CYCLOTOMIC EULER-MAHONIAN POLYNOMIALS

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Abstract. The cyclotomic Eulerian polynomials and the cyclotomic Mahonian polynomials have each been the subject of extensive studies in Combinatorics, with particular attention to their signed versions. In contrast, the joint study of cyclotomic Euler-Mahonian polynomials has received far less consideration. To the best of our knowledge, the only prior result in this direction is a formula due to Wachs for the signed Euler-Mahonian polynomials in the even case. In this paper, we focus on the cyclotomic Euler-Mahonian polynomials and derive a formula based on the Hadamard product. As corollaries, we obtain the I-analogue (where $I = \sqrt{-1}$) of Wachs' formula for signed Euler-Mahonian polynomials, as well as the previously missing odd case for the signed Euler-Mahonian polynomials.

1. Introduction

Let $A_n(t, q, p)$ denote the generating function for the symmetric group $\mathfrak{S}_n$ with respect to the trivariate statistic $(\mathrm{des}, \mathrm{maj}, \mathrm{inv})$, namely

$$A_n(t, q, p) = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{des}(\sigma)} q^{\mathrm{maj}(\sigma)} p^{\mathrm{inv}(\sigma)}.$$

The specialization $p = 1$, written $A_n(t, q) := A_n(t, q, 1)$, recovers the usual *Euler–Mahonian polynomials*. In addition, the polynomials $A_n(t, 1)$ and $A_n(1, q)$ are known as the *Eulerian polynomials* and the *Mahonian polynomials*, respectively.

For every positive integer m, let ξ_m denote a primitive m-th root of unity. By analogy with the case $m = 2$ (so that $\xi_2 = -1$), which gives rise to the *signed* polynomials, we call the specialization $p = \xi_m$ the *cyclotomic* polynomials of order m. The cyclotomic Eulerian and cyclotomic Mahonian polynomials have been widely studied individually in combinatorics, with particular emphasis on the signed Eulerian and signed Mahonian cases [1, 9, 10, 21, 25]. A more detailed historical overview of this line of work is provided in Section 2.

However, the cyclotomic Euler-Mahonian polynomials have received little attention so far. To the best of our knowledge, the only result in

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this area is due to Wachs [25], who derived a formula for the signed Euler-Mahonian polynomials in the even case, $A_{2n}(t,q,-1)$; see Theorem 7.

In this paper, we investigate the joint cyclotomic Euler-Mahonian polynomials. Let $(t;q)_n := (1-t)(1-tq)\cdots(1-tq^{n-1})$ denote the standard q-ascending factorial. Recall that the *Hadamard product* of two power series is given by

$$\sum_{\ell\geq 0} a_\ell t^\ell \otimes \sum_{\ell\geq 0} b_\ell t^\ell = \sum_{\ell\geq 0} a_\ell b_\ell t^\ell.$$

By employing the Hadamard product, we derive an explicit expression for the cyclotomic Euler-Mahonian polynomials. The main theorem is presented below.

Theorem 1. *Let ξ_m denote a primitive m-th root of unity. For any positive integer n, express n in the form $n = mk+i$ with $0 \leq i \leq m-1$. Then*

$$\frac{A_n(t,q,\xi_m)}{(t;q)_{n+1}} = \frac{A_i(t,q,\xi_m)}{(t;q)_{i+1}} \otimes \frac{A_k(t,q^m)}{(t;q^m)_{k+1}}. \tag{1}$$

Theorem 1 naturally specializes to several previously known identities for the polynomials $A_n(t,q,\xi_m)$ by setting specific values for the parameters t, q, m and i. In Section 3, we recover the following identities:

- Case $q = 1$ and $m = 2$: This recovers the results for the signed Eulerian polynomials, originally studied by Désarménien, Foata, Loday and Wachs [9, 10, 21, 25]. See Theorem 5.
- Case $t = 1$ and $m = 2$: This yields the identities for the signed Mahonian polynomials, originally studied by Gessel, Simion and Wachs [25]. See Theorem 6.
- case $t = 1$: This recovers the identity for the cyclotomic Mahonian polynomials, studied by Adin, Gessel and Roichman [1]. See Theorem 8.
- case $m = 2, i = 0$: This provides the even case of the signed Euler-Mahonian polynomials, studied by Wachs [25]. See Theorem 7.

We also derived several further corollaries, which are presented in Section 3. We highlight just two of them: one is an I-analogue (where $I = \sqrt{-1}$) of Wachs's formula for signed Euler–Mahonian polynomials, and the other provides the previously missing odd case for signed Euler–Mahonian polynomials.

Corollary 2. *Let $I = \sqrt{-1}$, we have*

$$A_{4k}(t,q,I) = \frac{(t;q)_{4k+1}}{(t;q^4)_{k+1}} A_k(t,q^4). \tag{2}$$

Corollary 3. *We have*

$$A_{2k+1}(t,q,-1) = (t;q)_{2k+2} \sum_{\ell \geq 0} [\ell+1]_q \, [\ell+1]_{q^2}^k t^\ell. \tag{3}$$

Section 2 reviews the basic preliminary concepts and the original results concerning the signed Eulerian and Mahonian polynomials. We present two proofs of the main theorem. The first follows the approach developed by Adin, Gessel, and Roichman [1], see Section 4. The second proof is more direct, see Section 5. Both proofs make use of Gessel's generating function formula for $A_n(t,q,p)$ [18].

