VANISHING PROPERTIES OF FOURIER COEFFICIENTS OF HOLOMORPHIC η-QUOTIENTS

KATHRIN BRINGMANN, GUONIU HAN, BERNHARD HEIM, AND BEN KANE

Abstract. In this paper, we study vanishing of Fourier coefficients of holomorphic η-quotients. We investigate examples of two different types: the first one involves integral weight CM newforms, while the second one involves half-integral weight η-quotients associated with sums of squares and Hurwitz class numbers.

1. Introduction and statement of results

Let $m \in \mathbb{N}$ and $\delta_j \in \mathbb{Z}$ for $1 \leq j \leq m$. We define

$$\prod_{j=1}^{m} \left(q^j; q^j\right)_\infty^{\delta_j} =: \sum_{n\geq 0} C_{1^{\delta_1}2^{\delta_2}\cdots m^{\delta_m}}(n) q^n,$$

where $(a;q)_n := \prod_{m=0}^{n-1}(1-aq^m)$ for $n \in \mathbb{N}_0 \cup \{\infty\}$ is the *q-Pochhammer symbol*. In this paper, we investigate when $C_{1^{\delta_1}2^{\delta_2}\cdots m^{\delta_m}}(n)$ vanishes. Specifically, define the *vanishing set*

$$\mathcal{S}_{1^{\delta_1}2^{\delta_2}\cdots m^{\delta_m}} := \left\{n \in \mathbb{N} : C_{1^{\delta_1}2^{\delta_2}\cdots m^{\delta_m}}(n) = 0\right\}.$$

A famous conjecture of Lehmer [12] states that $\mathcal{S}_{1^{24}} = \emptyset$, while for the partition generating function one sees that $\mathcal{S}_{1^{-1}} = \emptyset$. Since (see [13, Theorem 1.60])

$$\sum_{n\in\mathbb{Z}} q^{n^2} = \frac{\left(q^2;q^2\right)_\infty^5}{(q;q)_\infty^2 \left(q^4;q^4\right)_\infty^2},$$

questions related to whether integers are represented as sums of squares may be interpreted as determinations of $\mathcal{S}_{1^{\delta_1}2^{\delta_2}\cdots m^{\delta_m}}$. For example, in this language, Lagrange's four-squares theorem, i.e., that every $n \in \mathbb{N}$ may be written in the form $\sum_{j=1}^4 n_j^2 = n$ with $n_j \in \mathbb{Z}$ is equivalent to $S_{1^{-8}2^{20}4^{-8}} = \emptyset$. In another direction, Granville and Ono [7, Theorem 1] proved that for $t \geq 4$ and $n \in \mathbb{N}$, there always exist a so-called t-core partition of n, which is equivalent to showing that $S_{1^{-1}t^t}(n) = \emptyset$. To give another interesting example related to 3-core partitions, in [9, Theorem 1.1], Ono and the second author proved Conjecture 4.6 of [8], which states that[1]

$$\mathcal{S}_{1^8} = \mathcal{S}_{1^{-1}3^3} = \left\{n \in \mathbb{N} : \exists p \equiv 2 \pmod 3,\ \mathrm{ord}_p(3n+1) \text{ is odd}\right\}.$$

The second author later conjectured that $\mathcal{S}_{1^8} = \mathcal{S}_{1^2 3^2}$ as well, and this was proven by Clemm [2, Theorem 1 and Remark 2]. This set of three examples is not isolated, as we demonstrate now.

Theorem 1.1. *We have*

$$\mathcal{S}_{1^{-1}3^3 4^2} = \mathcal{S}_{1^4 2^{-2} 4^4} = \left\{n \in \mathbb{N} : \exists p \equiv 3 \pmod 4,\ \mathrm{ord}_p(3n+2) \text{ is odd}\right\}.$$

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[1] Here and throughout p denotes a prime.

Note that the vanishing set appearing in Theorem 1.1 precisely consists of those n for which $3n+2$ is not the norm of an element of $\mathbb{Z}[i]$. In the same way, the vanishing set in the next theorem is related to norms of elements in $\mathbb{Z}[\sqrt{-2}]$.

Theorem 1.2. *We have*

$$\begin{aligned}\mathcal{S}_{1^12^{-2}4^3} &= \mathcal{S}_{1^12^24^1} = \mathcal{S}_{1^32^{-1}4^2} = \mathcal{S}_{1^32^3} = \mathcal{S}_{1^72^{-3}4^2} \\ &= \{n \in \mathbb{N} : \exists p \equiv 5, 7 \pmod 8,\ \mathrm{ord}_p(8n+3) \text{ is odd}\}.\end{aligned}$$

Like in Theorem 1.1, the vanishing set in the next example is related to norms in $\mathbb{Z}[i]$, but has an additional congruence condition on n.

Theorem 1.3. *We have*

$$\mathcal{S}_{1^{-1}2^{10}3^{-1}4^{-4}} = \mathcal{S}_{1^72^{-2}3^{-1}} = \{n \in \mathbb{N} : n \equiv 2 \pmod 3 \ \textit{and}\ \exists p \equiv 3 \pmod 4, \mathrm{ord}_p(n) \text{ is odd}\}.$$

The behaviour of $\sum_{n\geq 0} C_{1^{\delta_1}\cdots m^{\delta_m}}(n)q^n$ is different depending on the parity of $\sum_{j=1}^m \delta_j$. This is demonstrated by the differing shape of the vanishing sets in the following theorem.

Theorem 1.4. *We have*

$$\mathcal{S}_{1^22^34^{-2}} = \mathcal{S}_{1^62^{-3}} = \left\{n \in \mathbb{N} : n = 4^k(8m+7),\ k, m \in \mathbb{N}_0\right\}.$$

The paper is organized as follows: In Section 2, we recall basic facts about modular forms. In Section 3, we study the space $S_2(\Gamma_0(36), \chi_{12})$, where $\chi_D(n) := \left(\frac{D}{n}\right)$ with $\left(\frac{\cdot}{\cdot}\right)$ the extended Legendre symbol. In Section 4, we prove Theorem 1.1, in Section 5 Theorem 1.2, in Section 6 Theorem 1.3, and in Section 7 Theorem 1.4.

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2. Preliminaries

2.1. **Modular forms.** Here we introduce modular forms, see e.g. [13] for more details. As usual, for d odd, we set

$$\varepsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4, \\ i & \text{if } d \equiv 3 \pmod 4. \end{cases}$$

For $\kappa \in \frac{1}{2}\mathbb{Z}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ ($4 \mid c$ if $\kappa \notin \mathbb{Z}$), define the weight κ *slash operator* by

$$f\big|_\kappa \gamma(z) := \begin{cases} \left(\frac{c}{d}\right) \varepsilon_d^{2\kappa} (cz+d)^{-\kappa} f(\gamma z) & \text{if } \kappa \in \mathbb{Z} + \frac{1}{2}, \\ (cz+d)^{-\kappa} f(\gamma z) & \text{if } \kappa \in \mathbb{Z}. \end{cases}$$

For $\kappa \in \frac{1}{2}\mathbb{Z}$, $N \in \mathbb{N}$ ($4 \mid N$ if $\kappa \notin \mathbb{Z}$), and a character $\chi \pmod N$, a function $f : \mathbb{H} \to \mathbb{C}$ is a *holomorphic modular form of weight* κ *on* $\Gamma_0(N)$ *with character* χ if the following conditions hold:

(1) The function f is holomorphic on $\mathbb{H}$.
(2) We have $f|_\kappa \gamma = \chi(d) f$ for $\gamma \in \Gamma_0(N)$.
(3) For $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, $(cz+d)^{-\kappa} f(\gamma z)$ is bounded as $z \to i\infty$.

We denote the corresponding space of such forms by $M_\kappa\left(\Gamma_0(N),\chi\right)$. We call the equivalence classes of $\Gamma_0(N)\backslash(\mathbb{Q}\cup\{i\infty\})$ the *cusps of* $\Gamma_0(N)$. If the function in (3) vanishes as $z\to i\infty$ for every cusp $\gamma(i\infty)$, then we call f a *cusp form*. We denote the corresponding space by $S_\kappa(\Gamma_0(N),\chi)$. We sometimes omit χ in the notation if it is trivial.

2.2. **Operators on modular forms.** For $f(z)=\sum_{n\in\mathbb{Z}}c(n)q^n$ (with $q:=e^{2\pi iz}$) and $\ell\in\mathbb{N}$, we define the *U-operator* and the *V-operator* as

$$f\mid U_\ell(z):=\sum_{n\in\mathbb{Z}}c(\ell n)q^n,\quad f\mid V_\ell(z):=f(\ell z).$$

Moreover, we define for $M\in\mathbb{N}$ and $m\in\mathbb{Z}$ the *sieving operator*

$$f\mid S_{M,m}(z):=\sum_{\substack{n\in\mathbb{Z}\\ n\equiv m\pmod M}}c(n)q^n.$$

The actions of these operators on half-integral weight modular forms may be found in [1, Lemma 2.3]. To state the result, let $\operatorname{rad}(n):=\prod_{p\mid n}p$ be the *radical* of $n\in\mathbb{N}$ and recall that the *conductor* of a character $\chi\pmod N$ is the minimal $N_\chi\mid N$ for which there exists a character $\psi\pmod N$ with $\chi(n)=\psi(n)$ for every $n\in\mathbb{Z}$ with $\gcd(n,N)=1$.

Lemma 2.1. *Suppose that $f\in M_\kappa(\Gamma_0(N),\chi)$ with $\kappa\in\mathbb{Z}+\frac12$ and $4\mid N$, and χ is a character of conductor $N_\chi\mid N$.*

(1) *We have $f|U_\delta\in M_\kappa(\Gamma_0(4\operatorname{lcm}(\frac N4,\operatorname{rad}(\delta))),\chi\chi_{4\delta})$.*
(2) *Suppose that $M\mid 24$ and $M\not\equiv 2\pmod 4$. Then $f|S_{M,m}\in M_\kappa(\Gamma_0(\operatorname{lcm}(N,M^2,MN_\chi)),\chi)$.*
(3) *We have $f|V_\delta\in M_\kappa(\Gamma_0(N\delta),\chi\chi_{4\delta})$.*

We also require the following lemma for integral-weight modular forms.

Lemma 2.2. *Let $N\in\mathbb{N}$, χ a character (mod N) with conductor $N_\chi\mid N$, $\kappa\in\mathbb{N}$, and $f\in M_\kappa(\Gamma_0(N),\chi)$. Then the following hold:*

(1) *For $\delta\in\mathbb{N}$ we have $f|V_\delta\in M_\kappa(\Gamma_0(N\delta),\chi)$.*
(2) *For $M\in\mathbb{N}$ with $M\mid 24$, $f|S_{M,m}\in M_\kappa(\Gamma_0(\operatorname{lcm}(N,M^2,MN_\chi)),\chi)$.*

We require the following lemma about products of half-integral weight modular forms.

