{P+1}]$. Hence, if the claim is false, then there exist infinitely many N such that $f_i(X_N) = 0$ for all $i \in [0, 2^{P+1}]$. This implies $f_0(X_{N-i})f_1(X_{N-i}) = 00$ for all $i \in [0, p]$. But this case can not happen. Since $\tau_2(0) = \tau_3(0) = 11, \sigma_k(1) \in \{10, 01\}$ for all k and $\sigma_k \in \{\tau_2, \tau_3\}$ for infinitely many k, the sequence $\{f_0(X_n)f_1(X_n)\}_n$ has infinitely many terms 01 or 10. Note that the sequence $\{f_0(X_n)f_1(X_n)\}_n$ is periodic with period P. So, we have $\{f_0(X_n)f_1(X_n) : j \leq n \leq j+P\} = \{0,1\}$ for all large enough j. This is a contradiction. Hence, our claim holds.

Fix i with $i \in [0, 2^{P+2}]$, then by this claim and Formula (3), we have $\sigma_n(a) = \sigma_{n+P}(a)$ for all $a \in \{0,1\}$ and for all large enough n. This completes the proof. □

Although the sequence $\phi(\sigma)$ contains a large class of automatic apwenian sequences, this map ϕ is not a bijection.

Example 4. *Let $\mathbf{p} = \{p(n)\}_{n\geq 0}$ be the period-doubling sequence which satisfies $p(2n) = 1, p(2n+1) = 1 - p(n)$ for all $n \geq 0$. Let $\rho : 1 \mapsto 11, 0 \mapsto 00$ be a 2-uniform morphism. Then it is easy to check from Formula (1) that the sequence $\rho(\mathbf{p})$ is apwenian. But, the first few terms of the sequence*

$$\{u(n)\}_{n\geq 0} = \rho(\mathbf{p}) = 1100111111001100\cdots,$$

show that $u(7)u(8)\cdots u(14) = X_3 \neq \sigma(X_2) = \sigma(u(3)u(4)u(5)u(6))$ for all $\sigma \in \{\tau_0, \tau_1, \tau_2, \tau_3\}$. Hence, by Formula (2), the sequence $\rho(\mathbf{p})$ can not be generated by any $\sigma \in \{\tau_0, \tau_1, \tau_2, \tau_3\}^{\mathbb{N}}$.

Let $\Sigma \subset \mathbb{N}$ be a finite set and σ be a k-uniform morphism over Σ. For any $a \in \Sigma$, let $\sigma[i](a)$ be the $(i+1)$-th element of $\sigma(a)$. If a k-uniform morphism σ satisfies that $a \equiv \sigma[0](a) + \sigma[1](a) + \cdots + \sigma[k-1](a)$ for all $a \in \Sigma$, then we call it *sum-equivalent.* Given a finite alphabet $\Sigma \subset \mathbb{N}$, define a finite set

$$\mathcal{A}(\Sigma) = \{\sigma : \sigma \text{ is a 2-uniform sum-equivalent morphism over } \Sigma\}.$$

It is clear that $\mathcal{A}(\{0,1\}) = \{\tau_0, \tau_1, \tau_2, \tau_3\}$. Assume $1 \in \Sigma$ and $\sigma = \sigma_0\sigma_1\sigma_2\cdots$ is a sequence over $\mathcal{A}(\Sigma)$. We can construct an apwenian sequence in the following way. Define $X_0 = 1$ and $X_{n+1} = \sigma_n(X_n)$ for all $n \geq 0$, then we obtain a sequence $\phi(\sigma)$ over Σ that

$$\phi(\sigma) = \lim_{n\to\infty} X_0X_1\cdots X_n = \prod_{n=0}^{\infty} X_n = X_0X_1X_2\cdots.$$

Let $\{u(n)\}_{n\geq 0} = \phi(\sigma)$, then $u(2n+1)u(2n+2) = \sigma_k(u(n))$ for all $n \geq 0$ with $k = \lfloor \log_2(n+1) \rfloor$. Since each σ_k is 2-uniform sum-equivalent, we have $u(2n+1) + u(2n+2) \equiv u(n)$ for all $n \geq 0$. Hence, the sequences $\phi(\sigma)$ are apwenian over Σ. Similarly, the "if" part of Theorem 1 also holds for $\mathcal{A}(\Sigma)$. That is, if $\sigma = \sigma_0\sigma_1\sigma_2\cdots$ is ultimately periodic over $\mathcal{A}(\Sigma)$, then the sequence $\phi(\sigma)$ is 2-automatic over Σ.

Example 5. *Let $\tau : 1 \mapsto 30, 0 \mapsto 31, 3 \mapsto 01$ be a 2-uniform morphism and $\sigma = \tau\tau\tau\cdots$ be a periodic sequence over $\mathcal{A}(\{0,1,3\})$. Then the sequence $\phi(\sigma)$ molulo 2 is just the sequence $\rho(\mathbf{p})$ defined in Example 4, i.e., $\phi(\sigma) \pmod 2 = \rho(\mathbf{p})$.*

To see this, we assume that $\{u(n)\}_{n\geq 0} = \phi(\sigma)$. Since $u(2n+1)u(2n+2) = \tau(u(n))$ for all $n \geq 0$ and $\tau[0](a) \in \{0,3\}, \tau[1](a) \in \{0,1\}$ for all $a \in \{0,1,3\}$, we have $u(2n) \in \{0,1\}$ and $u(2n+1) \in \{0,3\}$ for all $n \geq 0$. By the definition of τ, we see that if $a \in \{0,1\}$, we have $\tau[0](a) \equiv 1, \tau[1](a) \equiv 1 + a$, and if $a \in \{0,3\}$, we have $\tau[0](a) \equiv 1 + a, \tau[1](a) \equiv 1$. Hence, the fact $\tau(u(2n)) = u(4n+1)u(4n+2)$ tells us that for all $n \geq 0$,

$$u(4n+1) \equiv 1, \quad u(4n+2) \equiv 1 + u(2n). \tag{4}$$

Similarly, the fact $\tau(u(2n+1)) = u(4n+3)u(4n+4)$ and $u(0) = 1$ implies that for all $n \geq 0$,

$$u(4n+3) \equiv 1 + u(2n+1), \quad u(4n) \equiv 1. \tag{5}$$

Thus, by Formula (4) and (5), we see that the sequence $\{u(n) \pmod 2\}_{n\geq 0} = \rho(\mathbf{p})$, where $\rho : 1 \mapsto 11, 0 \mapsto 00$ and $\mathbf{p}$ is the period-doubling sequence.

Example 4 and Example 5 tell us that, although a 2-automatic apwenian sequence over $\{0,1\}$ can not be generated by a sequence σ over $\mathcal{A}(\{0,1\})$, it can be generated by a sequence σ over $\mathcal{A}(\{0,1,3\})$ with projection mod 2. Hence, we have the following conjecture.

