PERIODIC SIGN CHANGES FOR WEAKLY HOLOMORPHIC η-QUOTIENTS

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Abstract. In this paper, we study sign changes of weakly holomorphic modular forms which are given as η-quotients. We give representative examples for forms of negative weight, weight zero, and positive weight.

1. Introduction and statement of results

Let $m \in \mathbb{N}$ and $\delta_\ell \in \mathbb{Z}$ for $1 \leq \ell \leq m$. We define

$$\prod_{\ell=1}^{m} \left(q^\ell; q^\ell\right)_\infty^{\delta_\ell} =: \sum_{n \geq 0} C_{1^{\delta_1}2^{\delta_2}\cdots m^{\delta_m}}(n) q^n,$$

where $(a;q)_n := \prod_{j=0}^{n-1}(1-aq^j)$ for $n \in \mathbb{N}_0 \cup \{\infty\}$ is the *q-Pochhammer symbol*. The signs

$$s_{1^{\delta_1}2^{\delta_2}\cdots m^{\delta_m}}(n) := \operatorname{sgn}\left(C_{1^{\delta_1}2^{\delta_2}\cdots m^{\delta_m}}(n)\right)$$

were investigated by a number of authors. For example, letting $M(a,c;n)$ denote the number of partitions of n with crank $\equiv a \pmod{c}$, Andrews–Lewis [3, Conjecture 2] conjectured that[1]

$$\begin{aligned} M(0,3;3n) &> M(1,3;3n) \text{ for } n \in \mathbb{N}, \\ M(0,3;3n+1) &< M(1,3;3n+1) \text{ for } n \in \mathbb{N}, \\ M(0,3;3n+2) &< M(1,3;3n+2) \text{ for } n \in \mathbb{N} \setminus \{1,4,5\}. \end{aligned}$$

Since $M(0,3;n) - M(1,3;n) = C_{1^2 3^{-1}}(n)$, this conjecture can be repackaged as

$$s_{1^2 3^{-1}}(n) = \begin{cases} 1 & \text{if } 3 \mid n \text{ or } n = 5, \\ 0 & \text{if } n \in \{14, 17\}, \\ -1 & \text{otherwise.} \end{cases}$$

This conjecture was proven by Kane [9, Corollary 2]. To give another example, Andrews [2, Theorem 2.1] proved that for a prime p the signs $s_{1^1 p^{-1}}(n)$ of the Fourier coefficients of the infinite Borwein products $\frac{(q;q)_\infty}{(q^p;q^p)_\infty}$ are periodic in n with period p. In this paper, we investigate other cases where $s_{1^{\delta_1}2^{\delta_2}\cdots m^{\delta_m}}(n)$ is periodic with some period $M \in \mathbb{N}$ i.e., $s_{1^{\delta_1}2^{\delta_2}\cdots m^{\delta_m}}(n)$ only depends on $n \pmod{M}$. Techniques used to show these results vary depending on the sign of $\sum_{\ell=1}^m \delta_\ell$, so we give representative cases for each of the possibilities for the sign.

We first consider a case with $\sum_{\ell=1}^m \delta_\ell < 0$ and $M = 5$.

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[1] We have $M(0,3;3n+2) = M(1,3;3n+2)$ for $n \in \{4,5\}$ and $M(0,3;5) > M(1,3;5)$.

Theorem 1.1. *We have*

$$s_{1^15^{-2}}(n)=\begin{cases}1 & \text{if } n\equiv 0 \pmod{5},\\ -1 & \text{if } n\equiv 1,2 \pmod{5},\\ 0 & \text{if } n\equiv 3,4 \pmod{5}.\end{cases}$$

Remark. *As noted by Wang* [13, (1.4)], *Andrews showed in his proof of* [2, Theorem 2.1] *that the signs $s_{1^15^{-1}}(n)$ of the Fourier coefficients of $\frac{(q;q)_\infty}{(q^5;q^5)_\infty}$ are periodic with period 5, up to some exceptional n satisfying $s_{1^15^{-1}}(n)=0$. Using this, a straightforward argument shows that the signs of $s_{1^15^{-2}}(n)$ of the Fourier coefficients of $\frac{(q;q)_\infty}{(q^5;q^5)_\infty^2}=\frac{(q;q)_\infty}{(q^5;q^5)_\infty}\frac{1}{(q^5;q^5)_\infty}$ satisfy the same property. Proving that none of the Fourier coefficients in the congruence classes $n\equiv 0,1,2 \pmod{5}$ vanish requires a slightly more delicate analysis, however. Since we are only interested in purely periodic signs in this paper, we still investigate this case to demonstrate how to use the methods in this paper.*

Theorem 1.1 also has a combinatorial interpretation. To state it, for $n\in\mathbb{N}$, let $p_2(n)$ be the number of 2-colored partitions of n, setting $p_2(x):=0$ for $x\notin\mathbb{N}_0$, and for $m\in\mathbb{Z}$ let $P_5(m):=\frac{3m^2-m}{2}$ be the m-th generalized pentagonal number.

Corollary 1.2. *For $n\in\mathbb{N}$, we have*

$$\sum_{m\in\mathbb{Z}}(-1)^m p_2\left(\frac{n-P_5(m)}{5}\right)\begin{cases}>0 & \text{if } n\equiv 0 \pmod{5},\\ <0 & \text{if } n\equiv 1,2 \pmod{5},\\ =0 & \text{if } n\equiv 3,4 \pmod{5}.\end{cases}$$

We next treat a case with $\sum_{\ell=1}^m\delta_\ell=0$ and $M=4$.

Theorem 1.3. *We have*

$$s_{1^12^24^{-3}}(n)=\begin{cases}1 & \text{if } n\equiv 0,3 \pmod{4},\\ -1 & \text{if } n\equiv 1,2 \pmod{4}.\end{cases}$$

We finally consider a case with $\sum_{\ell=1}^m\delta_\ell>0$ and $M=9$.

Theorem 1.4. *We have*

$$s_{1^93^{-5}}(n)=\begin{cases}1 & \text{if } n\equiv 0,2,5,6,8 \pmod{9},\\ -1 & \text{if } n\equiv 1,3,4,7 \pmod{9}.\end{cases}$$

The paper is organized as follows. In Section 2, we give preliminary facts about modular forms and their Fourier coefficients, Kloosterman sums, and Bessel functions. In Section 3, we prove Theorem 1.1 and Corollary 1.2. Section 4 is devoted to the proof of Theorem 1.3. In Section 5 we show Theorem 1.4. Further related conjectures are given in Appendix A.

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2. Preliminaries

2.1. **Modular forms.** We set

$$\varepsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4, \\ i & \text{if } d \equiv 3 \pmod 4. \end{cases}$$

Suppose that $\kappa \in \frac{1}{2}\mathbb{Z}$ and Γ is a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, with $\Gamma \subseteq \Gamma_0(4)$ if $\kappa \in \mathbb{Z} + \frac{1}{2}$, containing $T := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, the weight κ *slash operator* is defined by

$$F\big|_\kappa \gamma(\tau) := \begin{cases} \left(\frac{c}{d}\right)^{2k} \varepsilon_d^{2\kappa} (c\tau + d)^{-\kappa} F(\gamma\tau) & \text{if } \kappa \in \mathbb{Z} + \frac{1}{2}, \\ (c\tau + d)^{-\kappa} F(\gamma\tau) & \text{if } \kappa \in \mathbb{Z}. \end{cases}$$

Here $\left(\frac{\cdot}{\cdot}\right)$ is the extended Legendre symbol. A holomorphic function $F : \mathbb{H} \to \mathbb{C}$ is called a *weight κ weakly holomorphic modular form on Γ with character χ* if for every $\gamma \in \Gamma$ we have

$$F|_\kappa \gamma = \chi(d) F$$

and for every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ there exists $n_0 \in \mathbb{Q}$ such that

$$(c\tau + d)^{-\kappa} F(\gamma\tau) e^{2\pi i n_0 \tau} \tag{2.1}$$

is bounded as $\tau \to i\infty$. We call the equivalence classes of $\Gamma \backslash (\mathbb{Q} \cup \{i\infty\})$ the *cusps of* Γ. We abuse notation and also call representatives $\varrho \in \mathbb{Q}$ of elements of $\Gamma \backslash (\mathbb{Q} \cup \{i\infty\})$ cusps. For a cusp ϱ we choose $\gamma_\varrho \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma_\varrho(i\infty) = \varrho$. If F is a weakly holomorphic modular form of weight κ on Γ with some character χ, then $F_\varrho(\tau) := (c\tau + d)^{-\kappa} F(\gamma_\varrho \tau)$ is invariant under T^{σ_ϱ} for some $\sigma_\varrho \in \mathbb{N}$. Hence F has a Fourier expansion (with $q := e^{2\pi i \tau}$)

$$F_\varrho(\tau) = \sum_{n \gg -\infty} c_{F,\varrho}(n) q^{\frac{n}{\sigma_\varrho}}.$$

Note that there are only finitely many negative n because of (2.1). We drop ϱ from the notation if $\varrho = i\infty$. The terms in the Fourier expansion with $n < 0$ are called the *principal part* of F at the cusp ϱ.

2.2. **Special modular forms.** Recall the transformation law of the *partition generating function*

$$P(q) := \sum_{n \geq 0} p(n) q^n = \frac{1}{(q;q)_\infty},$$

where $p(n)$ denotes the number of partitions of n, and the *Dedekind η-function* $\eta(\tau) := q^{\frac{1}{24}}(q;q)_\infty$. We take $\tau = \frac{1}{k}(h + iz)$ with $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$, so that $q = e^{\frac{2\pi i}{k}(h + iz)}$. Here $h, k \in \mathbb{N}$ satisfy $0 \leq h < k$ and $\gcd(h,k) = 1$, and for $hh' \equiv -1 \pmod k$ we set $q_1 := e^{\frac{2\pi i}{k}(h' + \frac{i}{z})}$. Let $\omega_{h,k}$ be defined through

$$P(q) = \omega_{h,k} \sqrt{z} e^{\frac{\pi}{12k}\left(\frac{1}{z} - z\right)} P(q_1). \tag{2.2}$$

Then we have (see [1, equation (5.2.4)])

$$\omega_{h,k} = \begin{cases} \left(\frac{-k}{h}\right) e^{-\pi i \left(\frac{1}{4}(2 - hk - h) + \frac{1}{12}\left(k - \frac{1}{k}\right)\left(2h - h' + h^2 h'\right)\right)} & \text{if } h \text{ is odd}, \\ \left(\frac{-h}{k}\right) e^{-\pi i \left(\frac{1}{4}(k-1) + \frac{1}{12}\left(k - \frac{1}{k}\right)\left(2h - h' + h^2 h'\right)\right)} & \text{if } k \text{ is odd}. \end{cases} \tag{2.3}$$

With $[a]_b$ the inverse of $a \pmod b$ for $\gcd(a,b) = 1$, P has the following transformation.

Lemma 2.1. *If* $\gcd(d,k)=g$, *then*

$$P\left(q^d\right)=\omega_{\frac{d}{g}h,\frac{k}{g}}\sqrt{\frac{dz}{g}}e^{\frac{\pi g}{12k}\left(\frac{g}{dz}-\frac{dz}{g}\right)}P\left(e^{\frac{2\pi i g}{k}\left(\left[\frac{d}{g}\right]_{\frac{k}{g}}h'+\frac{ig}{dz}\right)}\right).$$

Moreover, since $\eta(\tau)=\frac{q^{\frac{1}{24}}}{P(q)}$, (2.2) implies that, for any $h,h'\in\mathbb{Z}$ with $hh'\equiv-1\pmod{k}$,

$$\eta\left(\frac{1}{k}(h+iz)\right)=\omega_{h,k}^{-1}e^{\frac{\pi i(h-h')}{12k}}e^{-\frac{\pi i}{4}}(-iz)^{-\frac{1}{2}}\eta\left(\frac{1}{k}\left(h'+\frac{i}{z}\right)\right).$$

In particular, $\eta(24\tau)$ is a weight $\frac{1}{2}$ modular form on $\Gamma_0(576)$ with character χ_{12}, where $\chi_D(n):=\left(\frac{D}{n}\right)$ (see [12, Corollary 1.62]).

