THE $\frac{1}{2}$-CONJECTURE FOR q-BINOMIAL COEFFICIENTS WITH FRACTIONAL INDEX

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ABSTRACT. For a nonnegative integer k and a rational number $r \in \mathbb{Q}^+$, we define the generalized Gaussian binomial coefficient $\begin{bmatrix} r+k \\ k \end{bmatrix}_q = \frac{(q^{r+1};q)_k}{(q;q)_k}$. When $r = a/b$ with a, b coprime positive integers and $b \geq 2$, expanding $\begin{bmatrix} r+k \\ k \end{bmatrix}_q$ via the finite q-binomial theorem produces fractional powers of q, so that $\begin{bmatrix} r+k \\ k \end{bmatrix}_q$ is a *Puiseux series* in q with nonnegative exponents; concretely it lies in $\mathbb{Q}[[q^{1/b}]]$. The notion we single out is the *integer trace* of this expansion, the subseries consisting of those terms $c_r(d)\, q^d$ whose exponent d is an integer, with all fractional powers discarded. This projection is not standard, and there is no a priori reason for the surviving coefficients to behave coherently as r varies. Nonetheless, ordering the family by the coefficientwise partial order leads to the $\frac{1}{2}$-*Conjecture*: among all $r \in \mathbb{Q}^+$, the value $r = \frac{1}{2}$ maximizes the integer trace, in the sense that the coefficients of $\begin{bmatrix} 1/2+k \\ k \end{bmatrix}_q$ dominate those of $\begin{bmatrix} r+k \\ k \end{bmatrix}_q$ coefficientwise for every r. That so elementary a definition should single out $\frac{1}{2}$ this cleanly came as a surprise to us. We prove the conjecture in several special cases and provide further computational evidence.

1. INTRODUCTION

The *Gaussian binomial coefficients*, also called *q-binomial coefficients*, are central objects in combinatorics, number theory, and the theory of q-series (see, e.g., [1, 2, 3]). For integers $n \geq k \geq 0$ they are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;\, q)_n}{(q;\, q)_k\, (q;\, q)_{n-k}}, \tag{1}$$

where

$$(u;q)_n := \begin{cases} 1, & \text{if } n = 0, \\ (1-u)(1-uq)\cdots(1-uq^{n-1}), & \text{if } n \geq 1, \end{cases}$$

is the standard *q-shifted factorial* (or *q-Pochhammer symbol*) [2, p. 3]. It is classical that $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is a polynomial in q with nonnegative integer coefficients, and that it counts the number of k-dimensional subspaces of an n-dimensional vector space over $\mathbb{F}_q$ [3, Proposition 1.7.2].

Date: August 7, 2026.

2020 *Mathematics Subject Classification.* 05A20, 05A30.

Key words and phrases. Gaussian binomial coefficients, q-binomial coefficients, integer traces, Puiseux series.

By cancellation in (1), we may write

$$\begin{bmatrix} r+k \\ k \end{bmatrix}_q = \frac{(1-q^{r+1})(1-q^{r+2})\cdots(1-q^{r+k})}{(1-q)(1-q^2)\cdots(1-q^k)} \tag{2}$$

for nonnegative integers r and k. The right-hand side of (2) is meaningful for *any* $r \in \mathbb{R}$, and in particular for rational r. For example, setting $r = \frac{1}{2}$ gives

$$\begin{bmatrix} 1/2+k \\ k \end{bmatrix}_q = \frac{(1-q^{3/2})(1-q^{5/2})\cdots(1-q^{1/2+k})}{(1-q)(1-q^2)\cdots(1-q^k)},$$

which is a well-defined power series in $q^{1/2}$.

In this paper, we study the extension of (1) to rational numbers. Following the notation of (2), we study the family

$$\begin{bmatrix} r+k \\ k \end{bmatrix}_q = \frac{(q^{r+1};\, q)_k}{(q;\, q)_k}, \qquad r \in \mathbb{Q}^+, \tag{3}$$

which coincides with the classical Gaussian binomial coefficient when r is a positive integer. For $r = a/b$ with a, b coprime positive integers and $b \geq 2$, expanding $(q^{r+1};\, q)_k$ via the finite q-binomial theorem [1, Theorem 3.3] produces fractional powers of q, so $\begin{bmatrix} r+k \\ k \end{bmatrix}_q$ is a *Puiseux series* in q with nonnegative exponents; concretely it lies in $\mathbb{Q}[[q^{1/b}]]$. Recall that the *ring of Puiseux series* (with nonnegative exponents) over $\mathbb{Q}$ is

$$\mathcal{P} := \bigcup_{b \geq 1} \mathbb{Q}[[q^{1/b}]], \tag{4}$$

the set of formal series in q whose exponents are nonnegative rationals admitting a common denominator. As $r = a/b$ ranges over $\mathbb{Q}^+$ the denominator b is unbounded, so no single $\mathbb{Q}[[q^{1/b}]]$ contains the entire family $\{\begin{bmatrix} r+k \\ k \end{bmatrix}_q\}_{r \in \mathbb{Q}^+}$; the ring $\mathcal{P}$ provides a common ambient space for all of them.