Lemma 2.3. *Suppose $f_1\in M_{\kappa_1+\frac12}(\Gamma_0(N),\psi_1)$, $f_2\in M_{\kappa_2+\frac12}(\Gamma_0(N),\psi_2)$, $f_3\in M_{\kappa_3}(\Gamma_0(N),\psi_3)$ for some $\kappa_1,\kappa_2,\kappa_3\in\mathbb{N}_0$, $N\in\mathbb{N}$, and characters ψ_1, ψ_2, ψ_3 modulo a divisor of N. Then $f_1f_2\in M_{\kappa_1+\kappa_2+1}\left(\Gamma_0(N),\psi_1\psi_2\chi_{-4}^{\kappa_1+\kappa_2+1}\right)$ and $f_1f_3\in M_{\kappa_1+\kappa_3+\frac12}(\Gamma_0(N),\psi_1\psi_3\chi_{-4}^{\kappa_3})$.*

For $f_1\in M_{\kappa_1}(\Gamma,\chi_1)$ and $f_1\in M_{\kappa_2}(\Gamma,\chi_2)$, for some group $\Gamma\subseteq\mathrm{SL}_2(\mathbb{Z})$, and for $\ell\in\mathbb{N}_0$, the ℓ-th *Rankin–Cohen bracket* is defined by

$$[f_1,f_2]_\ell:=\frac{1}{(2\pi i)^\ell}\sum_{r=0}^{\ell}\frac{(-1)^r\Gamma(\kappa_1+\ell)\Gamma(\kappa_2+\ell)}{r!(\ell-r)!\Gamma(\kappa_1+r)\Gamma(\kappa_2+\ell-r)}f_1^{(r)}f_2^{(\ell-r)}.$$

By [3, Corollary 7.2] we have the following.

Lemma 2.4. *Suppose that, for $j\in\{1,2\}$, $f_j\in M_{\kappa_j+\frac12}(\Gamma_0(N),\psi_j)$ for some $\kappa_j\in\mathbb{N}_0$, $N\in\mathbb{N}$, and characters $\psi_j\pmod N$. Then, for $\ell\in\mathbb{N}_0$, $[f_1,f_2]_\ell\in M_{\kappa_1+\kappa_2+2\ell+1}(\Gamma_0(N),\psi_1\psi_2\chi_{-4}^{\kappa_1+\kappa_2+1})$.*

2.3. **Hecke eigenforms.** For $N, \kappa \in \mathbb{N}$, χ a character (mod N), and p a prime we define the *Hecke operator* T_p acting on $f(z) = \sum_{n\geq 0} c(n)q^n \in M_\kappa(\Gamma_0(N), \chi)$ by (see [13, Definition 2.1])

$$f\big|T_p(z) := \sum_{n\geq 0} \left(c(pn) + \chi(p)p^{\kappa-1} c\left(\frac{n}{p}\right)\right) q^n,$$

where $c(\alpha) := 0$ for $\alpha \in \mathbb{Q} \setminus \mathbb{N}_0$. We call a simultaneous eigenfunction under the Hecke operators T_p for $p \nmid N$ a *Hecke eigenform*. For $\kappa \in \mathbb{N}$, the space $S_\kappa(\Gamma_0(N), \chi)$ splits into the old space, spanned by the images of V_δ on $S_\kappa(\Gamma_0(M), \chi)$ for $\delta \mid \frac{N}{M}$ and $M \mid N$ with $M < N$, and the new space is the orthogonal complement of the old space in $S_\kappa(\Gamma_0(N), \chi)$ with respect to the *Petersson inner product* ($z = x + iy$, $f, g \in S_\kappa(\Gamma_0(N), \chi)$)

$$\langle f, g\rangle := \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]} \int_{\Gamma_0(N)\backslash\mathbb{H}} f(z)\overline{g(z)} y^\kappa \frac{dxdy}{y^2}.$$

Hecke eigenforms in the new space are called *newforms*. Those newforms whose Fourier expansions $\sum_{n\geq 1} c(n)q^n$ have $c(1) = 1$ are called *normalized newforms* (also known as *primitive forms*). We require the following.

Lemma 2.5. *Let $\kappa, N \in \mathbb{N}$ and χ a character (mod N). Suppose that $f(z) = \sum_{n\geq 1} c(n)q^n$ is a normalized newform in $S_\kappa(\Gamma_0(N), \chi)$. Then the Fourier coefficients $c(n)$ are multiplicative and for every $p \nmid N$ and $r \in \mathbb{N}$ we have*

$$c\left(p^r\right) = c(p)c\left(p^{r-1}\right) - \chi(p)p^{\kappa-1}c\left(p^{r-2}\right).$$

With $d(n)$ the number of divisors of n, Deligne [4] proved the following bound.

Theorem 2.6. *If $f(z) = \sum_{n\geq 1} c(n)q^n$ is a normalized newform in $S_\kappa(\Gamma_0(N), \chi)$, then*

$$|c(n)| \leq d(n)n^{\frac{\kappa-1}{2}}.$$

2.4. **Eisenstein series.** Let $\kappa \in \mathbb{N}$ and χ, ψ primitive characters. We require the modular properties of the Eisenstein series

$$E_{\kappa,\chi,\psi}(z) := \mathbb{1}_{\chi=\chi_1} L(1-\kappa, \psi) + \mathbb{1}_{\psi=\chi_1}\mathbb{1}_{\kappa=1} L(0, \chi) + 2\sum_{n\geq 1}\sum_{d|n} \chi\left(\frac{n}{d}\right)\psi(d)d^{\kappa-1}q^n,$$

where $L(s, \chi) := \sum_{n\geq 1} \chi(n)n^{-s}$ is defined for $\mathrm{Re}(s) > 1$ and is meromorphically continued to the whole complex plane. Moreover $\mathbb{1}_S := 1$ if a statement S is true, and $\mathbb{1}_S := 0$ if S is false. The following modular properties may be found in [5, Theorem 4.5.1, Theorem 4.6.2, Theorem 4.8.1].

Lemma 2.7. *Suppose that $\kappa, d \in \mathbb{N}$, χ and ψ are primitive characters of conductors N_χ and N_ψ, respectively, with $\chi(-1)\psi(-1) = (-1)^\kappa$.*

1. *If $\kappa \neq 2$, then $E_{\kappa,\chi,\psi}\big|V_d \in M_\kappa\left(\Gamma_0\left(N_\chi N_\psi d\right), \chi\psi\right)$.*
2. *If $(\chi, \psi) \neq (\chi_1, \chi_1)$, then $E_{2,\chi,\psi}\big|V_d \in M_2\left(\Gamma_0\left(N_\chi N_\psi d\right), \chi\psi\right)$. If $(\chi, \psi) = (\chi_1, \chi_1)$, then $E_{2,\chi_1,\chi_1} - dE_{2,\chi_1,\chi_1}\big|V_d \in M_2\left(\Gamma_0(d)\right)$.*

The subspace of modular forms formed by linear combinations of $E_{\kappa,\chi,\psi}\big|V_d$ is the *Eisenstein series subspace*. We split a modular form $f = E + g$ where E is contained in the Eisenstein series subspace and g is a cusp form. We call E the *Eisenstein series part* of f and g the *cuspidal part* of f.

2.5. **Valence formula.** In order to show identities between modular forms, we use the following lemma, which is a consequence of the valence formula.

Lemma 2.8. *Let $\kappa \in \frac{1}{2}\mathbb{N}$, $N \in \mathbb{N}$, and χ be a character (mod N). Let $f(z) = \sum_{n\geq 0} c(n)q^n \in M_\kappa(\Gamma_0(N), \chi)$. If $c(n) = 0$ for every $0 \leq n \leq N\frac{\kappa}{12}\prod_{p|N}(1 + \frac{1}{p})$, then $f \equiv 0$.*

2.6. **Modularity of eta-quotients.** Define the *Dedekind eta-function*

$$\eta(z) := q^{\frac{1}{24}} \prod_{n\geq 1} (1-q^n).$$

Note

$$\prod_{j=1}^{m} \eta(jz)^{\delta_j} = q^{\frac{1}{24}\sum_{j=1}^{m} j\delta_j} \prod_{j=1}^{m} \left(q^j;q^j\right)_{\infty}^{\delta_j}.$$

Thus, to investigate $\mathcal{S}_{1^{\delta_1}2^{\delta_2}\cdots m^{\delta_m}}$, we require the modularity of certain η-quotients, which may be found in [13, Theorems 1.64 and 1.65].

Lemma 2.9. *If $f(z) = \prod_{\delta|N} \eta(\delta z)^{r_\delta}$ is an η-quotient of weight $\kappa = \frac{1}{2}\sum_{\delta|N} r_\delta \in \mathbb{Z}$, with the additional properties that $\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24}$, $\sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24}$, and for every $d \mid N$ we have $\sum_{\delta|N} \frac{\gcd(d,\delta)^2 r_\delta}{\delta} \geq 0$, then $f \in M_\kappa(\Gamma_0(N), \chi_{(-1)^\kappa s})$, where $s := \prod_{\delta|N} \delta^{r_\delta}$.*

2.7. **Unary theta functions.** For a character χ and $j \in \{0,1\}$, as in [13, Definition 1.42], we define the *unary theta function*

$$\theta(\chi, j, z) := \sum_{n\in\mathbb{Z}} \chi(n) n^j q^{n^2}.$$

We also let $\Theta(z) := \theta(\chi_1, 0, z) = \sum_{n\in\mathbb{Z}} q^{n^2}$ be the standard *theta function* of Jacobi. Recall that χ is called *even* (resp. *odd*) if $\chi(-1) = 1$ (resp. $\chi(-1) = -1$). The modular properties of $\theta(\chi, j, z)$ can be found in [13, Theorem 1.44].

Lemma 2.10. *Suppose that χ is a primitive Dirichlet character with conductor N_χ.*

(1) *If χ is even, then $\theta(\chi, 0, z) \in M_{\frac{1}{2}}(\Gamma_0(4N_\chi^2), \chi)$.*

(2) *If χ is odd, then $\theta(\chi, 1, z) \in S_{\frac{3}{2}}(\Gamma_0(4N_\chi^2), \chi\chi_{-4})$.*

2.8. **Binary quadratic forms.** For[2] $a, b \in \mathbb{N}$, let

$$r_{(a,b)}(n) := \#\left\{\boldsymbol{n} \in \mathbb{Z}^2 : an_1^2 + bn_2^2 = n\right\}.$$

Jacobi [16, Proposition 10] obtained the following formula for $r_{(1,1)}(n)$.

Lemma 2.11. *We have, for $n \in \mathbb{N}$,*

$$r_{(1,1)}(n) = 4\sum_{d|n} \left(\frac{-4}{d}\right).$$

A similar expression for $r_{(1,2)}(n)$ is well-known (for example, see [6, (31.12)]).

Lemma 2.12. *We have*

$$\sum_{n\geq 0} r_{(1,2)}(n) q^n = 1 + 2\sum_{n\geq 1}\sum_{d|n} \left(\frac{-2}{d}\right) q^n.$$

In particular, $r_{(1,2)}(n) = 0$ if and only if there exists $p \equiv 5, 7 \pmod 8$ for which $\mathrm{ord}_p(n)$ is odd.

[2]Here and throughout the paper, we use bold letters for vectors.