Conjecture 1. *Let $\{u(n)\}_{n\geq 0}$ be a 2-automatic apwenian sequence over $\{0,1\}$. Then there exist a finite alphabet $\Sigma \subset \mathbb{N}$ and a sequence σ over $\mathcal{A}(\Sigma)$ such that*

$$\{u(n)\}_{n\geq 0} = \phi(\sigma) \pmod 2.$$

4. Automatic apwenian sequences over $\{0,1,2\}$

In this section, we consider the automatic apwenian sequences over the alphabet $\{0,1,2\}$. Let $p \geq 2$ be an integer. We define a p-uniform morphism over $\{0,1,2\}$ by

$$\sigma : 1 \mapsto u(0)u(1)\cdots u(p-1), \quad 0 \mapsto v(0)v(1)\cdots v(p-1), \quad 2 \mapsto w(0)w(1)\cdots w(p-1), \tag{6}$$

where $u(0) = 1$ and $u(i), v(i), w(i) \in \{0,1,2\}$ for all i. If $\sigma^\infty(1) = \{a(n)\}_{n\geq 0}$, then for all $n \geq 0$,

$$\sigma(a(n)) = a(np)a(np+1)\cdots a(np+p-1). \tag{7}$$

Clearly, the purely p-uniform morphic sequences $\sigma^\infty(1)$ defined in (6) are purely p-automatic sequences starting with 1 over $\{0,1,2\}$. It is interesting that the purely automatic apwenian sequence over $\{0,1,2\}$ is the period-doubling like sequence. We say a sequence $\{a(n)\}_{n\geq 0}$ is a *period-doubling like sequence* if $a(n) \equiv p(n)$ for all $n \geq 0$, where $\{p(n)\}_{n\geq 0}$ is the period-doubling sequence $\sigma^\infty(1)$ given by $\sigma : 1 \mapsto 10, 0 \mapsto 11$.

Theorem 2. *Let $\{a(n)\}_{n\geq 0}$ be a purely automatic sequences over $\{0,1,2\}$. Then the sequence $\{a(n)\}_{n\geq 0}$ is apwenian if and only if $\{a(n)\}_{n\geq 0}$ is a period-doubling like sequence.*

One direction is clear. This is because that if $\{a(n)\}_{n\geq 0}$ is a period-doubling like sequence over $\{0,1,2\}$, then $a(2n) = 1$ and $a(2n+1) \equiv 1 - a(n)$. So, the sequence $\{a(n)\}_{n\geq 0}$ is apwenian, i.e., $a(0) = 1$ and $a(n) \equiv a(2n+1) + a(2n+2)$ for all n. Now, we turn to the other direction. Assume the sequence $\{a(n)\}_{n\geq 0}$ is an apwenian sequence and can be generated by the morphism defined in (6). We will prove that $\{a(n)\}_{n\geq 0}$ is a period-doubling like sequence by the following two key steps.

(1) Show that $u(0) = v(0) = w(0) = 1$ (Lemma 6);

(2) Show that $u(p-1) \not\equiv v(p-1) \equiv w(p-1)$ (Lemma 8).

Before doing this, we need some lemmas. Since the exchange of even numbers in the alphabet do not change the sequence $\{a(n) \pmod 2\}_{n\geq 0}$, we always assume that 0 is the first even letter that appears in $\{a(n)\}_{n\geq 0}$, i.e., $\min\{n \geq 0 : a(n) = 0\} < \min\{n \geq 0 : a(n) = 2\}$.

Lemma 4. *Let $\{a(n)\}_{n\geq 0}$ be a purely automatic apwenian sequence over $\{0,1,2\}$, then $a(0)a(1)a(2) = 101$.*

Proof. Assume $\{a(n)\}_{n\geq 0}$ is defined in (6). Since $\{a(n)\}_{n\geq 0}$ is apwenian, we have $a(0) \equiv a(1) + a(2)$. Note that $a(0) = 1$. Hence, either $a(1)a(2) = 01$ or $a(1)a(2) = 10$. Now, we prove that the second case will not happen. Assume $a(1)a(2) = 10$. Then, we claim that for any $n \geq 1$,

1) $a(2^n - 1) = 1$,

2) $a(2^n p + 2^m - 1) + a(2^n p + 2^m) + \cdots + a(2^n p + 2^{m+1} - 2) \equiv 0$ $(0 \leq m \leq n-1)$,

3) $a(2^n p + 2^n - 1) + a(2^n p + 2^n) + \cdots + a(2^n p + 2^{n+1} - 2) \equiv 1$.

We prove this claim by induction on n. Since $a(0) = a(1) = 1$, by Formula (7), we have $1 = a(p) \equiv a(2p+1) + a(2p+2)$ and $a(p-1) = a(2p-1)$. Note that $a(p-1) \equiv a(2p-1) + a(2p)$. So, we have $a(2p) \equiv 0$. This implies that this claim holds for $n = 1$. Suppose that this claim holds for $n \leq k$, we consider the case $n = k+1$.

Note that $a(n) \equiv a(2n+1) + a(2n+2)$ for all n. Hence, by induction, we have $a(2^{k+1}p + 2^m - 1) + a(2^{k+1}p + 2^m) + \cdots + a(2^{k+1}p + 2^{m+1} - 2) \equiv 0$ for $m \in [1,k]$ and $a(2^{k+1}p + 2^{k+1} - 1) + a(2^{k+1}p + 2^{k+1}) + \cdots + a(2^{k+1}p + 2^{k+2} - 2) \equiv 1$. So, the assertion 2) holds for $m \in [1,k]$ and the assertion 3) holds. When $m = 1$, we have $a(2^{k+1}p + 1) + a(2^{k+1}p + 2) \equiv 0$. This implies that $a(2^{k+1}) \neq 1$ by Formula (7). So, we have $a(2^{k+1}) \equiv 0$. Hence, by induction, we have $a(2^{k+1} - 1) \equiv a(2^{k+1}) + a(2^k - 1) = 1$ which implies that the assertion 1) holds. Since $\sigma(a(2^k - 1)) = \sigma(a(2^{k+1} - 1)) = \sigma(1)$, by Formula (7), we have $a(2^k p - 1) = a(2^{k+1}p - 1)$. So, we have $a(2^{k+1}p) \equiv 0$. Hence, the assertion 2) holds for $m = 0$. This completes our claim.

Since the sequence $\{a(n)\}_{n\geq 0}$ defined in (6) is also be generated by σ^i for all $i \geq 1$, we always assume p is large enough. Then, by Formula (7), the claim shows that $\sigma(a(2^n))$ are different for each n. Hence, $a(2^n)$ are different which contradicts the fact that $a(n)$ takes finitely many values. □

Lemma 5. *Let $\{a(n)\}_{n\geq 0}$ be a purely automatic sequence over $\{0,1,2\}$ and $\{p(n)\}_{n\geq 0}$ be the period-doubling sequence. For any integer $N\geq 1$, if $a(n)\in\{0,1\}$ for all $n\leq 2N$, then $a(n)=p(n)$ for all $n\leq 2N$.*

Proof. By Lemma 4, we see that the conclusion holds for $N=1$. Now, assume the conclusion holds for $n\leq 2N$ and $a(n)\in\{0,1\}$ for all $n\leq 2N+2$. We need to prove that $a(2N+i)=p(2k+i)$ for $i=1,2$, i.e., $a(2N+1)=1-a(N)$ and $a(2N+2)=1$. Assume $\{a(n)\}_{n\geq 0}$ is defined in (6). By Formula (7), we note that if $a(n)\in\{0,1\}$, then $a(np)=1$. Hence, we have $a(2(N+1)p)=1$. So, we have $a((N+1)p-1)\equiv a(2(N+1)p-1)+a(2(N+1)p)\equiv a(2(N+1)p-1)+1\neq a(2(N+1)p-1)$. Since $a((N+1)p-1)$ is the last letter of $\sigma(a(N))$ and $a(2(N+1)p-1)$ is the last letter of $\sigma(a(2N+1))$, we have $a(N)\neq a(2N+1)$. Since $a(N),a(2N+1),a(2N+2)\in\{0,1\}$ and $a(N)\equiv a(2N+1)+a(2N+2)$, we have $a(2N+2)=1$ and $a(2N+1)=1-a(N)$. This completes the proof. ☐