Once $\begin{bmatrix} r+k \\ k \end{bmatrix}_q$ is viewed inside $\mathcal{P}$, a natural question is what to do with its fractional powers. For

$$A(q) = \sum_{\alpha \in \mathbb{Q}_{\geq 0}} a_\alpha q^\alpha \in \mathcal{P},$$

where the exponents occurring in A have a common denominator, define the *integer-trace projection* by

$$\mathrm{Tr}_{\mathbb{Z}}\, A(q) := \sum_{d \in \mathbb{Z}_{\geq 0}} a_d q^d \in \mathbb{Q}[[q]]. \tag{5}$$

Thus $\mathrm{Tr}_{\mathbb{Z}}$ discards every nonintegral power. We also write $A(q)\big|_{\mathbb{Z}} := \mathrm{Tr}_{\mathbb{Z}}\, A(q)$.

We use the usual coefficientwise partial order on all of $\mathbb{Q}[[q]]$, not only on series whose coefficients are nonnegative. For $F, G \in \mathcal{P}$, write

$$F \geq_q G \quad \Longleftrightarrow \quad \mathrm{Tr}_{\mathbb{Z}}(F-G) \in \mathbb{Q}_{\geq 0}[[q]]. \tag{6}$$

Equivalently, every coefficient of $\mathrm{Tr}_{\mathbb{Z}}\, F$ is at least the corresponding coefficient of $\mathrm{Tr}_{\mathbb{Z}}\, G$. This is a preorder on $\mathcal{P}$ and descends to the usual coefficientwise partial order on the integer traces in $\mathbb{Q}[[q]]$. We write $F =_q G$ when $\mathrm{Tr}_{\mathbb{Z}}\, F = \mathrm{Tr}_{\mathbb{Z}}\, G$. We

are not aware of this projection having been studied before. Since the integer-exponent terms arise from cancellations among fractional powers, one might expect the surviving coefficients to carry little structure as r varies.

The projection is linear but is not multiplicative in general. We shall use the following elementary compatibility whenever one factor has only integral exponents.

Lemma 1.1 (Integral-factor rule)**.** *For $A(q) \in \mathcal{P}$ and $B(q) \in \mathbb{Q}[[q]]$,*

$$\mathrm{Tr}_{\mathbb{Z}}\big(A(q)B(q)\big) = \big(\mathrm{Tr}_{\mathbb{Z}}\, A(q)\big)B(q). \tag{7}$$

Proof. Multiplication by a monomial q^j with $j \in \mathbb{Z}_{\geq 0}$ preserves the fractional part of every exponent of $A(q)$. Consequently, no nonintegral exponent of $A(q)$ can become integral after multiplication by a term of $B(q)$. Taking the Cauchy product therefore gives (7) coefficientwise. □

Here is a small example showing both the support selection and the possible signs.

Example 1.2. Let $D_4(q) = (1-q)(1-q^2)(1-q^3)(1-q^4)$. Direct expansion gives

$$\begin{aligned}
\mathrm{Tr}_{\mathbb{Z}} \begin{bmatrix} 1/2+4 \\ 4 \end{bmatrix}_q &= \frac{1+q^4+q^5+2q^6+q^7+q^8+q^{12}}{D_4(q)}, \\
\mathrm{Tr}_{\mathbb{Z}} \begin{bmatrix} 1/3+4 \\ 4 \end{bmatrix}_q &= \frac{1-q^7-q^8-q^9-q^{10}}{D_4(q)}, \\
\mathrm{Tr}_{\mathbb{Z}} \begin{bmatrix} 1/4+4 \\ 4 \end{bmatrix}_q &= \frac{1+q^{11}}{D_4(q)}.
\end{aligned}$$

For denominator 2, the contributing indices are $0, 2, 4$ and all signs are positive. For denominator 3, the index 3 contributes with a negative sign; for denominator 4, only 0 and 4 contribute, both positively but with sparser support. This is the basic pattern behind the conjecture.

Extensive computation shows otherwise. Among all positive rational parameters r, a single value, $r = \frac{1}{2}$, dominates every other in the coefficientwise partial order on integer traces. We call this statement the $\frac{1}{2}$-*Conjecture* (Conjecture 2.2). It is striking that so elementary an operation as deleting fractional powers should pick out $\frac{1}{2}$ in this way.