2. Preliminaries and Notation

Let $\mathfrak{S}_n$ denote the symmetric group on the set $\{1,2,\ldots,n\}$. For a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathfrak{S}_n$, the *inversion number*, *descent number*, and *major index* are defined respectively by

$$\begin{aligned} \operatorname{inv}\sigma &= \#\{(i,j) \mid 1 \leq i < j \leq n,\ \sigma_i > \sigma_j\}, \\ \operatorname{des}\sigma &= \#\{i \mid 1 \leq i \leq n-1,\ \sigma_i > \sigma_{i+1}\}, \\ \operatorname{maj}\sigma &= \sum\{i \mid 1 \leq i \leq n-1,\ \sigma_i > \sigma_{i+1}\}, \end{aligned}$$

where the notation "#" denotes the cardinality of a set. Recall that $A_n(t,q,p)$ denotes the generating polynomial for the symmetric group $\mathfrak{S}_n$ with respect to the statistics $(\operatorname{des}, \operatorname{maj}, \operatorname{inv})$,

$$A_n(t,q,p) = \sum_{\sigma \in \mathfrak{S}_n} t^{\operatorname{des}(\sigma)} q^{\operatorname{maj}(\sigma)} p^{\operatorname{inv}(\sigma)}.$$

We abbreviate $A_n(t,q) := A_n(t,q,1)$ for the Euler–Mahonian polynomials. The *q-ascending factorial* is defined by

$$\begin{aligned} (a;q)_n &= \begin{cases} 1, & \text{if } n = 0; \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{if } n \geq 1; \end{cases} \\ (a;q)_\infty &= \prod_{n \geq 0} (1-aq^n). \end{aligned}$$

For brevity, we shall also use throughout the paper the shorthand notation

$$(q)_n := (q;q)_n,$$

and likewise for any other variable, so that, for instance, $(p)_n = (p;p)_n$. Write $[n]_q = 1 + q + \cdots + q^{n-1}$ and $[n]_q! = [1]_q[2]_q\cdots[n]_q$. We define the q-binomial coefficient by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}.$$

It is a classical result [3, 22] that $A_n(1,q) = [n]_q!$ and that

$$\frac{1}{(t;q)_n} = \sum_{k\geq 0} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q t^k \quad \text{and} \quad \frac{1}{(t;q)_\infty} = \sum_{k\geq 0} \frac{t^k}{(q)_k}. \tag{4}$$

Carlitz [7, 8] obtained the following identity for the Euler–Mahonian polynomials.

Theorem 4 (Carlitz)**.** *For all $n \geq 0$,*

$$\frac{A_n(t,q)}{(t;q)_{n+1}} = \sum_{\ell\geq 0} t^\ell \big([\ell+1]_q\big)^n. \tag{5}$$

Désarménien and Foata established the following identities linking the Eulerian polynomials with the signed Eulerian polynomials, confirming a conjecture of Loday [9, 10, 21].

Theorem 5 (Loday, Désarménien, Foata)**.** *We have*

$$A_{2k}(t,1,-1) = (1-t)^k A_k(t,1), \tag{6}$$

$$A_{2k+1}(t,1,-1) = (1-t)^k A_{k+1}(t,1). \tag{7}$$

Wachs [25] provided a bijective proof of Theorem 5. In addition, her bijection leads to a formula of Gessel and Simion for the signed Mahonian polynomials.

Theorem 6 (Gessel, Simion, Wachs)**.** *We have*

$$A_n(1,q,-1) = \left(\frac{1-q}{1+q}\right)^{\lfloor \frac{n}{2} \rfloor} [n]_q!. \tag{8}$$

Moreover, Wachs's bijection gives a q-analogue of (6), corresponding to the even case of Theorem 5, for the signed Euler–Mahonian polynomials.

Theorem 7 (Wachs)**.** *We have*

$$A_{2k}(t,q,-1) = (tq;q^2)_k A_k(t,q^2). \tag{9}$$

As Wachs observed, her bijection does not yield a q-analogue in the odd case for signed Euler–Mahonian polynomials.

The cyclotomic Mahonian polynomials were studied by Adin, Gessel, and Roichman [1]. In their work, they derived a generalization of Theorem 6 of Gessel and Simion.

Theorem 8 (Adin, Gessel, Roichman)**.** *Let $n = mk + i$ with $0 \leq i \leq m-1$. Then*

$$A_n(1,q,\xi_m) = \frac{(q;q)_n}{(q;q)_i (1-q^m)^k} A_i(1,q,\xi_m). \tag{10}$$

The cyclotomic Eulerian polynomials have also been analyzed by Désarménien and Foata, who expressed $A_n(t,1,\xi_m)$ in terms of a two-index Eulerian polynomial; see [9, 10].

Since then, these signed enumeration identities have been broadened and developed along two principal lines. On the one hand, they have been extended from the symmetric group to other reflection groups by Adin–Gessel–Roichman [1], Reiner [24], Biagioli [4], Biagioli–Caselli [5], Brenti–Sentinelli [6], Eu–Fu–Lo [13], and Eu–Lin–Lo [16]. On the other hand, the focus has shifted from general permutations to various families of restricted permutations, including 321-avoiding permutations studied by Adin–Roichman [2] and Eu–Fu–Pan–Ting [15], simsun permutations examined by Eu–Fu–Pan [14], PRW permutations studied by Lin–Wang–Zeng [20], permutations with subsequence constraints considered by Eu–Fu–Hsu–Liao–Sun [12], and derangements analyzed by Ji–Zhang [19]. Finally, let us mention the recent work of Dong and Lin [11, Theorem 1.11], which provides a refinement of Désarménien–Foata's formula (6)–(7) by the peak statistic.