2.9. **Hurwitz class numbers.** For a discriminant $-D < 0$, we let[3] $H(D)$ denote the D-th *Hurwitz class number*, which counts the number of equivalence classes of positive-definite integral binary quadratic forms of discriminant $-D$, weighted by $\frac{1}{2}$ if the quadratic form is equivalent to a (constant) multiple of $n_1^2+n_2^2$ and weighted by $\frac{1}{3}$ if it is equivalent to a multiple of $n_1^2+n_1n_2+n_2^2$. For $\ell_1, \ell_2 \in \mathbb{N}$ with $\gcd(\ell_1,\ell_2)=1$ and ℓ_2 squarefree, we define

$$\mathcal{H}_{\ell_1,\ell_2} := \mathcal{H}\big| \left(U_{\ell_1\ell_2} - \ell_2 U_{\ell_1} \circ V_{\ell_2}\right),$$

with $\mathcal{H}$ denoting the class number generating function

$$\mathcal{H}(z) := \sum_{D\geq 0} H(D) q^D.$$

Using the modularity properties of $\mathcal{H}$ shown by Zagier [15] (see also [10, Chapter 2, Theorem 2]), the modularity of $\mathcal{H}_{\ell_1,\ell_2}$ was shown in [1, Lemma 2.6].

Lemma 2.13. *For $\ell_1, \ell_2 \in \mathbb{N}$ with $\gcd(\ell_1,\ell_2)=1$ and ℓ_2 squarefree, we have*

$$\mathcal{H}_{\ell_1,\ell_2} \in M_{\frac{3}{2}}(\Gamma_0(4\,\mathrm{rad}(\ell_1)\ell_2), \chi_{4\ell_1\ell_2}).$$

3. The space $S_2(\Gamma_0(36), \chi_{12})$

We require properties of the normalized newforms

$$g_1(z) = q + \sqrt{2}iq^2 - 2q^4 - \sqrt{2}iq^5 - 2\sqrt{2}iq^8 + O\left(q^{10}\right),$$
$$g_2(z) = q - \sqrt{2}iq^2 - 2q^4 + \sqrt{2}iq^5 + 2\sqrt{2}iq^8 + O\left(q^{10}\right),$$

which generate $S_2(\Gamma_0(36),\chi_{12})$. The Fourier coefficients of $g_j(z) = \sum_{n\geq 1} c_j(n)q^n$ satisfy $c_j(n) \in \mathbb{Q}(\sqrt{-2})$ and $c_2(n) = \overline{c_1(n)}$. Hence

$$g_1(z) + g_2(z) = 2\sum_{n\geq 1} \mathrm{Re}\left(c_1(n)\right) q^n, \quad g_1(z) - g_2(z) = 2i\sum_{n\geq 1} \mathrm{Im}\left(c_1(n)\right) q^n.$$

Using Lemma 2.2 (2) and Lemma 2.8, we obtain the following identities.

Lemma 3.1. *We have*

$$\frac{1}{2}\left(g_1+g_2\right) = g_1\big|S_{3,1}, \quad \frac{1}{2}\left(g_1-g_2\right) = g_1\big|S_{3,2}, \quad g_1\big|S_{3,0} = 0.$$

Using Lemmas 2.10 (2), 2.1 (3), 2.3, 2.8, and [13, Proposition 1.41], we obtain the following.

Lemma 3.2. *We have, for $j \in \{1,2\}$,*

$$g_j(z) = \frac{(-1)^{j+1}i}{2\sqrt{2}}\left(\Theta(z) - \Theta(9z)\right)\theta(\chi_{-3},1,z) + \frac{1}{2}\theta(\chi_{-3},1,z)\Theta(9z).$$

Specifically, we have

$$c_j(n) = \frac{(-1)^{j+1}i}{2\sqrt{2}} \sum_{\substack{\boldsymbol{n}\in\mathbb{Z}^2,\, 3\nmid n_1\\ n_1^2+n_2^2=n}} \chi_{-3}(n_2)n_2 + \frac{1}{2}\sum_{\substack{\boldsymbol{n}\in\mathbb{Z}^2,\, 3\nmid n_1\\ n_1^2+9n_2^2=n}} \chi_{-3}(n_1)n_1.$$

Define

$$\gamma_1(n) := \begin{cases} \frac{c_1(n)}{\sqrt{2}i} & \text{if } n\equiv 2 \pmod 3,\\ c_1(n) & \text{otherwise.}\end{cases}$$

A direct calculation using Lemma 3.2 gives the following.

[3]Note that throughout $H(0) := -\frac{1}{12}$.

Lemma 3.3. *We have $\gamma_1(n) \in \mathbb{Z}$, and moreover*

$$\gamma_1(n) = \begin{cases} \sum\limits_{\substack{\boldsymbol{n}\in\mathbb{N}^2 \\ n_1^2+n_2^2=n}} \chi_{-3}(n_2)n_2 & \text{if } n \equiv 2 \pmod 3, \\ \mathbb{1}_{n=\square}\chi_{-3}\left(\sqrt{n}\right)\sqrt{n} + 2 \sum\limits_{\substack{\boldsymbol{n}\in\mathbb{N}^2 \\ n_1^2+9n_2^2=n}} \chi_{-3}(n_1)n_1 & \text{if } n \equiv 1 \pmod 3, \\ 0 & \text{if } 3 \mid n. \end{cases}$$

We next use Lemma 3.2 to determine which Fourier coefficients $c_1(n)$ vanish.

Proposition 3.4. *We have $c_1(n) = 0$ if and only if there exists $p \equiv 3 \pmod 4$ for which $\mathrm{ord}_p(n)$ is odd or if $3 \mid n$.*

Proof. Since g_1 is a newform, it has multiplicative Fourier coefficients by Lemma 2.5 and the claim is equivalent to showing that $\gamma_1(p^r) = 0$ if and only if $p = 3$ or ($p \equiv 3 \pmod 4$ and r is odd). Note that the case $p = 3$ is already established in Lemma 3.3.

Suppose first that $p \equiv 3 \pmod 4$ and r is odd. Since $n_1^2 + n_2^2 = p^r$ does not have any integer solutions and p^r is not a square, Lemma 3.3 implies that

$$\gamma_1(p^r) = 0. \tag{3.1}$$

For the reverse direction, we claim that $\gamma_1(2^r) \neq 0$ and if $p \equiv 1 \pmod 4$, then[4]

$$\gamma_1\left(p^r\right) \not\equiv 0 \pmod p. \tag{3.2}$$

Note that (3.2) implies that $c_1(p^r) \neq 0$ in particular. We prove (3.2) by induction on $r \in \mathbb{N}_0$. Since $\gamma_1(1) = 1$ by Lemma 3.3, the claim is true for $r = 0$. In our induction below, we use Hecke relations to relate $\gamma_1(p^r)$ with $\gamma_1(p^{r-1})$ and $\gamma_1(p^{r-2})$, so we need an additional base case $r = 1$, which we next prove. First assume that $p = 2$. By Lemma 3.3, we have $\gamma_1(2) = 1$.

Next suppose that $p \equiv 1 \pmod 4$. By Lemma 2.11, we have 8 solutions in $\mathbb{Z}^2$ to $n_1^2 + n_2^2 = p$. Fixing one solution $\boldsymbol{a} \in \mathbb{N}^2$, we obtain the 8 solutions by[5] $(\pm a_1, \pm a_2)$ and $(\pm a_2, \pm a_1)$. Thus $\boldsymbol{n} \in \mathbb{N}^2$ satisfies $n_1^2 + n_2^2 = p$ if and only if

$$\boldsymbol{n} \in \{\boldsymbol{a}, (a_2, a_1)\}. \tag{3.3}$$

If $p \equiv 5 \pmod{12}$, then by (3.3) there are precisely two terms $\boldsymbol{n} = \boldsymbol{a}$ and $\boldsymbol{n} = (a_2, a_1)$ in the sum in Lemma 3.3 and thus

$$\gamma_1(p) = \chi_{-3}\left(a_1\right)a_1 + \chi_{-3}\left(a_2\right)a_2.$$

Since $a_1^2 + a_2^2 = p$, we have $a_j \leq \sqrt{p}$ and $p \geq 5$ implies that $p > 2\sqrt{p}$, so $|\gamma_1(p)| < p$. Since $\gamma_1(p) = 0$ is impossible, $\gamma_1(p) \not\equiv 0 \pmod p$.

If $p \equiv 1 \pmod{12}$, then exactly one of a_1 or a_2 is divisible by 3. Without loss of generality, assume that $3 \mid a_2$. Writing the terms in the sum from Lemma 3.3 as $n_1^2 + (3n_2)^2 = p$, we see from (3.3) that $(n_1, 3n_2) \in \{\boldsymbol{a}, (a_2, a_1)\}$. Since $3 \mid a_2$, we see that the sum has a single term $\boldsymbol{n} = (a_1, \frac{a_2}{3})$ and

$$\gamma_1(p) = 2\chi_{-3}(a_1)a_1.$$

Then $\frac{1}{2}|\gamma_1(p)| < p$. Since $a_1 \neq 0$, we have $\gamma_1(p) \neq 0$. Since $\gamma_1(p) \neq 0$ and $\frac{1}{2}|\gamma_1(p)| < p$, we see that (3.2) holds in this case as well. This completes the case $r = 1$ of (3.2).

Since $g_1 \in S_2(\Gamma_0(36), \chi_{12})$ is a normalized newform, Lemma 2.5 implies that

$$c_1\left(p^r\right) = c_1(p)c_1\left(p^{r-1}\right) - \chi_{12}(p)pc_1\left(p^{r-2}\right). \tag{3.4}$$

[4]Note that since $\gamma_1(p^r) \in \mathbb{Z}$ by Lemma 3.3, (3.2) makes sense as a congruence in the integers.

[5]Note that $a_1 = a_2$ implies $p = a_1^2 + a_2^2 = 2a_2^2$, which contradicts $p \equiv 1 \pmod 4$.

For $p=2$, we have $\gamma_1(2)=1$ by Lemma 3.3, and (3.4) implies that

$$\gamma_1(2^r) = \begin{cases} \gamma_1(2)\gamma_1\left(2^{r-1}\right) & \text{if } r \text{ is odd,} \\ -2\gamma_1(2)\gamma_1\left(2^{r-1}\right) & \text{if } r \text{ is even.} \end{cases}$$

Hence for $r \in \mathbb{N}$ we have $\gamma_1(2^r) = (-2)^{\lfloor \frac{r}{2} \rfloor} \neq 0$ by induction.

Next suppose that $p \equiv 1 \pmod 4$ and assume that (3.2) holds for $j \in \mathbb{N}$ with $j < r$. If $p \equiv 1 \pmod{12}$, then $p^j \equiv 1 \pmod 3$ for all $j \in \mathbb{N}_0$, so $\gamma_1(p^j) = c_1(p^j)$ and (3.4) implies

$$\gamma_1\left(p^r\right) = \gamma_1(p)\gamma_1\left(p^{r-1}\right) - \chi_{12}(p)p\gamma_1\left(p^{r-2}\right) \equiv \gamma_1(p)\gamma_1\left(p^{r-1}\right) \not\equiv 0 \pmod p,$$

where we use the inductive hypothesis (3.2) and $\gamma_1(p) \not\equiv 0 \pmod p$ in the last step.