The paper is organized as follows. Section 2 introduces the $\frac{1}{2}$-Conjecture (Conjecture 2.2) and develops the key structural observation that all integer-trace contributions from $\begin{bmatrix} 1/2+k \\ k \end{bmatrix}_q$ are nonnegative. Section 3 collects the basic properties of the integer trace order $\geq_q$ used throughout the proofs. Section 4 proves the conjecture for small values $k = 1, 2, 3$ by direct computation. Section 5 establishes the conjecture for the two infinite families $r = m + \frac{1}{2}$ (half-integers) and $r = m$ (positive integers). Finally, Section 6 presents computer-verified evidence for $r = 1/4$ and $r = 1/3$, together with stronger auxiliary conjectures on the sign of the numerator polynomials $H_k(q)$ and $U_k(q)$.

2. The $\frac{1}{2}$-conjecture

Let k be a nonnegative integer. We begin with the finite q-binomial theorem (see, e.g., [1, Theorem 3.3]):

$$(u;\,q)_k \;=\; \sum_{s=0}^{k}(-1)^s \begin{bmatrix} k \\ s \end{bmatrix}_q q^{\binom{s}{2}} u^s. \tag{8}$$

Applying (8) with $u = q^{r+1}$ gives

$$(q^{r+1};\,q)_k \;=\; \sum_{s=0}^{k}(-1)^s \begin{bmatrix} k \\ s \end{bmatrix}_q q^{E(s,r)}, \tag{9}$$

where

$$E(s,r) \;=\; \binom{s}{2} + (r+1)s \;=\; \frac{s(s+1)}{2} + rs. \tag{10}$$

Definition 2.1. For $r \in \mathbb{Q}^+$, the *integer support* is

$$S_r \;:=\; \big\{s \in \{0,1,\ldots,k\} \;\big|\; E(s,r) \in \mathbb{Z}\big\}.$$

The integer trace of $\begin{bmatrix} r+k \\ k \end{bmatrix}_q$ is determined entirely by the terms indexed by S_r, since only those contribute integer powers of q to the numerator. More precisely, the inverse denominator $(q;q)_k^{-1}$ belongs to $\mathbb{Q}[[q]]$, so the integral-factor rule gives

$$\mathrm{Tr}_{\mathbb{Z}} \begin{bmatrix} r+k \\ k \end{bmatrix}_q = \frac{\mathrm{Tr}_{\mathbb{Z}}\big((q^{r+1};q)_k\big)}{(q;q)_k}. \tag{11}$$

For example, when $r = 1/2$, the exponent formula (10) gives

$$E\big(s,\tfrac{1}{2}\big) \;=\; \frac{s(s+1)}{2} + \frac{s}{2} \;=\; \frac{s^2+2s}{2} \;=\; \frac{s(s+2)}{2}. \tag{12}$$

This is an integer if and only if $s(s+2) \equiv 0 \pmod 2$, which holds precisely when s is even. Hence

$$S_{1/2} \;=\; \{0,2,4,\ldots,2\lfloor k/2\rfloor\}. \tag{13}$$

Crucially, $(-1)^s = +1$ for every $s \in S_{1/2}$, so *every contribution to the integer trace of* $\begin{bmatrix} 1/2+k \\ k \end{bmatrix}_q$ *is nonnegative.*

For the general case $r = a/b$ with a, b coprime positive integers and $b \geq 2$, since $s(s+1)/2$ is always an integer, the condition $E(s, a/b) \in \mathbb{Z}$ reduces to $as/b \in \mathbb{Z}$, and by $\gcd(a,b) = 1$ this is equivalent to $b \mid s$. Hence

$$S_{a/b} \;=\; \big\{0,\, b,\, 2b,\, \ldots,\, b\lfloor k/b\rfloor\big\}. \tag{14}$$

The integrality condition $b \mid s$ is periodic in s with minimal period b.

Two regimes arise according to the parity of b:

- If b is odd (so $b \geq 3$), then $S_{a/b}$ contains odd values of s (e.g. $s = b$), so the signs $(-1)^s$ in (9) alternate and the integer-trace numerator has both positive and negative contributions.
- If b is even, then every $s \in S_{a/b}$ is a multiple of b and hence even, so $(-1)^s = +1$ and the integer-trace numerator has nonnegative coefficients, just as for $r = \frac{1}{2}$.

Among all $b \geq 2$, the choice $b = 2$ minimizes b and so yields the densest integer support: $S_{1/2} = \{0, 2, 4, \ldots\} \cap [0, k]$ has $\lfloor k/2 \rfloor + 1$ elements, the maximum among all $S_{a/b}$. This combination of nonnegative contributions and maximal support is what motivates Conjecture 2.2.

Write the integer trace expansion as

$$\begin{bmatrix} r+k \\ k \end{bmatrix}_q \Big|_{\mathbb{Z}} = \sum_{d \geq 0} c_r(d)\, q^d. \tag{15}$$

The denominator $(q;\, q)_k^{-1}$ has the well-known expansion [1, Theorem 1.1]:

$$\frac{1}{(q;\, q)_k} = \sum_{n \geq 0} p_k(n)\, q^n, \tag{16}$$

where $p_k(n) \geq 0$ counts the number of partitions of n into at most k parts (equivalently, into parts each at most k). See also [3, Eq. (1.76)].