2.3. **Zuckerman and exact formulas.** Let F be a weakly holomorphic modular form of weight $\kappa\in-\frac{1}{2}\mathbb{N}_0$ for some congruence subgroup $\Gamma\subseteq\mathrm{SL}_2(\mathbb{Z})$. Suppose that for $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$ with $c\neq0$ and $\varrho=\frac{a}{c}\in\mathbb{Q}$ we have the transformation law

$$F(\gamma\tau)=\chi(\gamma)(c\tau+d)^{\kappa}F_{\varrho}(\tau).$$

Now let $\gamma=\gamma_{h,k}\in\mathrm{SL}_2(\mathbb{Z})$ with $a=h$ and $c=k$. Note that $h'=-d$ satisfies the congruence $hh'\equiv-1\pmod{k}$. Taking $\tau=\frac{1}{k}(h'+\frac{i}{z})$, we obtain the transformation

$$F\left(\frac{1}{k}(h+iz)\right)=\chi(\gamma_{h,k})(-iz)^{-\kappa}F_{\varrho}\left(\frac{1}{k}\left(h'+\frac{i}{z}\right)\right).$$

Let F have the Fourier expansion at $i\infty$

$$F(\tau)=\sum_{n\gg-\infty}a(n)q^{n+\alpha}$$

and Fourier expansions at each rational number $0\leq\frac{h}{k}<1$ (with $\gcd(h,k)=1$)

$$(k\tau-h')^{-\kappa}F\left(\gamma_{h,k}\tau\right)=\sum_{n\gg-\infty}a_{h,k}(n)q^{\frac{n+\alpha_{h,k}}{c_k}}.$$

Furthermore, let I_α denote the usual I-Bessel function. In this framework, the relevant theorem of Zuckerman [16, Theorem 1], which was extended to a larger class of functions which include weight zero weakly holomorphic modular forms by Ono and the first author [7, Theorem 1.1], may be stated as follows.

Theorem 2.2. *Assume the notation and hypotheses above. If* $n+\alpha>0$, *then we have*

$$a(n)=2\pi(n+\alpha)^{\frac{\kappa-1}{2}}\sum_{k\geq1}\frac{1}{k}\sum_{\substack{0\leq h<k\\ \gcd(h,k)=1}}\chi(\gamma_{h,k})e^{-\frac{2\pi i(n+\alpha)h}{k}}$$
$$\times\sum_{m+\alpha_{h,k}<0}a_{h,k}(m)e^{\frac{2\pi i}{kc_k}(m+\alpha_{h,k})h'}\left(\frac{|m+\alpha_{h,k}|}{c_k}\right)^{\frac{1-\kappa}{2}}I_{-\kappa+1}\left(\frac{4\pi}{k}\sqrt{\frac{(n+\alpha)|m+\alpha_{h,k}|}{c_k}}\right).$$

2.4. **Kloosterman sums.** We define the Kloosterman sums

$$K_k(n,m) := \sum_{h \pmod{k}^*} e^{\frac{2\pi i}{k}(nh+mh')}.$$

We require Weil's [15] bound (see [8, (11.16)] for the statement in this form).

Lemma 2.3. *We have, for $k \in \mathbb{N}$ and $n, m \in \mathbb{Z}$,*

$$K_k(n,m) \leq \sqrt{\gcd(n,m,k)}\, d(k)\sqrt{k},$$

where $d(k)$ denotes the number of divisors of k.

2.5. **Bessel functions.** We require certain bounds for the I-Bessel functions. Upper bounds for $I_\kappa(x)$ are well-known and may be found, for example, in [6, Lemma 2.2 (1) and (3)]. A lower bound for $I_1(x)$ for x sufficiently large was given in [5, Lemma 2.4], and a lower bound for $I_{\frac{3}{2}}(x)$ follows by a similar argument using [11, (10.47.7) and (10.49.9)].

Lemma 2.4.

(1) *For $0 \leq x < 1$, $\kappa \in \mathbb{R}$ with $\kappa > -\frac{1}{2}$, we have*

$$I_\kappa(x) \leq \frac{2^{1-\kappa}x^\kappa}{\Gamma(\kappa+1)}.$$

(2) *For $x \geq 1$ and $\kappa \in \mathbb{R}$ with $\kappa > -\frac{1}{2}$, we have*

$$I_\kappa(x) \leq \sqrt{\frac{2}{\pi x}}e^x.$$

(3) *For $\kappa \in \{1, \frac{3}{2}\}$ and $x \geq 3$, we have*

$$I_\kappa(x) \geq \frac{e^x}{4\sqrt{x}}.$$

3. Proof of Theorem 1.1 and Corollary 1.2

For ease of notation, we abbreviate $c_1(n) := C_{1^15^{-2}}(n)$, $s_1(n) := \operatorname{sgn}(c_1(n))$, and set

$$f_1(q) := \frac{(q;q)_\infty}{(q^5;q^5)^2} = \frac{P^2\left(q^5\right)}{P(q)} = \sum_{n\geq 0} c_1(n)q^n. \tag{3.1}$$

3.1. **The case $n \equiv 3, 4 \pmod 5$.** Using (see Theorem 1.60 of [12])

$$(q;q)_\infty = \sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{n(3n+1)}{2}},$$

we see that the Fourier coefficients of f_1 are not supported on exponents that are congruent to $3, 4 \pmod 5$. This gives Theorem 1.1 for $n \equiv 3, 4 \pmod 5$.

3.2. **The exact formula.** We next use Theorem 2.2 to obtain an exact formula for $c_1(n)$. To state the result, set

$$\chi_1(h,k) := \frac{\omega^2_{h,\frac{k}{5}}}{\omega_{h,k}}.$$

Note that $\chi_1(h+k,k) = \chi_1(h,k)$, so χ_1 only depends on $h \pmod{k}$.

Lemma 3.1. *For $n \in \mathbb{N}$, we have*

$$c_1(n) = \frac{2 \cdot 3^{\frac{3}{4}}\pi}{(8n-3)^{\frac{3}{4}}} \sum_{\substack{k \geq 1 \\ 5|k}} \frac{1}{k} \sum_{h \pmod{k}^*} \chi_1(h,k) e^{-\frac{2\pi i n h}{k}} I_{\frac{3}{2}}\left(\frac{\pi}{2k}\sqrt{3\,(8n-3)}\right).$$

Proof. In order to use Theorem 2.2 for $F_1(\tau) := q^{-\frac{3}{8}} f_1(q)$, we need to determine the growth of F_1 at the cusp $\frac{h}{k}$. For $5 \nmid k$, plugging (2.2) into (3.1) we see that, for $z \to 0$,

$$f_1(q) = 5\frac{\omega^2_{5h,k}}{\omega_{h,k}} \sqrt{z} e^{-\frac{\pi}{20kz} - \frac{3\pi z}{4k}} \frac{P\left(e^{\frac{2\pi i}{k}\left([5]_k h' + \frac{i}{5z}\right)}\right)^2}{P\left(q_1\right)} \ll \sqrt{z} e^{-\frac{\pi}{20k}\mathrm{Re}\left(\frac{1}{z}\right)} \to 0.$$

So F_1 has no principal part at $\frac{h}{k}$ if $5 \nmid k$.

Similarly, for $5 \mid k$ we obtain

$$f_1(q) = \chi_1(h,k)\sqrt{z} e^{\frac{3\pi}{4k}\left(\frac{1}{z}-z\right)}(1 + O(q_1)). \qquad \square$$

We split the formula in Lemma 3.1 into a main term

$$M_1(n) := \frac{4 \cdot 3^{\frac{3}{4}}\pi}{5\,(8n-3)^{\frac{3}{4}}} I_{\frac{3}{2}}\left(\frac{\pi}{10}\sqrt{3(8n-3)}\right)\left(\cos\left(\frac{4\pi n}{5}\right) - \cos\left(\frac{2\pi}{5}(n-2)\right)\right)$$

and an error term

$$E_1(n) := \frac{2 \cdot 3^{\frac{3}{4}}\pi}{(8n-3)^{\frac{3}{4}}} \sum_{\substack{k > 5 \\ 5|k}} \frac{1}{k} \sum_{h \pmod{k}^*} \chi_1(h,k) e^{-\frac{2\pi i n h}{k}} I_{\frac{3}{2}}\left(\frac{\pi}{2k}\sqrt{3(8n-3)}\right).$$

Lemma 3.2. *We have*

$$c_1(n) = M_1(n) + E_1(n).$$

Moreover, if $|M_1(n)| > |E_1(n)|$, then $s_1(n)$ agrees with the value claimed in Theorem 1.1.

Proof. The first identity follows from a straightford calculation showing that $M_1(n)$ is the term with $k = 5$ of Lemma 3.1.

Now suppose that $|M_1(n)| > |E_1(n)|$. Then the first identity implies that

$$s_1(n) = \mathrm{sgn}\left(M_1(n)\right) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod{5}, \\ -1 & \text{if } n \equiv 1, 2 \pmod{5}. \end{cases}$$

This matches what is claimed in Theorem 1.1. $\square$

3.3. **Bounding the main and error terms.** We next bound the main term and error term from Lemma 3.2.

Lemma 3.3.
(1) *For* $n \equiv 0, 1, 2 \pmod 5$, *we have*

$$|M_1(n)| \geq \frac{4 \cdot 3^{\frac{3}{4}}\pi}{5(8n-3)^{\frac{3}{4}}} I_{\frac{3}{2}}\left(\frac{\pi}{10}\sqrt{3(8n-3)}\right)\left(1-\cos\left(\frac{2\pi}{5}\right)\right).$$

(2) *We have*

$$|E_1(n)| \leq \frac{4\sqrt{6}\pi^{\frac{3}{2}}}{5\,(8n-3)^{\frac{1}{4}}} + \frac{3^{\frac{5}{4}}\pi^2}{5(8n-3)^{\frac{1}{4}}} I_{\frac{3}{2}}\left(\frac{\pi}{20}\sqrt{8n-3}\right).$$

Proof. (1) Directly from the definition, we have

$$|M_1(n)| \geq \frac{4 \cdot 3^{\frac{3}{4}}\pi}{5(8n-3)^{\frac{3}{4}}} I_{\frac{3}{2}}\left(\frac{\pi}{10}\sqrt{3(8n-3)}\right) \min_{n\in\{0,1,2\}}\left|\cos\left(\frac{4\pi n}{5}\right) - \cos\left(\frac{2\pi}{5}(n-2)\right)\right|.$$

The minimum occurs for $n = 2$, giving the claim.
(2) Making the change of variables $k \mapsto 5k$ and taking the absolute value inside the sum, we conclude that

$$|E_1(n)| \leq \frac{2 \cdot 3^{\frac{3}{4}}\pi}{(8n-3)^{\frac{3}{4}}} \sum_{k\geq 2} I_{\frac{3}{2}}\left(\frac{\pi}{10k}\sqrt{3(8n-3)}\right). \tag{3.2}$$

We split the sum over k in (3.2) into terms with k small and k large. Using Lemma 2.4 (1) we conclude that the contribution to (3.2) from $k > \frac{\pi}{10}\sqrt{3(8n-3)}$ is bounded by

$$\frac{2\sqrt{3}\pi^2}{5\sqrt{5}} \sum_{k>\frac{\pi}{10}\sqrt{3(8n-3)}} k^{-\frac{3}{2}} \leq \frac{2\sqrt{3}\pi^2}{5\sqrt{5}} \int_{\frac{\pi}{10}\sqrt{3(8n-3)}}^{\infty} x^{-\frac{3}{2}}dx = \frac{4\sqrt{6}\pi^{\frac{3}{2}}}{5\,(8n-3)^{\frac{1}{4}}}. \tag{3.3}$$

Estimating the Bessel function against the term with $k = 2$, the contribution from $k \leq \frac{\pi}{10}\sqrt{3(8n-3)}$ can be bounded by

$$\frac{2 \cdot 3^{\frac{3}{4}}\pi}{(8n-3)^{\frac{3}{4}}} \sum_{2\leq k\leq\frac{\pi}{10}\sqrt{3(8n-3)}} I_{\frac{3}{2}}\left(\frac{\pi}{10k}\sqrt{3(8n-3)}\right) \leq \frac{3^{\frac{5}{4}}\pi^2}{5(8n-3)^{\frac{1}{4}}} I_{\frac{3}{2}}\left(\frac{\pi}{20}\sqrt{8n-3}\right). \tag{3.4}$$