For $p \equiv 5 \pmod{12}$, we have $p^r \equiv 1 \pmod 3$ if r is even and $p^r \equiv 2 \pmod 3$ if r is odd, so we split into the cases r even and r odd. For r even, we have $r-1$ odd and $r-2$ even, so (3.4) implies that

$$\gamma_1\left(p^r\right) = \sqrt{2}i\gamma_1(p)\sqrt{2}i\gamma_1\left(p^{r-1}\right) - \chi_{12}(p)p\gamma_1\left(p^{r-2}\right) \equiv -2\gamma_1(p)\gamma_1\left(p^{r-1}\right) \not\equiv 0 \pmod p,$$

where we use the inductive hypothesis, $p \neq 2$, and $\gamma_1(p) \not\equiv 0 \pmod p$ in the last step.

If $p \equiv 5 \pmod{12}$ and r is odd, then $r-1$ is even and $r-2$ is odd, so (3.4) implies that

$$\gamma_1\left(p^r\right) \equiv \gamma_1(p)\gamma_1\left(p^{r-1}\right) \not\equiv 0 \pmod p.$$

We finally inductively show that $c(p^r) \neq 0$ for $3 < p \equiv 3 \pmod 4$ and r even. The base case $r=0$ is established by $c_1(1)=1$. Suppose that $r \geq 2$ is even. Since $r-1$ is odd, we have $c_1(p^{r-1}) = 0$ by (3.1). Hence in this case (3.4) simplifies as

$$c_1\left(p^r\right) = -\chi_{12}(p)pc_1\left(p^{r-2}\right) \neq 0$$

by induction. In the last step, we use the fact that $\chi_{12}(p) \neq 0$ because $p \neq 3$. □

4. Proof of Theorem 1.1

4.1. **The case $1^{-1}3^34^2$.** In this subsection, we prove half of Theorem 1.1.

Proposition 4.1. *We have*

$$\mathcal{S}_{1^{-1}3^34^2} = \left\{n \in \mathbb{N} : \exists p \equiv 3 \pmod 4,\ \mathrm{ord}_p(3n+2) \text{ is odd}\right\}.$$

Using Lemmas 2.2 (2), 2.9, and 2.8, we first relate the eta-quotient to g_1 and the newform

$$g_3(z) = q + 3\sqrt{2}iq^5 + 4q^{13} - 3\sqrt{2}iq^{17} + O\left(q^{25}\right) \in S_2\left(\Gamma_0(144), \chi_{12}\right).$$

Lemma 4.2. *We have*

$$\frac{\eta^3(9z)\eta^2(12z)}{\eta(3z)} = -\frac{i}{\sqrt{2}}g_1\big|S_{12,2}(z) + \frac{i}{\sqrt{2}}g_1\big|S_{12,8}(z) - \frac{i}{3\sqrt{2}}g_3\big|S_{6,5}(z).$$

In addition to Lemma 3.3, we require a formula for the Fourier coefficients $c_3(n)$ of g_3. Using Lemmas 2.10, 2.1 (3), 2.3, 2.2 (2), and 2.8, it is not hard to show the following formula for g_3 and its Fourier coefficients.

Lemma 4.3. *We have*

$$\begin{aligned} g_3(z) = &\left(\theta\left(\chi_{-3}, 1, 4z\right)\Theta(z)\right)\big|S_{12,1} + \frac{1}{2}\left(\theta\left(\chi_{-3}, 1, z\right)\Theta(4z)\right)\big|S_{12,1} \\ &+ \frac{i}{\sqrt{2}}\left(\theta\left(\chi_{-3}, 1, 4z\right)\Theta(z)\right)\big|S_{12,5} + \frac{i}{2\sqrt{2}}\left(\theta\left(\chi_{-3}, 1, z\right)\Theta(4z)\right)\big|S_{12,5}. \end{aligned}$$

In particular, the n-th Fourier coefficient $c_3(n)$ of g_3 is

$$\begin{cases} \mathbb{1}_{n=\square}\chi_{-3}(\sqrt{n})\sqrt{n}+4\sum\limits_{\substack{\boldsymbol{m}\in\mathbb{N}^2\\ 4m_1^2+9m_2^2=n}}\chi_{-3}(m_1)\,m_1+2\sum\limits_{\substack{\boldsymbol{m}\in\mathbb{N}^2\\ m_1^2+36m_2^2=n}}\chi_{-3}(m_1)\,m_1 & \text{if } n\equiv 1 \pmod{12},\\ 2\sqrt{2}i\sum\limits_{\substack{\boldsymbol{m}\in\mathbb{N}^2\\ 4m_1^2+m_2^2=n}}\chi_{-3}(m_1)\,m_1+\sqrt{2}i\sum\limits_{\substack{\boldsymbol{m}\in\mathbb{N}^2\\ m_1^2+4m_2^2=n}}\chi_{-3}(m_1)\,m_1 & \text{if } n\equiv 5 \pmod{12},\\ 0 & \text{otherwise.}\end{cases}$$

A computation similar to the case of Proposition 3.4 gives a classification for those $n\in\mathbb{N}$ for which $c_3(n)$ vanishes.

Proposition 4.4. *For $n\in\mathbb{N}$, we have that $c_3(n)$ vanishes if and only if $\gcd(n,6)>1$ or if there exists a prime $p\equiv 3 \pmod 4$ with $\mathrm{ord}_p(n)$ odd.*

Lemma 4.2 and Proposition 4.4 now directly imply Proposition 4.1.

4.2. **The case $1^4 2^{-2} 4^4$.** In this subsection, we prove the other half of Theorem 1.1.

Proposition 4.5. *We have*

$$\mathcal{S}_{1^4 2^{-2} 4^4}=\{n\in\mathbb{N}: \exists p\equiv 3 \pmod 4,\ \mathrm{ord}_p(3n+2) \text{ is odd}\}.$$

We first relate the eta-quotient to a normalized newform

$$g_4(z)=q-2q^2+4q^4+8q^5-8q^8+O\left(q^{10}\right)\in S_3\left(\Gamma_0(36),\chi_{-4}\right).$$

Using Lemmas 2.9, 2.2, and 2.8, we obtain the following.

Lemma 4.6. *We have*

$$\frac{\eta^4(3z)\eta^4(12z)}{\eta^2(6z)}=-\frac{1}{2}g_4\big|S_{3,2}(z).$$

We find the following formulas for g_4, letting

$$\Theta_1(z):=2\sum_{\boldsymbol{n}\in\mathbb{Z}^2}\left(\frac{-3}{n_1n_2}\right)n_1n_2q^{n_1^2+n_2^2},\quad \Theta_2(z):=\sum_{\boldsymbol{n}\in\mathbb{Z}^2}n_1^2q^{n_1^2+9n_2^2},\quad \Theta_3(z):=9\sum_{\boldsymbol{n}\in\mathbb{Z}^2}n_2^2q^{n_1^2+9n_2^2}.$$

Lemma 4.7. *We have*

$$g_4=-\frac{\Theta_1}{4}+\frac{\Theta_2}{2}-\frac{\Theta_3}{2},\quad g_4\big|S_{3,0}=0,\quad g_4\big|S_{3,1}=\frac{\Theta_2}{2}-\frac{\Theta_3}{2},\quad g_4\big|S_{3,2}=-\frac{\Theta_1}{4}.$$

Moreover, we have for the Fourier coefficients c_4 of g_4

$$c_4(n)=\begin{cases}\mathbb{1}_{n=\square}n+2\sum_{\substack{\boldsymbol{n}\in\mathbb{N}^2\\ n_1^2+9n_2^2=n}}\left(n_1^2-9n_2^2\right) & \text{if } n\equiv 1 \pmod 3,\\ -2\sum_{\substack{\boldsymbol{n}\in\mathbb{N}^2\\ n_1^2+n_2^2=n}}\left(\frac{-3}{n_1n_2}\right)n_1n_2 & \text{if } n\equiv 2 \pmod 3,\\ 0 & \text{if } n\equiv 0 \pmod 3.\end{cases}$$

Proof. Using Lemma 2.2 (2) the left-hand sides of the first four claimed identities are weight 3 cusp forms on $\Gamma_0(36)$ with character χ_{-4}. We next write $\Theta_1(z)=2\theta^2(\chi_{-3},1,z)$. Using Lemmas 2.10 (2) and 2.3, we conclude that $\Theta_1\in S_3\left(\Gamma_0(36),\chi_{-4}\right)$.

By Lemma 2.1 (3) and Lemma 2.4, we have

$$\Theta_2-\Theta_3=2\left[\Theta,\Theta|V_9\right]_1\in M_3\left(\Gamma_0(36),\chi_{-4}\right).$$

Thus we conclude that the right-hand sides of the first four identities are in $M_3(\Gamma_0(36),\chi_{-4})$. Lemma 2.8 then gives the identities for the modular form, the identity for $c_4(n)$ following by a direct calculation, picking off the Fourier coefficients. □

To prove Proposition 4.5 we require the following proposition; its proof is similar to that of Proposition 3.4.

Proposition 4.8. *We have* $c_4(n) = 0$ *if and only if* $3 \mid n$ *or there exists a prime* $p \equiv 3 \pmod 4$ *for which* $\mathrm{ord}_p(n)$ *is odd.*

Lemma 4.6 and Proposition 4.8 now directly imply Proposition 4.5.

5. Proof of Theorem 1.2

5.1. **The case** $1^1 2^{-2} 4^3$**.** The goal of this subsection is the claimed evaluation of the first set appearing in Theorem 1.2.

Proposition 5.1. *We have*

$$\mathcal{S}_{1^1 2^{-2} 4^3} = \{n \in \mathbb{N} : \exists p \equiv 5, 7 \pmod 8,\ \mathrm{ord}_p(8n+3) \text{ is odd}\}.$$

We first relate $C_{1^1 2^{-2} 4^3}(n)$ to $r_{1,2}(n)$. Using Lemmas 2.9, 2.10 (1), 2.3, 2.1 (3), 2.8, and [13, Proposition 1.41] yields the following.

Lemma 5.2. *We have*

$$\frac{\eta(8z)\eta^3(32z)}{\eta^2(16z)} = \frac{1}{4}\sum_{n\geq 0}(-1)^n r_{(1,2)}(8n+3)q^{8n+3}.$$

Proposition 5.1 now directly follows from Lemmas 5.2 and 2.12.

5.2. **The case** $1^1 2^2 4^1$**.** The goal of this subsection is the claimed evaluation of the second set appearing in Theorem 1.2.

Proposition 5.3. *We have*

$$\mathcal{S}_{1^1 2^2 4^1} = \{n \in \mathbb{N} : \exists p \equiv 5, 7 \pmod 8,\ \mathrm{ord}_p(8n+3) \text{ is odd}\}.$$

We first relate the eta-quotient to CM newforms (see [14, Section 1] for a definition). Using Lemmas 2.10, 2.1 (3), 2.3, 2.2 (2), 2.9, and 2.8 gives the following by a direct calculation.