Combining (9) and (16), and applying Lemma 1.1 as in (11), yields the convolution formula

$$c_r(d) = \sum_{s \in S_r} (-1)^s \sum_{n \geq 0} \beta_{k,s}(n)\, p_k\big(d - n - E(s, r)\big), \tag{17}$$

where $\beta_{k,s}(n)$ denotes the coefficient of q^n in $\begin{bmatrix} k \\ s \end{bmatrix}_q$. Since $\begin{bmatrix} k \\ s \end{bmatrix}_q$ is a polynomial in q with nonnegative integer coefficients, we have $\beta_{k,s}(n) \geq 0$ for all n, and likewise $p_k(n) \geq 0$. The only possible source of negativity in (17) is therefore the sign $(-1)^s$.

This analysis leads to our main conjecture.

Conjecture 2.2 (The $\frac{1}{2}$-Conjecture)**.** *For every $r \in \mathbb{Q}^+$ and every integer $k \geq 1$,*

$$\begin{bmatrix} 1/2+k \\ k \end{bmatrix}_q \geq_q \begin{bmatrix} r+k \\ k \end{bmatrix}_q. \tag{18}$$

The conjecture asserts that $r = \frac{1}{2}$ is a *global maximizer* of the integer traces in the coefficientwise partial order on $\mathbb{Q}[[q]]$.

The following special case deserves individual attention.

Conjecture 2.3 ($\frac{1}{2}$ vs. $\frac{1}{4}$)**.** *For all $k \geq 1$,*

$$\begin{bmatrix} 1/2+k \\ k \end{bmatrix}_q \geq_q \begin{bmatrix} 1/4+k \\ k \end{bmatrix}_q.$$

Conjecture 2.3 is a special case of Conjecture 2.2 with $r = 1/4$, but is singled out because of the proximity of $1/4$ to $1/2$ and because the computation of both sides at the level of numerator polynomials already exhibits rich combinatorial structure (see Section 6).

3. Properties of the integer trace order

The following properties of the integer trace order $\geq_q$ on $\mathcal{P}$ are elementary but important. We record them for use in the proofs of Sections 5 and 6.

Lemma 3.1. *For series in $\mathcal{P}$, the integer trace order satisfies the following properties.*

(1) *$F(q) \geq_q G(q)$ if and only if $F(q) - G(q) \geq_q 0$.*

(2) *If* $F(q) \geq_q G(q)$ *and* $c \in \mathbb{Q}^+$, *then* $cF(q) \geq_q cG(q)$.
(3) *If* $F_1(q) \geq_q G_1(q)$ *and* $F_2(q) \geq_q G_2(q)$, *then* $F_1(q)+F_2(q) \geq_q G_1(q)+G_2(q)$.
(4) *If* $F_1(q) \geq_q F_2(q)$ *and* $F_2(q) \geq_q F_3(q)$, *then* $F_1(q) \geq_q F_3(q)$.
(5) *If* $F(q) \geq_q G(q)$ *and* $k_1, k_2, \ldots, k_\ell$ *are positive integers, then*

$$\frac{F(q)}{(1-q^{k_1})(1-q^{k_2})\cdots(1-q^{k_\ell})} \geq_q \frac{G(q)}{(1-q^{k_1})(1-q^{k_2})\cdots(1-q^{k_\ell})}.$$

(6) *If* $n, m \in \mathbb{N}$ *with* $n \leq m$, *then* $\dfrac{q^n - q^m}{1-q} \geq_q 0$.

Proof. Properties (1)–(4) follow directly from the coefficientwise order on the traces in $\mathbb{Q}[[q]]$. Notice in particular that property (3) compares the sum with $G_1 + G_2$.

(5) Put

$$B(q) = \prod_{i=1}^{\ell} \frac{1}{1-q^{k_i}} \in \mathbb{Q}_{\geq 0}[[q]].$$

By Lemma 1.1,

$$\mathrm{Tr}_{\mathbb{Z}}\big((F-G)B\big) = \mathrm{Tr}_{\mathbb{Z}}(F-G)B.$$

If $F \geq_q G$, then both factors on the right have nonnegative coefficients, and so does their Cauchy product. Property (1) now gives the claim.

(6) We compute

$$\frac{q^n - q^m}{1-q} \;=\; q^n + q^{n+1} + \cdots + q^{m-1} \geq_q 0.$$

□

4. Partial results: small k

In this section, we prove the $\frac{1}{2}$-conjecture when k is small.

Proposition 4.1. *The* $\frac{1}{2}$*-conjecture is true for* $k = 1, 2, 3$.