Lemma 6. *Let $\{a(n)\}_{n\geq 0}$ be an apwenian sequence defined in (6), then $u(0)=v(0)=w(0)=1$.*

Proof. By Lemma 4, we see that $u(0)=a(0)=a(2)=1$ and $a(1)=0$. So, by Formula (7), we have $a(2p+i)=a(i)$ for all $i\in[0,p-1]$ and $v(0)=a(p)$. Hence, we have

$$v(0)=a(p)\equiv a(2p+1)+a(2p+2)=a(1)+a(2)\equiv a(0)=1.$$

Now, we prove that $w(0)\equiv 0$ can not happen. If $w(0)\equiv 0$, by Formula (7), we see that $a(n)=2$ if and only if $a(np)=w(0)\equiv 0$. Note that $a(np)\equiv a(2np+1)+a(2np+2)$ for all n, and $a(np+1)+a(np+2)\equiv 1$ when $a(n)\in\{0,1\}$. Hence, we have

$$a(n)=2 \text{ if and only if } a(2n)=2. \tag{8}$$

This implies that $\min\{n: a(n)=2\}$ is odd. Suppose that N is the least number such that $a(2N+1)=2$. Then $a(j)\in\{0,1\}$ for all $j\leq 2N$, and $a(N)=a(2N+2)\in\{0,1\}$. By Lemma 5, we have $a(n)=p(n)$ for all $n\leq 2N$, where $\{p(n)\}_{n\geq 0}$ is the period-doubling sequence.

(1) Case 1: p is even. Assume $p=2q$. Then, by Formula (7), we see that for any $n\geq 0$,

$$\sigma(a(n))=a(2nq)a(2nq+1)\cdots a(2nq+2q-1). \tag{9}$$

Since $a(2N)=p(2N)=1=a(0)$ and $\sigma(a(2N+1))=\sigma(2)$ is starting with an even number $w(0)$, by Formula (9), we have $a(4Nq+2q-1)=a(2q-1)$ and $a(4Nq+2q)=w(0)\equiv 0$. Hence, we have $a(2Nq+q-1)\equiv a(4Nq+2q-1)+a(4Nq+2q)\equiv a(2q-1)$.

- If $a(2N+2)=1$, then $a(N)=1$. Since $a(N)=a(0)$ and $\sigma(0)=\sigma(a(1))$ is starting with 1, by Formula (9), we have $a(2Nq+q-1)=a(q-1)$ and $a(2q)=1$. So, $a(2Nq+q-1)=a(q-1)\equiv a(2q-1)+a(2q)\equiv a(2q-1)+1$. This is a contradiction.
- If $a(2N+2)=0$, then $a(N)=0$.
  - When $N=1$. We have $a(3)=2, a(4)=0$. Then $a(6q)=0$ and $a(3q-1)\equiv a(6q-1)+a(6q)\equiv a(6q-1)=a(2q-1)$. Since $a(0)=a(2)=1$ and $a(1)=a(4)=0$, by Formula (9), we have $a(5q-1)=a(q-1)=a(2q-1)+1$ and $a(10q-1)=a(4q-1)=a(2q-1)+a(2q)=a(2q-1)+1$. So, we have $a(10q)\equiv a(5q-1)+a(10q-1)\equiv 0$. This implies that $a(5)=2$. Hence, $a(12q-1)=a(8q-1)$. Note that $a(8q)=1$ and $a(6q-1)=a(2q-1)$. We see that $a(12q-1)=a(8q-1)\equiv a(4q-1)+a(8q)=a(2q-1)$ and $a(12q)\equiv a(12q-1)+a(6q-1)\equiv 0$. This implies that $a(6)=2$. So, we have $a(5)=a(6)=2$. This contradicts the fact that $1=a(2)\equiv a(5)+a(6)$.
  - When $N\geq 2$. Since $a(2)=1$ and $a(n)=p(n)\in\{0,1\}$ for all $n\leq 2N$, we have $a(3)=p(3)=a(4)=p(4)=1$. Since $a(6q)=1$, we have $a(3q-1)\equiv a(6q-1)+a(6q)=a(6q-1)+1=a(2q-1)+1$. Note that $\sigma(a(N))=\sigma(a(1))=\sigma(0)$. Hence, by Formula (9), we have $a(2Nq+q-1)=a(3q-1)\equiv a(2q-1)+1$, which is a contradiction.

(2) **Case 2**: p is odd. We will obtain a contradiction that there exists an odd number $M\geq 1$ such that $a(M)=1$ and $a(2n+1)a(2n+2)\neq 10$ for all $n\geq 0$. In fact, if $a(M)=a(0)=1$ for some odd number $M\geq 1$, by Formula (7), we have $a(Mp)a(Mp+1)=a(0)a(1)=10$. Note that Mp is odd. Hence, there exists n such that $a(2n+1)a(2n+2)=10$, which is a contradiction.

- If $a(2N+2)=1$, then $a(N)=1$. By Lemma 4, we see that $N\geq 2$. Hence, $a(3)=p(3)=1$. This implies that there exists an odd number $M\geq 1$ such that $a(M)=1$. Denote $a(p-1)=x$ and $\bar{x}=x+1$. Then $a(2p-1)\equiv a(p-1)+a(2p)=x+1=\bar{x}$. Since $a(N)=a(2N+2)=1, a(2N+1)=2$ and $\sigma(2)$ is starting with an even number $w(0)$, we have $a(2(N+1)p-1)\equiv a((N+1)p-1)+a(2(N+1)p)=a(p-1)+1=\bar{x}$. Hence, we have

$$\sigma(1)=1Ux, \quad \sigma(0)\equiv 1V\bar{x}, \quad \sigma(2)\equiv 0W\bar{x}, \tag{10}$$

  where U,V,W are finite words with length $p-2$. Now, we prove that $a(2n+1)a(2n+2)\neq 10$ for all $n\geq 0$. If $a(2n+1)a(2n+2)=10$ for some n, then $a(n)=1$. Thus, we have the following cases.

  - When $a(n)=1, a(n+1)\in\{0,1\}$. Note from Formula (8) that $a(2(n+1))\in\{0,1\}$. By Formula (7), we have $a((n+1)p-1)=x$ and $a(2(n+1)p)=1$. Hence, $a(2(n+1)p-1)\equiv a((n+1)p-1)+a(2(n+1)p)=x+1=\bar{x}$. Since $a(2(n+1)p-1)$ is the last letter of $\sigma(a(2n+1))$, by Formula (10), we see that $a(2n+1)\in\{0,2\}$. So, we have $a(2n+2)=1$ which implies $a(2n+1)a(2n+2)\neq 10$.
  - When $a(n)=1, a(n+1)=2$. Similarly, from Formula (8), we have $a(2(n+1))=2$. Hence, we have $a(2n+1)=1$, i.e., $a(2n+1)a(2n+2)=12\neq 10$.