Lemma 5.4. *We have*

$$\eta(8z)\eta^2(16z)\eta(32z) = \frac{1}{2\sqrt{2}} g_5\big|S_{8,3},$$

where

$$g_5(z) := \frac{1}{2}\theta\left(\chi_{-8}, 1, z\right)\theta\left(\chi_1, 0, 8z\right) + \frac{1}{\sqrt{2}}\theta\left(\chi_8, 0, z\right)\theta\left(\chi_{-4}, 1, 2z\right) \in S_2\left(\Gamma_0(256)\right)$$

is a normalized newform. Moreover, we have for the Fourier coefficients of c_5 *of* g_5

$$c_5(n) = \begin{cases} \mathbb{1}_{n=\square}\left(\frac{-2}{\sqrt{n}}\right)\sqrt{n} + 2\sum_{\substack{\boldsymbol{n}\in\mathbb{N}^2\\ n_1^2+8n_2^2=n}}\left(\frac{-2}{n_1}\right)n_1 & \text{if } n \equiv 1 \pmod 8,\\ 2\sqrt{2}\sum_{\substack{\boldsymbol{n}\in\mathbb{N}^2\\ n_1^2+2n_2^2=n}}\left(\frac{2}{n_1}\right)\left(\frac{-4}{n_2}\right)n_2 & \text{if } n \equiv 3 \pmod 8,\\ 0 & \text{otherwise.}\end{cases}$$

Proposition 5.3 now follows similar to Proposition 4.8.

5.3. **The case $1^32^{-1}4^2$.** The goal of this subsection is the claimed evaluation of the third set appearing in Theorem 1.2.

Proposition 5.5. *We have*

$$\mathcal{S}_{1^32^{-1}4^2} = \left\{n \in \mathbb{N} : \exists p \equiv 5, 7 \pmod 8,\ \mathrm{ord}_p(8n+3) \text{ is odd}\right\}.$$

We first use Lemmas 2.9, 2.2 (3), and 2.8 to relate the eta-quotient to a normalized newform

$$g_6(z) = q + 2iq^3 - q^9 + O\left(q^{10}\right) \in S_2(\Gamma_0(64), \chi_8).$$

Lemma 5.6. *We have*

$$\frac{\eta^3(8z)\eta^2(32z)}{\eta(16z)} = -\frac{i}{2} g_6\big|S_{8,3}(z).$$

We next compute the Fourier coefficients $c_6(n)$ of g_6. Using Lemmas 2.10, 2.1 (3), 2.3, 2.2 (2), and 2.8 directly yields the following.

Lemma 5.7. *We have*

$$g_6(z) = \frac{1}{2}\left(\theta\left(\chi_{-4}, 1, z\right)\Theta(2z)\right)\big|S_{8,1} + \frac{i}{2}\left(\theta\left(\chi_{-4}, 1, z\right)\Theta(2z)\right)\big|S_{8,3}.$$

Moreover, we have for the Fourier coefficients c_6 of g_6

$$c_6(n) = \begin{cases} \mathbb{1}_{n=\square}\left(\frac{-1}{\sqrt{n}}\right)\sqrt{n} + 2\sum_{\substack{\boldsymbol{m}\in\mathbb{N}^2\\ m_1^2+2m_2^2=n}}\left(\frac{-1}{m_1}\right)m_1 & \text{if } n \equiv 1 \pmod 8,\\ 2i\sum_{\substack{\boldsymbol{m}\in\mathbb{N}^2\\ m_1^2+2m_2^2=n}}\left(\frac{-1}{m_1}\right)m_1 & \text{if } n \equiv 3 \pmod 8.\end{cases}$$

Proposition 5.5 now follows similarly to Proposition 4.8.

5.4. **The case 1^32^3.** The goal of this subsection is the claimed evaluation of the fourth set appearing in Theorem 1.2.

Proposition 5.8. *We have*

$$\mathcal{S}_{1^32^3} = \left\{n \in \mathbb{N} : \exists p \equiv 5, 7 \pmod 8,\ \mathrm{ord}_p(8n+3) \text{ is odd}\right\}.$$

Using Lemmas 2.9, 2.2 (2), and 2.8, we relate the eta-quotient to a normalized newform

$$g_7(z) = q + 4\sqrt{2}q^3 + 23q^9 + O\left(q^{10}\right) \in S_3(\Gamma_0(128), \chi_{-8}).$$

Lemma 5.9. *We have*

$$\eta^3(8z)\eta^3(16z) = \frac{1}{4\sqrt{2}} g_7\big|S_{8,3}(z).$$

We next find a formula for the Fourier coefficients of g_7. To state the formula, set

$$\Theta_4(z) := \sum_{\boldsymbol{m}\in\mathbb{Z}^2}\left(\frac{-4}{m_1m_2}\right)m_1m_2q^{m_1^2+2m_2^2}, \quad \Theta_5(z) := \sum_{\substack{\boldsymbol{m}\in\mathbb{Z}^2\\ 2\nmid m_1}}(-1)^{m_2}\left(m_1^2 - 8m_2^2\right)q^{m_1^2+8m_2^2}.$$

A direct calculation using [13, Theorem 1.60], Lemma 2.9, [13, Proposition 1.41], and Lemma 2.8 gives the following.

Lemma 5.10. *We have*

$$g_7 = \sqrt{2}\Theta_4 + \frac{\Theta_5}{2}.$$

In particular, we have

$$c_7(n) = \begin{cases} \mathbb{1}_{n=\square} n + 2\sum_{\substack{\boldsymbol{m}\in\mathbb{N}^2\\ m_1^2+8m_2^2=n}} (-1)^{m_2}\left(m_1^2-8m_2^2\right) & \textit{if } n\equiv 1 \pmod 8,\\ 4\sqrt{2}\sum_{\substack{\boldsymbol{m}\in\mathbb{N}^2\\ m_1^2+2m_2^2=n}} \left(\frac{-1}{m_1m_2}\right) m_1m_2 & \textit{if } n\equiv 3 \pmod 8,\\ 0 & \textit{otherwise.}\end{cases}$$

Proposition 5.8 now follows similar to Proposition 3.4.

5.5. **The case** $1^7 2^{-3} 4^2$**.** The goal of this subsection is the claimed evaluation of the fifth set appearing in Theorem 1.2.

Proposition 5.11. *We have*

$$\mathcal{S}_{1^7 2^{-3} 4^2} = \{n\in\mathbb{N} : \exists p\equiv 5,7 \pmod 8,\ \mathrm{ord}_p(8n+3) \text{ is odd}\}.$$

Using Lemmas 2.9, 2.2 (2), and 2.8, we relate the eta-quotient to a normalized newform

$$g_8(z) = q + 2q^3 - 5q^9 + O\left(q^{10}\right) \in S_3(\Gamma_0(32), \chi_{-8}).$$

Lemma 5.12. *We have*

$$\frac{\eta^7(8z)\eta^2(32z)}{\eta^3(16z)} = \frac{1}{2} g_8 \big| S_{8,3}(z).$$

We next obtain the following formula for the Fourier coefficients of g_8, using Lemmas 5.10, 2.1 (3), and 2.8, setting

$$\Theta_6(z) := \sum_{\substack{\boldsymbol{n}\in\mathbb{Z}^2\\ 2\nmid n_1}} \left(n_1^2-8n_2^2\right) q^{n_1^2+8n_2^2}, \quad \Theta_7(z) := \sum_{\substack{\boldsymbol{n}\in\mathbb{Z}^2\\ 2\nmid n_1,n_2}} \left(n_1^2-2n_2^2\right) q^{n_1^2+2n_2^2},$$

Lemma 5.13. *We have*

$$g_8 = \frac{\Theta_6}{2} - \frac{\Theta_7}{2}.$$

In particular, we have for the Fourier coefficients c_8 *of* g_8

$$c_8(n) = \begin{cases} \mathbb{1}_{n=\square} n + 2\sum_{\substack{\boldsymbol{m}\in\mathbb{N}^2\\ m_1^2+8m_2^2=n}} \left(m_1^2-8m_2^2\right) & \textit{if } n\equiv 1 \pmod 8,\\ -2\sum_{\substack{\boldsymbol{m}\in\mathbb{N}^2\\ m_1^2+2m_2^2=n}} \left(m_1^2-2m_2^2\right) & \textit{if } n\equiv 3 \pmod 8,\\ 0 & \textit{otherwise.}\end{cases}$$

Proposition 5.11 now follows similarly to Proposition 3.4.

6. Proof of Theorem 1.3

6.1. **The case** $1^{-1}2^{10}3^{-1}4^{-4}$**.** Define

$$f_1(z) := \frac{\eta^{10}(2z)}{\eta(z)\eta(3z)\eta^4(4z)}.$$

Since f_1 is not a cusp form, it has a non-trivial Eisenstein series part. In order to investigate the Eisenstein series part, we define

$$\mathcal{E}_1(z) := 1 + \sum_{n\geq 1}\left(\frac{1}{2}\sum_{d\mid n}\left(\frac{12}{d}\right) d + \frac{1}{2}\sum_{d\mid n}\left(\frac{-3}{\frac{n}{d}}\right)\left(\frac{-4}{d}\right) d + 2\sum_{d\mid n}\left(\frac{-4}{\frac{n}{d}}\right)\left(\frac{-3}{d}\right) d + 2\sum_{d\mid n}\left(\frac{12}{\frac{n}{d}}\right) d\right.$$

$$-\frac{3}{2}\sum_{d\mid\frac{n}{3}}\left(\frac{12}{d}\right)d+\frac{9}{2}\sum_{d\mid\frac{n}{3}}\left(\frac{-3}{\frac{n}{3d}}\right)\left(\frac{-4}{d}\right)d+6\sum_{d\mid\frac{n}{3}}\left(\frac{-4}{\frac{n}{3d}}\right)\left(\frac{-3}{d}\right)d-18\sum_{d\mid\frac{n}{3}}\left(\frac{12}{\frac{n}{3d}}\right)d\Bigg)q^n.$$

A direct calculation gives the following identity for $\mathcal{E}_1$ in terms of Eisenstein series.

Lemma 6.1. *We have*

$$\mathcal{E}_1=\frac{1}{4}E_{2,\chi_1,\chi_{12}}+\frac{1}{4}E_{2,\chi_{-3},\chi_{-4}}+E_{2,\chi_{-4},\chi_{-3}}+E_{2,\chi_{12},\chi_1}-\frac{3}{4}E_{2,\chi_1,\chi_{12}}\big|V_3+\frac{9}{4}E_{2,\chi_{-3},\chi_{-4}}\big|V_3$$
$$+3E_{2,\chi_{-4},\chi_{-3}}\big|V_3-9E_{2,\chi_{12},\chi_1}\big|V_3\in M_2\left(\Gamma_0(36),\chi_{12}\right).$$

Recall the newforms g_1 and g_2, defined in Section 3, which span the space $S_2(\Gamma_0(36),\chi_{12})$. Using Lemmas 6.1 and 2.8 we obtain an identity for f_1.

Lemma 6.2. *We have*

$$f_1=\mathcal{E}_1-2\left(1+\sqrt{2}i\right)g_1-2\left(1-\sqrt{2}i\right)g_2.$$

We first rewrite the Fourier coefficients of $\mathcal{E}_1$. A direct calculation gives the following.

