General method to derive the relationship between two sets of Zernike coefficients corresponding to different aperture sizes

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Zernike polynomials have been widely used to describe the aberrations in wavefront sensing of the eye. The Zernike coefficients are often computed under different aperture sizes. For the sake of comparison, the same aperture diameter is required. Since no standard aperture size is available for reporting the results, it is important to develop a technique for converting the Zernike coefficients obtained from one aperture size to another size. By investigating the properties of Zernike polynomials, we propose a general method for establishing the relationship between two sets of Zernike coefficients computed with different aperture sizes. © 2006 Optical Society of America

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1. INTRODUCTION

In the past decades, interest in wavefront sensing of the human eye has increased rapidly in the field of ophthalmic optics. Several techniques have been developed for measuring aberrations of the eye.[1,2] In general, these techniques typically represent the aberrations as a wavefront error map at the corneal or pupil plane. Zernike polynomials, due to their properties such as orthogonality and rotational invariance, have been extensively used for fitting corneal surfaces.[3–6] Moreover, the lower terms of the Zernike polynomial expansion can be related to known types of aberrations such as defocus, astigmatism, coma, and spherical aberration.[7] When the Zernike coefficients are computed, an aperture radius describing the circular area in which the Zernike polynomials are defined must be specified. Such a specification is usually affected by the measurement conditions and by variation in natural aperture size across the human population. Since the Zernike coefficients are often obtained under different aperture sizes, the values of the expansion coefficients cannot be directly compared. Unfortunately, this type of comparison is exactly what needs to be done in repeatability and epidemiological studies. To solve this problem, a technique for converting a set of Zernike coefficients from one aperture size to another is required.

Recently, Schwiegerling[8] proposed a method to derive the relationship between the sets of Zernike coefficients for two different aperture sizes, but he did not provide a full demonstration for his results. Campbell[9] developed an algorithm based on matrix representation to find a new set of Zernike coefficients from an original set when the aperture size is changed. The advantage of Campbell's method is its easy implementation. In this paper, by investigating the properties of Zernike polynomials, we present a general method for establishing the relationship between two sets of Zernike coefficients computed with different aperture sizes. An explicit and rigorous demonstration of the method is given in detail. It is shown that the results derived from the proposed method are much more simple than those obtained by Schwiegerling, and moreover, our method can be easily implemented.

2. BACKGROUND

Zernike polynomials have been successfully used in many scientific research fields such as image analysis,[10] pattern recognition,[11] and astronomical telescopes.[12] Some efficient algorithms for fast computation of Zernike moments defined by Eq. (7) below have also been reported.[13–15] Recently, Zernike polynomials have been applied to describe the aberrations in the human eye.[1] There are several different representations of Zernike polynomials in the literature. We adopt the standard Optical Society of

America notation. The Zernike polynomial of order n with index m describing the azimuthal frequency of the azimuthal component is defined as

$$Z_n^m(\rho,\theta)=\begin{cases} N_n^m R_n^m(\rho)\cos(m\theta) & \text{for } m\geq 0 \\ -N_n^m R_n^m(\rho)\sin(m\theta) & \text{for } m<0 \end{cases}, \quad |m|\leq n,\ n-|m| \text{ even}, \tag{1}$$

where the radial polynomial $R_n^m(\rho)$ is given by

$$R_n^m(\rho)=\sum_{s=0}^{(n-|m|)/2}\frac{(-1)^s(n-s)!}{s!\,[(n+|m|)/2-s]!\,[(n-|m|)/2-s]!}\rho^{n-2s}, \tag{2}$$

and N_n^m is the normalization factor given by

$$N_n^m=\sqrt{\frac{2(n+1)}{1+\delta_{m,0}}}. \tag{3}$$

Here $\delta_{m,0}$ is the Kronecker symbol.