Proof. For $k = 1$ and any $r \in \mathbb{Q}^+$, we compare $\begin{bmatrix} 1/2+k \\ k \end{bmatrix}_q$ and $\begin{bmatrix} r+k \\ k \end{bmatrix}_q$ directly:

$$\begin{bmatrix} 1/2+k \\ k \end{bmatrix}_q - \begin{bmatrix} r+k \\ k \end{bmatrix}_q \;=\; \frac{(1-q^{3/2})-(1-q^{r+1})}{1-q} \;=\; \frac{q^{r+1}-q^{3/2}}{1-q} \;=_q\; \frac{q^{r+1}}{1-q}.$$

Now

$$q^{r+1} \;=_q\; \begin{cases} 0, & \text{if } r \notin \mathbb{N}, \\ q^{r+1}, & \text{if } r \in \mathbb{N}, \end{cases}$$

so $q^{r+1} \geq_q 0$, and therefore $q^{r+1}/(1-q) \geq_q 0$ by Lemma 3.1(5).

For $k = 2$ and $r \in \mathbb{Q}^+$,

$$\begin{aligned} \begin{bmatrix} 1/2+k \\ k \end{bmatrix}_q - \begin{bmatrix} r+k \\ k \end{bmatrix}_q &= \frac{(1-q^{3/2})(1-q^{5/2})-(1-q^{r+1})(1-q^{r+2})}{(1-q)(1-q^2)} \\ &= \frac{(q^{r+1}+q^{r+2}+q^4)-(q^{3/2}+q^{5/2}+q^{2r+3})}{(1-q)(1-q^2)} \\ &=_q \frac{q^{r+1}+q^{r+2}+q^4-q^{2r+3}}{(1-q)(1-q^2)}. \end{aligned}$$

Since

$$q^{r+1}+q^{r+2}+q^4-q^{2r+3} \;=_q\; \begin{cases} q^4, & \text{if } r\notin \frac{1}{2}\mathbb{N}, \\ q^4-q^{2r+3}, & \text{if } r\in\frac{1}{2}\mathbb{N} \text{ and } r\notin\mathbb{N}, \\ (q^{r+1}-q^{2r+3})+q^{r+2}+q^4, & \text{if } r\in\mathbb{N}, \end{cases}$$

we obtain in each case

$$\frac{q^{r+1}+q^{r+2}+q^4-q^{2r+3}}{(1-q)(1-q^2)} \;\geq_q\; 0$$

by Lemma 3.1(6).

For $k=3$ and $r\in\mathbb{Q}^+$,

$$\begin{aligned} &\begin{bmatrix} 1/2+k \\ k \end{bmatrix}_q - \begin{bmatrix} r+k \\ k \end{bmatrix}_q \\ &= \frac{(1-q^{3/2})(1-q^{5/2})(1-q^{7/2})-(1-q^{r+1})(1-q^{r+2})(1-q^{r+3})}{(1-q)(1-q^2)(1-q^3)} \\ &= \frac{V}{(1-q)(1-q^2)(1-q^3)}, \end{aligned}$$

where

$$V \;=_q\; q^4+q^5+q^6+q^{r+1}+q^{r+2}+q^{r+3}-q^{2r+3}-q^{2r+4}-q^{2r+5}+q^{3r+6}.$$

If $r\notin\frac{1}{2}\mathbb{N}$ and $r\notin\frac{1}{3}\mathbb{N}$, then $V=_q q^4+q^5+q^6\geq_q 0$.

If $r\in\frac{1}{2}\mathbb{N}$ and $r\notin\mathbb{N}$, then $V=_q (q^4-q^{2r+3})+(q^5-q^{2r+4})+(q^6-q^{2r+5})$, so $V/(1-q)\geq_q 0$ by Lemma 3.1(6).

If $r\in\frac{1}{3}\mathbb{N}$ and $r\notin\mathbb{N}$, then $V=_q q^4+q^5+q^6+q^{3r+6}\geq_q 0$.

If $r\in\mathbb{N}$, then

$$V \;=_q\; q^4+q^5+q^6+(q^{r+1}-q^{2r+3})+(q^{r+2}-q^{2r+4})+(q^{r+3}-q^{2r+5})+q^{3r+6},$$

so $V/(1-q)\geq_q 0$ by Lemma 3.1(6). □

5. Partial results: half-integers and positive integers

Using Lemma 3.1, we prove Conjecture 2.2 in the cases where r is a half-integer $m+\frac{1}{2}$ $(m\geq 0)$ or a positive integer $m\geq 1$.