Lemma 6.3. *Suppose that $\nu_2,\nu_3\in\mathbb{N}_0$ and $m\in\mathbb{N}$ with $\gcd(m,6)=1$. Then the $2^{\nu_2}3^{\nu_3}m$-th Fourier coefficient of $\mathcal{E}_1$ is*

$$\Bigg(\frac{1}{2}+(-1)^{\nu_2+\nu_3}\frac{3^{\nu_3}}{2}\left(\frac{-3}{m}\right)+(-1)^{\nu_2+\nu_3}2^{\nu_2+1}\left(\frac{-1}{m}\right)+2^{\nu_2+1}3^{\nu_3}\left(\frac{3}{m}\right)$$
$$-\mathbb{1}_{3\mid n}\left(\frac{3}{2}+(-1)^{\nu_2+\nu_3}\frac{3^{\nu_3+1}}{2}\left(\frac{-3}{m}\right)+(-1)^{\nu_2+\nu_3}3\cdot 2^{\nu_2+1}\left(\frac{-1}{m}\right)+2^{\nu_2+1}3^{\nu_3+1}\left(\frac{3}{m}\right)\right)\Bigg)$$
$$\times\prod_{p\mid m}\frac{1-\left(\left(\frac{3}{p}\right)p\right)^{\mathrm{ord}_p(m)+1}}{1-\left(\frac{3}{p}\right)p}.$$

We next use Lemma 6.3 to show the following.

Corollary 6.4. *For $n\not\equiv 2\pmod 3$, the n-th Fourier coefficient of f_1 does not vanish.*

Proof. We first bound the cuspidal part of f_1. Recalling that $c_j(n)$ denotes the n-th Fourier coefficient of g_j, the n-th Fourier coefficient of the cuspidal part of f_1 is

$$-2(c_1(n)+c_2(n))+2\sqrt{2}i(c_2(n)-c_1(n)). \tag{6.1}$$

By Lemma 3.1, we have $c_1(n)+c_2(n)=0$ unless $n\equiv 1\pmod 3$ and $c_2(n)-c_1(n)=0$ unless $n\equiv 2\pmod 3$. Thus we conclude from Theorem 2.6 that the absolute value of (6.1) is

$$\begin{cases}2|c_1(n)+c_2(n)|\le 4d(n)\sqrt{n} & \text{if } n\equiv 1\pmod 3,\\ 2\sqrt{2}|c_2(n)-c_1(n)|\le 4\sqrt{2}d(n)\sqrt{n} & \text{if } n\equiv 2\pmod 3,\\ 0 & \text{if } 3\mid n.\end{cases}$$

We next look at the Eisenstein series part. For ease of notation, we abbreviate $\nu_p:=\mathrm{ord}_p(n)$. For a prime p and $\nu\in\mathbb{N}_0$ we define

$$F_p(\nu):=\begin{cases}\frac{2^{\nu+2}-1}{3} & \text{if } p=2,\\ \frac{p^{\nu+1}-1}{p-1} & \text{if } p\equiv\pm1\pmod{12},\\ \frac{p^{\nu+1}+1}{p+1} & \text{if } p\equiv\pm5\pmod{12},\ 2\mid\nu,\\ \frac{p^{\nu+1}-1}{p+1} & \text{if } p\equiv\pm5\pmod{12},\ 2\nmid\nu.\end{cases}$$

Note that for p odd

$$F_p(\nu) = \left| \frac{1 - \left(\left(\frac{3}{p}\right) p\right)^{\nu+1}}{1 - \left(\frac{3}{p}\right) p} \right|. \tag{6.2}$$

A direct calculation using Lemma 6.3 with $\nu = \nu_2$ and $\nu_3 = 0$ then shows that for $n \equiv 1 \pmod 3$ the n-th Fourier coefficient of f_1 is non-zero if

$$G_1(n) := \prod_{p|n} \frac{F_p(\nu_p)}{(\nu_p + 1)p^{\frac{\nu_p}{2}}} > \frac{4}{3}. \tag{6.3}$$

We therefore next determine those n for which $G_1(n) \leq \frac{4}{3}$. For this, for each prime p and $\nu \in \mathbb{N}$ we determine certain constants $\mathcal{C}_p(\nu)$ for which

$$G_1(p^\nu) = \frac{F_p(\nu)}{(\nu+1)p^{\frac{\nu}{2}}} \geq \mathcal{C}_p(\nu).$$

We first consider the case $p \neq 2$. Bounding against the worst case for $F_p(\nu)$, we have

$$G_1(p^\nu) \geq \frac{p^{\nu+1} - 1}{(\nu+1)(p+1)p^{\frac{\nu}{2}}}.$$

A direct calculation shows that

$$g_\nu(x) := \frac{x^{\nu+1} - 1}{(x+1)x^{\frac{\nu}{2}}}$$

is increasing for $x > 0$, so $g_\nu(x) > a(\nu+1)$ implies that for $p \geq x$ we have

$$G_1(p^\nu) \geq \frac{g_\nu(p)}{\nu+1} \geq \frac{g_\nu(x)}{\nu+1} > a. \tag{6.4}$$

Define for $x \geq 3$ and $a \in \mathbb{R}_{\geq 1}$

$$f_{a,\nu}(x) := x^{\nu+1} - 1 - a(\nu+1)(x+1)x^{\frac{\nu}{2}}.$$

Using induction on ν, one can show that if $f_{a,\nu}(x) \geq 0$, then $f_{a,\nu+j}(x) \geq 0$ for all $j \in \mathbb{N}_0$. Hence if $f_{a,\mu}(x) \geq 0$ for some $\mu \in \mathbb{N}$, then $g_\nu(x) \geq a(\nu+1)$ for all $\nu \in \mathbb{N}_0$ with $\nu \geq \mu$. Combining with (6.4), we see that if $f_{a,\mu}(x) \geq 0$, then for all $p \geq x$ and $\nu \in \mathbb{N}_0$ with $\nu \geq \mu$

$$G_1(p^\nu) \geq \frac{g_\nu(p)}{\nu+1} \geq \frac{g_\nu(x)}{\nu+1} > a. \tag{6.5}$$

Directly computing

$$f_{2.1,1}(20) \geq 0, \qquad f_{2.1,2}(8) \geq 0, \qquad f_{2.1,3}(5) \geq 0, \tag{6.6}$$

we conclude from (6.5) that

$$G_1(p^\nu) \geq 2.1 \text{ for } p \geq 23,\ \nu \in \mathbb{N}, \qquad G_1(p^\nu) \geq 2.1 \text{ for } p \in \{11, 13, 17, 19\}, \nu \geq 2,$$
$$G_1(p^\nu) \geq 2.1 \text{ for } p \in \{5, 7\}, \nu \geq 3.$$

One also directly checks that for $\nu \geq 6$ we have

$$G_1(2^\nu) > \frac{2\sqrt{5}}{3}.$$

We therefore conclude that if $G_1(n) \leq \frac{4}{3}$, then

$$n = \prod_{p \leq 19} p^{\nu_p}$$

with $0 \le \nu_2 \le 5$, $0 \le \nu_5, \nu_7 \le 2$, and $0 \le \nu_p \le 1$ for $11 \le p \le 19$. Checking all cases explicitly with a computer, we conclude that $G_1(n) > \frac{4}{3}$ for $n > 1120$. It was verified with a computer that for $n \le 1120$ the n-th Fourier coefficient of f_1 is positive, so we conclude the claim for $n \equiv 1 \pmod 3$.

Next assume $3 \mid n$. In this case, we have $c_1(n) = c_2(n) = 0$ by Lemma 3.1, so the n-th Fourier coefficient of f_1 agrees with the n-th Fourier coefficient of $\mathcal{E}_1$, and we only need to show that the n-th Fourier coefficient of $\mathcal{E}_1$ does not vanish. Then the first factor in Lemma 6.3, plugging in $\nu = \nu_2$ and $\mu = \nu_3$ and abbreviating $m := \prod_{\substack{p \mid n \\ p \nmid 6}} p^{\nu_p}$, equals

$$-1 + (-1)^{\nu_2+\nu_3+1} 3^{\nu_3} \left(\frac{-3}{m}\right) + (-1)^{\nu_2+\nu_3+1} 2^{\nu_2+2} \left(\frac{-1}{m}\right) - 2^{\nu_2+2} 3^{\nu_3} \left(\frac{3}{m}\right). \tag{6.7}$$

By the triangle inequality, we may bound the absolute value of (6.7) from below by

$$2^{\nu_2+2}3^{\nu_3} - 1 - 3^{\nu_3} - 2^{\nu_2+2} = 2^{\nu_2+2}(3^{\nu_3} - 1) - 1 - 3^{\nu_3} \ge 4(3^{\nu_3} - 1) - 1 - 3^{\nu_3} = 3^{\nu_3+1} - 5 > 0.$$

So in particular this factor does not vanish. The other factors in Lemma 6.3 satisfy

$$\frac{1 - \left(\left(\frac{3}{p}\right) p\right)^{\nu_p+1}}{1 - \left(\frac{3}{p}\right) p} \neq 0,$$

so the n-th Fourier coefficient of $\mathcal{E}_1$ does not vanish for $3 \mid n$. □

Let $a_1(n) := C_{1^{-1}2^{10}3^{-1}4^{-4}}(n)$. Using Lemmas 6.3 and 6.2 and then simplifying with Lemma 3.1, we obtain the following.

Lemma 6.5. *For $n \equiv 2 \pmod 3$, we have $a_1(n) = 0$ if and only if $c_1(n) = 0$.*

Lemma 6.5 and Proposition 3.4 directly give the easy direction of Theorem 1.3 for $a_1(n)$.

Lemma 6.6. *If $n \equiv 2 \pmod 3$ and if there exists $p \equiv 3 \pmod 4$ with $\mathrm{ord}_p(n)$ odd, then $a_1(n) = 0$.*

We may now conclude the first half of Theorem 1.3, using Corollary 6.4 and Proposition 3.4.

Theorem 6.7. *We have $a_1(n) = 0$ if and only if $n \equiv 2 \pmod 3$ and there exists a prime $p \equiv 3 \pmod 4$ for which $\mathrm{ord}_p(n)$ is odd.*

6.2. **The case $1^7 2^{-2} 3^{-1}$.** Since

$$f_2(z) := \frac{\eta^7(z)}{\eta^2(2z)\eta(3z)}$$

is not a cusp form, we define a corresponding Eisenstein series

$$\begin{aligned}\mathcal{E}_2(z) := 1 + \sum_{n \ge 1} \Bigg(& -\frac{1}{2} \sum_{d \mid n} \left(\frac{12}{d}\right) d - \frac{1}{2} \sum_{d \mid n} \left(\frac{-3}{\frac{n}{d}}\right) \left(\frac{-4}{d}\right) d + \sum_{d \mid n} \left(\frac{-4}{\frac{n}{d}}\right) \left(\frac{-3}{d}\right) d + \sum_{d \mid n} \left(\frac{12}{\frac{n}{d}}\right) d \\ & + \frac{3}{2} \sum_{d \mid \frac{n}{3}} \left(\frac{12}{d}\right) d - \frac{9}{2} \sum_{d \mid \frac{n}{3}} \left(\frac{-3}{\frac{n}{3d}}\right) \left(\frac{-4}{d}\right) d + 3 \sum_{d \mid \frac{n}{3}} \left(\frac{-4}{\frac{n}{3d}}\right) \left(\frac{-3}{d}\right) d - 9 \sum_{d \mid \frac{n}{3}} \left(\frac{12}{\frac{n}{3d}}\right) d + \sum_{d \mid \frac{n}{2}} \left(\frac{12}{d}\right) d \\ & - \sum_{d \mid \frac{n}{2}} \left(\frac{-3}{\frac{n}{2d}}\right) \left(\frac{-4}{d}\right) d + 4 \sum_{d \mid \frac{n}{2}} \left(\frac{-4}{\frac{n}{2d}}\right) \left(\frac{-3}{d}\right) d - 4 \sum_{d \mid \frac{n}{2}} \left(\frac{12}{\frac{n}{2d}}\right) d - 3 \sum_{d \mid \frac{n}{6}} \left(\frac{12}{d}\right) d - 9 \sum_{d \mid \frac{n}{6}} \left(\frac{-3}{\frac{n}{6d}}\right) \left(\frac{-4}{d}\right) d \\ & + 12 \sum_{d \mid \frac{n}{6}} \left(\frac{-4}{\frac{n}{6d}}\right) \left(\frac{-3}{d}\right) d + 36 \sum_{d \mid \frac{n}{6}} \left(\frac{12}{\frac{n}{6d}}\right) d \Bigg) q^n.\end{aligned}$$

A direct calculation shows the following.