Equations (2) and (3) show that both the radial polynomial $R_n^m(\rho)$ and the normalization factor N_n^m are symmetric about m, i.e., $R_n^m(\rho)=R_n^{-m}(\rho)$, $N_n^m=N_n^{-m}$, for $m\geq 0$. Thus, for the study of these polynomials, we can only consider the case where $m\geq 0$. Let $n=m+2k$ with $k\geq 0$; Eq. (2) can be rewritten as

$$\begin{aligned} R_{m+2k}^m(\rho)&=\sum_{s=0}^{k}\frac{(-1)^s(m+2k-s)!}{s!\,(k-s)!\,(m+k-s)!}\rho^{m+2k-2s} \\ &=\sum_{s=k}^{0}\frac{(-1)^{k-s}(m+k+s)!}{s!\,(k-s)!\,(m+s)!}\rho^{m+2s} \\ &\quad\text{(making the change of variables} = k-s) \\ &=\sum_{s=0}^{k}c_{k,s}^m\rho^{m+2s}, \end{aligned} \tag{4}$$

where

$$C_{k,s}^m=(-1)^{k-s}\frac{(m+k+s)!}{s!\,(k-s)!\,(m+s)!}. \tag{5}$$

Since the Zernike polynomials are orthogonal over the unit circle, the polar coordinates (r,θ) must be scaled to the normalized polar coordinates (ρ,θ) by setting $\rho=r/r_{\max}$, where $r_{\max}$ denotes the maximum radial extent of the wavefront error surface. The wavefront error, $W(r,\theta)$, can thus be represented by a finite set of the Zernike polynomials as

$$W(r,\theta)=\sum_{n=0}^{N}\sum_{m}a_{n,m}Z_n^m(r/r_{\max},\theta), \tag{6}$$

where N denotes the maximum order used in the representation, and $a_{n,m}$ are the Zernike coefficients given by

$$a_{n,m}=\int_0^{r_{\max}}\int_0^{2\pi}Z_n^m(r/r_{\max},\theta)W(r,\theta)r\mathrm{d}r\mathrm{d}\theta. \tag{7}$$

Equation (7) shows clearly that the coefficients $a_{n,m}$ depend on the choice of $r_{\max}$. This dependence makes it difficult to compare two wavefront error measures obtained under different aperture sizes. To surmount this difficulty, it is necessary to develop a method that is capable of computing the Zernike coefficients for a given aperture size r_2 based on the expansion coefficients for a different aperture size r_1. Without loss of generality, we assume that r_1 takes a value of 1, and the problem can be formulated as follows.

Assume that the wavefront error can be expressed as

$$W(r,\theta)=\sum_{n=0}^{N}\sum_{m}a_{n,m}Z_n^m(r,\theta), \tag{8}$$

where the coefficients $a_{n,m}$ are known. The same wavefront error must be represented as

$$W(r,\theta)=\sum_{n=0}^{N}\sum_{m}b_{n,m}Z_n^m(\lambda r,\theta), \tag{9}$$

where λ is a parameter taking a positive value. We need to find the coefficient conversion relationships between two sets of coefficients $\{b_{n,m}\}$ and $\{a_{n,m}\}$.

3. METHODS AND RESULTS

In this section, we propose a general method that allows a new set of Zernike coefficients $\{b_{n,m}\}$ corresponding to an arbitrary aperture size to be found from an original set of coefficients $\{a_{n,m}\}$. As indicated by Schwiegerling,[8] the new coefficients $b_{n,m}$ depend only on the coefficients $a_{n,m}$ that have the same azimuthal frequency m. Thus, we consider a subset of terms in Eq. (8), all of which have the same azimuthal frequency m:

$$W_m(r,\theta)=\begin{cases}\left(\sum_{k=0}^{K}a_{m+2k,m}N_{m+2k}^m R_{m+2k}^m(r)\right)\cos(m\theta), & \text{for } m\geq 0\\ -\left(\sum_{k=0}^{K}a_{-m+2k,m}N_{-m+2k}^{-m}R_{-m+2k}^{-m}(r)\right)\sin(m\theta), & \text{for } m<0\end{cases}, \tag{10}$$

where K is given by

$$K=\begin{cases}(N-|m|)/2, & \text{if } N \text{ and } m \text{ have the same parity}\\ (N-1-|m|)/2, & \text{otherwise}\end{cases}. \tag{11}$$