- If $a(2N+2)=0$, then $a(N)=0$. Similarly, denote $a(p-1)=x$ and $\bar{x}=x+1$. Then $a(2p-1)\equiv a(p-1)+a(2p)=x+1=\bar{x}$. Since $a(N)=a(2N+2)=0, a(2N+1)=2$ and $\sigma(2)$ is starting with an even number $w(0)$, we have $a(2(N+1)p-1)\equiv a((N+1)p-1)+a(2(N+1)p)=a(2p-1)+1\equiv x$. Hence, we have

$$\sigma(1)=1Ux, \quad \sigma(0)\equiv 1V\bar{x}, \quad \sigma(2)\equiv 0Wx, \tag{11}$$

  where U,V,W are finite words with length $p-2$. Now, we show that $a(2n+1)a(2n+2)\neq 10$ for all $n\geq 0$. If $a(2n+1)a(2n+2)=10$ for some n, then $a(n)=1$. We have the following cases.

  - When $a(n)=1, a(n+1)\in\{0,1\}$. Similarly, from Formula (8), we have $a(2(n+1))\in\{0,1\}$. By Formula (7), we have $a((n+1)p-1)=x$ and $a(2(n+1)p)=1$. Hence, $a(2(n+1)p-1)\equiv a((n+1)p-1)+a(2(n+1)p)=x+1=\bar{x}$. Since $a(2(n+1)p-1)$ is the last letter of $\sigma(a(2n+1))$, by Formula (11), we see that $a(2n+1)=0$. So, we have $a(2n+2)=1$ and $a(2n+1)a(2n+2)=01\neq 10$.
  - When $a(n)=1, a(n+1)=2$. Similarly, from Formula (8), we have $a(2(n+1))=2$. Hence, we have $a(2n+1)=1$, i.e., $a(2n+1)a(2n+2)=12\neq 10$.

  To end this proof, we need to find the odd number $M\geq 1$ satisfying $a(M)=1$. Consider the number N.

  - When $N=1$. Then $a(3)=2$. Note from Formula (8) that if $a(n)a(n+1)=12$, then $a(2n+1)a(2n+2)=12$. Hence, $a(2)a(3)=12$ implies that $a(5)a(6)=12$. Thus, we have $a(5)=1$.
  - When $N\geq 2$. Then $a(3)=p(3)=1$.

Therefore, our assumption that $w(0)\equiv 0$ is false. Hence, $w(0)=1$. □

Lemma 7. *Let $\{a(n)\}_{n\geq 0}$ be an apwenian sequence and $\{p(n)\}_{n\geq 0}$ be the period-doubling sequence. Let $k\geq 1$ be an integer. Then $a(n)\equiv p(n)$ for all $n\in[0,2^{k+1}]$ if and only if $a(2^{k+1})=1$ and $a(2^k+n)\equiv a(2^k+2^{k-1}+n)\equiv a(n)$ for all $n\in[0,2^{k-1}-1]$.*

Proof. Let $\{p(n)\}_{n\geq 0}$ be the period-doubling sequence generated by the morphsim ψ with $\psi(1)=10, \psi(0)=11$. If $a(n)\equiv p(n)$ for all $n\in[0,2^{k+1}]$, then $a(2^{k+1})=1$ and $a(0)a(1)\cdots a(2^{k+1}-1)\equiv\psi^{k+1}(1)$. Since $\psi^{k+1}(1)=\psi^k(10)=\psi^{k-1}(1011)=\psi^{k-1}(1)\psi^{k-1}(0)\psi^{k-1}(1)\psi^{k-1}(1)$, we have $a(2^k+n)\equiv a(2^k+2^{k-1}+n)\equiv a(n)$ for all $n\in[0,2^{k-1}-1]$.

Now consider the 'if' part. Assume $a(2^{k+1})=1$ and $a(2^k+n)\equiv a(2^k+2^{k-1}+n)\equiv a(n)$ for all $n\in[0,2^{k-1}-1]$. We claim that $a(2^{k-1}+n)\equiv a(2^{k-1}+2^{k-2}+n)\equiv a(n)$ for all $n\in[0,2^{k-2}-1]$. To prove this claim, we consider the following two cases.

- If $0 \le n \le 2^{k-2} - 2$, then $0 \le 2n + 1 < 2n + 2 \le 2^{k-1} - 2 < 2^{k-1} - 1$. Hence, by the hypothesis, we have

$$\begin{aligned} a(2^k + 2n + 1) &\equiv a(2^k + 2^{k-1} + 2n + 1) \equiv a(2n + 1), \\ a(2^k + 2n + 2) &\equiv a(2^k + 2^{k-1} + 2n + 2) \equiv a(2n + 2). \end{aligned}$$

  So, by the apwenian property of the sequence $\{a(n)\}_{n\ge0}$, we have

$$\begin{aligned} a(2^{k-1} + n) &\equiv a(2^k + 2n + 1) + a(2^k + 2n + 2) \equiv a(2n + 1) + a(2n + 2) \equiv a(n), \\ a(2^{k-1} + 2^{k-2} + n) &\equiv a(2^k + 2^{k-1} + 2n + 1) + a(2^k + 2^{k-1} + 2n + 2) \\ &\equiv a(2n + 1) + a(2n + 2) \equiv a(n). \end{aligned}$$

- If $n = 2^{k-2} - 1$, then

$$\begin{aligned} a(2^k + 2n + 1) &= a(2^k + 2^{k-1} - 1) \equiv a(2^{k-1} - 1), \\ a(2^k + 2n + 2) &= a(2^k + 2^{k-1}) \equiv a(0). \end{aligned}$$

  Note that $a(2^{k-1}) \equiv a(2^k + 1) + a(2^k + 2) \equiv a(1) + a(2) \equiv a(0) = 1$. Hence,

$$\begin{aligned} a(2^{k-1} + 2^{k-2} - 1) &\equiv a(2^k + 2^{k-1} - 1) + a(2^k + 2^{k-1}) \\ &\equiv a(2^{k-1} - 1) + a(0) \\ &\equiv a(2^{k-1} - 1) + a(2^{k-1}) \\ &\equiv a(2^{k-2} - 1). \\ a(2^{k-1} + 2^{k-2} + 2^{k-2} - 1) &\equiv a(2^k + 2^{k-1} + 2^{k-1} - 1) + a(2^k + 2^{k-1} + 2^{k-1}) \\ &\equiv a(2^{k-1} - 1) + a(2^{k+1}) \\ &\equiv a(2^{k-1} - 1) + 1 \\ &\equiv a(2^{k-1} - 1) + a(2^{k-1}) \\ &\equiv a(2^{k-2} - 1). \end{aligned}$$

Therefore, this claim holds. Repeating the argument of this claim, we know that for any $\ell \in [1, k]$ and $n \in [0, 2^{\ell-1} - 1]$, $a(2^\ell + n) \equiv a(2^\ell + 2^{\ell-1} + n) \equiv a(n)$. This implies that $a([10w]_2) = a([11w]_2) = a([w]_2)$ for any $w \in \{0, 1\}^*$ with $|w| \le k - 1$. Note that $a([10]_2) = a(2) = 1 = a(0) = a([00]_2)$. Hence, $a(2n) = 1$ for all $n \le 2^k$. So, $a(2n + 1) \equiv a(n) + a(2n + 2) \equiv a(n) + 1$ for all $n \le 2^k$. Thus, $a(n) \equiv p(n)$ for all $n \in [0, 2^{k+1}]$. □

Lemma 8. *Let $\{a(n)\}_{n\ge0}$ be an apwenian sequence defined in (6), then $u(p-1) \not\equiv v(p-1) \equiv w(p-1)$.*

Proof. Denote $u(p - 1) = a(p - 1) = x$ and $\bar{x} = x + 1$. By Lemma 4, we have $v(p - 1) = a(2p - 1) \equiv a(p - 1) + a(2p) = x + 1 = \bar{x}$. Now, assume $v(p - 1) \not\equiv w(p - 1)$. Then, by Lemma 6, we have

$$\sigma(1) = 1Ux, \quad \sigma(0) \equiv 1V\bar{x}, \quad \sigma(2) \equiv 1Wx, \tag{12}$$

where U, V, W are finite words with length $p - 2$.