Lemma 6.8. *Suppose that $\nu_2,\nu_3\in\mathbb{N}_0$ and $m\in\mathbb{N}$ with $\gcd(m,6)=1$. Then the $2^{\nu_2}3^{\nu_3}m$-th Fourier coefficient of $\mathcal{E}_2$ equals*

$$\Bigg(-\frac{1}{2}-\frac{3^{\nu_3}}{2}\left(\frac{-3}{2^{\nu_2}m}\right)\left(\frac{-1}{3^{\nu_3}}\right)+2^{\nu_2}\left(\frac{-3}{2^{\nu_2}}\right)\left(\frac{-1}{3^{\nu_3}m}\right)+2^{\nu_2}3^{\nu_3}\left(\frac{3}{m}\right)$$
$$+\mathbb{1}_{3|n}\left(\frac{3}{2}-\frac{1}{2}3^{\nu_3+1}\left(\frac{-3}{2^{\nu_2}m}\right)\left(\frac{-1}{3^{\nu_3-1}}\right)+3\cdot 2^{\nu_2}\left(\frac{-3}{2^{\nu_2}}\right)\left(\frac{-1}{3^{\nu_3-1}m}\right)-2^{\nu_2}3^{\nu_3+1}\left(\frac{3}{m}\right)\right)$$
$$+\mathbb{1}_{2|n}\left(1-3^{\nu_3}\left(\frac{-3}{2^{\nu_2-1}m}\right)\left(\frac{-1}{3^{\nu_3}}\right)+2^{\nu_2+1}\left(\frac{-3}{2^{\nu_2-1}}\right)\left(\frac{-1}{3^{\nu_3}m}\right)-2^{\nu_2+1}3^{\nu_3}\left(\frac{3}{m}\right)\right)$$
$$+\mathbb{1}_{6|n}\left(-3-3^{\nu_3+1}\left(\frac{-3}{2^{\nu_2-1}m}\right)\left(\frac{-1}{3^{\nu_3-1}}\right)+2^{\nu_2+1}3\left(\frac{-3}{2^{\nu_2-1}}\right)\left(\frac{-1}{3^{\nu_3-1}m}\right)+2^{\nu_2+1}3^{\nu_3+1}\left(\frac{3}{m}\right)\right)\Bigg)$$
$$\times\prod_{p|m}\frac{1-\left(\left(\frac{3}{p}\right)p\right)^{\mathrm{ord}_p(m)+1}}{1-\left(\frac{3}{p}\right)p}.$$

We directly obtain the following lemma.

Lemma 6.9. *We have*

$$\begin{aligned}\mathcal{E}_2=&-\frac{1}{4}E_{2,\chi_1,\chi_{12}}-\frac{1}{4}E_{2,\chi_{-3},\chi_{-4}}+\frac{1}{2}E_{2,\chi_{-4},\chi_{-3}}+\frac{1}{2}E_{2,\chi_{12},\chi_1}\\&+\frac{3}{4}E_{2,\chi_1,\chi_{12}}|V_3-\frac{9}{4}E_{2,\chi_{-3},\chi_{-4}}|V_3+\frac{3}{2}E_{2,\chi_{-4},\chi_{-3}}|V_3-\frac{9}{2}E_{2,\chi_{12},\chi_1}|V_3\\&+\frac{1}{2}E_{2,\chi_1,\chi_{12}}|V_2-\frac{1}{2}E_{2,\chi_{-3},\chi_{-4}}|V_2+2E_{2,\chi_{-4},\chi_{-3}}|V_2-2E_{2,\chi_{12},\chi_1}|V_2\\&-\frac{3}{2}E_{2,\chi_1,\chi_{12}}|V_6-\frac{9}{2}E_{2,\chi_{-3},\chi_{-4}}|V_6+6E_{2,\chi_{-4},\chi_{-3}}|V_6+18E_{2,\chi_{12},\chi_1}|V_6.\end{aligned}$$

Using Lemmas 2.9, 6.9, 2.7, and 2.8, we obtain an identity for f_2.

Lemma 6.10. *We have*

$$f_2=\mathcal{E}_2-4\left(1-\sqrt{2}i\right)g_1-4\left(1+\sqrt{2}i\right)g_2.$$

We next classify those n for which $a_2(n):=C_{1^7 2^{-2}3^{-1}}(n)=0$.

Theorem 6.11. *We have $a_2(n)=0$ if and only if $n\equiv 2\pmod 3$ and there exists $p\equiv 3\pmod 4$ for which $\mathrm{ord}_p(n)$ is odd.*

Proof. As above, we write $n=2^{\nu_2}3^{\nu_3}m$ with $\gcd(m,6)=1$. For $\nu_3=0$, Lemma 6.8 implies that the $2^{\nu_2}m$-th Fourier coefficient of $\mathcal{E}_2$ is $\prod_{p|m}\frac{1-((\frac{3}{p})p)^{\mathrm{ord}_p(m)+1}}{1-(\frac{3}{p})p}$ times (note $(\frac{-3}{2})=-1$)

$$-\frac{1}{2}\left(1+\left(\frac{-3}{n}\right)\right)+2^{\nu_2}\left(\frac{-1}{m}\right)\left(\frac{-3}{2^{\nu_2}}\right)\left(1+\left(\frac{-3}{n}\right)\right)$$
$$+\mathbb{1}_{2|n}\left(1+\left(\frac{-3}{n}\right)+2^{\nu_2+1}\left(\frac{-3}{2^{\nu_2-1}}\right)\left(\frac{-1}{m}\right)\left(1+\left(\frac{-3}{n}\right)\right)\right).\qquad(6.8)$$

In particular, if $n\equiv 2\pmod 3$, then $(\frac{-3}{n})=-1$ and we see that the Fourier coefficient of $\mathcal{E}_2$ vanishes. Thus, for $n\equiv 2\pmod 3$, Lemma 6.10 implies that

$$a_2(n)=-4\left(c_1(n)+c_2(n)\right)+4\sqrt{2}i\left(c_1(n)-c_2(n)\right).$$

Using Lemma 3.1, one easily obtains that for $n\equiv 2\pmod 3$

$$a_2(n)=8\sqrt{2}ic_1(n).$$

Therefore $a_2(n) = 0$ if and only if $c_1(n) = 0$ for $n \equiv 2 \pmod 3$. By Proposition 3.4, since $n \equiv 2 \pmod 3$ (and hence $3 \nmid n$ in particular) this occurs if and only if there exists a prime $p \equiv 3 \pmod 4$ for which $\mathrm{ord}_p(n)$ is odd. This gives the claim for $n \equiv 2 \pmod 3$.

For $n \equiv 1 \pmod 3$, we have $(\frac{-3}{n}) = 1$, so, after simplifying (6.8), Lemma 6.8 implies that the $2^{\nu_2}m$-th Fourier coefficient of $\mathcal{E}_2$ is

$$\prod_{p|m} \frac{1-\left(\left(\frac{3}{p}\right)p\right)^{\mathrm{ord}_p(m)+1}}{1-\left(\frac{3}{p}\right)p}\left(-1+2^{\nu_2+1}\left(\frac{-1}{m}\right)\left(\frac{-3}{2^{\nu_2}}\right)+2\mathbb{1}_{2|n}+2^{\nu_2+2}\mathbb{1}_{2|n}\left(\frac{-3}{2^{\nu_2-1}}\right)\left(\frac{-1}{m}\right)\right).$$

We then note that

$$\left|-1+2^{\nu_2+1}\left(\frac{-1}{m}\right)\left(\frac{-3}{2^{\nu_2}}\right)+\mathbb{1}_{2|n}\left(2+2^{\nu_2+2}\left(\frac{-3}{2^{\nu_2-1}}\right)\left(\frac{-1}{m}\right)\right)\right|$$
$$= \begin{cases} \left|2\left(\frac{-1}{m}\right)-1\right| \geq 1 & \text{if } 2\nmid n,\\ \left|\left(\frac{-1}{m}\right)\left(\frac{-3}{2^{\nu_2}}\right)\left(2^{\nu_2+1}-2^{\nu_2+2}\right)+1\right| \geq 2^{\nu_2+1}-1 & \text{if } 2\mid n.\end{cases}$$

Combining with (6.2), the absolute value of the $2^{\nu_2}m$-th Fourier coefficient of $\mathcal{E}_2$ is bounded from below by

$$\left(2^{\nu_2+1}-1\right)\prod_{p|m} F_p(\mathrm{ord}_p(m)). \tag{6.9}$$

Plugging Lemma 3.1 in to evaluate the cuspidal part of Lemma 6.10, we conclude for $n \equiv 1 \pmod 3$ that $a_2(n) \neq 0$ if the expression in (6.9) is greater than $8|c_1(n)|$. Using Theorem 2.6, the absolute value of the Fourier coefficient of the cuspidal part is bounded from above by $8d(n)\sqrt{n}$. We conclude that $a_2(n) \neq 0$ if

$$\left(2^{\nu_2+1}-1\right)\prod_{p|m} F_p(\mathrm{ord}_p(m)) > 8d(n)\sqrt{n}.$$

Defining

$$G_2(n) := \frac{2^{\mathrm{ord}_2(n)+1}-1}{(\mathrm{ord}_2(n)+1)2^{\frac{\mathrm{ord}_2(n)}{2}}}\prod_{\substack{p|n\\ p\neq 2}} \frac{F_p(\mathrm{ord}_p(n))}{(\mathrm{ord}_p(n)+1)p^{\frac{\mathrm{ord}_p(n)}{2}}}$$

and rearranging, we conclude that if $G_2(n) > 8$ then $a_2(n) \neq 0$. By construction, G_2 is multiplicative, and $G_2(n) = G_1(n)$ for odd $n \in \mathbb{N}$ (see (6.3)), so by (6.5) we can obtain a bound on $\mathrm{ord}_p(n)$ for p odd after evaluating $f_{a,\nu}(x)$. As in (6.6), we have

$$f_{10,1}(402) \geq 0, \quad f_{10,2}(31) \geq 0, \quad f_{10,3}(13) \geq 0, \quad f_{10,4}(8) \geq 0,$$
$$f_{10,5}(6) \geq 0, \quad f_{10,6}(5) \geq 0.$$