3. Specializations

In this section, we specialize our main theorem by assigning specific values to the parameters t, q, m and i. In doing so, we recover several identities already stated in Section 2, and we further obtain additional special cases presented as corollaries. We begin by rewriting (1) in a different form. For each i with $0 \le i \le m-1$, let $C_i(\ell; q, \xi_m)$ $(\ell \ge 0)$ be the coefficients defined by the expansion

$$\frac{A_i(t,q,\xi_m)}{(t;q)_{i+1}} = \sum_{\ell \ge 0} C_i(\ell; q, \xi_m) t^\ell. \tag{11}$$

Using Carlitz's formula (5), the main theorem is equivalent to

$$\frac{A_n(t,q,\xi_m)}{(t;q)_{n+1}} = \sum_{\ell \ge 0} C_i(\ell; q, \xi_m)\, ([\ell+1]_{q^m})^k t^\ell, \tag{12}$$

providing $n = mk + i$ with $0 \le i \le m-1$.

3.1. **Case $i = 0$.** In this case we have $n = mk$. Since $A_0(t,p,q) = 1$, the main theorem gives

$$\frac{A_n(t,q,\xi_m)}{(t;q)_{n+1}} = \frac{1}{1-t} \otimes \frac{A_k(t,q^m)}{(t;q^m)_{k+1}} = \frac{A_k(t,q^m)}{(t;q^m)_{k+1}}.$$

Thus, in this situation we obtain an explicit expression without resorting to the Hadamard product.

Corollary 9. *We have*

$$A_{mk}(t,q,\xi_m) = \frac{(t;q)_{mk+1}}{(t;q^m)_{k+1}} A_k(t,q^m). \tag{13}$$

Taking $m=2$, i.e., $\xi_2=-1$ in (13), we recover Wachs' identity stated in Theorem 7 for the signed Euler–Mahonian polynomials. Similarly, setting $m=4$, namely $\xi_4=I=\sqrt{-1}$ in (13), we obtain Corollary 2.

3.2. **Case $i=1$.** Here we have $n=mk+1$. Since $A_1(t,q,p)=1$, we get

$$\frac{A_i(t,q,\xi_m)}{(t;q)_{i+1}}=\frac{1}{(t;q)_2}=\sum_{\ell\geq 0}[\ell+1]_q t^\ell.$$

Applying the main theorem, we obtain:

Corollary 10. *We have*

$$A_{mk+1}(t,q,\xi_m)=(t;q)_{mk+2}\sum_{\ell\geq 0}[\ell+1]_q\,[\ell+1]_{q^m}^k t^\ell. \tag{14}$$

For $m=2$, this yields Corollary 3, which supplies the missing odd case for the signed Euler–Mahonian polynomials.

3.3. **Case $i=2$.** In this situation we assume $m\geq 3$ and $n=mk+2$. Since $A_2(t,q,p)=1+tpq$, we obtain

$$\begin{aligned}\frac{A_2(t,q,\xi_m)}{(t;q)_3}&=\frac{1+tq\xi_m}{(t;q)_3}\\&=(1+tq\xi_m)\sum_{\ell\geq 0}\begin{bmatrix}2+\ell\\ \ell\end{bmatrix}_q t^\ell\\&=\sum_{\ell\geq 0}\frac{(1-q^{\ell+2})+q\xi_m(1-q^\ell)}{1-q^2}\,[\ell+1]_q\,t^\ell.\end{aligned}$$

Applying the main theorem (12), we obtain:

Corollary 11. *For $n=mk+2$ we have*

$$\frac{A_{mk+2}(t,q,\xi_m)}{(t;q)_{mk+3}}=\sum_{\ell\geq 0}\frac{(1-q^{\ell+2})+q\xi_m(1-q^\ell)}{1-q^2}\,[\ell+1]_q\,[\ell+1]_{q^m}^k\,t^\ell.$$

3.4. **Case $t=1$.** This specializes to the Mahonian polynomial case. Recall first the following useful result, the "$t=1$ Lemma" [17, p. 163].

Lemma 12. *Let $(a_\ell)_{\ell\geq 0}$ be a sequence of formal series with limit a, in the sense that $o(a-a_\ell)\to\infty$ as $\ell\to\infty$. Define*

$$b(t):=(1-t)\sum_{\ell\geq 0}a_\ell\,t^\ell.$$

Then $b(1)=a$.

We now apply the "$t=1$ Lemma" to identity (12). Multiplying both sides by $1-t$ gives

$$\frac{A_n(t,q,\xi_m)(1-t)}{(t;q)_{n+1}}=(1-t)\sum_{\ell\geq 0}C_i(\ell;q,\xi_m)\frac{(1-q^{(\ell+1)m})^k}{(1-q^m)^k}\,t^\ell.$$

Setting $t = 1$ and invoking Lemma 12, we obtain

$$\frac{A_n(1, q, \xi_m)}{(q;q)_n} = \frac{1}{(1-q^m)^k} \lim_{\ell\to\infty} C_i(\ell; q, \xi_m). \tag{15}$$

Now choose $n = mk + i$ with $k = 0$, that is, $n = i$. Then

$$\frac{A_i(1, q, \xi_m)}{(q;q)_i} = \lim_{\ell\to\infty} C_i(\ell; q, \xi_m). \tag{16}$$

Substituting (16) into (15) yields

$$A_n(1, q, \xi_m) = \frac{(q;q)_n}{(q;q)_i(1-q^m)^k} A_i(1, q, \xi_m).$$

This is precisely Theorem 8 of Adin, Gessel, and Roichman. Furthermore, when $m = 2$, we recover Theorem 6 due to Gessel–Simion.