Similarly, the subset of terms in Eq. (9) with the same azimuthal frequency m can be expressed as

$$W_m(r,\theta)=\begin{cases}\left(\sum_{k=0}^{K} b_{m+2k,m}N_{m+2k}^{m}R_{m+2k}^{m}(\lambda r)\right)\cos(m\theta), & \text{for } m\geq 0\\ -\left(\sum_{k=0}^{K} b_{-m+2k,m}N_{-m+2k}^{-m}R_{-m+2k}^{-m}(\lambda r)\right)\sin(m\theta), & \text{for } m<0.\end{cases} \quad (12)$$

By equating Eqs. (10) and (12), the sine and cosine dependence immediately cancels, and this leads to the following relation:

$$\sum_{k=0}^{K} b_{m+2k,m}N_{m+2k}^{m}R_{m+2k}^{m}(\lambda r)=\sum_{k=0}^{K} a_{m+2k,m}N_{m+2k}^{m}R_{m+2k}^{m}(r). \quad (13)$$

Note that we have taken into account only the case of $m \geq 0$; the case where $m<0$ can be treated in a similar manner. Let

$$\bar{R}_{m+2k}^{m}(r)=N_{m+2k}^{m}R_{m+2k}^{m}(r). \quad (14)$$

Equation (13) can be rewritten as

$$\sum_{k=0}^{K} b_{m+2k,m}\bar{R}_{m+2k}^{m}(\lambda r)=\sum_{k=0}^{K} a_{m+2k,m}\bar{R}_{m+2k}^{m}(r). \quad (15)$$

To solve Eq. (15), we will use the following basic results.

Lemma 1. Let a function $f(r)$ be expressed as

$$f(r)=\sum_{n=0}^{K} a_n P_n(r)=\sum_{n=0}^{K} b_n P_n(\lambda r), \quad (16)$$

where $P_n(r)$ is a polynomial of order n given by

$$P_n(r)=\sum_{k=0}^{n} c_{n,k}r^k, \quad c_{n,n}\neq 0; \quad (17)$$

then we have

$$b_i=\frac{1}{\lambda^i}\left[a_i+\sum_{n=i+1}^{K}\left(\sum_{k=i}^{n}\frac{c_{n,k}d_{k,i}}{\lambda^{k-i}}\right)a_n\right], \quad i=0,1,2,\ldots,K, \quad (18)$$

from which $C_K=(c_{n,k})$, with $0\leq k\leq n\leq K$, is a $(K+1)\times(K+1)$ lower triangular matrix, and $D_K=(d_{n,k})$ is the inverse matrix of C_K.

The proof of Lemma 1 is deferred to Appendix A.

We are interested in a special case of Lemma 1 for which each polynomial order n can be expressed as $n=m+qk$ where m and q are given positive integers, $k=0,1,\ldots,K$. The corresponding result is described in the following corollary.

Corollary. Given the positive integer numbers m, q, and K, let $P_n^m(r)$ be a set of polynomials defined as

$$P_n^m(r)=P_{m+qk}^m(r)=\sum_{s=0}^{k} c_{k,s}^m r^{m+qs}, \quad k=0,1,2,\ldots,K. \quad (19)$$

Let $f(r)$ be a function that can be represented as

$$f(r)=\sum_{k=0}^{K} a_{m+qk,m}P_{m+qk}^m(r)=\sum_{k=0}^{K} b_{m+qk,m}P_{m+qk}^m(\lambda r); \quad (20)$$

then we have

$$b_{m+qk}=\frac{1}{\lambda^{m+qk}}\left[a_{m+qk}+\sum_{i=k+1}^{K}\left(\sum_{j=k}^{i}\frac{c_{i,j}^m d_{j,k}^m}{\lambda^{(j-k)q}}\right)a_{m+qi}\right], \quad k=0,1,2,\ldots,K, \quad (21)$$

from which $D_K^m=(d_{i,j}^m)$ is the inverse matrix of $C_K^m=(c_{i,j}^m)$; both matrices are a $(K+1)\times(K+1)$ lower triangle matrix.