By Formula (12), we note that $a(np) = 1$ for all $n \ge 0$, and $a(np+p-1) \equiv x$ if and only if $a(n) \in \{1, 2\}$. Hence, $a(n) \in \{1, 2\}$ if and only if $a(2(n+1)p-1) \equiv a(np+p-1) + a(2(n+1)p) = a(np+p-1) + 1 = x+1 = \bar{x}$. Note that $a(2(n+1)p-1)$ is the last letter of $\sigma(a(2n+1))$. Hence, by Formula (12), we have

$$a(n) \in \{1, 2\} \text{ if and only if } a(2n + 1) = 0. \tag{13}$$

Since $\{a(n)\}_{n\ge0}$ is apwenian, by Formula (13), we see that $a(n) = 1$ implies $a(2n + 1)a(2n + 2) = 01$. If p is odd, then, by Lemma 4 and Formula (7), we have $a(p)a(p + 1) \equiv 10 \ne 01$. This is a contradiction. Hence, p is even. Assume $p = 2q$. Then, $\sigma(a(n)) = a(2qn)a(2qn + 1) \cdots a(2qn + 2q - 1)$ for all $n \ge 0$.

Now, we show firstly that $a(2n) \in \{1, 2\}$ for all n. Since $a(0) = a(2) = 1$ and $a(1) = 0$, by Lemma 4, we see that $a(q-1) \equiv a(3q-1) \equiv \bar{x}$. Hence, $a(2qn+q-1) \equiv \bar{x}$ when $a(n) \in \{0, 1\}$. Note from Formula (13) that $\min\{n : a(n) = 2\}$ is odd. Assume N is the least number satisfying $a(2N+1) = 2$. Then $a(n) \in \{0, 1\}$ for all $n \le 2N$. By Lemma 5, we have $a(2N) = p(2N) = 1$. Hence, $a(4N + 1)a(4N + 2) = 01$. So, by

Formula (7) and the apwenian property of $\{a(n)\}_{n\geq 0}$, we have $a(2q(2N+1)+q-1) \equiv \bar{x}$. That is, $a(2qn+q-1) \equiv \bar{x}$ when $a(n)=2$. Thus, we have $a(2qn+q-1) \equiv \bar{x}$ for all $n \geq 0$. Since $a(2qn)=1$ for all $n \geq 0$, we have $a(2q2n+2q-1) \equiv a(2qn+q-1)+a(2q2n+2q) \equiv \bar{x}+1 = x$ for all $n \geq 0$. This implies that the last letter of $\sigma(a(2n))$ equals to x molulo 2. By Formula (12), we have $a(2n) \in \{1,2\}$ for all n.

Define a coding ρ from $\{0,1,2\}$ to $\{0,1\}$ by $\rho(0)=0, \rho(1)=\rho(2)=1$. Then, $\rho(a(2n))=1$ for all n. Moreover, by Formula (13), we have $\rho(a(2n+1)) = 1-\rho(a(n))$. Hence, $\{\rho(a(n))\}_{n\geq 0}$ is the period-doubling sequence which is 2-automatic. So, by Lemma 1, the p-automatic sequence $\{a(n)\}_{n\geq 0}$ must satisfy that $p=2^k$ for some k.

Then, we claim that $a(2^k+i) \equiv a(2^k+2^{k-1}+i) \equiv a(i)$ for all $i \in [0, 2^{k-1}-1]$. Since $a(1)=0$, by Formula (13), we see that $a(3) \in \{1,2\}$. Hence, we have the following cases.

- If $a(3)=1$, then $\sigma(a(0)) = \sigma(a(2)) = \sigma(a(3)) = \sigma(1)$. Since $p=2^k$, by Formula (7), we have $a(i) = a(2\cdot 2^k+i) = a(3\cdot 2^k+i)$ for all $i \in [0, 2^k-1]$. Note that $a(np) = a(n\cdot 2^k) = 1$ for all $n \geq 0$. Hence, we have $a(i) = a(2\cdot 2^k+i) = a(3\cdot 2^k+i)$ for any $i \in [0, 2^k]$. Now fix i satisfying $i \in [0, 2^{k-1}-1]$. Since $0 \leq 2i+1 < 2i+2 \leq 2^k$, we have

  $$a(2^k+i) \equiv a(2\cdot 2^k+2i+1)+a(2\cdot 2^k+2i+2) = a(2i+1)+a(2i+2) \equiv a(i),$$
  $$a(2^k+2^{k-1}+i) \equiv a(3\cdot 2^k+2i+1)+a(3\cdot 2^k+2i+2) = a(2i+1)+a(2i+2) \equiv a(i).$$

  Hence, we have $a(2^k+i) \equiv a(2^k+2^{k-1}+i) \equiv a(i)$ for all $i \in [0, 2^{k-1}-1]$.

- If $a(3)=2$, then by Formula (13), we have $a(7)=a(5)=a(1)=0$. So, $a(6)=a(2)=a(0)=1$. Hence, by Formula (7), we have $a(i) = a(2\cdot 2^k+i) = a(6\cdot 2^k+i)$ and $a(2^k+i) = a(7\cdot 2^k+i)$ for all $i \in [0, 2^k]$. When $i \in [0, 2^{k-1}-1]$, then $0 \leq 2i+1 < 2i+2 \leq 2^k$ and

  $$a(3\cdot 2^k+i) \equiv a(6\cdot 2^k+2i+1)+a(6\cdot 2^k+2i+2) \equiv a(2i+1)+a(2i+2) \equiv a(i).$$

  When $i \in [2^{k-1}, 2^k-1]$, then $2^k+1 \leq 2i+1 < 2i+2 \leq 2\cdot 2^k$ and

  $$\begin{aligned} a(3\cdot 2^k+i) &\equiv a(6\cdot 2^k+2i+1)+a(6\cdot 2^k+2i+2) \\ &= a(7\cdot 2^k+2i+1-2^k)+a(7\cdot 2^k+2i+2-2^k) \\ &= a(2^k+2i+1-2^k)+a(2^k+2i+2-2^k) \\ &= a(2i+1)+a(2i+2) \equiv a(i). \end{aligned}$$

  Thus, we see that $a(3\cdot 2^k+i) \equiv a(i)$ for all $i \in [0, 2^k-1]$, i.e., $\sigma(2) \equiv \sigma(1)$. So, we have $\sigma(a(0)) = \sigma(a(2)) \equiv \sigma(a(3))$. By a similar discussion of case 'a(3)=1', we have $a(2^k+i) \equiv a(2^k+2^{k-1}+i) \equiv a(i)$ for all $i \in [0, 2^{k-1}-1]$.