3.5. **Case $q = 1$.** This corresponds to the Eulerian polynomial setting. By substituting $q = 1$ into Corollaries 9, 10 and 11, we obtain:

Corollary 13. *We have*

$$\begin{aligned}
A_{mk}(t, 1, \xi_m) &= (1-t)^{(m-1)k} A_k(t, 1),\\
A_{mk+1}(t, 1, \xi_m) &= (1-t)^{(m-1)k} A_{k+1}(t, 1),\\
A_{mk+2}(t, 1, \xi_m) &= (1-t)^{(m-1)k} \Big(\tfrac{1+\xi_m}{2} A_{k+2}(t, 1)\\
&\qquad + \tfrac{1-\xi_m}{2} (1-t) A_{k+1}(t, 1) \Big).
\end{aligned}$$

Proof. The first equality follows immediately from setting $q = 1$ in (13). For the second, substitute $q = 1$ into (14); then, using Carlitz's formula (5), we obtain

$$\begin{aligned}
A_{mk+1}(t, 1, \xi_m) &= (1-t)^{mk+2} \sum_{\ell\geq 0} (\ell+1)^{k+1} t^\ell\\
&= (1-t)^{mk+2} \frac{A_{k+1}(t, 1)}{(1-t)^{k+2}}.
\end{aligned}$$

For the third identity, substituting $q = 1$ in Corollary 11 gives

$$\begin{aligned}
A_{mk+2}(t, 1, \xi_m) &= (1-t)^{mk+3} \sum_{\ell\geq 0} \frac{(\ell+2) + \ell\,\xi_m}{2} (\ell+1)^{k+1} t^\ell\\
&= (1-t)^{mk+3} \sum_{\ell\geq 0} \left(\tfrac{1+\xi_m}{2} (\ell+1)^{k+2} + \tfrac{1-\xi_m}{2} (\ell+1)^{k+1} \right) t^\ell\\
&= (1-t)^{mk+3} \left(\frac{1+\xi_m}{2} \frac{A_{k+2}(t, 1)}{(1-t)^{k+3}} \right.\\
&\qquad \left. + \frac{1-\xi_m}{2} \frac{A_{k+1}(t, 1)}{(1-t)^{k+2}} \right).
\end{aligned}$$

□

In the special case $m = 2$, this reduces to the theorem of Loday, Désarménien and Foata, Theorem 5.

3.6. **Case** $m = 4$**.** In this situation, $\xi_m = I = \sqrt{-1}$. Substituting $m = 4$ and $\xi_m = I$ into the previously discussed cases $t = 1$ and $q = 1$ respectively, we obtain

$$\begin{aligned}
A_{4k}(t,1,I) &= (1-t)^{3k} A_k(t,1),\\
A_{4k+1}(t,1,I) &= (1-t)^{3k} A_{k+1}(t,1),\\
A_{4k+2}(t,1,I) &= (1-t)^{3k}\Big(\tfrac{1+I}{2} A_{k+2}(t,1)\\
&\qquad + \tfrac{1-I}{2}(1-t)A_{k+1}(t,1)\Big);
\end{aligned}$$

and

$$\begin{aligned}
A_{4n}(1,q,I) &= \frac{(1-q)^{4n}}{(1-q^4)^n}[4n]_q!,\\
A_{4n+1}(1,q,I) &= \frac{(1-q)^{4n+1}}{(1-q)(1-q^4)^n}[4n+1]_q!,\\
A_{4n+2}(1,q,I) &= \frac{(1+qI)}{(1-q)(1-q^2)}\frac{(1-q)^{4n+2}}{(1-q^4)^n}[4n+2]_q!,\\
A_{4n+3}(1,q,I) &= \frac{(1-q-q^2)+Iq(1+q-q^2)}{(1-q)(1-q^2)(1-q^3)}\frac{(1-q)^{4n+3}}{(1-q^4)^n}[4n+3]_q!.
\end{aligned}$$

4. First proof of the main theorem

Recall the following generating function for the trivariate statistic $(\mathrm{des}, \mathrm{maj}, \mathrm{inv})$ obtained by Gessel [18]:

Theorem 14 (Gessel)**.** *We have*

$$\sum_{n\geq 0} \frac{A_n(t,q,p)u^n}{(p;p)_n (t;q)_{n+1}} = \sum_{\ell\geq 0} \frac{t^\ell}{\prod_{i=0}^{\ell}\prod_{j\geq 0}(1-uq^ip^j)}. \tag{17}$$

We now adopt the approach developed by Adin, Gessel, and Roichman [1], phrased in terms of cyclotomic polynomials. Consider the sequence of polynomials $f_0(p), f_1(p), \ldots$ in p, defined through the *Eulerian generating function*

$$F(u;p) = \sum_{n=0}^{\infty} f_n(p)\frac{u^n}{(p;p)_n}. \tag{18}$$

Let $\Phi_m(p)$ denote the *cyclotomic polynomial* of order m in the variable p. For polynomials $f(p)$ and $g(p)$ with rational coefficients, we write $f(p) \equiv g(p) \pmod{\Phi_m(p)}$ if and only if $f(\xi_m) = g(\xi_m)$. Let

$$F(u;p) = \sum_{n=0}^{\infty} f_n(p)\frac{u^n}{(p;p)_n} \text{ and } G(u;p) = \sum_{n=0}^{\infty} g_n(p)\frac{u^n}{(p;p)_n}$$

be two Eulerian generating functions. We then define the congruence $F(u;p) \equiv G(u;p)$ to mean that the corresponding coefficients satisfy $f_n(p) \equiv g_n(p) \pmod{\Phi_m(p)}$ for all $n \geq 0$. The key properties of these

congruences are summarized in the following lemma, due to Adin, Gessel, and Roichman [1]:

Lemma 15. *Let* $v_i := u^i/(p)_i$.

(1) *If* $0 \le i, j < m$ *and* $i + j \ge m$, *then* $v_i v_j \equiv 0$.
(2) *If* $0 \le i < m$, *then* $v_{mk+i} \equiv v_m^k v_i/k!$.
(3) *We have* $(p)_{m-1} \equiv m$.