Both Lemma 1 and Corollary are valid for any type of polynomials. To apply them, an essential step consists of finding the inverse matrix D_K or D_K^m when the original matrix C_K or C_K^m is known. For the purpose of this paper, we are particularly interested in the use of Zernike polynomials. For the radial polynomials $R_{m+2k}^m(r)$ defined by Eq. (4), we have the following proposition.

Proposition 1. For the lower triangular matrix C_K^m whose elements $c_{k,s}^m$ are defined by Eq. (5), the elements of the inverse matrix D_K^m are given as follows:

$$d_{k,s}^m=\frac{(m+2s+1)k!\,(m+k)!}{(k-s)!\,(m+k+s+1)!}. \quad (22)$$

The proof of Proposition 1 is deferred to Appendix A.

For the normalized radial polynomials $\bar{R}_{m+2k}^m(r)$ defined by Eq. (14), it can be rewritten as

$$\bar{R}_{m+2k}^m(r)=N_{m+2k}^m R_{m+2k}^m(r)=\sqrt{\frac{2(m+2k+1)}{1+\delta_{m,0}}}R_{m+2k}^m(r)=\sum_{s=0}^{k}\bar{c}_{k,s}^m r^{m+2s}, \quad (23)$$

where

$$\bar{c}_{k,s}^m=\sqrt{\frac{2(m+2k+1)}{1+\delta_{m,0}}}c_{k,s}^m=(-1)^{k-s}\sqrt{\frac{2(m+2k+1)}{1+\delta_{m,0}}}\frac{(m+k+s)!}{s!\,(k-s)!\,(m+s)!}. \quad (24)$$

Since the normalization factor N_{m+2k}^m depends only on m and k, by using Proposition 1, we can easily derive the following result without proof.

Proposition 2. For the lower triangular matrix $\bar{C}_K^m$ whose elements $\bar{c}_{k,s}^m$ are defined by Eq. (24), the elements of the inverse matrix $\bar{D}_K^m$ are given as follows:

$$\bar{d}_{k,s}^m = \sqrt{\frac{1+\delta_{m,0}}{2(m+2s+1)}}\, d_{k,s}^m = \sqrt{\frac{(1+\delta_{m,0})(m+2s+1)}{2}}\, \frac{k\,!\,(m+k)!}{(k-s)\,!\,(m+k+s+1)!}. \tag{25}$$

We are now ready to establish the relationship between the two sets of Zernike coefficients $\{b_{m,m}, b_{m+2,m}, b_{m+4,m}, \ldots, b_{m+2K,m}\}$ and $\{a_{m,m}, a_{m+2,m}, a_{m+4,m}, \ldots, a_{m+2K,m}\}$ that appear in Eq. (13). Applying the Corollary to the normalized radial polynomials $\bar{R}_{m+2k}^m(r)$ with $q=2$ and using Eqs. (24) and (25), we have Theorem 1.

Theorem 1. For given integers m and K, and real positive number λ, let $\{b_{m,m}, b_{m+2,m}, b_{m+4,m}, \ldots, b_{m+2K,m}\}$ and $\{a_{m,m}, a_{m+2,m}, a_{m+4,m}, \ldots, a_{m+2K,m}\}$ be two sets of Zernike coefficients corresponding to the aperture sizes 1 and λ, respectively; we then have

$$\begin{aligned} b_{m+2k,m} &= \frac{1}{\lambda^{m+2k}}\left[a_{m+2k,m} + \sum_{i=k+1}^{K}\sum_{j=k}^{i}\left(\frac{\bar{c}_{i,j}^m \bar{d}_{j,k}^m}{\lambda^{2(j-k)}}\right)a_{m+2i,m}\right] \\ &= \frac{1}{\lambda^{m+2k}}\left[a_{m+2k,m} + \sum_{i=k+1}^{K} C(m,k,i)a_{m+2i,m}\right], \\ &\qquad k = 0,1,\ldots,K, \end{aligned} \tag{26}$$

where

$$C(m,k,i) = \sqrt{(m+2i+1)(m+2k+1)} \times \sum_{j=k}^{i}\frac{(-1)^{i-j}}{\lambda^{2(j-k)}}\frac{(m+i+j)!}{(i-j)\,!\,(j-k)\,!\,(m+j+k+1)!} \quad \text{for } i = k+1, k+2, \ldots, 0. \tag{27}$$