Therefore, our claim holds. Then, by Lemma 7, we have $a(n) \equiv p(n)$ for all $n \in [0, 2^{k+1}]$. Note that the sequence $\{a(n)\}_{n\geq 0}$ defined in (6) is also generated by the morphism σ^i for all $i \geq 1$. So, for all $i \geq 1$, $a(n) \equiv p(n)$ for all $n \in [0, 2\cdot 2^{ki}]$. Let i tends to infinity, we conclude that $\{a(n) \pmod 2\}_{n\geq 0}$ is the period-doubling sequence. Hence, if $a(n)=0$, we have $a(2n+1)=a(2n+2)=1$. But, the minimality of N shows that $a(N)=0$ and $a(2N+1)=a(2N+2)=2$. This is a contradiction. Thus, the assumption that $v(p-1) \not\equiv w(p-1)$ is false. □

Proof of Theorem 2. Let $\{a(n)\}_{n\geq 0}$ be an apwenian sequence defined in (6). By Lemma 6, we know that $a(np)=1$ for all $n \geq 0$. Hence, for any $n \geq 0$, $a((n+1)p-1) \equiv a(2(n+1)p-1)+a(2(n+1)p) \equiv a(2(n+1)p-1)+1$. So, we have $a((n+1)p-1) \not\equiv a(2(n+1)p-1)$. Note that $a((n+1)p-1)$ is the last letter of $\sigma(a(n))$ and $a(2(n+1)p-1)$ is the last letter of $\sigma(a(2n+1))$. Hence, by Lemma 8, we see that $a(n) \not\equiv a(2n+1)$, i.e., $a(2n+1) \equiv 1+a(n)$. Thus, $a(2n)=1$ for all $n \geq 0$. This implies that $\{a(n)\}_{n\geq 0}$ is the period-doubling like sequence. □

When $\Sigma = \{0,1\}$, the authors in [9] shows that the only purely automatic apwenian sequence over Σ is the period-doubling sequence. When $\Sigma = \{0,1,2\}$, Theorem 2 shows that the only purely automatic apwenian sequence over Σ is the period-doubling like sequence. A natural extension is that how many purely automatic apwenian sequences are there over other alphabets Σ. The numerical experiment suggests that, when Σ contains only one odd number, the purely automatic apwenian sequence over Σ is also just the period-doubling like sequence. We propose the following conjecture.

Conjecture 2. *Let $\Sigma = \{1, n_1, n_2, \cdots, n_{k-1}\}$ and n_i be even numbers for all $i \in [1, k-1]$. Let $\{a(n)\}_{n\geq 0}$ be a purely automatic sequence over Σ. Then the sequence $\{a(n)\}_{n\geq 0}$ is apwenian if and only if $\{a(n)\}_{n\geq 0}$ is a period-doubling like sequence.*

5. Automatic apwenian sequences over other alphabets

In this section, we list some automatic apwenian sequences over other alphabets. Firstly, we consider the 2-automatic apwenian sequences in terms of 2-uniform morphisms. Next, we give two 3-automatic apwenian sequences.

5.1. 2-automatic apwenian sequences

Assume $\Sigma = \{n_1, n_2, \cdots, n_k\} \subset \mathbb{N}$ satisfying $n_1 < n_2 < \cdots < n_k$. If the morphism σ is defined on Σ, we denote the morphism σ by a k-tuple $(\sigma(n_1), \sigma(n_2), \cdots, \sigma(n_k))$ for convenience. For example, the morphism $\sigma = (12, 21)$ defined over $\{1, 2\}$ means that $1 \mapsto 12, 2 \mapsto 21$, and the morphism $\rho = (1, 0, 1)$ defined over $\{1, 2, 3\}$ means that $1 \mapsto 1, 2 \mapsto 0, 3 \mapsto 1$.

Recall that a sequence $\mathbf{u} \in \{0, 1\}^\infty$ is 2-automatic if and only if there exist an alphabet $\Sigma \subset \mathbb{N}^+$ and a 2-uniform morphism σ on Σ such that $\mathbf{u} = \sigma^\infty(a) (\bmod 2)$ for some $a \in \Sigma$. For example, the characteristic sequence $\mathbf{u}$ of the powers of 2 can be generated by the morphism $\sigma = (13, 22, 23)$ on $\Sigma = \{1, 2, 3\}$, i.e., $\mathbf{u} = \sigma^\infty(1) \pmod 2$.

Unless otherwise stated, we assume that $1 \in \Sigma \subset \mathbb{N}^+$ and the 2-uniform morphism σ is prolongable on 1. Let σ be a 2-uniform morphism on Σ. For any $a \in \Sigma$ and $i \in \{0, 1\}$, let $\sigma[i](a)$ be the $(i+1)$-th element of $\sigma(a)$. Then $\sigma[i]$ are codings on Σ and $\sigma(a) = \sigma[0](a)\sigma[1](a)$ for all $a \in \Sigma$.

Theorem 3. *Let σ be a 2-uniform morphism on Σ. The sequence $\sigma^\infty(1)(\bmod 2)$ is apwenian if and only if for any $k \geq 0$*

$$\sigma[1]^k \circ \sigma[0] \equiv \sigma[0]^{k+1} \circ \sigma[1] + \sigma[1]^{k+1} \circ \sigma[0]. \tag{14}$$

Proof. Let $\sigma^\infty(1) = \{u(n)\}_{n \geq 0}$. Then $\{u(n)\}_{n \geq 0}$ is a fixed point of σ. Hence, for any $n \geq 0$, we have $\sigma(u(n)) = u(2n)u(2n+1)$. Since $\sigma(a) = \sigma[0](a)\sigma[1](a)$ for all $a \in \Sigma$, it follows that

$$u(2n) = \sigma[0](u(n)), \quad u(2n+1) = \sigma[1](u(n)) \ (n \geq 0).$$

Hence, for any $k \geq 0$, $u(2^{k+1}n + 2^k - 1) = \sigma[1]^k \circ \sigma[0](u(n))$ and $u(2^{k+2}n + 2^{k+1}) = \sigma[0]^{k+1} \circ \sigma[1](u(n))$.

Note that $u(0) = 1$, and the sequence $\{u(n) (\text{mod } 2)\}_{n \geq 0}$ is apwenian if and only if $u(n) \equiv u(2n+1) + u(2n+2)$ for all $n \geq 0$. Hence, for any $k \geq 0$ and $n \geq 0$,

$$\begin{aligned} \sigma[1]^k \circ \sigma[0](u(n)) &\equiv u(2^{k+1}n + 2^k - 1) \\ &\equiv u(2^{k+2}n + 2^{k+1} - 1) + u(2^{k+2}n + 2^{k+1}) \\ &\equiv \sigma[1]^{k+1} \circ \sigma[0](u(n)) + \sigma[0]^{k+1} \circ \sigma[1](u(n)). \end{aligned}$$

So, we obtain Formula (14).

Conversely, if Formula (14) holds, then $u(2^{k+1}n+2^k-1) \equiv u(2^{k+2}n+2^{k+1}-1)+u(2^{k+2}n+2^{k+1})$ for all $n, k \geq 0$. Since for every integer $m \geq 0$, there exist integers $n \geq 0, k \geq 0$ such that $m+1 = 2^k(2n+1)$ i.e., $m = 2^{k+1}n + 2^k - 1$. Hence, for all $m \geq 0$, we have $u(m) \equiv u(2m+1) + u(2m+2)$. This completes the proof. □

Remark 1. In fact, the sequence $\sigma^\infty(1)(\bmod 2)$ is apwenian if and only if Formula (14) holds for finitely many terms k. This is because that there are finite codings on Σ and the sequences $\{\sigma[0]^k \circ \sigma[1](a)\}_k$, $\{\sigma[1]^k \circ \sigma[0](a)\}_k$ are ultimately periodic.