Now we are prepared to prove the main theorem, making use of Theorem 14.

Proof of Theorem 1. Define

$$L(\ell) = \frac{1}{\prod_{i=0}^{\ell} \prod_{j\ge 0}(1 - uq^i p^j)},$$

so that

$$\sum_{n\ge 0} \frac{A_n(t,q,p)v_n}{(t;q)_{n+1}} = \sum_{\ell \ge 0} L(\ell) t^\ell. \tag{19}$$

We compute

$$\begin{aligned} L(\ell) &= \prod_{i=0}^{\ell} \prod_{j\ge 0} \exp\left(\log \frac{1}{(1-uq^ip^j)}\right) = \prod_{d\ge 1} \exp\left(\sum_{i=0}^{\ell}\sum_{j\ge 0} \frac{(uq^ip^j)^d}{d}\right) \\ &= \prod_{d\ge 1} \exp\left(\frac{(1-q^{(\ell+1)d})u^d}{d(1-q^d)(1-p^d)}\right) \\ &\equiv \prod_{d=1}^{m-1} \exp(L'(d)) \times \exp(L'(m)) \times \prod_{d\ge m+1} \exp(L'(d)), \end{aligned}$$

where

$$L'(d) := \frac{(1-q^{(\ell+1)d})(p)_{d-1}}{d(1-q^d)} v_d.$$

We now examine these three factors separately. For the third factor, we immediately obtain

$$\prod_{d\ge m+1} \exp(L'(d)) \equiv 1.$$

For the second factor, using Lemma 15 (2) and (3), we have

$$\begin{aligned} \exp(L'(m)) &= \exp\left(\frac{(1-q^{(\ell+1)m})(p)_{m-1}v_m}{m(1-q^m)}\right) \equiv \exp\left(\frac{(1-q^{(\ell+1)m})v_m}{1-q^m}\right) \\ &= \sum_{k\ge 0} \frac{(1-q^{(\ell+1)m})^k v_m^k}{(1-q^m)^k k!} \equiv \sum_{k\ge 0} ([\ell+1]_{q^m})^k v_{mk}. \end{aligned}$$

By Lemma 15 (1), the first factor is a polynomial in u. Thus we may write

$$\prod_{d=1}^{m-1} \exp(L'(d)) \equiv \sum_{i=0}^{m-1} C_i(\ell;q,p)\, v_i.$$

Combining all three factors, we obtain

$$L(\ell) \equiv \left(\sum_{i=0}^{m-1} C_i(\ell;q,p)\, v_i\right)\left(\sum_{k\geq 0}([\ell+1]_{q^m})^k v_{mk}\right).$$

Therefore, using identity (19), we get

$$\sum_{n\geq 0} \frac{A_n(t,q,p)\, v_n}{(t;q)_{n+1}} \equiv \sum_{\ell\geq 0}\left(\sum_{i=0}^{m-1} C_i(\ell;q,p)\, v_i \sum_{k\geq 0}([\ell+1]_{q^m})^k v_{mk}\right) t^\ell.$$

Now, comparing coefficients of v_n on both sides for $n = mk+i$ and using Lemma 15 (2), we obtain $v_{mk}v_i/v_{mk+i} = 1$, and hence

$$\frac{A_n(t,q,p)}{(t;q)_{n+1}} \equiv \sum_{\ell\geq 0} C_i(\ell;q,p)\,([\ell+1]_{q^m})^k\, t^\ell. \tag{20}$$

First, substituting $k=0$ and $n=i$ for $0\leq i\leq m-1$ into (20), we obtain

$$\sum_{\ell\geq 0} C_i(\ell;q,p)\, t^\ell \equiv \frac{A_i(t,q,p)}{(t;q)_{i+1}}.$$

Next, applying Carlitz's formula (5), we have

$$\sum_{\ell\geq 0}([\ell+1]_{q^m})^k\, t^\ell = \frac{A_k(t,q^m)}{(t;q^m)_{k+1}}.$$

Therefore, identity (20) can be rewritten as

$$\frac{A_n(t,q,p)}{(t;q)_{n+1}} \equiv \frac{A_i(t,q,p)}{(t;q)_{i+1}} \otimes \frac{A_k(t,q^m)}{(t;q^m)_{k+1}}.$$

This completes the proof of the main theorem. □

5. Second proof of the main theorem

Throughout this section, the summation condition

$$\sum_{n_0,\ldots,n_\ell \vdash n}$$

will be a shorthand notation for a sum ranging over all nonnegative integers n_j $(0\leq j\leq \ell)$ satisfying $n_0+n_1+\cdots+n_\ell = n$. Recall that the q-multinomial coefficient is defined by

$$\begin{bmatrix} n_0+n_1+\cdots+n_\ell \\ n_0, n_1, \ldots, n_\ell \end{bmatrix}_q = \frac{(q;q)_{n_0+n_1+\cdots+n_\ell}}{(q;q)_{n_0}(q;q)_{n_1}\cdots(q;q)_{n_\ell}}.$$

Starting from Gessel's identity (17) and using the second expansion in (4), with t and q replaced by uq^i and p, we obtain

$$\begin{aligned}[t^\ell]\sum_{n\geq 0}\frac{A_n(t,q,p)u^n}{(p;p)_n(t;q)_{n+1}} &= \prod_{i=0}^{\ell}\frac{1}{\prod_{j\geq 0}(1-(uq^i)p^j)}\\ &= \prod_{i=0}^{\ell}\sum_{n_i\geq 0}\frac{(uq^i)^{n_i}}{(p)_{n_i}}\\ &= \sum_{n\geq 0}\sum_{n_0,\ldots,n_\ell\vdash n} q^{n_1+2n_2+\cdots+\ell n_\ell}\begin{bmatrix} n\\ n_0,\ldots,n_\ell\end{bmatrix}_p\frac{u^n}{(p;p)_n}.\end{aligned}$$

By comparing the coefficient of $v_n = u^n/(p)_n$, we deduce the following result. This formulation has the benefit that it can be directly specialized to $p = \xi_m$, in contrast to the original version of Gessel's theorem.