Proposition 5.1. *The $\frac{1}{2}$-conjecture holds for half-integer parameters: for every integer $m\geq 0$,*

$$\begin{bmatrix} 1/2+k \\ k \end{bmatrix}_q \geq_q \begin{bmatrix} m+1/2+k \\ k \end{bmatrix}_q. \tag{19}$$

Proof. We prove (19) by induction on m. The base case $m=0$ is trivial. For the inductive step it suffices to show

$$\begin{bmatrix} m+1/2+k \\ k \end{bmatrix}_q \geq_q \begin{bmatrix} m+3/2+k \\ k \end{bmatrix}_q. \tag{20}$$

Set $r = m + \frac{1}{2}$. We compute the difference $\begin{bmatrix} r+k \\ k \end{bmatrix}_q - \begin{bmatrix} r+1+k \\ k \end{bmatrix}_q$ directly:

$$\begin{aligned} &\begin{bmatrix} r+k \\ k \end{bmatrix}_q - \begin{bmatrix} r+1+k \\ k \end{bmatrix}_q \\ &= \frac{(1-q^{r+2})\cdots(1-q^{r+k})}{(1-q)(1-q^2)\cdots(1-q^k)}\left[(1-q^{r+1})-(1-q^{r+1+k})\right] \\ &= \frac{(1-q^{r+2})\cdots(1-q^{r+k})\cdot\left(q^{r+1+k}-q^{r+1}\right)}{(1-q)(1-q^2)\cdots(1-q^k)} \\ &= \frac{(-q^{r+1})\,(1-q^{r+2})\cdots(1-q^{r+k})}{(1-q)(1-q^2)\cdots(1-q^{k-1})}. \end{aligned} \tag{21}$$

We claim that the right-hand side of (21) is $\geq_q 0$.

Since $r = m + \frac{1}{2}$ is a half-integer, the exponents $r+1, r+2, \ldots, r+k$ are half-integers as well. Expand $(-q^{r+1})(1-q^{r+2})\cdots(1-q^{r+k})$ by choosing a subset $J \subseteq \{2,3,\ldots,k\}$ of the nonconstant terms. If $|J| = j$, the corresponding monomial has sign $(-1)^{j+1}$ and exponent

$$(r+1) + \sum_{i\in J}(r+i) = (j+1)r + I_J, \qquad I_J := 1 + \sum_{i\in J} i \in \mathbb{Z}.$$

Because $r = m + \frac{1}{2}$, this exponent is integral exactly when $j+1$ is even (equivalently, j is odd). Every monomial that survives the integer-trace projection therefore has sign $(-1)^{j+1} = +1$. After collecting equal powers, all coefficients in the integer trace remain nonnegative. Therefore

$$(-q^{r+1})(1-q^{r+2})\cdots(1-q^{r+k}) \;\geq_q\; 0,$$

and Lemma 3.1(5), applied with denominator $(1-q)(1-q^2)\cdots(1-q^{k-1})$, yields $\begin{bmatrix} r+k \\ k \end{bmatrix}_q - \begin{bmatrix} r+1+k \\ k \end{bmatrix}_q \geq_q 0$.

By transitivity and induction, $\begin{bmatrix} 1/2+k \\ k \end{bmatrix}_q \geq_q \begin{bmatrix} m+1/2+k \\ k \end{bmatrix}_q$ for all $m \geq 0$. □

Proposition 5.2. *The $\frac{1}{2}$-conjecture holds for positive integer parameters: for every positive integer m,*

$$\begin{bmatrix} 1/2+k \\ k \end{bmatrix}_q \;\geq_q\; \begin{bmatrix} m+k \\ k \end{bmatrix}_q. \tag{22}$$

Proof. We use the integer partition interpretations of the two sides.

Step 1: A lower bound for $\begin{bmatrix} 1/2+k \\ k \end{bmatrix}_q$. By definition,

$$\begin{bmatrix} 1/2+k \\ k \end{bmatrix}_q = \frac{(1-q^{3/2})(1-q^{5/2})\cdots(1-q^{1/2+k})}{(1-q)(1-q^2)\cdots(1-q^k)}.$$

To extract the integer trace, we expand the numerator using (9) with $r = \frac{1}{2}$:

$$(1-q^{3/2})\cdots(1-q^{1/2+k}) \;=\; \sum_{s=0}^{k}(-1)^s q^{s(s+2)/2}\begin{bmatrix} k \\ s \end{bmatrix}_q.$$

Only even values of s contribute integer powers, giving

$$(1-q^{3/2})\cdots(1-q^{1/2+k})\big|_{\mathbb{Z}} \;=\; \sum_{\ell=0}^{\lfloor k/2\rfloor} q^{2\ell(\ell+1)} \begin{bmatrix} k \\ 2\ell \end{bmatrix}_q \;=\; 1+\sum_{\ell=1}^{\lfloor k/2\rfloor} q^{2\ell(\ell+1)} \begin{bmatrix} k \\ 2\ell \end{bmatrix}_q \;\geq_q\; 1.$$

Therefore, by Lemma 3.1(5),

$$\operatorname{Tr}_{\mathbb{Z}} \begin{bmatrix} 1/2+k \\ k \end{bmatrix}_q \;=\; \frac{(1-q^{3/2})\cdots(1-q^{1/2+k})\big|_{\mathbb{Z}}}{(q;\,q)_k} \;\geq_q\; \frac{1}{(q;\,q)_k}. \tag{23}$$

Here the equality is precisely Lemma 1.1, applied to the inverse denominator $(q;q)_k^{-1} \in \mathbb{Q}[[q]]$.