Using Theorem 3, we can check for some 2-automatic sequences whether they are apwenian or not. We give some examples.

Example 6 (The period-doubling sequence). *Let $\Sigma = \{1, 2\}$ and $\sigma = (12, 11)$. Then the fixed point $\sigma^\infty(1) \pmod 2$ is the* period-doubling sequence. *Since $\sigma[0] = (1, 1)$ and $\sigma[1] = (2, 1)$, we have*

$$\sigma[0]^k \circ \sigma[1] = \begin{cases} (2,1) & k = 0, \\ (1,1) & k \geq 1, \end{cases} \quad \text{and} \quad \sigma[1]^k \circ \sigma[0] = \begin{cases} (1,1) & k \text{ is even}, \\ (2,2) & k \text{ is odd}. \end{cases}$$

By Theorem 3, we see that the period-doubling sequence is apwenian.

Example 7 (**The characteristic sequence of the powers of 2**). *Let* $\Sigma = \{1,2,3\}$ *and* $\sigma = (13,22,23)$. *Then the fixed point* $\sigma^\infty(1) \pmod 2$ *is* the characteristic sequence of the powers of 2. *Let* $\{v(n)\}_{n\geq 1} = \sigma^\infty(1) \pmod 2$, *then* $v(n) = 1$ *if* $n = 2^k$ *for some* $k \geq 0$ *otherwise* $v(n) = 0$. *Since* $\sigma[0] = (1,2,2)$ *and* $\sigma[1] = (3,2,3)$, *we have*

$$\sigma[0]^k \circ \sigma[1] = \begin{cases} (3,2,3) & k = 0, \\ (2,2,2) & k \geq 1, \end{cases} \quad \text{and} \quad \sigma[1]^k \circ \sigma[0] = \begin{cases} (1,2,2) & k = 0, \\ (3,2,2) & k \geq 1. \end{cases}$$

By Theorem 3, we see that the characteristic sequence of the powers of 2 is apwenian.

Example 8. *Let* $\Sigma = \{1,2,3\}$ *and* $\sigma = (13,13,22)$. *Then* $\sigma[0] = (1,1,2)$ *and* $\sigma[1] = (3,3,2)$. *Hence, for any* $k \geq 0$, *we have*

$$\sigma[0]^k \circ \sigma[1] = \begin{cases} (3,3,2) & k = 0, \\ (2,2,1) & k = 1, \\ (1,1,1) & k \geq 2, \end{cases} \quad \text{and} \quad \sigma[1]^k \circ \sigma[0] = \begin{cases} (1,1,2) & k = 0, \\ (3,3,3) & k \text{ is odd}, \\ (2,2,2) & k \geq 2 \text{ is even}. \end{cases}$$

By Theorem 3, we see that the fixed point $\sigma^\infty(1) \pmod 2$ *is apwenian.*

Example 9 (**The characteristic sequence of the integers of the form** $(2^k - 1)$). *Let* $\Sigma = \{1,2,3,4\}$ *and* $\sigma = (12,32,44,44)$. *Then the fixed point* $\sigma^\infty(1) \pmod 2$ *is* the characteristic sequence of the integers of the form $(2^k - 1)$. *Let* $\{v(n)\}_{n\geq 1} = \sigma^\infty(1) \pmod 2$, *then* $v(n) = 1$ *if* $n = 2^k - 1$ *for some* $k \geq 1$ *otherwise* $v(n) = 0$. *Since* $\sigma[0] = (1,3,4,4)$ *and* $\sigma[1] = (2,2,4,4)$, *we have*

$$\sigma[0]^k \circ \sigma[1] = \begin{cases} (2,2,4,4) & k = 0, \\ (3,3,4,4) & k = 1, \\ (4,4,4,4) & k \geq 2, \end{cases} \quad \text{and} \quad \sigma[1]^k \circ \sigma[0] = \begin{cases} (1,3,4,4) & k = 0, \\ (2,4,4,4) & k \geq 1. \end{cases}$$

By Theorem 3, we see that the characteristic sequence of the integers of the form $(2^k - 1)$ *is apwenian.*

Example 10 (**The Rudin-Shapiro sequence**). *Let* $\Sigma = \{1,2,3,4\}$ *and* $\sigma = (13,43,12,42)$. *Then the fixed point* $\sigma^\infty(1) \pmod 2$ *is the* Rudin-Shapiro sequence *on* $\{0,1\}$. *Since* $\sigma[0] = (1,4,1,4)$ *and* $\sigma[1]1 = (3,3,2,2)$, *we have*

$$\sigma[0]^k \circ \sigma[1] = \begin{cases} (3,3,2,2) & k = 0, \\ (1,1,4,4) & k \geq 1. \end{cases} \quad \text{and} \quad \sigma[1]^k \circ \sigma[0] = \begin{cases} (1,4,1,4) & k = 0, \\ (3,2,3,2) & k \text{ is odd}, \\ (2,3,2,3) & k \geq 2 \text{ is even}. \end{cases}$$

By Theorem 3, we see that the Rudin-Shapiro sequence is not apwenian.

Example 11 (**The Baum-Sweet sequence**). *Let* $\Sigma = \{1,2,3,4\}$ *and* $\sigma = (13,34,23,44)$. *Then the fixed point* $\sigma^\infty(1) \pmod 2$ *is the* Baum-Sweet sequence. *Since* $\sigma[0] = (1,4,1,4)$ *and* $\sigma[1] = (3,3,2,2)$, *we have*

$$\sigma[0]^k \circ \sigma[1] = \begin{cases} (3,4,3,4) & k \text{ is even}, \\ (2,4,2,4) & k \text{ is odd}. \end{cases} \quad \text{and} \quad \sigma[1]^k \circ \sigma[0] = \begin{cases} (1,3,2,4) & k = 0, \\ (3,3,4,4) & k \geq 1. \end{cases}$$

By Theorem 3, we see that the Baum-Sweet sequence is not apwenian.

Example 12. *Let* $\Sigma = \{1,2,3,4\}$ *and* $\sigma = (13,14,24,23)$. *Then* $\sigma[0] = (1,1,2,2)$ *and* $\sigma[1] = (3,4,4,3)$. *Hence, for any* $k \geq 0$, *we have*

$$\sigma[0]^k \circ \sigma[1] = \begin{cases} (3,4,4,3) & k = 0, \\ (2,2,2,2) & k \geq 1, \\ (1,1,1,1) & k \geq 2, \end{cases} \quad \text{and} \quad \sigma[1]^k \circ \sigma[0] = \begin{cases} (1,1,2,2) & k = 0, \\ (3,3,4,4) & k = 2n+1, \\ (4,4,3,3) & k = 2n+2. \end{cases}$$

By Theorem 3, we see that the fixed point $\sigma^\infty(1) \pmod 2$ *is apwenian.*

Given a 2-uniform morphism σ on Σ, Theorem 3 provides a method to verify whether the sequence $\sigma^\infty(1) \pmod 2$ is apwenian or not. In the following table we list all 2-automatic apwenian sequences o 2 for $\Sigma = \{1,2\}, \{1,2,3\}, \{1,2,3,4\}, \{1,2,3,5\}$ and $\{1,2,3,4,6\}$.