Theorem 16. *We have*

$$[t^\ell]\frac{A_n(t,q,p)}{(t;q)_{n+1}} = \sum_{n_0,\cdots,n_\ell\vdash n} q^{n_1+2n_2+\cdots+\ell n_\ell}\begin{bmatrix} n\\ n_0,\cdots,n_\ell\end{bmatrix}_p. \tag{21}$$

We now present a self-contained proof of (21), based on MacMahon's method [22]. Fix a positive integer n. Let $\mathcal{W}_n = \{0,1,2,\ldots\}^n$ denote the set of all words over the nonnegative integers of length n, and let $\mathcal{P}_n$ be the set of all partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ with largest part $\lambda_1 \leq n$. Let $\mathcal{D}_n$ denote the set of all weakly decreasing words in $\mathcal{W}_n$, and, for $\overline{w} \in \mathcal{D}_n$, let $\mathfrak{S}(\overline{w})$ denote the set of all rearrangements of $\overline{w}$, so that $\mathcal{W}_n$ is the disjoint union of the sets $\mathfrak{S}(\overline{w})$ with $\overline{w} \in \mathcal{D}_n$. For any word (or permutation, or partition) $w = x_1x_2\cdots x_n$, define $\max(w) = \max\{x_1,x_2,\ldots,x_n\}$ and $\mathrm{sum}(w) = x_1+x_2+\cdots+x_n$. If w is a word, define

$$\begin{aligned}\mathrm{inv}\, w &= \#\{(i,j) \mid 1\leq i<j\leq n,\ x_i > x_j\},\\ \mathrm{coinv}\, w &= \#\{(i,j) \mid 1\leq i<j\leq n,\ x_i < x_j\}.\end{aligned}$$

For a partition λ, let $\mathrm{len}(\lambda)$ denote its length, and write λ^c for its conjugate partition.

Proposition 17. *There is a bijection between $\mathcal{W}_n$ and $\mathfrak{S}_n\times\mathcal{P}_n$ sending $w \mapsto (\sigma,\lambda)$ such that*

[P1] $\max(w) = \mathrm{des}(\sigma) + \mathrm{len}(\lambda)$;
[P2] $\mathrm{sum}(w) = \mathrm{maj}(\sigma) + \mathrm{sum}(\lambda)$;
[P3] $\mathrm{coinv}(w) = \mathrm{inv}(\sigma)$.

The above bijection is not new: it is essentially MacMahon's classical correspondence [22] (see also [8, 17]). We reproduce its construction here in order to keep the second proof self-contained. Properties [P1] and [P2] are well known, since they are exactly what is needed for the

classical Carlitz-type identities. Property [P3], however, which is the key to the p-refinement obtained in the present paper, does not seem to have been recorded in the literature, to the best of our knowledge.

Proof. We describe the bijection by means of an example. Let $n = 8$ and $w = 45412225 \in \mathcal{W}_n$. Place w in the first row and $\mathrm{id} = 12345678$ in the second row:

$$\begin{aligned} w &= 4\ 5\ 4\ 1\ 2\ 2\ 2\ 5 \\ \mathrm{id} &= 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8 \end{aligned}$$

We now repeatedly swap adjacent columns $\left(\begin{smallmatrix} a & b \\ x & y \end{smallmatrix}\right)$ whenever $a < b$, and continue this process until the top row becomes a weakly decreasing word $\overline{w}$:

$$\begin{aligned} \overline{w} &= 5\ 5\ 4\ 4\ 2\ 2\ 2\ 1 \\ \sigma &= 2\ 8\ 1\ 3\ 5\ 6\ 7\ 4 \\ z &= 2\ 2\ 1\ 1\ 1\ 1\ 1\ 0 \\ \mu &= 3\ 3\ 3\ 3\ 1\ 1\ 1\ 1 \end{aligned}$$

At this stage, the bottom row is the permutation σ. Next, define z by $z_j = \#\{i \mid j \le i < n,\ \sigma_i > \sigma_{i+1}\}$ for $j < n$, and $z_n = 0$. Then set $\mu_j = \overline{w}_j - z_j$, and let $\lambda = \mu^c$ be the conjugate of the partition μ. In our running example, this gives $\lambda = 844$.

We check [P1] and [P2] using the facts that $\mathrm{des}(\sigma) = z_1$ and $\mathrm{maj}(\sigma) = \mathrm{sum}(z)$. For [P3], observe that at every intermediate stage, if the current top word is w' and the bottom word is σ', then the quantity $\mathrm{coinv}(w') + \mathrm{inv}(\sigma')$ remains invariant under the column swaps. Thus $\mathrm{coinv}(w) + \mathrm{inv}(\mathrm{id}) = \mathrm{coinv}(\overline{w}) + \mathrm{inv}(\sigma)$. This implies [P3] since $\mathrm{inv}(\mathrm{id}) = 0$ and $\mathrm{coinv}(\overline{w}) = 0$. □

Second proof of Theorem 16. By Proposition 17, we obtain

$$\sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{des}(\sigma)} q^{\mathrm{maj}(\sigma)} p^{\mathrm{inv}(\sigma)} \times \sum_{\lambda \in \mathcal{P}_n} t^{\mathrm{len}(\lambda)} q^{\mathrm{sum}(\lambda)} = \sum_{w \in \mathcal{W}_n} t^{\max(w)} q^{\mathrm{sum}(w)} p^{\mathrm{coinv}(w)}.$$