The relationship established in Theorem 1 is explicit, and the coefficient $b_{m+2k,m}$ depends only on the set of coefficients $\{a_{m+2k,m}, a_{m+2(k+1),m}, \ldots, a_{m+2K,m}\}$; thus, it is more simple than that given by Schwiegerling.[8] Note also that even though the above results were demonstrated for the case $m \geq 0$, they remain valid for $m < 0$ due to the symmetry property of the radial polynomials $R_n^m(r)$ about m.

Table 1 shows the conversion relationship between the coefficients $b_{n,m}$ and $a_{n,m}$ for Zernike polynomial expansions up to 45 terms (up to order 8). The results are the same as those given by Schwiegerling[8] except for $b_{1,m}$.

As correctly indicated by Schwiegerling,[8] an interesting feature can be observed from Table 1: For a given radial polynomial order n, the conversion from the original to the new coefficients has the same form regardless of the azimuthal frequency m. This can be demonstrated as follows.

Theorem 2. Let $C(m,k,i)$ defined by Eq. (27) be the coefficient of $a_{m+2i,m}$ in the expansion of $b_{m+2k,m}$ given by Eq. (26), and let $C(m+2l, k-l, i-l)$ be the coefficient of $a_{m+2i,m+2l}$ in the expansion of $b_{m+2k,m+2l}$ where l is an integer number less than or equal to k; then we have

$$C(m,k,i) = C(m+2l, k-l, i-l). \tag{28}$$

Proof. From Eq. (27), we have

$$\begin{aligned} C(m+2l,k-l,i-l) &= \sqrt{(m+2i+1)(m+2k+1)}\sum_{j=k-t}^{i-l}\frac{(-1)^{i-l-j}}{\lambda^{2(j-k+l)}}\frac{(m+l+i+j)!}{(i-l-j)\,!\,(j-k+l)\,!\,(m+l+j+k+1)!} \\ &= \sqrt{(m+2i+1)(m+2k+1)}\sum_{j=k-l}^{i}\frac{(-1)^{i-j}}{\lambda^{2(j-k)}}\frac{(m+i+j)!}{(i-j)\,!\,(j-k)\,!\,(m+j+k+1)!}. \end{aligned} \tag{29}$$

Table 1. Coefficient Conversion Relationships for Zernike Polynomial Expansions up to Order 8

n	m	New Expansion Coefficients $b_{n,m}$
0	0	$b_{0,m}=a_{0,m}-\sqrt{3}(1-\frac{1}{\lambda^2})a_{2,m}+\sqrt{5}(1-\frac{3}{\lambda^2}+\frac{2}{\lambda^4})a_{4,m}-\sqrt{7}(1-\frac{6}{\lambda^2}+\frac{10}{\lambda^4}-\frac{5}{\lambda^6})a_{6,m}+3(1-\frac{10}{\lambda^2}+\frac{30}{\lambda^4}-\frac{35}{\lambda^6}+\frac{14}{\lambda^8})a_{8,m}$
1	$-1,1$	$b_{1,m}=\frac{1}{\lambda}[a_{1,m}-2\sqrt{2}(1-\frac{1}{\lambda^2})a_{3,m}+\sqrt{3}(3-\frac{8}{\lambda^2}+\frac{5}{\lambda^4})a_{5,m}-4(2-\frac{10}{\lambda^2}+\frac{15}{\lambda^4}-\frac{7}{\lambda^6})a_{7,m}]$
2	$-2,0,2$	$b_{2,m}=\frac{1}{\lambda^2}[a_{2,m}-\sqrt{15}(1-\frac{1}{\lambda^2})a_{4,m}+\sqrt{21}(2-\frac{5}{\lambda^2}+\frac{3}{\lambda^4})a_{6,m}-\sqrt{3}(10-\frac{45}{\lambda^2}+\frac{63}{\lambda^4}-\frac{28}{\lambda^6})a_{8,m}]$
3	$-3,-1,1,3$	$b_{3,m}=\frac{1}{\lambda^3}[a_{3,m}-2\sqrt{6}(1-\frac{1}{\lambda^2})a_{5,m}+2\sqrt{2}(5-\frac{12}{\lambda^2}+\frac{7}{\lambda^4})a_{7,m}]$
4	$-4,-2,0,2,4$	$b_{4,m}=\frac{1}{\lambda^4}[a_{4,m}-2\sqrt{35}(1-\frac{1}{\lambda^2})a_{6,m}+3\sqrt{5}(3-\frac{7}{\lambda^2}+\frac{4}{\lambda^4})a_{8,m}]$
5	$-5,-3,-1,1,3,5$	$b_{5,m}=\frac{1}{\lambda^5}[a_{5,m}-4\sqrt{3}(1-\frac{1}{\lambda^4})a_{7,m}]$
6	$-6,-4,-2,0,2,4,6$	$b_{6,m}=\frac{1}{\lambda^6}[a_{6,m}-3\sqrt{7}(1-\frac{1}{\lambda^2})a_{8,m}]$
7	$-7,-5,-3,-1,1,3,5,7$	$b_{7,m}=\frac{1}{\lambda^7}a_{7,m}$
8	$-8,-6,-4,-2,0,2,4,6,8$	$b_{8,m}=\frac{1}{\lambda^8}a_{8,m}$