Step 2: An upper bound for $\begin{bmatrix} m+k \\ k \end{bmatrix}_q$. By the classical combinatorial interpretation of the q-binomial coefficient [1, Theorem 1.1],

$$\frac{1}{(q;\,q)_k} \;=\; \sum_{\lambda \in P_k} q^{|\lambda|}, \tag{24}$$

where P_k is the set of all partitions whose parts are each at most k. On the other hand,

$$\begin{bmatrix} m+k \\ k \end{bmatrix}_q \;=\; \sum_{\lambda \in P_{m,k}} q^{|\lambda|}, \tag{25}$$

where $P_{m,k}$ is the set of partitions with at most m parts, each of size at most k.

Since $P_{m,k} \subseteq P_k$, we have

$$\frac{1}{(q;\,q)_k} \;\geq_q\; \begin{bmatrix} m+k \\ k \end{bmatrix}_q. \tag{26}$$

Combining (23) and (26) yields (22). □

6. Further examples

We verify the $\frac{1}{2}$-conjecture for $r = 1/3$ and $r = 1/4$ in the range $1 \leq k \leq 150$ with the help of a computer.

The following observation makes every computer check in this section a finite, directly inspectable certificate.

Lemma 6.1 (Prefix-sum certificate)**.** *Let $A(q) = \sum_{e=0}^{D} a_e q^e \in \mathbb{Q}[q]$. Then*

$$\frac{A(q)}{1-q} \geq_q 0 \quad\Longleftrightarrow\quad \sum_{e=0}^{d} a_e \geq 0 \quad \text{for every } 0 \leq d \leq D. \tag{27}$$

If these equivalent conditions hold, then for every $k \geq 1$,

$$\frac{A(q)}{(q;q)_k} \geq_q 0. \tag{28}$$

Proof. For every $d \geq 0$,

$$[q^d]\frac{A(q)}{1-q} = \sum_{e=0}^{\min(d,D)} a_e.$$

For $d \geq D$ this coefficient is the total sum $\sum_{e=0}^{D} a_e$, so it is enough to check the $D+1$ prefix sums in (27). Finally,

$$\frac{A(q)}{(q;q)_k} = \frac{A(q)}{1-q} \prod_{j=2}^{k} \frac{1}{1-q^j},$$

and every remaining factor has nonnegative coefficients. □

Proposition 6.2. *For every integer k with $1 \leq k \leq 150$,*

$$\begin{bmatrix} 1/2+k \\ k \end{bmatrix}_q \geq_q \begin{bmatrix} 1/4+k \\ k \end{bmatrix}_q. \tag{29}$$

Proof. By identity (9) with $r = 1/2$ and $r = 1/4$,

$$\begin{aligned} &\begin{bmatrix} 1/2+k \\ k \end{bmatrix}_q - \begin{bmatrix} 1/4+k \\ k \end{bmatrix}_q \\ &= \frac{\displaystyle\sum_{j=0}^{k} (-1)^j q^{j/2+j(j+1)/2} \begin{bmatrix} k \\ j \end{bmatrix}_q - \sum_{j=0}^{k} (-1)^j q^{j/4+j(j+1)/2} \begin{bmatrix} k \\ j \end{bmatrix}_q}{(1-q)\cdots(1-q^k)} \\ &=_q \frac{H_k(q)}{(1-q)(1-q^2)\cdots(1-q^k)}, \end{aligned}$$

where

$$H_k(q) \;=\; \sum_{\ell=1}^{\lfloor k/2 \rfloor} q^{2\ell(\ell+1)} \begin{bmatrix} k \\ 2\ell \end{bmatrix}_q - \sum_{\ell=1}^{\lfloor k/4 \rfloor} q^{\ell(8\ell+3)} \begin{bmatrix} k \\ 4\ell \end{bmatrix}_q. \tag{30}$$

We list the values of $H_k(q)$ for $k = 1, 2, 3, 4$:

$$\begin{aligned} H_1(q) &= 0; \\ H_2(q) &= q^4; \\ H_3(q) &= q^4 + q^5 + q^6; \\ H_4(q) &= q^4 + q^5 + 2q^6 + q^7 + q^8 - q^{11} + q^{12}. \end{aligned}$$

In the first three cases $H_k(q)$ has nonnegative coefficients, so $\begin{bmatrix} 1/2+k \\ k \end{bmatrix}_q - \begin{bmatrix} 1/4+k \\ k \end{bmatrix}_q \geq_q 0$ for $k = 1, 2, 3$ by Lemma 3.1(5).