Σ	sequence (mod 2)	σ
$\{1,2\}$	$10111010101110111011\cdots$	$(12,11)$
$\{1,2,3\}$	$11001111110011001100\cdots$	$(13,13,22)$
	$11010001000000010000\cdots$	$(13,22,23)$
$\{1,2,3,4\}$	$11001001110110001100\cdots$	$(13,14,24,23)$
	$10100010000000100000\cdots$	$(12,32,44,44)$
	$10100100000100000000\cdots$	$(12,34,43,44)$
$\{1,2,3,5\}$	$11010111011011010110\cdots$	$(15,23,12,21)$
	$11010111011101010111\cdots$	$(13,25,25,33)$
	$10111100101011111011\cdots$	$(12,35,11,22)$
	$10111100110011111100\cdots$	$(12,35,35,22)$
	$11001111101010101011\cdots$	$(15,33,32,22)$
$\{1,2,3,4,6\}$	$11001001110000001100\cdots$	$(13,14,26,26,43)$
	$10100100000101100110\cdots$	$(12,34,43,62,43)$
	$11010001011000010001\cdots$	$(13,34,43,62,43)$

5.2. 3-automatic apwenian sequences

Many 2-automatic apwenian sequences have been found. An interesting question, asked in [1], is that

"are there PLCP/apwenian sequences that are d-automatic for some d not a power of 2?"

In the following, we give two 3-automatic apwenian sequences over $\{1,2,3\}$.

Example 13. *Let $\sigma = (121, 132, 132)$ be a morphism on $\Sigma = \{1,2,3\}$. Then the fixed points $\sigma^\infty(1)$* (mod 2) *is apwenian.*

Proof. Let $\{u(n)\}_{n\geq 0} = \sigma^\infty(1)$. It is easy to check that the sequence $\{u(n)\}_{n\geq 0}$ can be generated by the following recurrences:

$$\begin{gathered} u(3n) = u(9n+2) = 1; \\ u(9n+1) = u(9n+5) = 2; \\ u(9n+4) = 3; \\ u(9n+7) = u(3n+1); \\ u(9n+8) = u(3n+2). \end{gathered} \tag{15}$$

Now, we prove that the sequence $\{u(n)\}_{n\geq 0}$ satisfies $u(n) \equiv u(2n+1) + u(2n+2)$ for all $n \geq 0$ by induction on n. The first few terms of $\{u(n)\}_{n\geq 0}$ can be checked directly. Aassume that $u(n) \equiv u(2n+1) + u(2n+2)$ for all $n \leq 9k-1$ for some $k \geq 1$. We will check the equation holds for $n \leq 9k+8$.

- By Formula (15), we see that $u(9k) = 1, u(18k+1) = 2$ and $u(18k+2) = 1$. Hence, $u(9k) \equiv u(18k+1) + u(18k+2)$.
- By Formula (15), we see that $u(9k+1) = 2, u(18k+3) = 1$ and $u(18k+4) = 3$. Hence, $u(9k+1) \equiv u(18k+3) + u(18k+4)$.
- By Formula (15), we see that $u(9k+2) = 1, u(18k+5) = 2$ and $u(18k+6) = 1$. Hence, $u(9k+2) \equiv u(18k+5) + u(18k+6)$.
- By Formula (15), we see that $u(9k+3) = 1, u(18k+7) = u(6k+1)$ and $u(18k+8) = u(6k+2)$. By induction, $u(6k+1) + u(6k+2) \equiv u(3k) = 1$. Hence, $u(9k+3) \equiv u(18k+7) + u(18k+8)$.
- By Formula (15), we see that $u(9k+4) = 3, u(18k+9) = 1$ and $u(18k+10) = 2$. Hence, $u(9k+4) \equiv u(18k+9) + u(18k+10)$.
- By Formula (15), we see that $u(9k+5) = 2, u(18k+11) = 1$ and $u(18k+12) = 1$. Hence, $u(9k+5) \equiv u(18k+11) + u(18k+12)$.

- By Formula (15), we see that $u(9k+6) = 1, u(18k+13) = 3$ and $u(18k+14) = 2$. Hence, $u(9k+6) \equiv u(18k+13) + u(18k+14)$.
- By Formula (15), we see that $u(9k+7) = u(3k+1), u(18k+15) = 1$ and $u(18k+16) = u(6k+4)$. By induction, $u(3k+1) \equiv u(6k+3) + u(6k+4) = 1 + u(6k+4)$.. Hence, $u(9k+7) \equiv u(18k+15) + u(18k+16)$.
- By Formula (15), we see that $u(9k+8) = u(3k+2), u(18k+17) = u(6n+5)$ and $u(18k+18) = 1$. By induction, $u(3k+2) \equiv u(6k+5) + u(6k+6) = u(6k+5) + 1$.. Hence, $u(9k+8) \equiv u(18k+17) + u(18k+18)$.

Therefore, the sequence $\sigma^\infty(1) \pmod 2 = \{u(n) (\bmod\ 2)\}_{n\geq 0}$ is apwenian. □

Example 14. *Let* $\tau = \{132, 121, 121\}$ *be a morphism on* $\Sigma = \{1, 2, 3\}$. *Then the fixed points* $\tau^\infty(1) \pmod 2$ *is apwenian.*

Proof. Let $\{u(n)\}_{n\geq 0} = \tau^\infty(1)$. Then the sequence $\{u(n)\}_{n\geq 0}$ can be generated by the following recurrences:

$$\begin{aligned} u(3n) = u(9n+5) = u(27n+8) = 1; \\ u(9n+2) = u(9n+4) = u(27n+7) = u(27n+17) = 2; \\ u(9n+1) = u(27n+16) = 3; \\ u(27n+25) = u(3n+1); \\ u(27n+26) = u(3n+2). \end{aligned} \tag{16}$$

Using Formula (16), we can prove similarly that the sequence $\{u(n)\}_{n\geq 0}$ satisfies $u(n) \equiv u(2n+1) + u(2n+2)$ for all $n \geq 0$ by induction on n. This implies that the sequence $\tau^\infty(1) \pmod 2 = \{u(n) (\bmod\ 2)\}_{n\geq 0}$ is apwenian. We omit the details here. □

In the following table, we list all k-automatic apwenian sequences modulo 2 over $\Sigma = \{1, 2, 3\}$ for $k \in [2, 9]$. Two different 3-automatic apwenian sequences are also found in the last row.

k	sequence (mod 2)	σ
2	$10111010101110111011\cdots$	$(12, 13, 12)$
	$11001111110011001100\cdots$	$(13, 13, 22)$
	$11010001000000010000\cdots$	$(13, 22, 23)$
3	$10111010110111011010\cdots$	$(121, 132, 132)$
	$11010110111010111011\cdots$	$(132, 121, 121)$
4	$11010111011011010110\cdots$	$(1321, 2112, 2113)$
5	None	
6	None	
7	None	
8	None	
9	$10111011010111010110\cdots$	$(121132132, 121132121, 121132121)$
	$11010111011010110111\cdots$	$(132121132, 132121121, 132121121)$

Acknowledgement

The authors are grateful to the referee for their careful reading and helpful comments.

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