Thus (6.5) yields

$$\begin{aligned} &G_2\left(p^\nu\right) > 10 \text{ for } p \geq 402,\ \nu \in \mathbb{N}, && G_2\left(p^\nu\right) > 10 \text{ for } 31 \leq p < 402,\ \nu \geq 2,\\ &G_2\left(p^\nu\right) > 10 \text{ for } 13 \leq p < 31,\ \nu \geq 3, && G_2\left(p^\nu\right) > 10 \text{ for } p = 11,\ \nu \geq 4,\\ &G_2\left(p^\nu\right) > 10 \text{ for } p = 7, \nu \geq 5, && G_2\left(p^\nu\right) > 10 \text{ for } p = 6,\ \nu \geq 6. \end{aligned} \tag{6.10}$$

Moreover, if $\#\{p \text{ prime} : p\|n\} \geq 7$, then bounding against the worst-case choice of 7 primes gives $G_2(n) \geq 8$. Hence we conclude from (6.10) and a direct computation of $G_2(2^\nu)$ that

$$n = 2^{\nu_2}\prod_{3<p\leq 401} p^{\nu_p} \tag{6.11}$$

with $0 \leq \nu_2 \leq 12$, $0 \leq \nu_5 \leq 5$, $0 \leq \nu_7 \leq 4$, $0 \leq \nu_{11} \leq 3$, $0 \leq \nu_{13} \leq 2$, $0 \leq \nu_p \leq 1$ for $p \geq 17$, and $\#\{p : \nu_p = 1\} \leq 6$. We used a computer to evaluate $G_2(n)$ for every such $n \equiv 1 \pmod 3$ of the type (6.11), and find that $G_2(n) > 8$ for $n > 309400$ with $n \equiv 1 \pmod 3$. It was verified with a computer (running code that completed in a few hours on a standard desktop computer) that $C_{1^7 2^{-2} 3^{-1}}(n) \neq 0$ for $n \leq 309400$ with $n \equiv 1 \pmod 3$, yielding the claim for $n \equiv 1 \pmod 3$.

For $3 \mid n$, Lemma 3.1 implies that the Fourier coefficient of the cusp form appearing on the right-hand side of Lemma 6.10 vanishes, so by Lemma 6.10, $a_2(n) = 0$ if and only if the n-th Fourier coefficient of $\mathcal{E}_2$ vanishes. By Lemma 6.8, this is the case if and only if

$$\begin{aligned}
&-\frac{1}{2} - \frac{3^{\nu_3}}{2}\left(\frac{-3}{2^{\nu_2}m}\right)\left(\frac{-1}{3^{\nu_3}}\right) + 2^{\nu_2}\left(\frac{-3}{2^{\nu_2}}\right)\left(\frac{-1}{3^{\nu_3}m}\right) + 2^{\nu_2}3^{\nu_3}\left(\frac{3}{m}\right)\\
&\quad + \left(\frac{3}{2} - \frac{1}{2}3^{\nu_3+1}\left(\frac{-3}{2^{\nu_2}m}\right)\left(\frac{-1}{3^{\nu_3-1}}\right) + 3\cdot 2^{\nu_2}\left(\frac{-3}{2^{\nu_2}}\right)\left(\frac{-1}{3^{\nu_3-1}m}\right) - 2^{\nu_2}3^{\nu_3+1}\left(\frac{3}{m}\right)\right)\\
&\quad + \mathbb{1}_{2\mid n}\left(1 - 3^{\nu_3}\left(\frac{-3}{2^{\nu_2-1}m}\right)\left(\frac{-1}{3^{\nu_3}}\right) + 2^{\nu_2+1}\left(\frac{-3}{2^{\nu_2-1}}\right)\left(\frac{-1}{3^{\nu_3}m}\right) - 2^{\nu_2+1}3^{\nu_3}\left(\frac{3}{m}\right)\right.\\
&\quad \left. -3 - 3^{\nu_3+1}\left(\frac{-3}{2^{\nu_2-1}m}\right)\left(\frac{-1}{3^{\nu_3-1}}\right) + 2^{\nu_2+1}3\left(\frac{-3}{2^{\nu_2-1}}\right)\left(\frac{-1}{3^{\nu_3-1}m}\right) + 2^{\nu_2+1}3^{\nu_3+1}\left(\frac{3}{m}\right)\right)
\end{aligned} \tag{6.12}$$

vanishes. If $2 \nmid n$, then $\nu_2 = 0$ and (6.12) simplifies as

$$\left(1 - 2\left(\frac{-1}{n}\right)\right) + 3^{\nu_3}\left(\frac{3}{m}\right)\left(\left(\frac{-1}{n}\right) - 2\right).$$

This vanishes if and only if

$$3^{\nu_3}\left(\frac{3}{m}\right)\left(\left(\frac{-1}{n}\right) - 2\right) = -1 + 2\left(\frac{-1}{n}\right). \tag{6.13}$$

Since 3 divides the left-hand side of (6.13) (we have $\nu_3 \geq 1$ because $3 \mid n$), it must divide the right-hand side, which can only occur if $(\frac{-1}{n}) = -1$, in which case the right-hand side equals -3. But then (6.13) simplifies to $3^{\nu_3}(\frac{3}{m})(-3) = -3$, which can only occur if $\nu_3 = 0$, leading to a contradiction. Hence for $3 \mid n$ and $2 \nmid n$ we conclude that $a_2(n) \neq 0$.

Finally suppose that $2 \mid n$. Combining terms with the same Legendre symbols, (6.12) becomes (note that $(\frac{-3}{2}) = (\frac{-1}{3}) = -1$)

$$-1 + 2^{\nu_2+1}3^{\nu_3}\left(-\frac{1}{2^{\nu_2+1}}\left(\frac{-3}{2^{\nu_2}m}\right)\left(\frac{-1}{3^{\nu_3}}\right) + \frac{1}{3^{\nu_3}}\left(\frac{-3}{2^{\nu_2}}\right)\left(\frac{-1}{3^{\nu_3}m}\right) + \left(\frac{3}{m}\right)\right). \tag{6.14}$$

Since $\nu_2, \nu_3 \in \mathbb{N}$, the absolute value of (6.14) is bounded from below by

$$2^{\nu_2+1}3^{\nu_3}\left(1 - \frac{1}{4} - \frac{1}{3}\right) - 1 = \frac{5}{12}2^{\nu_2+1}3^{\nu_3} - 1 \geq \frac{5}{12}\cdot 4\cdot 3 - 1 = 4.$$

We conclude that (6.12) does not vanish, and hence $C_{1^7 2^{-2} 3^{-1}}(n) \neq 0$, for all n with $3 \mid n$. □

7. Proof of Theorem 1.4

7.1. **The case $1^2 2^3 4^{-2}$.** In this subsection, we prove the claimed evaluation of the first set appearing in Theorem 1.4.

Proposition 7.1. *We have*

$$S_{1^2 2^3 4^{-2}} = \left\{n \in \mathbb{N} : n = 4^k(8m+7) \text{ for some } k, m \in \mathbb{N}_0\right\}.$$

We first obtain a formula for

$$f_3(z) := \frac{\eta^2(z)\eta^3(2z)}{\eta^2(4z)}.$$

We have the following lemma.

Lemma 7.2. *We have*

$$f_3 = \mathcal{H}_{1,2}\big|U_2\big|\left(12S_{4,0} - 4S_{4,1} - 4S_{4,2} + 12S_{4,3}\right).$$

Proof. By [13, Proposition 1.41], [13, Theorem 1.60], Lemmas 2.3, and 2.9, we have

$$f_3(z) = \frac{\eta^4(z)}{\eta^2(2z)}\Theta(z) \in M_{\frac{3}{2}}(\Gamma_0(8)).$$

For the right-hand side, observe that by Lemma 2.13, $\mathcal{H}_{1,2} \in M_{\frac{3}{2}}(\Gamma_0(8), \chi_8)$. Applying U_2 gives, by Lemma 2.1 (1), an element of $M_{\frac{3}{2}}(\Gamma_0(8))$. Finally, by Lemma 2.1 (2), $S_{4,a}$ ($a \in \{1,2,3\}$) gives an element of $M_{\frac{3}{2}}(\Gamma_0(16))$. Thus both sides are modular forms of weight $\frac{3}{2}$ on $\Gamma_0(16)$. So we have to check 3 Fourier coefficients, which were checked with a computer. $\square$

We are now ready to prove Proposition 7.1.

Proof of Proposition 7.1. By Lemma 7.2, we have $C_{1^2 2^3 4^{-2}}(n) = 0$ if and only if the n-th Fourier coefficient of $\mathcal{H}_{1,2}|U_2$ vanishes. However, by [1, Lemma 4.1], we have $\Theta^3 = 12\mathcal{H}_{1,2}|U_2$, so $C_{1^2 2^3 4^{-2}}(n) = 0$ if and only if $r_{(1,1,1)}(n) = 0$. By Legendre's three-square theorem [11]

$$r_{(1,1,1)}(n) = 0 \Leftrightarrow n = 4^k(8m+7) \text{ for some } k, m \in \mathbb{N}_0. \tag{7.1}$$

This is the claim. $\square$

7.2. **The case $1^6 2^{-3}$.** In this subsection, we prove the claimed identity for the other set appearing in Theorem 1.4.

Proposition 7.3. *We have*

$$S_{1^6 2^{-3}} = \left\{n \in \mathbb{N} : n = 4^k(8m+7) \textit{ for some } k, m \in \mathbb{N}_0\right\}.$$

Proof. Using $\frac{\eta(z)^2}{\eta(2z)} = \sum_{n\in\mathbb{Z}}(-1)^n q^{n^2}$ (see [13, Theorem 1.60]), one directly obtains

$$\frac{\eta^6(z)}{\eta^3(2z)} = \left(\sum_{n\in\mathbb{Z}}(-1)^n q^{n^2}\right)^3 = \sum_{n_1,n_2,n_3\in\mathbb{Z}}(-1)^{n_1+n_2+n_3} q^{n_1^2+n_2^2+n_3^2} = \sum_{n\geq 0}(-1)^n r_{(1,1,1)}(n) q^n.$$

By (7.1), we have

$$r_{(1,1,1)}(n) = 0 \Leftrightarrow n = 4^k(8m+7) \text{ for some } k, m \in \mathbb{N}_0.$$ $\square$

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University of Cologne, Department of Mathematics and Computer Science, Weyertal 86-90, 50931 Cologne, Germany
Email address: kbringma@math.uni-koeln.de

I.R.M.A., UMR 7501, Université de Strasbourg et CNRS, 7 rue René Descartes, F-67084 Strasbourg, France
Email address: guoniu.han@unistra.fr

University of Cologne, Department of Mathematics and Computer Science, Weyertal 86-90, 50931 Cologne, Germany
Email address: bheim@uni-koeln.de

The University of Hong Kong, Department of Mathematics, Pokfulam, Hong Kong
Email address: bkane@hku.hk