Comparing Eqs. (27) and (29), we obtain the result of the theorem.

Another interesting feature was also observed that is summarized in the following theorem.

Theorem 3. For a fixed value of N, let $N=m+2K=m'+2K'$; from Theorem 1, we have

$$b_{m+2(K-l),m}=\frac{1}{\lambda^{N-2l}}\left[a_{m+2(K-l),m}+\sum_{i=K-l+1}^{K}C(m,K-l,i)a_{m+2_i,m}\right]=\frac{1}{\lambda^{N-2l}}\left[a_{m+2(K-l),m}+\sum_{i=0}^{l-1}C(m,K-l,i+K-l+1)a_{N+2i-2l+2,m}\right],\quad l=0,1,\ldots,K, \tag{30}$$

$$b_{m'+2(K'-l),m}=\frac{1}{\lambda^{N-2l}}\left[a_{m'+2(K'-l),m'}+\sum_{i=K'-l+1}^{K'}C(m',K'-l,i)a_{m'+2_i,m'}\right]=\frac{1}{\lambda^{N-2l}}\left[a_{m'+2(K'-l),m'}+\sum_{i=0}^{l-1}C(m',K'-l,i+K'-l+1)a_{N+2i-2l+2,m'}\right],$$
$$l=0,1,\ldots,K. \tag{31}$$

Then

$$C(m,K-l,i+K-l+1)=C(m',K'-l,i+K'-l+1).$$
$$\text{for } i=0,1,\ldots,l-1,\ l=0,1,\ldots,\min(K,K'). \tag{32}$$

Proof. From Eq. (27), we have

$$C(m,K-l,i+K-l+1)=\sqrt{(m+2i+2K-2l+3)(m+2K-2l+1)}\times\sum_{j=k-l}^{i+K-l+1}\frac{(-1)^{i+K+1-l-j}}{\lambda^{2(j-K+l)}}\frac{(m+i+K-l+j+1)!}{(i+K-l+1-j)!\,(j-K+l)!\,(m+j+K-l+1)!}$$
$$=\sqrt{(N+2i-2l+3)(N-2l+1)}\sum_{j=0}^{i+1}\frac{(-1)^{i+1-j}}{\lambda^{2j}}\frac{(N+i-2l+j+1)!}{j!\,(i+1-j)!\,(N+j-l+1)!}. \tag{33}$$

Similarly,

$$C(m',K'-l,i+K'-l+1)=\sqrt{(m'+2i+2K'-2l+3)(m+2K'-2l+1)}\times\sum_{j=k'-l}^{i+K'-l+1}\frac{(-1)^{i+K'+1-l-j}}{\lambda^{2(j-K'+l)}}\frac{(m'+i+K'-l+j+1)!}{(i+K'-l+1-j)!\,(j-K'+l)!\,(m'+j+K'-l+1)!}$$
$$=\sqrt{(N+2i-2l+3)(N-2l+1)}\sum_{j=0}^{i+1}\frac{(-1)^{i+1-j}}{\lambda^{2j}}\frac{(N+i-2l+j+1)!}{j!\,(i+1-j)!\,(N+j-l+1)!}. \tag{34}$$

Comparison of Eqs. (33) and (34) shows that Eq. (32) is valid.