Adding (3.3) and (3.4) gives the claim. □

3.4. **Proof of Theorem 1.1.** We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Comparing Lemma 3.3 (1) with Lemma 3.3 (2), for $n \equiv 0, 1, 2 \pmod 5$ we conclude by Lemma 3.2 that $s_1(n)$ matches the value claimed in Theorem 1.1 if

$$\frac{4 \cdot 3^{\frac{3}{4}}\pi}{5(8n-3)^{\frac{3}{4}}} I_{\frac{3}{2}}\left(\frac{\pi}{10}\sqrt{3(8n-3)}\right)\left(1-\cos\left(\frac{2\pi}{5}\right)\right) > \frac{2^{\frac{5}{2}}\sqrt{3}\pi^{\frac{3}{2}}}{5\,(8n-3)^{\frac{1}{4}}} + \frac{3^{\frac{5}{4}}\pi^2}{5(8n-3)^{\frac{1}{4}}} I_{\frac{3}{2}}\left(\frac{\pi}{20}\sqrt{8n-3}\right).$$

Rearranging, this is equivalent to

$$\frac{\sqrt{8n-3}}{\left(1-\cos\left(\frac{2\pi}{5}\right)\right) I_{\frac{3}{2}}\left(\frac{\pi}{10}\sqrt{3(8n-3)}\right)}\left(\frac{\sqrt{2\pi}}{3^{\frac{1}{4}}}+\frac{\sqrt{3}\pi}{4} I_{\frac{3}{2}}\left(\frac{\pi}{20}\sqrt{8n-3}\right)\right)<1.$$

We next use Lemma 2.4 (2) for $n \geq 6$ to bound

$$I_{\frac{3}{2}}\left(\frac{\pi}{20}\sqrt{8n-3}\right) \leq \frac{2\sqrt{10}}{\pi(8n-3)^{\frac{1}{4}}} e^{\frac{\pi}{20}\sqrt{8n-3}}.$$

Moreover, by Lemma 2.4 (3) for $n \geq 5$, we have

$$I_{\frac{3}{2}}\left(\frac{\pi}{10}\sqrt{3(8n-3)}\right) \geq \frac{e^{\frac{\pi}{10}\sqrt{3(8n-3)}}}{2\cdot 3^{\frac{1}{4}}\sqrt{2\pi}(8n-3)^{\frac{1}{4}}}.$$

Thus we want

$$\frac{2\cdot 3^{\frac{1}{4}}\sqrt{2\pi}(8n-3)^{\frac{3}{4}} e^{-\frac{\pi}{10}\sqrt{3(8n-3)}}}{1-\cos\left(\frac{2\pi}{5}\right)}\left(\frac{\sqrt{2\pi}}{3^{\frac{1}{4}}}+\sqrt{\frac{15}{2}}\frac{e^{\frac{\pi}{20}\sqrt{8n-3}}}{(8n-3)^{\frac{1}{4}}}\right)<1. \tag{3.5}$$

Estimating $4.8xe^{-x} \leq 1$ for $x \geq 2.5$, we see that for $n \geq 34$ the left-hand side of (3.5) can be bounded from above by

$$\frac{\sqrt{8n-3}\,2\cdot 3^{\frac{1}{4}}\sqrt{2\pi}}{\left(1-\cos\left(\frac{2\pi}{5}\right)\right) e^{\frac{\pi}{20}\sqrt{8n-3}(2\sqrt{3}-1)}}\left(\frac{20\sqrt{2\pi}}{4.8\cdot 3^{\frac{1}{4}}\pi\,(8n-3)^{\frac{1}{4}}}+\sqrt{\frac{15}{2}}\right)<\frac{33\sqrt{8n-3}}{e^{\frac{\pi}{20}(2\sqrt{3}-1)\sqrt{8n-3}}}.$$

Using $xe^{-x} \leq \frac{1}{86}$ for $x \geq 6.3$, we obtain that (3.5) holds for $n \geq 34$. For $n < 34$, we evaluate $s_1(n)$ with a computer to determine that it agrees with the value claimed in Theorem 1.1. $\square$

We next prove Corollary 1.2.

Proof of Corollary 1.2. First recall that

$$\frac{1}{(q^5;q^5)_\infty^2}=\sum_{n\geq 0} p_2(n) q^{5n}.$$

Hence [12, Theorem 1.60] implies that

$$\frac{(q;q)_\infty}{(q^5;q^5)_\infty^2}=\sum_{n\geq 0}\sum_{m\in\mathbb{Z}}(-1)^m p_2\left(\frac{n-P_5(m)}{5}\right) q^n.$$

The result follows by Theorem 1.1. $\square$

4. Proof of Theorem 1.3

We abbreviate $c_2(n) := C_{1^1 2^2 4^{-3}}(n)$, $s_2(n) := \operatorname{sgn}(c_2(n))$, and let

$$f_2(q) := \frac{(q;q)_\infty \left(q^2;q^2\right)_\infty^2}{(q^4;q^4)_\infty^3} = \frac{P^3\left(q^4\right)}{P(q)P^2\left(q^2\right)}.$$

4.1. **The exact formula.** We next obtain an exact formula for $c_2(n)$. Let

$$\chi_2(h,k) := \frac{\omega_{h,\frac{k}{4}}^3}{\omega_{h,k}\omega_{h,\frac{k}{2}}^2}.$$

Note that $\chi_2(h+k,k) = \chi_2(h,k)$, so $\chi_2(h,k)$ only depends on $h \pmod{k}$.

Lemma 4.1. *We have*

$$c_2(n) = \frac{2\sqrt{7}\pi}{\sqrt{24n-7}} \sum_{\substack{k\geq 1\\ 4|k}} \frac{1}{k} \sum_{h \pmod{k}^*} \chi_2(h,k) e^{-\frac{2\pi i n h}{k}} I_1\left(\frac{\pi}{6k}\sqrt{7\,(24n-7)}\right).$$

Proof. Lemma 2.1 and (2.2) imply that $f_2(q) = O(e^{-\frac{5\pi}{48kz}})$ as $q \to e^{\frac{2\pi i h}{k}}$ with k odd. For $2\|k$ Lemma 2.1 and (2.2) again imply that $f_2(q) = O(e^{-\frac{\pi}{6kz}})$ as $q \to e^{\frac{2\pi i h}{k}}$. Finally, for $4 \mid k$, taking $g = d$ in Lemma 2.1 and (2.2) imply that

$$f_2(q) = \chi_2(h,k) e^{\frac{7\pi}{12k}\left(\frac{1}{z}-z\right)}(1 + O(q_1)). \tag{4.1}$$

Setting $F_2(\tau) := q^{-\frac{7}{24}} f_2(q)$ and plugging (4.1) into Theorem 2.2 gives the claim. $\square$

4.2. **The multiplier system.** We next rewrite the multiplier system χ_2.

Lemma 4.2. *We have*

$$\chi_2(h,k) = e^{\frac{2\pi i}{24k}\left(\left(-5\frac{k^2}{2}+7\right)h - \left(5\frac{k^2}{4}+7\right)h'\right)},$$

where we choose h' to satisfy $hh' \equiv -1 \pmod{8\gcd(k,3)k}$.

Proof. Noting that we may choose $[-h]_{\frac{k}{2}} = [-h]_{\frac{k}{4}} = h'$, we may show that

$$\chi_2(h,k) = e^{2\pi i A},$$

where

$$A := \frac{1}{96k}\left(\left(\left(5k^2+28\right)h^2 - 5k^2 - 28\right)h' + \left(-5k^2+56\right)h\right).$$

The claim follows by a direct computation distinguishing whether $3 \mid k$ or $3 \nmid k$. $\square$

4.3. **The main term.** Define

$$M_2(n) := \frac{\sqrt{7}\pi \cos\left(\frac{\pi}{2}\left(n+\frac{1}{4}\right)\right)}{\sqrt{24n-7}} I_1\left(\frac{\pi}{24}\sqrt{7(24n-7)}\right),$$
$$E_2(n) := \frac{2\sqrt{7}\pi}{\sqrt{24n-7}} \sum_{\substack{k\geq 5\\ 4|k}} \frac{1}{k} \sum_{h \pmod{k}^*} \chi_2(h,k) e^{-\frac{2\pi i n h}{k}} I_1\left(\frac{\pi}{6k}\sqrt{7(24n-7)}\right).$$

Lemma 4.3. *We have*

$$c_2(n) = M_2(n) + E_2(n).$$

Moreover, if $|M_2(n)| > |E_2(n)|$, then $s_2(n)$ agrees with the claimed value in Theorem 1.3.

Proof. For the first identity, we need to show that $M_2(n)$ equals the term with $k = 4$ of Lemma 4.1, which is

$$\frac{\sqrt{7}\pi}{2\sqrt{24n-7}} \sum_{h\in\{1,3\}} \chi_2(h,4) i^{-nh} I_1\left(\frac{\pi}{24}\sqrt{7(24n-7)}\right).$$

Using Lemma 4.2, the sum on h becomes

$$\sum_{h\in\{1,3\}} e^{\frac{2\pi i}{32}(-11h-9h')} e^{-\frac{\pi i h n}{2}}.$$

From this it is not hard to see that the main term is as claimed.

To see the second statement, note that if $|M_2(n)| > |E_2(n)|$, then

$$s_2(n) = \operatorname{sgn}(M_2(n)) = \operatorname{sgn}\left(\cos\left(\frac{\pi}{2}\left(n+\frac{1}{4}\right)\right)\right) = \begin{cases} 1 & \text{if } n \equiv 0,3 \pmod{4}, \\ -1 & \text{if } n \equiv 1,2 \pmod{4}. \end{cases}$$

This matches the claim in Theorem 1.3. □

4.4. **Kloosterman sums.** Let

$$A_k(n) := \sum_{\substack{0\le h<k \\ \gcd(h,k)=1 \\ hh'\equiv -1 \pmod{8\gcd(k,3)k}}} \chi_2(h,k) e^{-\frac{2\pi i n h}{k}}.$$

Lemma 4.4.

(1) *We have, for* $3 \mid k$,

$$A_k(n) = \frac{1}{24} K_{24k}\left(-5\frac{k^2}{2} + 7 - 24n, -5\frac{k^2}{4} - 7\right).$$

(2) *We have, for* $3 \nmid k$,

$$A_k(n) = \frac{1}{8} K_{8k}\left(\frac{1}{3}\left(-5\frac{k^2}{2}+7\right) - 8n, \frac{1}{3}\left(-5\frac{k^2}{4} - 7\right)\right).$$

Proof. Using Lemma 4.2, we rewrite

$$A_k(n) = \frac{1}{8\gcd(k,3)} \sum_{h \pmod{8\gcd(k,3)k)^*}} e^{\frac{2\pi i}{24k}\left(\left(-5\frac{k^2}{2}+7-24n\right)h - \left(5\frac{k^2}{4}+7\right)h'\right)}.$$

Distinguishing whether $3 \mid k$ or $3 \nmid k$ gives the claim. □

We can now use Lemma 4.4 to explicitly bound $A_k(n)$.