For $k = 4$, the polynomial $H_4(q)$ already contains the term $-q^{11}$, so $H_4(q)$ itself is not coefficientwise nonnegative. Nevertheless, its successive coefficient sums are all nonnegative (the coefficient -1 in degree 11 is absorbed by the preceding positive mass). Lemma 6.1 therefore proves the required inequality for $k = 4$.

For every $1 \leq k \leq 150$, our symbolic computation expands $H_k(q)$ with exact integer coefficients and checks the finite certificate

$$\sum_{e=0}^{d} [q^e] H_k(q) \geq 0 \qquad (0 \leq d \leq \deg H_k). \tag{31}$$

Lemma 6.1 converts (31) directly into $H_k(q)/(q;q)_k \geq_q 0$, which proves the proposition. Thus the computation does not rely on a heuristic pairing: the recorded prefix sums are an exact finite certificate for every k in the stated range. □

The corresponding code and data are available on the first author's personal webpage:

https://irma.math.unistra.fr/~guoniu/halfconj.html

This allows for independent verification of the computations.

Proposition 6.3. *For every integer k with $1 \le k \le 150$,*

$$\begin{bmatrix} 1/2+k \\ k \end{bmatrix}_q \ge_q \begin{bmatrix} 1/3+k \\ k \end{bmatrix}_q. \tag{32}$$

Proof. By identity (9) with $r = 1/2$ and $r = 1/3$,

$$\begin{aligned} &\begin{bmatrix} 1/2+k \\ k \end{bmatrix}_q - \begin{bmatrix} 1/3+k \\ k \end{bmatrix}_q \\ &= \frac{\displaystyle\sum_{j=0}^{k} (-1)^j q^{j/2+j(j+1)/2} \begin{bmatrix} k \\ j \end{bmatrix}_q - \sum_{j=0}^{k} (-1)^j q^{j/3+j(j+1)/2} \begin{bmatrix} k \\ j \end{bmatrix}_q}{(1-q)(1-q^2)\cdots(1-q^k)} \\ &=_q \frac{U_k(q)}{(1-q)(1-q^2)\cdots(1-q^k)}, \end{aligned}$$

where

$$U_k(q) = \sum_{\ell=1}^{\lfloor k/2 \rfloor} q^{2\ell(\ell+1)} \begin{bmatrix} k \\ 2\ell \end{bmatrix}_q - \sum_{\ell=1}^{\lfloor k/3 \rfloor} (-1)^\ell q^{\ell(9\ell+5)/2} \begin{bmatrix} k \\ 3\ell \end{bmatrix}_q. \tag{33}$$

We list the values of $U_k(q)$ for $k = 1, 2, 3, 4$:

$$\begin{aligned} U_1(q) &= 0; \\ U_2(q) &= q^4; \\ U_3(q) &= q^4 + q^5 + q^6 + q^7; \\ U_4(q) &= q^4 + q^5 + 2q^6 + 2q^7 + 2q^8 + q^9 + q^{10} + q^{12}. \end{aligned}$$

As in the proof of Proposition 6.2, the program uses exact integer arithmetic to verify

$$\sum_{e=0}^{d} [q^e] U_k(q) \ge 0 \qquad (0 \le d \le \deg U_k) \tag{34}$$

for every $1 \le k \le 150$. Lemma 6.1 then gives $U_k(q)/(q;q)_k \ge_q 0$, proving the proposition. The code and data are available on the same webpage. □

We next explain the lower limit $k = 19$ in the two stronger conjectures below.

Remark 6.4 (The observed threshold 19). The stronger property in question is coefficientwise nonnegativity of the *numerator polynomial itself*, rather than nonnegativity only after division by $1 - q$. For $1 \le k \le 18$, exact computation gives

polynomial	values of $k \le 18$ having a negative coefficient
$H_k(q)$	$4, 5, 6, \ldots, 18$
$U_k(q)$	$6, 8, 10, 12, 18$.

Both $H_k(q)$ and $U_k(q)$ are coefficientwise nonnegative for every tested value $19 \le k \le 150$. Thus 19 is the smallest observed point after which neither sequence has another

failure in the computed range. This is an empirical threshold, not a consequence of the prefix-sum proof: even at the exceptional smaller values, the prefix sums in (31) and (34) are nonnegative, which is enough for the two propositions. The conjectures assert that the stronger numerator positivity seen from $k = 19$ onward persists for all larger k.

Empirical evidence suggests the following two stronger statements.

Conjecture 6.5. *For all $k \geq 19$, we have $H_k(q) \geq_q 0$, where $H_k(q)$ is defined by (30).*

Conjecture 6.6. *For all $k \geq 19$, we have $U_k(q) \geq_q 0$, where $U_k(q)$ is defined by (33).*

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