Table 2 shows the case of $N=m+2K=7$ for different values of m and K.

Table 2. Coefficient Conversion Relationships for Different Values of m and K Where $N=m+2K=7$

m	K	New Expansion Coefficients $b_{n,m}$
5	1	$b_{7,5}=\frac{1}{\lambda^7}a_{7,5}$
		$b_{5,5}=\frac{1}{\lambda^5}[a_{5,5}-4\sqrt{3}(1-\frac{1}{\lambda^2})a_{7,5}]$
3	2	$b_{7,3}=\frac{1}{\lambda^7}a_{7,3}$
		$b_{5,3}=\frac{1}{\lambda^5}[a_{5,3}-4\sqrt{3}(1-\frac{1}{\lambda^2})a_{7,3}]$
		$b_{3,3}=\frac{1}{\lambda^3}[a_{3,3}-2\sqrt{6}(1-\frac{1}{\lambda^2})a_{5,3}+2\sqrt{2}(5-\frac{12}{\lambda^2}+\frac{7}{\lambda^4})a_{7,3}]$
1	3	$b_{7,1}=\frac{1}{\lambda^7}a_{7,1}$
		$b_{5,1}=\frac{1}{\lambda^5}[a_{5,1}-4\sqrt{3}(1-\frac{1}{\lambda^2})a_{7,1}]$
		$b_{3,1}=\frac{1}{\lambda^3}[a_{3,1}-2\sqrt{6}(1-\frac{1}{\lambda^2})a_{5,1}+2\sqrt{2}(5-\frac{12}{\lambda^2}+\frac{7}{\lambda^4})a_{7,1}]$
		$b_{1,1}=\frac{1}{\lambda}[a_{1,1}-2\sqrt{2}(1-\frac{1}{\lambda^2})a_{3,1}+\sqrt{3}(3-\frac{8}{\lambda^2}+\frac{5}{\lambda^4})a_{5,1}-4(2-\frac{10}{\lambda^2}+\frac{15}{\lambda^4}-\frac{7}{\lambda^6})a_{7,1}]$

4. CONCLUSION

We have developed a method that is suitable to determine a new set of Zernike coefficients from an original set when the aperture size is changed. An explicit and rigorous demonstration of the proposed approach was given, and some useful features have been observed and proved. The new algorithm allows a fair comparison of aberrations, described in terms of Zernike expansion coefficients that were computed with different aperture sizes. The proposed method is simple, and can be easily implemented.

Note that the formulas derived in this paper are mathematically correct for all values of $\lambda = r_1/r_2$, where r_1 and r_2 represent the original and new aperture sizes. But for application purposes, it is still recommended to make r_2 less than r_1. In the case where r_2 is greater than r_1, the wavefront error data must be extrapolated outside the region of the original fit. It is worth mentioning that such a process could produce erroneous results since the Zernike polynomials are no longer orthogonal in this region and they have high-frequency variations in the peripheries.[8]

APPENDIX A

Proof of Lemma 1. Equation (16) can be expressed in matrix form as

$$f(r) = (a_0,a_1,a_2,\ldots,a_K)\begin{bmatrix} P_0(r) \\ P_1(r) \\ P_2(r) \\ \vdots \\ P_K(r) \end{bmatrix} = (b_0,b_1,b_2,\ldots,b_K)\begin{bmatrix} P_0(\lambda r) \\ P_1(\lambda r) \\ P_2(\lambda r) \\ \vdots \\ P_K(\lambda r) \end{bmatrix}. \tag{A1}$$