Lemma 4.5. *We have*

$$|A_k(n)| \le \frac{\sqrt{7k}}{2\sqrt{2}} d(24k).$$

4.5. **Bounding the error term and finishing the proof of Theorem 1.3.** We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. Dividing the error terms $E_2(n)$ by the main term $M_2(n)$, Lemma 4.3 implies that Theorem 1.3 follows if

$$\frac{|E_2(n)|}{|M_2(n)|} < 1.$$

Applying the triangle inequality and then using Lemma 4.5, we bound

$$\frac{|E_2(n)|}{|M_2(n)|} \leq \frac{2}{\left|\cos\left(\frac{\pi}{2}\left(n+\frac{1}{4}\right)\right)\right| I_1\left(\frac{\pi}{24}\sqrt{7(24n-7)}\right)} \sum_{\substack{k\geq 5\\ 4|k}} \frac{|A_k(n)|}{k} I_1\left(\frac{\pi}{6k}\sqrt{7(24n-7)}\right)$$

$$\leq \frac{\sqrt{7}}{2\sqrt{2}\cos\left(\frac{3\pi}{8}\right) I_1\left(\frac{\pi}{24}\sqrt{7(24n-7)}\right)} \sum_{k\geq 2} \frac{d(96k)}{\sqrt{k}} I_1\left(\frac{\pi}{24k}\sqrt{7(24n-7)}\right). \tag{4.2}$$

Note that

$$d(96k) \leq d(96)d(k) = 12d(k).$$

Hence, using Lemma 2.4 (1), the contribution to the right-hand side of (4.2) from $k > \frac{\pi}{24}\sqrt{7(24n-7)}$ can be bounded by

$$\frac{7\pi\zeta^2\left(\frac{3}{2}\right)\sqrt{24n-7}}{4\sqrt{2}\cos\left(\frac{3\pi}{8}\right) I_1\left(\frac{\pi}{24}\sqrt{7(24n-7)}\right)}.$$

We next consider the contribution to the right-hand side of (4.2) from the terms with $2 \leq k \leq \frac{\pi}{24}\sqrt{7(24n-7)}$. Since $k \geq 2$, we may trivially bound

$$I_1\left(\frac{\pi}{24k}\sqrt{7(24n-7)}\right) \leq I_1\left(\frac{\pi}{48}\sqrt{7(24n-7)}\right),$$

which we may pull out of the sum on k. Since divisors of k may be paired as $(d, \frac{k}{d})$ with $d \leq \frac{k}{d}$, we have at most $\sqrt{k}$ such pairs, and hence we may trivially bound $d(k) \leq 2\sqrt{k}$. Hence the overall contribution to the right-hand side of (4.2) from the terms with $2 \leq k \leq \frac{\pi}{24}\sqrt{7(24n-7)}$ may be bounded from above by

$$\frac{7\pi\sqrt{24n-7}I_1\left(\frac{\pi}{48}\sqrt{7(24n-7)}\right)}{2\sqrt{2}\cos\left(\frac{3\pi}{8}\right) I_1\left(\frac{\pi}{24}\sqrt{7(24n-7)}\right)}.$$

Combining the two errors, we need

$$\frac{7\pi\zeta^2\left(\frac{3}{2}\right)\sqrt{24n-7}}{4\sqrt{2}\cos\left(\frac{3\pi}{8}\right) I_1\left(\frac{\pi}{24}\sqrt{7(24n-7)}\right)} + \frac{7\pi\sqrt{24n-7}I_1\left(\frac{\pi}{48}\sqrt{7(24n-7)}\right)}{2\sqrt{2}\cos\left(\frac{3\pi}{8}\right) I_1\left(\frac{\pi}{24}\sqrt{7(24n-7)}\right)} < 1. \tag{4.3}$$

We next bound (4.3) against elementary functions similarly to (3.5). Using Lemma 2.4 (2),(3) for $\frac{\pi}{24}\sqrt{7(24n-7)} \geq 3$ the left-hand side of (4.3) may for $n \geq 99$ be estimated against

$$\frac{7^{\frac{5}{4}}\pi^{\frac{3}{2}}\zeta^2\left(\frac{3}{2}\right)(24n-7)^{\frac{3}{4}}}{4\sqrt{3}\cos\left(\frac{3\pi}{8}\right)} e^{-\frac{\pi}{24}\sqrt{7(24n-7)}} + \frac{14\sqrt{2\pi}\sqrt{24n-7}}{\cos\left(\frac{3\pi}{8}\right)} e^{-\frac{\pi}{48}\sqrt{7(24n-7)}} < 1.$$

Theorem 1.3 now follows after checking the claim for $n \leq 98$ with a computer. ∎

5. Proof of Theorem 1.4

5.1. **General transformations.** We abbreviate $c_3(n) := C_{1^9 3^{-5}}(n)$ and $s_3(n) := \operatorname{sgn}(c_3(n))$. We determine the transformation law of

$$f_3(q) := \frac{(q;q)_\infty^9}{(q^3;q^3)_\infty^5} = \frac{P^5\left(q^3\right)}{P^9(q)} = \sum_{n\geq 0} c_3(n) q^n.$$

For $3 \mid k$, we set

$$\chi_3(h,k) := \frac{\omega_{h,\frac{k}{3}}^5}{\omega_{h,k}^9}. \tag{5.1}$$

Lemma 2.1 with $d = g$ and (2.2) yield

$$f_3(q) = \chi_3(h,k) z^{-2} e^{\frac{\pi}{2k}\left(\frac{1}{z}-z\right)} f_3(q_1). \tag{5.2}$$

For $3 \nmid k$, we note that we may choose h' with $3 \mid h'$, so that

$$[3]_k h' \equiv \frac{h'}{3} \pmod{k}.$$

Hence for $3 \nmid k$ and $3 \mid h'$, Lemma 2.1 and (2.2) yield

$$f_3(q) = \frac{\omega_{3h,k}^5}{\omega_{h,k}^9} 3^{\frac{5}{2}} z^{-2} e^{-\frac{11\pi}{18kz} - \frac{\pi z}{2k}} \frac{P^5\left(q_1^{\frac{1}{3}}\right)}{P^9(q_1)}. \tag{5.3}$$

In particular, this vanishes as $z \to 0$.

5.2. **The Circle Method.** Using the Residue Theorem, we have

$$c_3(n) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f_3(q)}{q^{n+1}} dq = \sum_{\substack{0\leq h<k<N \\ \gcd(h,k)=1}} e^{-\frac{2\pi i n h}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} f_3\left(e^{\frac{2\pi i}{k}(h+iz)}\right) e^{\frac{2\pi n z}{k}} d\Phi = \sum\nolimits_1 + \sum\nolimits_3,$$

where $z = \frac{k}{n} + ik\Phi$, $\vartheta'_{h,k} := \frac{1}{k(k+k_1)}$, $\vartheta''_{h,k} := \frac{1}{k(k+k_2)}$ with $\frac{h_1}{k_1} < \frac{h}{k} < \frac{h_2}{k_2}$ are consecutive fractions in the Farey sequence of order $N := \lfloor\sqrt{n}\rfloor$ and where $\sum_d$ is the restriction of the sum $\sum_{0\leq h<k\leq 0}$ to those k satisfying $\gcd(k,3) = d$. Assuming that $d \mid h'$, we may use (5.2) and (5.3) to obtain

$$\sum\nolimits_3 = \sum_{\substack{0\leq h<k<N \\ \gcd(h,k)=1 \\ 3\mid k}} \chi_3(h,k) e^{-\frac{2\pi i n h}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{-2} e^{\frac{\pi}{2k}\left(\frac{1}{z}-z\right)} f_3(q_1) e^{\frac{2\pi n z}{k}} d\Phi,$$

$$\sum\nolimits_1 = 3^{\frac{5}{2}} \sum_{\substack{0\leq h<k<N \\ \gcd(h,k)=1 \\ 3\nmid k}} \frac{\omega_{3h,k}^5}{\omega_{h,k}^9} e^{-\frac{2\pi i n h}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{-2} e^{-\frac{11\pi}{18kz} - \frac{\pi z}{2k}} \frac{P^5\left(q_1^{\frac{1}{3}}\right)}{P^9(q_1)} e^{\frac{2\pi n z}{k}} d\Phi.$$

We require the well-known bounds

$$\vartheta'_{h,k}, \vartheta''_{h,k} \leq \frac{1}{k(N+1)}, \quad \operatorname{Re}\left(\frac{1}{z}\right) \geq \frac{k}{2}, \quad |z| \geq \frac{k}{n}. \tag{5.4}$$

Next we split-off the principal parts. Note that (5.3) implies that these only occur if $3 \mid k$. We write

$$f_3(q_1) = 1 + (f_3(q_1) - 1) .$$

We call the contribution from the constant term $\sum_{3,1}$ and the remaining sum $\sum_{3,2}$. We have

$$\sum\nolimits_{3,1} = -i \sum_{\substack{0 \le h < k < N \\ \gcd(h,k)=1 \\ 3|k}} \frac{\chi_3(h,k)}{k} e^{-\frac{2\pi i n h}{k}} \int_{\frac{k}{n} - ik\vartheta'_{h,k}}^{\frac{k}{n} + ik\vartheta''_{h,k}} z^{-2} e^{\frac{\pi}{2kz} + \frac{\pi z}{2k}(4n-1)} dz. \tag{5.5}$$

We now approximate the integral in (5.5) in the following lemma.

Lemma 5.1. *We have*

$$\left| \int_{\frac{k}{n} - ik\vartheta'_{h,k}}^{\frac{k}{n} + ik\vartheta''_{h,k}} z^{-2} e^{\frac{\pi}{2kz} + \frac{\pi z}{2k}(4n-1)} dz - 2\pi i \sqrt{4n-1} I_1 \left(\frac{\pi}{k} \sqrt{4n-1} \right) \right| \le 4588024 \frac{n^{\frac{3}{2}}}{k^2}.$$

Proof. We follow the argument given in [10, pp. 404–405]. Let $A := \frac{\pi}{2k}(4n-1)$ and $B := \frac{\pi}{2k}$. From §6.22, equation (1) of [14], we obtain

$$2\pi i \sqrt{4n-1} I_1 \left(\frac{\pi}{k} \sqrt{4n-1} \right) = \left(\int_{-\infty}^{-\frac{k}{n}} + \int_{-\frac{k}{n}}^{-\frac{k}{n} - ik\vartheta'_{h,k}} + \int_{-\frac{k}{n} - ik\vartheta'_{h,k}}^{\frac{k}{n} - ik\vartheta'_{h,k}} + \int_{\frac{k}{n} - ik\vartheta'_{h,k}}^{\frac{k}{n} + ik\vartheta''_{h,k}} + \int_{\frac{k}{n} + ik\vartheta''_{h,k}}^{-\frac{k}{n} + ik\vartheta''_{h,k}} \right.$$
$$\left. + \int_{-\frac{k}{n} + ik\vartheta''_{h,k}}^{-\frac{k}{n}} + \int_{-\frac{k}{n}}^{-\infty} \right) z^{-2} \exp\left(Az + \frac{B}{z} \right) dz =: J_1 + J_2 + \ldots + J_7.$$

Note that J_4 is the integral appearing on the left-hand side of the lemma, so, subtracting that from the other side, it remains to bound the absolute values of the other integrals.

For J_2, we have $\mathrm{Re}(z) = -\frac{k}{n} < 0$ and $\mathrm{Re}(\frac{1}{z}) = \frac{\mathrm{Re}(z)}{|z|^2} < 0$. Thus, we may estimate

$$|J_2| \le \int_{-\frac{k}{n}}^{-\frac{k}{n} - ik\vartheta'_{h,k}} |z|^{-2} d|z| = \int_0^{k\vartheta'_{h,k}} \frac{1}{\frac{k^2}{n^2} + y^2} dy.$$

We bound this by setting $y = 0$ in the integrand and get

$$|J_2| \le \frac{n^2}{k^2} k\vartheta'_{h,k} \le \frac{n^2}{k^2(N+1)} \le \frac{n^{\frac{3}{2}}}{k^2}.$$

We have the same bound for $|J_6|$.

For J_3, we make the change of variables $z = x - ik\vartheta'_{h,k}$. Note that here $-\frac{k}{n} \le \mathrm{Re}(z) = x \le \frac{k}{n}$ and $\mathrm{Re}(\frac{1}{z}) \le 4k$. Thus, on J_3 we have

$$\left| \exp\left(Az + \frac{B}{z} \right) \right| \le \exp(4\pi),$$

so

$$|J_3| \le \exp(4\pi) \int_{-\frac{k}{n}}^{\frac{k}{n}} \frac{1}{|x - ik\vartheta'_{h,k}|^2} dx \le \exp(4\pi) \int_{-\frac{k}{n}}^{\frac{k}{n}} \frac{1}{x^2 + \left(k\vartheta'_{h,k}\right)^2} dx.$$

We bound the integral by setting $x = 0$ in the integrand and get $|J_3| \le 8 \exp(4\pi) k$. We bound J_5 in the same way.