Hence

$$A_n(t,q,p) \times \frac{1}{(1-tq)\cdots(1-tq^n)} = \sum_{\ell \ge 0} t^\ell \sum_{\substack{w \in \mathcal{W}_n \\ \max(w) = \ell}} q^{\mathrm{sum}(w)} p^{\mathrm{coinv}(w)},$$

or equivalently

$$\frac{A_n(t,q,p)}{(1-t)(1-tq)\cdots(1-tq^n)} = \sum_{\ell \ge 0} t^\ell \sum_{\substack{w \in \mathcal{W}_n \\ \max(w) \le \ell}} q^{\mathrm{sum}(w)} p^{\mathrm{coinv}(w)}.$$

Observe that, when $p = 1$, the preceding identity reduces to Carlitz's formula (5). Extracting the coefficient of t^ℓ from both sides, we find

$$[t^\ell]\frac{A_n(t,q,p)}{(t;q)_{n+1}} = \sum_{\substack{w\in\mathcal{W}_n\\ \max(w)\le\ell}} q^{\mathrm{sum}(w)}p^{\mathrm{coinv}(w)} = \sum_{\substack{\overline{w}\in\mathcal{D}_n\\ \max(\overline{w})\le\ell}} q^{\mathrm{sum}(\overline{w})}\sum_{w\in\mathfrak{S}(\overline{w})} p^{\mathrm{coinv}(w)}.$$

Write $\overline{w} = \ell^{n_\ell}\cdots 2^{n_2}1^{n_1}0^{n_0}$, so that $n = n_0+n_1+\cdots+n_\ell$ and $\mathrm{sum}(\overline{w}) = n_1 + 2n_2 + \cdots + \ell n_\ell$. By the classical combinatorial interpretation of the q-multinomial coefficient [3, 22], the statistic coinv is distributed on the set $\mathfrak{S}(\overline{w})$ of all rearrangements of $\overline{w}$ according to

$$\sum_{w\in\mathfrak{S}(\overline{w})} p^{\mathrm{coinv}(w)} = \begin{bmatrix} n \\ n_0,\cdots,n_\ell \end{bmatrix}_p.$$

Hence

$$[t^\ell]\frac{A_n(t,q,p)}{(t;q)_{n+1}} = \sum_{n_0,\cdots,n_\ell\vdash n} q^{n_1+2n_2+\cdots+\ell n_\ell}\begin{bmatrix} n \\ n_0,\cdots,n_\ell \end{bmatrix}_p. \qquad \square$$

We also recall the multinomial version of the q-Lucas theorem [1, 23].

Theorem 18 (Olive)**.** *Let $n_0,\ldots,n_\ell$ be nonnegative integers and write $n_0 = mk_0 + i_0,\ldots,n_\ell = mk_\ell + i_\ell$ with $0 \le i_0,\ldots,i_\ell \le m-1$. Then*

$$\begin{bmatrix} n_0+\cdots+n_\ell \\ n_0,\cdots,n_\ell \end{bmatrix}_{\xi_m} = \binom{k_0+\cdots+k_\ell}{k_0,\cdots,k_\ell}\begin{bmatrix} i_0+\cdots+i_\ell \\ i_0,\cdots,i_\ell \end{bmatrix}_{\xi_m}.$$

In addition, if $i_0+\cdots+i_\ell \ge m$, then

$$\begin{bmatrix} i_0+\cdots+i_\ell \\ i_0,\cdots,i_\ell \end{bmatrix}_{\xi_m} = 0.$$

Now we are ready to give the second proof of our main theorem, which is very short.

Second proof of Theorem 1. Let $n = mk + i$ with $0 \le i \le m-1$. For nonnegative integers $n_0, n_1, \ldots, n_\ell$ satisfying $n_0+\cdots+n_\ell = n$, write $n_j = mk_j + i_j$ with $0 \le i_j \le m-1$. Applying the first part of the q-Lucas theorem (Theorem 18), we obtain

$$q^{n_1+2n_2+\cdots+\ell n_\ell}\begin{bmatrix} n \\ n_0,\ldots,n_\ell \end{bmatrix}_{\xi_m} = q^{m(k_1+2k_2+\cdots+\ell k_\ell)}\binom{k}{k_0,\ldots,k_\ell}\times q^{i_1+2i_2+\cdots+\ell i_\ell}\begin{bmatrix} i \\ i_0,\ldots,i_\ell \end{bmatrix}_{\xi_m}.$$

We now sum this identity over all nonnegative integers n_j $(0 \le j \le \ell)$ with $n_0 + n_1 + \cdots + n_\ell = n$. By the second part of Theorem 18,

$$q^{n_1+2n_2+\cdots+\ell n_\ell} \begin{bmatrix} n \\ n_0, \ldots, n_\ell \end{bmatrix}_{\xi_m} = 0 \quad \text{if } i_0 + i_1 + \cdots + i_\ell \ge m,$$

so the sum effectively runs only over those sequences $(n_0, \ldots, n_\ell)$ for which $i_0 + \cdots + i_\ell \le m-1$. Equivalently, these sequences are characterized by the conditions $k_0 + \cdots + k_\ell = k$ and $i_0 + \cdots + i_\ell = i$. Thus,

$$\begin{aligned} &\sum_{n_0,\ldots,n_\ell \vdash n} q^{n_1+2n_2+\cdots+\ell n_\ell} \begin{bmatrix} n \\ n_0, \ldots, n_\ell \end{bmatrix}_{\xi_m} \\ &\quad = \sum_{k_0,\ldots,k_\ell \vdash k} q^{m(k_1+2k_2+\cdots+\ell k_\ell)} \binom{k}{k_0, \ldots, k_\ell} \\ &\qquad \times \sum_{i_0,\ldots,i_\ell \vdash i} q^{i_1+2i_2+\cdots+\ell i_\ell} \begin{bmatrix} i \\ i_0, \ldots, i_\ell \end{bmatrix}_{\xi_m}. \end{aligned}$$

Combining this relation with Theorem 16 completes the proof of the main theorem. □

6. Concluding remarks

We end the paper with three questions suggested by the referees, which we believe deserve further investigation.