Using Eq. (17), we have

$$\begin{bmatrix} P_0(r) \\ P_1(r) \\ P_2(r) \\ \vdots \\ P_K(r) \end{bmatrix} = C_K \begin{bmatrix} 1 \\ r \\ r^2 \\ \vdots \\ r^K \end{bmatrix}, \tag{A2}$$

$$\begin{bmatrix} P_0(\lambda r) \\ P_1(\lambda r) \\ P_2(\lambda r) \\ \vdots \\ P_K(\lambda r) \end{bmatrix} = C_K \begin{bmatrix} 1 \\ \lambda r \\ \lambda^2 r^2 \\ \vdots \\ \lambda^K r^K \end{bmatrix} = C_K \,\mathrm{diag}(1,\lambda,\lambda^2,\ldots,\lambda^K)\begin{bmatrix} 1 \\ r \\ r^2 \\ \vdots \\ r^K \end{bmatrix}. \tag{A3}$$

Substitution of Eqs. (A2) and (A3) into Eq. (A1) yields

$$(a_0,a_1,a_2,\ldots,a_K)C_K\begin{bmatrix} 1 \\ r \\ r^2 \\ \vdots \\ r^K \end{bmatrix} = (b_0,b_1,b_2,\ldots,b_K)C_K \,\mathrm{diag}(1,\lambda,\lambda^2,\ldots,\lambda^K)\begin{bmatrix} 1 \\ r \\ r^2 \\ \vdots \\ r^K \end{bmatrix}. \tag{A4}$$

Thus

$$\begin{aligned}(b_0,b_1,b_2,\ldots,b_K) &= (a_0,a_1,a_2,\ldots,a_K)C_K\,(\mathrm{diag}(1,\lambda,\lambda^2,\ldots,\lambda^K))^{-1}C_K^{-1} \\ &= (a_0,a_1,a_2,\ldots,a_K)C_K\,\mathrm{diag}(1,\lambda^{-1},\lambda^{-2},\ldots,\lambda^{-K})D_K.\end{aligned} \tag{A5}$$

Equation (18) can be easily obtained by expanding Eq. (A5).

Proof of Proposition 1. To prove the proposition, we need to demonstrate the following relation:

$$\sum_{s=l}^{k} c_{k,s}^m d_{s,l}^m = \delta_{k,l}, \quad 0 \le l \le k \le K. \tag{A6}$$

For $k=l$, by using Eqs. (5) and (22), we have

$$c_{k,k}^m d_{k,k}^m = \frac{(m+2k)!}{k!\,(m+k)!} \times \frac{(m+2k+1)k!\,(m+k)!}{(m+2k+1)!} = 1. \tag{A7}$$

For $l<k$, we have

$$\sum_{s=l}^{k} c_{k,s}^m d_{s,l}^m = \sum_{s=l}^{k} \frac{(-1)^{k-s}(m+2l+1)(m+k+s)!}{(s-l)!\,(k-s)!\,(m+s+l+1)!} = (-1)^k(m+2l+1)\sum_{s=l}^{k} F(m,k,l,s), \tag{A8}$$

where

$$F(m,k,l,s) = \frac{(-1)^s(m+k+s)!}{(s-l)!\,(k-s)!\,(m+s+l+1)!}. \tag{A9}$$

Let

$$G(m,k,l,s) = \frac{(-1)^{s+1}(m+k+s)!}{(s-l)!\,(k+1-s)!\,(m+l+s)!}\,\frac{(k+1-s)(s-l)}{(k-l)(m+k+l+1)}. \tag{A10}$$

It can then be easily verified that

$$F(m,k,l,s) = G(m,k,l,s+1) - G(m,k,l,s). \tag{A11}$$

Thus

$$\sum_{s=l}^{k} F(m,k,l,s) = \sum_{s=l}^{k} [G(m,k,l,s+1) - G(m,k,l,s)] = G(m,k,l,k+1) - G(m,k,l,l) = 0. \tag{A12}$$

We deduce from Eq. (A8) that

$$\sum_{s=l}^{k} c_{k,s}^m d_{s,l}^m = 0 \quad \text{for } l<k. \tag{A13}$$

The proof is now complete.

Note that the proof of Proposition 1 was inspired by a technique proposed by Petkovsek *et al.*[16]

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