Finally, we combine

$$J_1 + J_7 = 0.$$

Using that $1 \leq k \leq \sqrt{n}$, we easily obtain the claim. $\square$

Plugging Lemma 5.1 into (5.5), we obtain

$$\sum\nolimits_{3,1} = 2\pi\sqrt{4n-1} \sum_{\substack{0\leq h<k<N \\ \gcd(h,k)=1 \\ 3|k}} \frac{\chi_3(h,k)}{k} e^{-\frac{2\pi inh}{k}} I_1\left(\frac{\pi}{k}\sqrt{4n-1}\right) + E(n), \tag{5.6}$$

where $E(n)$ may be bounded against

$$|E(n)| \leq 4588024 n^{\frac{3}{2}} \sum_{\substack{1\leq k<N \\ 3|k}} \frac{1}{k^2} \leq \frac{4588024}{9}\zeta(2) n^{\frac{3}{2}} = \frac{2294012\pi^2}{27} n^{\frac{3}{2}}. \tag{5.7}$$

We define

$$M_3(n) := \frac{2\pi}{3\gcd(n,3)} \alpha_n \sqrt{4n-1} I_1\left(\frac{\pi}{3\gcd(n,3)}\sqrt{4n-1}\right), \tag{5.8}$$

$$E_3(n) := 2\pi\sqrt{4n-1} \sum_{\substack{0\leq h<k<N \\ \gcd(h,k)=1 \\ 3|k \\ k\geq 3\ell_n}} \frac{\chi_3(h,k)}{k} e^{-\frac{2\pi inh}{k}} I_1\left(\frac{\pi}{k}\sqrt{4n-1}\right),$$

where

$$\alpha_n := \begin{cases} -2\sin\left(\frac{2\pi n}{3}\right) & \text{if } 3\nmid n, \\ 2\left(\sin\left(\frac{2\pi}{9}(4-2n)\right) - \sin\left(\frac{2\pi}{9}(5-n)\right) - \sin\left(\frac{2\pi}{9}(5-4n)\right)\right) & \text{if } 3 \mid n, \end{cases} \tag{5.9}$$

$$\ell_n := \begin{cases} 2 & \text{if } 3\nmid n, \\ 4 & \text{if } 3 \mid n. \end{cases}$$

We obtain the following formula for $c_3(n)$.

Lemma 5.2. *We have*

$$c_3(n) = M_3(n) + E_3(n) + E(n) + \sum\nolimits_{3,2} + \sum\nolimits_{1}.$$

Moreover, if $|M_3(n)| > |E_3(n)| + |E(n)| + |\sum_{3,2}| + |\sum_1|$, *then* $s_3(n) = \operatorname{sgn}(\alpha_n)$, *which agrees with the claimed value of* $s_3(n)$ *in Theorem 1.4.*

Proof. Recall (5.6). For $3 \nmid n$ the first identity is equivalent to showing that $M_3(n)$ is the term with $k = 3$ of the sum in (5.6), while for $3 \mid n$ it is equivalent to showing that the terms with $k = 3$ and $k = 6$ vanish and $M_3(n)$ equals the term with $k = 9$ in (5.6).

We first evaluate the term coming from $k = 3$. Using (2.3) and (5.1) of $\chi_3(h,k)$, the sum on h for this term equals

$$\sum_{h\in\{1,2\}} \chi_3(h,3) e^{-\frac{2\pi ihn}{3}} = -2\sin\left(\frac{2\pi n}{3}\right).$$

This yields the first identity for $3 \nmid n$, and for $n \equiv 0 \pmod 3$ this vanishes.

For $k=6$, the sum on h becomes

$$\sum_{h\in\{1,5\}} \chi_3(h,6)e^{-\frac{2\pi i h n}{6}} = -2\sin\left(\frac{\pi n}{3}\right).$$

This also vanishes if $3 \mid n$.

Finally, plugging (2.3) into the sum on h for $k=9$ and $3 \mid n$ gives

$$\sum_{h\in\{1,2,4,5,7,8\}} \chi_3(h,9)e^{-\frac{2\pi i h n}{9}} = \alpha_n.$$

This yields the first identity for $3 \mid n$ as well.

Next suppose that $|M_3(n)| > |E_3(n)| + |E(n)| + |\sum_{3,2}| + |\sum_1|$. In this case, we have $s_3(n) = \operatorname{sgn}(\alpha_n)$, and one easily checks that

$$\operatorname{sgn}(\alpha_n) = \begin{cases} 1 & \text{if } n \equiv 0,2,5,6,8 \pmod 9, \\ -1 & \text{if } n \equiv 1,3,4,7 \pmod 9, \end{cases}$$

which matches the value claimed in Theorem 1.4. □

5.3. **Error bounds.** By Lemma 5.2, we need to bound $|M_3(n)|$ from below and $|E_3(n)|$, $|E(n)|$, $|\sum_{3,2}|$, and $|\sum_1|$ from above. The estimate for $|E(n)|$ is given in (5.7).

5.3.1. *Lower bounds for the main term.* Since the definition of $M_3(n)$ depends on $\gcd(n,3)$, we distinguish whether $3 \nmid n$ or $3 \mid n$. If $3 \nmid n$, then the definitions (5.8) and (5.9) give

$$|M_3(n)| = \frac{4\pi}{3}\sin\left(\frac{2\pi}{3}\right)\sqrt{4n-1}\,I_1\left(\frac{\pi}{3}\sqrt{4n-1}\right). \tag{5.10}$$

If $n \equiv 0 \pmod 3$, then the definitions (5.8) and (5.9) imply that

$$|M_3(n)| = \frac{2\pi}{9}|\alpha_n|\sqrt{4n-1}\,I_1\left(\frac{\pi}{9}\sqrt{4n-1}\right) \geq \frac{4\pi}{3}\sin\left(\frac{\pi}{9}\right)\sqrt{4n-1}\,I_1\left(\frac{\pi}{9}\sqrt{4n-1}\right). \tag{5.11}$$

5.3.2. *Bound of tails from* (5.6). Applying the bound $I_1(\frac{\pi}{k}\sqrt{4n-1}) \leq I_1(\frac{\pi}{3\ell_n}\sqrt{4n-1})$ uniformly for $k \geq 3\ell_n$ with $3 \mid k$ and trivially bounding the sum over h in $E_3(n)$ yields the following upper bound for $|E_3(n)|$.

Lemma 5.3. *We have*

$$|E_3(n)| \leq \frac{4\pi n}{3} I_1\left(\frac{\pi}{3\ell_n}\sqrt{4n-1}\right).$$

5.3.3. *Bounding* $\sum_1$. We require the following bound.

Lemma 5.4. *Assuming the notation above, we have*

$$\left|\frac{P^5\left(q_1^{\frac{1}{3}}\right)}{P^9(q_1)}\right| \leq 72.$$

Proof. We define

$$\frac{P^5(q)}{P^9(q^3)} =: \sum_{n\geq 0}\beta(n)q^n.$$

It is not hard to see that $\beta(n) \geq 0$. Thus

$$\left|\frac{P^5\left(q_1^{\frac{1}{3}}\right)}{P^9\left(q_1\right)}\right| \leq \sum_{n\geq 0} \beta(n)|q_1|^{\frac{n}{3}}.$$

By (5.4), we have $|q_1| = e^{-\frac{2\pi}{k}\mathrm{Im}\left(\frac{1}{z}\right)} < e^{-\pi}$, and hence

$$\left|\frac{P^5\left(q_1^{\frac{1}{3}}\right)}{P^9\left(q_1\right)}\right| \leq \sum_{n\geq 0} \beta(n)e^{-\frac{\pi n}{3}} = \frac{P^5\left(e^{-\frac{\pi}{3}}\right)}{P^9\left(e^{-\pi}\right)} \leq P^5\left(e^{-\frac{\pi}{3}}\right).$$

We now use the well-known estimate (for example, see [4, Theorem 14.5])

$$p(n) < e^{\pi\sqrt{\frac{2n}{3}}}. \tag{5.12}$$

Thus, for $n \geq 10$

$$p(n)e^{-\frac{\pi n}{3}} < e^{\pi\sqrt{\frac{2n}{3}}-\frac{\pi n}{3}} \leq e^{-\frac{\pi n}{13.5}}.$$

Thus

$$P\left(e^{-\frac{\pi}{3}}\right) \leq 1 + \sum_{n=1}^{9} p(n)e^{-\frac{\pi n}{3}} + \frac{e^{-\frac{10\pi}{13.5}}}{1-e^{-\frac{\pi}{13.5}}}.$$

Explicitly plugging in $p(n)$ for $1 \leq n \leq 9$ and raising to the fifth power then yields

$$P^5\left(e^{-\frac{\pi}{3}}\right) \leq 72.$$

This gives the claim. □

Using (5.4) and Lemma 5.4, we overall bound (noting that $N+1 > \sqrt{n}$)

$$\begin{aligned}
\left|\sum\nolimits_1\right| &\leq 3^{\frac{5}{2}} \cdot 72 \sum_{\substack{1\leq k<N \\ 3\nmid k}} k\left(|\vartheta'_{h,k}| + |\vartheta''_{h,k}|\right) \max_{-\vartheta'_{h,k}\leq\Phi\leq\vartheta''_{h,k}} \frac{1}{|z|^2} e^{-\frac{11\pi}{18k}\mathrm{Re}\left(\frac{1}{z}\right)-\frac{\pi}{2k}\mathrm{Re}(z)+\frac{2\pi n\mathrm{Re}(z)}{k}} \\
&\leq 3^{\frac{5}{2}} \cdot 72 \sum_{1\leq k<N} \frac{2}{N+1}\left(\frac{k}{n}\right)^{-2} e^{-\frac{11\pi}{18\cdot 2}+2\pi} \leq 3^{\frac{5}{2}} \cdot 144 e^{-\frac{11\pi}{36}+2\pi} n^{\frac{3}{2}} \sum_{1\leq k<N} \frac{1}{k^2} \\
&\leq 3^{\frac{5}{2}} \cdot 144 e^{-\frac{11\pi}{36}+2\pi}\zeta(2)n^{\frac{3}{2}} \leq 757137 n^{\frac{3}{2}}. \qquad (5.13)
\end{aligned}$$

5.3.4. *Bounding* $\sum_{3,2}$.

Lemma 5.5. *Assuming the notation above, we have*

$$|f_3(q_1) - 1|\,|q_1|^{-\frac{1}{4}} \leq \frac{6}{5}.$$

Proof. It is not hard to see that

$$\left|\prod_{n\geq 1} \frac{(1-q^n)^9}{(1-q^{3n})^5} - 1\right| \leq \left|P^9(|q|)P^5\left(|q|^3\right) - 1\right|.$$

Now we again use that $|q_1| < e^{-\pi}$ by (5.4). Using (5.12), for $n \geq 6$ we have

$$p(n)e^{-\pi n} < e^{\pi\sqrt{\frac{2n}{3}}-\pi n} \leq e^{-\frac{2\pi n}{3}}.$$

Thus

$$P\left(e^{-\pi}\right) \leq 1 + \sum_{n=1}^{5} p(n)e^{-\pi n} + \frac{e^{-4\pi}}{1-e^{-\frac{2\pi}{3}}}.$$

Explicitly plugging in $p(n)$ for $1 \leq n \leq 5$ then yields

$$1 \leq P^9\left(e^{-\pi}\right) \leq 1.52.$$

Similarly, (5.12) gives, for $n \geq 1$,

$$p(n)e^{-3\pi n} < e^{\pi\sqrt{\frac{2n}{3}}-3\pi n} \leq e^{-2\pi n}.$$

Thus

$$1 \leq P^5\left(e^{-3\pi}\right) \leq \left(\sum_{n\geq 0} e^{-2\pi n}\right)^5 = \frac{1}{\left(1-e^{-2\pi}\right)^5} \leq 1.01.$$

Therefore

$$\left(P^9\left(e^{-\pi}\right)P^5\left(e^{-3\pi}\right)-1\right)e^{\frac{\pi}{4}} < \frac{6}{5}.$$

The claim now follows. □

Using Lemma 5.5, we are now able to bound the contribution from $\sum_{3,2}$.