Question 19. *Wachs* [25] *proved the even case* (9) *by means of an explicit sign-reversing involution on permutations. Is there a direct combinatorial or bijective proof of the missing odd case* (3)*, or of the order-*4 *case* (2)*?*

Question 20. *Dong and Lin* [11, Theorem 1.11] *obtained a refinement of the signed Eulerian identities of Theorem 5 by the peak statistic. Is there a peak-statistic refinement of the main theorem, Theorem 1?*

Question 21. *The signed Mahonian and signed Eulerian polynomials have been extensively extended to the classical Weyl groups and, more generally, to the colored permutation groups* $G(r, 1, n)$ [1, 4, 16, 24]*. Can the Hadamard product framework developed here be adapted to the cyclotomic Euler–Mahonian polynomials of these groups?*

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References

[1] Ron M. Adin, Ira M. Gessel, and Yuval Roichman. Signed Mahonians. *J. Combin. Theory Ser. A*, 109(1):25–43, 2005.

[2] Ron M. Adin and Yuval Roichman. Equidistribution and sign-balance on 321-avoiding permutations. *Sém. Lothar. Combin.*, 51:Art. B51d, 14, 2004/05.

[3] George E. Andrews. *The theory of partitions*, volume Vol. 2 of *Encyclopedia of Mathematics and its Applications*. Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1976.

[4] Riccardo Biagioli. Signed Mahonian polynomials for classical Weyl groups. *European J. Combin.*, 27(2):207–217, 2006.

[5] Riccardo Biagioli and Fabrizio Caselli. Invariant algebras and major indices for classical Weyl groups. *Proc. London Math. Soc. (3)*, 88(3):603–631, 2004.

[6] Francesco Brenti and Paolo Sentinelli. Odd and even major indices and one-dimensional characters for classical Weyl groups. *Ann. Comb.*, 24(4):809–835, 2020.

[7] L. Carlitz. q-Bernoulli and Eulerian numbers. *Trans. Amer. Math. Soc.*, 76:332–350, 1954.

[8] L. Carlitz. A combinatorial property of q-Eulerian numbers. *Amer. Math. Monthly*, 82:51–54, 1975.

[9] Jacques Désarménien. Fonctions symétriques associées à des suites classiques de nombres. *Ann. Sci. École Norm. Sup. (4)*, 16(2):271–304, 1983.

[10] Jacques Désarménien and Dominique Foata. The signed Eulerian numbers. *Discrete Math.*, 99(1-3):49–58, 1992.

[11] Yao Dong and Zhicong Lin. Counting and signed counting permutations by descent-based statistics. *J. Algebraic Combin.*, 60(1):145–189, 2024.

[12] Sen-Peng Eu, Tung-Shan Fu, Hsiang-Chun Hsu, Hsin-Chieh Liao, and Wei-Liang Sun. Signed Mahonian identities on permutations with subsequence restrictions. *J. Combin. Theory Ser. A*, 170:105131, 24, 2020.

[13] Sen-Peng Eu, Tung-Shan Fu, and Yuan-Hsun Lo. Signed Mahonian on parabolic quotients of colored permutation groups. *Adv. in Appl. Math.*, 132:Paper No. 102269, 28, 2022.

[14] Sen-Peng Eu, Tung-Shan Fu, and Yeh-Jong Pan. A refined sign-balance of simsun permutations. *European J. Combin.*, 36:97–109, 2014.

[15] Sen-Peng Eu, Tung-Shan Fu, Yeh-Jong Pan, and Chien-Tai Ting. Two refined major-balance identities on 321-avoiding involutions. *European J. Combin.*, 49:250–264, 2015.

[16] Sen-Peng Eu, Zhicong Lin, and Yuan-Hsun Lo. Signed Euler-Mahonian identities. *European J. Combin.*, 91:Paper No. 103209, 24, 2021.

[17] Dominique Foata and Guo-Niu Han. *The q-series in Combinatorics; permutation statistics*. (on line), 2004.

[18] Ira M. Gessel. *Generating functions and enumeration of sequences*. M.I.T. doctoral thesis, 1977.

[19] Kathy Q. Ji and Dax T. X. Zhang. Signed Mahonian polynomials on derangements in classical Weyl groups. *European J. Combin.*, 124:Paper No. 104083, 20, 2025.

[20] Zhicong Lin, David G. L. Wang, and Jiang Zeng. Around the q-binomial-Eulerian polynomials. *European J. Combin.*, 78:105–120, 2019.

[21] Jean-Louis Loday. Opérations sur l'homologie cyclique des algèbres commutatives. *Invent. Math.*, 96(1):205–230, 1989.

[22] P. A. MacMahon. The Indices of Permutations and the Derivation Therefrom of Functions of a Single Variable Associated with the Permutations of any Assemblage of Objects. *Amer. J. Math.*, 35(3):281–322, 1913.

[23] Gloria Olive. Generalized powers. *Amer. Math. Monthly*, 72:619–627, 1965.
[24] Victor Reiner. Descents and one-dimensional characters for classical Weyl groups. *Discrete Math.*, 140(1-3):129–140, 1995.
[25] Michelle L. Wachs. An involution for signed Eulerian numbers. *Discrete Math.*, 99(1-3):59–62, 1992.

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