Lemma 5.6. *We have*

$$\left|\sum\nolimits_{3,2}\right| \leq 235n^{\frac{3}{2}}.$$

Proof. Trivially estimating the sum on h and using Lemma 5.5, we bound

$$\left|\sum\nolimits_{3,2}\right| \leq \sum_{\substack{1\leq k<N\\ 3|k}} k\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} \frac{1}{|z|^2}e^{\frac{\pi}{2k}\left(\mathrm{Re}\left(\frac{1}{z}\right)-\mathrm{Re}(z)\right)}\left|f_3(q_1)-1\right| e^{\frac{2\pi n\mathrm{Re}(z)}{k}}d\Phi$$
$$\leq \frac{6}{5}\sum_{\substack{1\leq k<N\\ 3|k}} k\left(\left|\vartheta'_{h,k}\right|+\left|\vartheta''_{h,k}\right|\right)\max_{-\vartheta'_{h,k}\leq\Phi\leq\vartheta''_{h,k}}\frac{1}{|z|^2}e^{\frac{\pi}{2k}\left(\mathrm{Re}\left(\frac{1}{z}\right)-\mathrm{Re}(z)\right)}|q_1|^{\frac{1}{4}}e^{\frac{2\pi n\mathrm{Re}(z)}{k}}. \quad (5.14)$$

By (5.4), we have $|z|^{-2} < (\frac{k}{n})^{-2}$ and $k|\vartheta'_{h,k}| \leq \frac{1}{N+1}$. Moreover, we have

$$e^{\frac{\pi}{2k}\mathrm{Re}\left(\frac{1}{z}\right)}|q_1|^{\frac{1}{4}} = 1.$$

Plugging back into (5.14) and using that $\mathrm{Re}(z) = \frac{k}{n}$ (by (5.4)) then yields

$$\left|\sum\nolimits_{3,2}\right| \leq \frac{12e^{2\pi}n^2}{5(N+1)}\sum_{\substack{1\leq k<N\\ 3|k}}\frac{1}{k^2} \leq \frac{4}{15}e^{2\pi}\zeta(2)n^{\frac{3}{2}} \leq 235n^{\frac{3}{2}}.$$ □

5.4. **Finishing the proof.**

5.4.1. $3 \nmid n$. Assume that $3 \nmid n$. Plugging (5.7), (5.10), Lemma 5.3, (5.13), and Lemma 5.6 into Lemma 5.2, we conclude that $s_3(n)$ is as claimed in Theorem 1.4 if

$$\left(\frac{4\pi}{3}\sin\left(\frac{2\pi}{3}\right)\sqrt{4n-1}I_1\left(\frac{\pi}{3}\sqrt{4n-1}\right)\right)^{-1} \times\left(\frac{4\pi n}{3}I_1\left(\frac{\pi}{6}\sqrt{4n-1}\right)+757137n^{\frac{3}{2}}+235n^{\frac{3}{2}}+\frac{2294012\pi^2}{27}n^{\frac{3}{2}}\right)<1. \quad (5.15)$$

We first estimate the left-hand side of (5.15) against

$$\left(\frac{4\pi}{3}\sin\left(\frac{2\pi}{3}\right)\sqrt{4n-1}I_1\left(\frac{\pi}{3}\sqrt{4n-1}\right)\right)^{-1}\left(\frac{4\pi n}{3}I_1\left(\frac{\pi}{6}\sqrt{4n-1}\right)+1595928n^{\frac{3}{2}}\right). \quad (5.16)$$

For $n \geq 6$, we may use Lemma 2.4 (2), (3) to bound (5.16) from above by

$$\frac{\sqrt{3}e^{-\frac{\pi}{3}\sqrt{4n-1}}}{\sqrt{\pi}\sin\left(\frac{2\pi}{3}\right)(4n-1)^{\frac{1}{4}}}\left(\frac{8ne^{\frac{\pi}{6}\sqrt{4n-1}}}{\sqrt{3}(4n-1)^{\frac{1}{4}}}+1595928n^{\frac{3}{2}}\right).$$

Bounding as we do for (3.5) yields that this is less than 1 for $n \geq 89$, and for $n < 89$ we directly evaluate $s_3(n)$ with a computer, verifying Theorem 1.4 for $3 \nmid n$.

5.4.2. $3 \mid n$. Next assume that $3 \mid n$. Plugging (5.7), (5.11), Lemma 5.3, (5.13), and Lemma 5.6 into Lemma 5.2, we conclude that $s_3(n)$ agrees with the value in the claim of Theorem 1.4 if

$$\left(\frac{4\pi}{3}\sin\left(\frac{\pi}{9}\right)\sqrt{4n-1}I_1\left(\frac{\pi}{9}\sqrt{4n-1}\right)\right)^{-1} \times\left(\frac{4\pi n}{3}I_1\left(\frac{\pi}{12}\sqrt{4n-1}\right)+757137n^{\frac{3}{2}}+235n^{\frac{3}{2}}+\frac{2294012\pi^2}{27}n^{\frac{3}{2}}\right)<1. \quad (5.17)$$

We first bound the left-hand side of (5.17) from above by

$$\left(\frac{4\pi}{3}\sin\left(\frac{\pi}{9}\right)\sqrt{4n-1}I_1\left(\frac{\pi}{9}\sqrt{4n-1}\right)\right)^{-1}\left(\frac{4\pi n}{3}I_1\left(\frac{\pi}{12}\sqrt{4n-1}\right)+1595928n^{\frac{3}{2}}\right). \quad (5.18)$$

For $n \geq 19$, (5.18) may be bounded from above by

$$\frac{1}{\sqrt{\pi}\sin\left(\frac{\pi}{9}\right)(4n-1)^{\frac{1}{4}}}e^{-\frac{\pi}{9}\sqrt{4n-1}}\left(\frac{8\sqrt{2}ne^{\frac{\pi}{12}\sqrt{4n-1}}}{\sqrt{3}(4n-1)^{\frac{1}{4}}}+1595928n^{\frac{3}{2}}\right).$$

Arguing as we do for verifying (3.5), one can check that this is less than 1 for $n \geq 1173$. Confirming Theorem 1.4 for $n < 1173$ directly with a computer, we conclude Theorem 1.4.

Appendix A. Further computational evidence for purely periodic sign changes

For each sign pattern listed below, we list further choices of $1^{\delta_1}2^{\delta_2}\cdots m^{\delta_m}$ for which computational data indicates that $s_{1^{\delta_1}2^{\delta_2}\cdots m^{\delta_m}}(n)$ satisfies the given purely periodic sign pattern. We verified these conjectures up to order q^{200000}. Note that the list is not exhaustive.

Table 1. The list of conjectures.

period/ sign pattern	case				
2/ $+-$	$1^{2}2^{-3}3^{-3}$ $1^{3}2^{-2}3^{3}$	$1^{2}2^{-2}3^{4}$ $1^{4}2^{-4}3^{-4}$	$1^{3}2^{-2}$ $1^{4}2^{-3}3^{4}$	$1^{3}2^{-2}3^{1}$	$1^{3}2^{-2}3^{2}$
3/ $++-$	$1^{-1}2^{5}3^{-3}4^{2}$ $1^{-1}2^{4}3^{-2}$	$1^{-1}2^{3}3^{-3}4^{1}$ $1^{-1}2^{5}3^{-4}4^{3}$	$1^{-1}2^{3}3^{-2}$ $1^{-1}2^{5}3^{-3}$	$1^{-1}2^{4}3^{-4}4^{2}$ $1^{-1}2^{5}3^{-3}4^{1}$	$1^{-1}2^{4}3^{-3}4^{1}$
3/ $+-+$	$1^{3}2^{-2}3^{-3}$ $1^{4}3^{-2}4^{3}$	$1^{3}2^{-1}3^{-2}$ $1^{4}3^{-3}4^{4}$	$1^{4}2^{-2}3^{-3}$ $1^{4}3^{-2}4^{1}$	$1^{4}2^{-1}3^{-2}$ $1^{4}3^{-2}4^{2}$	$1^{4}3^{-3}$
3/ $+0-$	$2^{2}3^{-1}4^{-1}$	$2^{3}3^{-1}$			
3/ $+-0$	$1^{3}3^{-2}$	$1^{1}2^{-1}3^{-2}4^{1}$	$1^{2}2^{-1}3^{-2}$		
3/ $+--$	$1^{1}2^{1}3^{-2}$ $1^{2}2^{3}3^{-3}$ $1^{3}2^{4}3^{-2}$	$1^{1}2^{2}3^{-2}$ $1^{2}2^{4}3^{-4}$ $1^{4}2^{3}3^{-2}$	$1^{1}2^{3}3^{-4}$ $1^{3}2^{1}3^{-2}$ $1^{4}2^{4}3^{-2}$	$1^{2}2^{1}3^{-2}$ $1^{3}2^{2}3^{-2}$	$1^{2}2^{2}3^{-2}$ $1^{3}2^{3}3^{-2}$
4/ $+++-$	$1^{-1}2^{1}3^{3}4^{-3}$	$1^{-3}2^{8}4^{-5}5^{-1}$	$1^{-3}2^{8}3^{1}4^{-7}$	$1^{-1}2^{1}3^{3}4^{-4}5^{-2}$	$1^{-1}3^{4}4^{-3}5^{1}$
4/ $++--$	$1^{-1}2^{3}3^{1}4^{-3}$	$1^{-1}2^{3}3^{2}4^{-4}$	$1^{-1}2^{4}4^{-4}$	$1^{-1}2^{4}3^{1}4^{-3}$	$1^{-1}2^{4}3^{2}4^{-4}$
4/ $++0+$	$1^{-1}2^{2}4^{-2}5^{1}$				
4/ $+--+$	$1^{1}3^{-1}4^{-3}$ $1^{1}2^{3}3^{-1}4^{-3}$ $1^{2}4^{-2}$ $1^{2}2^{2}4^{-2}$ $1^{3}2^{1}4^{-2}$	$1^{1}2^{1}3^{-1}4^{-3}$ $1^{1}2^{3}4^{-3}$ $1^{2}3^{1}4^{-2}$ $1^{2}2^{3}3^{-1}4^{-3}$ $1^{3}2^{2}4^{-2}$	$1^{1}2^{1}4^{-3}$ $1^{1}2^{4}3^{-1}4^{-4}$ $1^{2}2^{1}3^{-1}4^{-3}$ $1^{2}2^{3}4^{-3}$ $1^{3}2^{3}4^{-3}$	$1^{1}2^{2}3^{-1}4^{-3}$ $1^{1}2^{4}4^{-3}$ $1^{2}2^{1}4^{-2}$ $1^{2}2^{4}3^{-1}4^{-4}$ $1^{3}2^{4}4^{-3}$	$1^{1}2^{2}4^{-3}$ $1^{2}3^{-1}4^{-3}$ $1^{2}2^{2}3^{-1}4^{-3}$ $1^{2}2^{4}4^{-3}$
4/ $+---$	$1^{1}3^{1}4^{-3}$				
4/ $+-00$	$1^{2}2^{-1}4^{-1}$				
4/ $+--0$	$1^{2}3^{2}4^{-3}$				
4/ $+-++$	$1^{3}2^{-1}3^{-1}4^{-2}$	$1^{3}2^{-1}4^{-2}5^{1}$	$1^{4}2^{-1}4^{-3}$	$1^{4}4^{-6}$	$1^{4}2^{1}4^{-9}$
4/ $+-0+$	$1^{3}3^{-1}4^{-4}$				
4/ $+-+0$	$1^{4}2^{-2}4^{-1}$				
5/ $+++--$	$1^{-2}2^{3}3^{5}5^{-5}$ $1^{-2}2^{4}3^{6}5^{-7}$	$1^{-2}2^{3}3^{6}5^{-7}$ $1^{-2}2^{4}3^{7}5^{-8}$	$1^{-2}2^{4}3^{3}5^{-5}$ $1^{-1}2^{1}3^{3}5^{-3}$	$1^{-2}2^{4}3^{4}5^{-5}$	$1^{-2}2^{4}3^{5}5^{-6}$
5/ $++++-$	$1^{-3}2^{5}3^{4}5^{-9}$	$1^{-2}2^{3}3^{2}4^{2}5^{-4}$	$1^{-2}2^{3}3^{3}4^{1}5^{-4}$	$1^{-1}2^{1}4^{3}5^{-2}$	$1^{-1}2^{1}3^{1}4^{2}5^{-2}$
5/ $+++-+$	$1^{-3}2^{8}4^{-4}5^{-4}$	$1^{-1}3^{4}5^{-2}$	$1^{-1}2^{1}3^{3}4^{-2}5^{-5}$		
5/ $++--+$	$1^{-1}2^{4}4^{-2}5^{-3}$				
5/ $++---$	$1^{-1}2^{3}3^{1}5^{-3}$ $1^{-1}2^{4}3^{2}5^{-3}$ $1^{-1}2^{5}3^{3}5^{-3}$ $1^{-1}2^{6}3^{5}5^{-2}$	$1^{-1}2^{3}3^{2}5^{-3}$ $1^{-1}2^{4}3^{3}5^{-3}$ $1^{-1}2^{5}3^{4}5^{-3}$ $1^{-1}2^{6}3^{6}5^{-3}$	$1^{-1}2^{3}3^{3}5^{-3}$ $1^{-1}2^{4}3^{4}5^{-4}$ $1^{-1}2^{5}3^{5}5^{-4}$ $1^{-1}2^{7}3^{6}5^{-3}$	$1^{-1}2^{4}5^{-4}$ $1^{-1}2^{5}3^{1}5^{-3}$ $1^{-1}2^{6}3^{3}5^{-3}$ $1^{-1}2^{7}3^{7}5^{-5}$	$1^{-1}2^{4}3^{1}5^{-3}$ $1^{-1}2^{5}3^{2}5^{-2}$ $1^{-1}2^{6}3^{4}5^{-3}$
5/ $++0+0$	$1^{-1}2^{2}5^{-1}$				
5/ $++-00$	$1^{-1}2^{3}4^{-1}5^{-2}$				
5/ $+0-0-$	$2^{1}5^{-1}$				
5/ $+-+--$	$1^{1}2^{-2}4^{4}5^{-3}$				

5/ +−+−+	$1^{1}2^{-2}5^{-8}$	$1^{1}2^{-2}4^{1}5^{-6}$	$1^{1}2^{-2}4^{2}5^{-5}$	$1^{2}2^{-3}4^{2}5^{-7}$	$1^{2}2^{-2}5^{-7}$
	$1^{2}2^{-2}4^{1}5^{-5}$	$1^{2}2^{-2}4^{2}5^{-3}$	$1^{2}2^{-2}4^{3}5^{-3}$	$1^{3}2^{-3}4^{3}5^{-4}$	$1^{3}2^{-2}5^{-7}$
	$1^{3}2^{-2}4^{1}5^{-3}$	$1^{3}2^{-2}4^{2}5^{-3}$	$1^{3}2^{-2}4^{3}5^{-2}$	$1^{3}2^{-2}4^{4}5^{-2}$	$1^{4}2^{-3}4^{4}5^{-3}$
	$1^{5}2^{-3}4^{5}5^{-3}$	$1^{10}2^{-3}5^{-5}$			
5/ +−−+−	$1^{1}3^{-2}5^{-6}$	$1^{1}3^{-1}5^{-4}$	$1^{1}3^{-1}4^{1}5^{-4}$	$1^{1}2^{1}3^{-1}5^{-5}$	$1^{2}3^{-2}4^{1}5^{-6}$
	$1^{2}3^{-1}5^{-4}$	$1^{2}3^{-1}4^{1}5^{-3}$	$1^{2}3^{-1}4^{2}5^{-3}$	$1^{3}2^{1}3^{-1}5^{-4}$	$1^{3}2^{1}5^{-3}$
	$1^{3}2^{2}5^{-8}$				
5/ +−−−+	$1^{1}3^{1}4^{-1}5^{-4}$	$1^{1}3^{1}5^{-3}$	$1^{1}3^{2}5^{-3}$	$1^{1}3^{3}4^{-1}5^{-3}$	$1^{1}3^{3}5^{-2}$
	$1^{1}3^{4}5^{-3}$	$1^{2}3^{3}5^{-3}$			
5/ +−−++	$1^{1}2^{1}4^{-1}5^{-3}$	$1^{1}2^{2}4^{-2}5^{-3}$	$1^{1}2^{2}4^{-1}5^{-3}$	$1^{1}2^{2}5^{-5}$	$1^{1}2^{3}4^{-1}5^{-3}$
	$1^{2}4^{-1}5^{-4}$	$1^{2}5^{-3}$	$1^{2}3^{1}5^{-4}$	$1^{2}2^{1}5^{-5}$	$1^{2}2^{2}4^{-1}5^{-3}$
	$1^{2}2^{2}5^{-2}$	$1^{2}2^{3}5^{-5}$			
5/ +−00+	$1^{2}2^{-1}5^{-2}$				
5/ +−++−	$1^{4}3^{-1}5^{-3}$	$1^{4}4^{1}5^{-3}$			
5/ +−+++	$1^{3}2^{-1}5^{-2}$	$1^{3}2^{-1}4^{1}5^{-3}$	$1^{3}2^{-1}3^{1}5^{-3}$	$1^{4}2^{-1}3^{1}4^{1}5^{-4}$	$1^{4}3^{2}4^{1}5^{-5}$
5/ +−0+0	$1^{3}5^{-1}$				
6/ +−+−++	$1^{1}2^{-8}3^{-6}4^{7}5^{5}$	$1^{1}2^{-8}3^{-6}4^{8}5^{5}$	$1^{1}2^{-7}3^{-6}4^{6}5^{5}$	$1^{2}2^{-6}3^{-8}4^{4}$	$1^{2}2^{-6}3^{-7}4^{4}$
	$1^{8}2^{-6}3^{-7}4^{-1}5^{1}$	$1^{5}2^{-6}3^{-8}5^{-1}$	$1^{5}2^{-6}3^{-7}4^{1}$	$1^{6}2^{-7}3^{-10}4^{-1}$	$1^{6}2^{-7}3^{-9}5^{1}$
	$1^{6}2^{-6}3^{-8}4^{-1}5^{2}$	$1^{6}2^{-4}3^{-3}4^{1}5^{-1}$	$1^{7}2^{-7}3^{-10}4^{-2}5^{3}$	$1^{7}2^{-6}3^{-6}4^{1}5^{-1}$	$1^{7}2^{-5}3^{-5}$
6/ +−+++−	$1^{1}2^{-5}3^{-6}4^{-3}5^{5}$	$1^{1}2^{-5}3^{-6}4^{-2}5^{5}$	$1^{1}2^{-5}3^{-6}4^{-1}5^{5}$	$1^{1}2^{-4}3^{-6}4^{-3}5^{5}$	$1^{1}2^{-4}3^{-6}4^{-2}5^{5}$
	$1^{1}2^{-4}3^{-6}4^{-1}5^{5}$	$1^{1}2^{-3}3^{-6}4^{-2}5^{5}$			
6/ +−++++	$1^{1}2^{-5}3^{-6}4^{2}5^{5}$	$1^{1}2^{-4}3^{-5}4^{4}$			
6/ +−+−+0	$1^{1}2^{-4}3^{-3}4^{4}$	$1^{2}2^{-5}3^{-6}4^{2}$	$1^{3}2^{-3}3^{-1}4^{3}$	$1^{8}2^{-7}3^{-8}$	$1^{9}2^{-5}3^{-3}4^{1}$
6/ +−+0+0	$1^{1}2^{-3}3^{-3}4^{2}$				
6/ +−+0−+	$1^{1}2^{-3}3^{-3}4^{5}$	$1^{2}2^{-4}3^{-6}4^{3}$	$1^{4}2^{-3}3^{-4}4^{2}$	$1^{8}2^{-1}3^{-8}$	
6/ +−+−−+	$1^{2}2^{-3}3^{-3}4^{4}$	$1^{2}2^{-3}3^{-2}4^{4}$	$1^{3}2^{-3}3^{-3}4^{3}$	$1^{4}2^{-3}3^{-3}4^{2}$	
6/ +−++0−	$1^{1}2^{-2}3^{-3}$				
6/ +−++−−	$1^{2}2^{-3}3^{-5}5^{2}$	$1^{2}2^{-2}3^{-4}5^{1}$	$1^{2}2^{-2}3^{-3}5^{1}$	$1^{3}2^{-2}3^{-3}5^{2}$	$1^{3}2^{-2}3^{-2}5^{2}$
	$1^{4}2^{1}3^{-6}$	$1^{4}2^{1}3^{-5}$	$1^{4}2^{1}3^{-4}$	$1^{5}2^{1}3^{-3}4^{-1}$	$1^{5}2^{2}3^{-6}$
	$1^{5}2^{2}3^{-5}$	$1^{5}2^{2}3^{-4}$	$1^{5}2^{2}3^{-3}$		
6/ +−+00−	$1^{1}2^{-2}3^{-2}4^{1}5^{1}$				
6/ +−0+0−	$1^{1}2^{-1}3^{-3}4^{-2}$	$1^{1}2^{-1}3^{-2}4^{-1}5^{1}$			
6/ +−−+−0	$1^{1}3^{-3}4^{2}$				
6/ +−−++−	$1^{1}2^{4}3^{-5}$	$1^{1}2^{4}3^{-4}$			
6/ +−+0−0	$1^{2}2^{-4}3^{-6}$	$1^{4}2^{-3}3^{-4}4^{-1}$			
6/ +−+0−−	$1^{2}2^{-3}3^{-4}5^{2}$				
6/ +−++−0	$1^{2}2^{-2}3^{-6}4^{-1}$	$1^{4}3^{-4}4^{-1}$	$1^{5}2^{2}3^{-7}$		
6/ +−−+−+	$1^{2}2^{1}3^{-2}4^{2}$				
6/ +−+−0+	$1^{3}2^{-6}3^{-9}$	$1^{5}2^{-5}3^{-6}5^{1}$			
6/ +−+−−−	$1^{3}2^{-5}3^{-7}5^{2}$	$1^{3}2^{-4}3^{-6}4^{-1}5^{3}$	$1^{4}2^{-3}3^{-3}5^{1}$		
6/ +−+−00	$1^{4}2^{-4}3^{-4}4^{1}$				

6/ +−0+−−	$1^4 2^2 3^{-4} 4^1$				
6/ +−+−−0	$1^6 2^{-3} 3^{-2}$				
8/ +−−+++−−	$1^1 2^4 3^1 4^{-3}$				
8/ +−++0−0+	$1^4 4^{-5}$				
8/ +−++−−−+	$1^4 4^{-4}$ $1^4 2^1 4^{-4}$	$1^4 4^{-3}$ $1^4 2^1 4^{-3}$	$1^4 4^{-2}$	$1^4 2^1 4^{-6}$	$1^4 2^1 4^{-5}$
8/ +−0+0−0+	$1^4 2^2 4^{-10}$				
8/ +−0+−−0+	$1^4 2^2 4^{-9}$ $1^4 2^2 4^{-4}$	$1^4 2^2 4^{-8}$ $1^4 2^2 4^{-3}$	$1^4 2^2 4^{-7}$ $1^4 2^2 4^{-2}$	$1^4 2^2 4^{-6}$	$1^4 2^2 4^{-5}$
8/ +−−+−−++	$1^4 2^3 4^{-4}$	$1^4 2^3 4^{-3}$	$1^4 2^4 4^{-4}$	$1^4 2^4 4^{-3}$	
9/ +−+−−+−−+	$1^9 3^{-10}$				
9/ +−+−−+0−+	$1^9 3^{-9}$				
9/ +−+−−++−+	$1^9 3^{-8}$	$1^9 3^{-7}$	$1^9 3^{-6}$	$1^9 3^{-4}$	
10/ ++−0−0+−−−	$1^{-1} 2^3 4^1 5^{-3}$				
10/ +0+−+−+−+0	$2^{-2} 3^3 5^{-5}$				
10/ +−+−++0−0+	$1^1 2^{-2} 5^{-5}$				
10/ +−0−++−−0+	$1^1 2^{-1} 4^{-1} 5^{-5}$				
10/ +−+−+−+−++	$1^2 2^{-4} 5^{-10}$	$1^3 2^{-4} 4^1 5^{-7}$			
10/ +−+−+0−0−+	$1^2 2^{-3} 5^{-10}$				
10/ +−+−+−−−−+	$1^2 2^{-3} 5^{-9}$	$1^2 2^{-3} 5^{-8}$			
10/ +−+−++−−++	$1^2 2^{-2} 3^{-1} 5^{-5}$				
10/ +−+−++−−−+	$1^2 2^{-2} 4^1 5^{-3}$				
10/ +−+−+00−00	$1^2 2^{-2} 4^1 5^{-2}$				
10/ +−−+0+−−0−	$1^2 4^1 5^{-2}$				
10/ +−−+++−−+−	$1^2 2^1 5^{-3}$	$1^2 2^1 5^{-2}$			
10/ +−−++0−−+0	$1^2 2^2 4^{-1} 5^{-2}$				
10/ +−+−+−−+++	$1^3 2^{-4} 3^{-2} 5^{-10}$ $1^4 2^{-3} 3^{-1} 4^1 5^{-5}$	$1^3 2^{-3} 3^{-1} 5^{-7}$	$1^3 2^{-2} 5^{-4}$	$1^3 2^{-2} 5^{-3}$	$1^4 2^{-3} 3^{-1} 4^1 5^{-4}$
10/ +−−+−0−−+0	$1^3 2^2 5^{-7}$				
10/ +−−+−−−−++	$1^3 2^2 5^{-6}$	$1^3 2^2 5^{-5}$	$1^3 2^2 5^{-4}$		
10/ +−+0+0−+−+	$1^4 2^{-2} 4^1 5^{-4}$				
10/ +−++−0−++0	$1^4 2^{-1} 5^{-4}$				
10/ +−+−+−+0+0	$1^5 2^{-3} 4^1 5^{-1}$				
10/ +−+−−+++−−	$1^8 2^{-1} 5^{-3}$	$1^8 2^{-1} 5^{-2}$			
10/ +−+−+−+−−−	$1^9 2^{-6} 3^{-3} 5^{-8}$	$1^{10} 2^{-6} 3^{-3} 4^1 5^{-5}$			
12/ +++−++−+++++	$1^{-3} 2^8 3^{-1} 4^{-4}$				
12/ ++0−0+−−0+0−	$1^{-3} 2^9 3^1 4^{-6}$				

12/ ++−−++−++−−+	$1^{-3}2^{10}3^{-3}4^{-8}$	$1^{-2}2^{7}3^{-2}4^{-6}$	$1^{-2}2^{8}3^{-2}4^{-7}$	$1^{-2}2^{8}3^{-2}4^{-6}$	$1^{-2}2^{9}3^{-2}4^{-7}$
	$1^{-2}2^{9}3^{-1}4^{-5}5^{1}$	$1^{-2}2^{10}3^{-2}4^{-7}$	$1^{-2}2^{10}3^{-1}4^{-5}5^{1}$		
12/ ++−−++−−+−−+	$1^{-3}2^{10}3^{-2}4^{-7}5^{1}$				
12/ ++−0+0−++0+0	$1^{-2}2^{6}3^{-2}4^{-4}$				
12/ ++−−++−++−−−	$1^{-2}2^{7}3^{-1}4^{-5}5^{1}$				
12/ ++−−+−−++−−−	$1^{-2}2^{7}3^{-1}4^{-4}5^{1}$	$1^{-1}2^{6}3^{-1}4^{-5}$	$1^{-1}2^{6}3^{-1}4^{-4}$	$1^{-1}2^{7}3^{-1}4^{-4}$	
12/ ++−−++−0+−−0	$1^{-2}2^{8}3^{-2}4^{-8}$				
12/ ++−++−−+−−+−	$1^{-1}2^{4}3^{-2}4^{-3}5^{1}$				
12/ ++−++−−++−++	$1^{-1}2^{5}3^{-3}4^{-6}$				
12/ ++−++−0+−0+−	$1^{-1}2^{5}3^{-3}4^{-3}$				
12/ ++−−+−−++−−+	$1^{-1}2^{8}3^{-3}4^{-8}5^{-2}$				
12/ +−−++−++−0+−	$1^{1}3^{-3}4^{-4}$				
12/ +−−+−−++−++−	$1^{1}3^{-2}4^{-1}5^{3}$	$1^{2}3^{-4}4^{-5}$	$1^{2}3^{-4}4^{-4}5^{1}$	$1^{2}3^{-3}4^{-3}5^{2}$	$1^{2}3^{-2}4^{-2}5^{3}$
	$1^{2}2^{1}3^{-4}4^{-4}$	$1^{2}2^{1}3^{-3}4^{-4}$	$1^{2}2^{2}3^{-3}4^{-3}$	$1^{3}2^{1}3^{-4}4^{-5}$	$1^{3}2^{2}3^{-4}4^{-4}$
	$1^{3}2^{2}3^{-3}4^{-4}$	$1^{3}2^{3}3^{-3}4^{-3}$	$1^{3}2^{3}3^{-2}4^{-3}$	$1^{4}2^{3}3^{-3}4^{-4}$	$1^{4}2^{4}3^{-3}4^{-3}$
	$1^{4}2^{4}3^{-2}4^{-3}$				
12/ +−−++−++−−++	$1^{1}2^{1}3^{-3}4^{-6}$	$1^{1}2^{1}3^{-3}4^{-5}$	$1^{1}2^{2}3^{-3}4^{-5}$	$1^{2}2^{2}3^{-2}4^{-4}$	$1^{2}2^{3}3^{-2}4^{-5}$
	$1^{2}2^{3}3^{-2}4^{-4}$	$1^{2}2^{4}3^{-2}4^{-4}$	$1^{2}2^{5}3^{-2}4^{-4}$		
12/ +−−++−++−+−−	$1^{1}2^{1}3^{-3}4^{-4}5^{-1}$	$1^{1}2^{2}3^{-4}4^{-4}5^{-2}$	$1^{1}2^{2}3^{-3}4^{-4}5^{-2}$	$1^{1}2^{2}3^{-3}4^{-3}5^{-1}$	$1^{2}2^{4}3^{-5}4^{-7}5^{-6}$
	$1^{1}2^{2}3^{-2}4^{-3}5^{-1}$	$1^{1}2^{2}3^{-2}4^{-2}$	$1^{2}2^{3}3^{-4}4^{-5}5^{-3}$	$1^{2}2^{3}3^{-3}4^{-4}5^{-2}$	$1^{2}2^{3}3^{-2}4^{-3}5^{-1}$
12/ +−−+0−++−+0−	$1^{1}2^{1}3^{-3}4^{-3}$	$1^{1}2^{1}3^{-2}4^{-2}5^{1}$	$1^{1}2^{2}3^{-3}4^{-2}$		
12/ +−−++−−+−−++	$1^{1}2^{3}3^{-3}4^{-7}$	$1^{1}2^{4}3^{-3}4^{-6}$	$1^{1}2^{4}3^{-2}4^{-5}5^{1}$	$1^{2}2^{5}3^{-2}4^{-6}$	
12/ +−−+−−+−−++−	$1^{2}3^{-4}4^{-3}5^{2}$	$1^{2}3^{-3}4^{-2}5^{3}$	$1^{3}2^{1}3^{-5}4^{-5}$	$1^{3}2^{1}3^{-4}4^{-4}5^{1}$	$1^{3}2^{2}3^{-3}4^{-2}5^{2}$
	$1^{4}2^{3}3^{-4}4^{-4}$				
12/ +−−+−−++−−−+	$1^{2}3^{-3}4^{-4}5^{-1}$	$1^{3}2^{1}3^{-2}4^{-3}5^{-1}$	$1^{3}2^{2}3^{-2}4^{-3}5^{-1}$		
12/ +−−+−−++−−++	$1^{2}2^{1}3^{-4}4^{-6}5^{-2}$	$1^{3}2^{2}3^{-6}4^{-9}5^{-5}$	$1^{3}2^{3}3^{-3}4^{-5}5^{-2}$		
12/ +−−++−++−−−+	$1^{2}2^{1}3^{-2}4^{-4}$				
12/ +−++−−+−−+−+	$1^{3}2^{-1}3^{-3}4^{-1}$	$1^{3}2^{-1}3^{-2}4^{-1}$	$1^{4}2^{-1}3^{-3}4^{-2}$	$1^{4}2^{-1}3^{-2}4^{-1}5^{1}$	$1^{4}3^{-3}4^{-2}5^{-1}$
	$1^{4}3^{-4}4^{-4}5^{-3}$	$1^{4}3^{-4}4^{-3}5^{-2}$	$1^{4}3^{-3}4^{-3}5^{-2}$		
12/ +−0+0−0+0−0+	$1^{3}3^{-1}4^{-3}$				
12/ +−−+−−+−−+0−	$1^{3}2^{2}3^{-5}4^{-4}$	$1^{4}2^{4}3^{-4}4^{-3}$			
12/ +−++−0+−−+−0	$1^{4}2^{-1}3^{-4}4^{-2}$				
12/ +−++−−+−−+−−	$1^{5}3^{-3}4^{-2}$				
12/ +−++−++−−+−−	$1^{5}3^{-4}4^{-2}$	$1^{6}3^{-4}4^{-3}$	$1^{6}3^{-3}4^{-2}5^{1}$	$1^{6}2^{1}3^{-4}4^{-2}$	$1^{6}2^{1}3^{-3}4^{-2}$
	$1^{7}3^{-4}4^{-4}$	$1^{7}2^{1}3^{-4}4^{-3}$	$1^{7}2^{1}3^{-3}4^{-3}$	$1^{8}2^{2}3^{-3}4^{-3}$	
12/ +−++−++−++−−	$1^{6}2^{-2}3^{-6}4^{-5}$	$1^{6}2^{-2}3^{-5}4^{-4}5^{1}$	$1^{6}2^{-2}3^{-4}4^{-3}5^{2}$	$1^{6}2^{-1}3^{-5}4^{-3}5^{1}$	$1^{7}2^{1}3^{-5}4^{-3}$
	$1^{6}2^{-1}3^{-4}4^{-2}5^{2}$	$1^{7}3^{-5}4^{-4}$			
12/ +−++−++−−+−+	$1^{6}2^{1}3^{-3}4^{-3}5^{-1}$				
12/ +−+−−++−+++−	$1^{7}2^{-3}3^{-3}4^{-4}$	$1^{7}2^{-3}3^{-3}4^{-3}$	$1^{7}2^{-3}3^{-3}4^{-2}$		
12/ +−+−−++−++−+	$1^{8}2^{-3}3^{-4}4^{-2}$				

12/ $+-+--+--++-+$	$1^8 2^{-3} 3^{-5} 4^{-2}$ $1^9 2^{-3} 3^{-5} 4^{-3}$ $1^9 2^{-3} 3^{-4} 4^{-2} 5^1$ $1^9 2^{-2} 3^{-5} 4^{-2}$ $1^9 2^{-2} 3^{-4} 4^{-2}$ $1^{10} 2^{-2} 3^{-5} 4^{-3}$ $1^{10} 2^{-2} 3^{-4} 4^{-2} 5^1$ $1^{10} 3^{-5} 4^{-3} 5^{-2}$
12/ $+-+-+++-+0+-$	$1^9 2^{-4} 3^{-3} 4^{-4}$
12/ $+-+-+++-+-+-$	$1^9 2^{-4} 3^{-3} 4^{-3}$ $1^9 2^{-4} 3^{-3} 4^{-2}$
12/ $+-+--+--+--+$	$1^{10} 2^{-4} 3^{-6} 4^{-5}$ $1^{10} 2^{-3} 3^{-6} 4^{-4}$ $1^{10} 2^{-2} 3^{-6} 4^{-3}$ $1^{10} 2^{-2} 3^{-5} 4^{-2} 5^1$
15/ $+0-+0-+--+-0+-0$	$2^3 3^{-2} 5^{-3}$
15/ $+--+-0+-0+-0+-0$	$1^1 3^{-3} 5^{-2}$

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