AN ADJOINT METHOD FOR UNDERWATER GEOACOUSTIC INVERSIONS BASED ON LOCAL AND NONLOCAL IMPEDANCE CONDITIONS

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ABSTRACT. The adjoint method of optimal control is applied to a parabolic approximation of the wave equation (a Schrödinger-type equation) in order to solve an inverse problem for the geoacoustical properties of the seabed. The physical configuration is one of shallow water and wideband signals. The aim is to calculate an equivalent medium which reproduces the acoustic transmission losses in the waveguide. Both local and nonlocal impedance boundary conditions are considered, the latter giving a model that is very close to the "ground truth".

1. INTRODUCTION

Bottom properties are essential in underwater acoustics for the prediction of acoustic losses and for sonar applications. The methods used in the acoustics community for solving inverse problems are usually based on a signal processing approach. These methods are very sensitive to the physical context and are not easily transportable from one physical configuration to another.

A data assimilation approach, based on the formulation of an adjoint problem should provide us with a robust inversion scheme which will be well-suited to a "rapid environmental assessment" system, where we would like to estimate the seabed properties in real time. We propose an optimal control approach with the impedance boundary condition playing the role of the control function. The result will be an equivalent medium that is not necessarily the ground truth, but that reproduces faithfully the acoustic losses. This same optimal control approach can also be applied to a more complicated initial boundary value problem, favored in underwater acoustics, where an absorbing layer is used instead of an impedance boundary condition. Here the controls will be the geoacoustic properties of the spongy layer.

We base our study on the paraxial approximation of the wave equation since this is well-suited to the waveguide geometry of our physical context. An inverse problem is then posed as a data assimilation problem where we seek to minimize the least-squares difference between measurements of the acoustic field on a line of hydrophones and the simulated field based on an initially unknown impedance boundary condition on the seafloor. Questions of existence of solutions to the inverse problem are tricky. The numerical implementation should combine robust methods that give maximal accuracy without being too computationally expensive.

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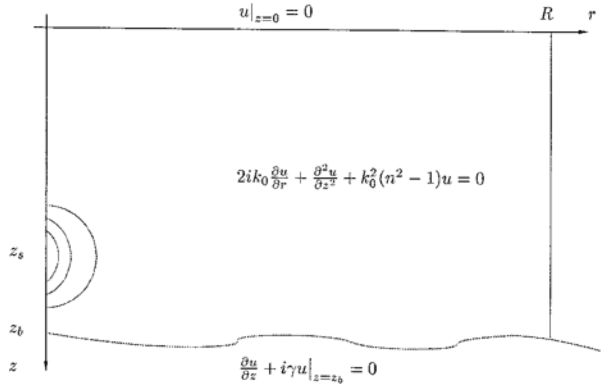

FIGURE 2.1. A waveguide for underwater acoustics.

The paraxial approximation for the solution of the wave equation is well documented - see []. The use of adjoint methods for the soltion of inverse problems originated with [LeDim] and has been applied to numerous data assimilation problems in oceanography [], in tatata [] and in tititi []. As far as existence goes, there has been work on the Schrödinger equation from the control theoretical viewpoint in [], [] and []. Furthermore, [Fern] has proved a convergence result for the exact gradient plus minimization algorithm applied to an inverse problem for the wave equation. The local impedance boundary condition is well known in the underwater acoustics field [Papadak], [Lee]. The use of a nonlocal condition is more recent, and was introduced by [Yev].

In this paper we first present the physical problem and its mathematical formulation by a paraxial approximation. We then define an optimal control problem and compute the exact gradient of the cost function by using an adjoint method. Next, we consider existence questions for the control and for the proposed algorithm. Finally we present the discrete formulation and some revealing numerical simulations.

2. A PARABOLIC EQUATION FOR WAVE PROPAGATION.

We consider a waveguide in the r-z (eventually (r, θ, z)) plane, where r denotes the range (horizontal distance from the origin) and z denotes the depth below the surface. A point source emits a signal from the depth $z = z_s$ which propagates in the waveguide and undergoes reflections from the surface $z = 0$ and from the seafloor $z = z_b$ where it is partially absorbed - see Figure 2.1. The signals are measured on one or more geophones situated on the line $r = R$.

Starting from the wave equation for the acoustic pressure,

$$\nabla^2 p + \frac{1}{c^2(r,z)} p_{tt} = 0\,,$$

we obtain a Helmholtz equation for a monochromatic wave of frequency $\omega = 2\pi f$,

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{\partial^2 p}{\partial z^2} + k_0^2 n^2 p = 0,$$

where $k_0 = \omega/c_0$, $n = c_0/c(z)$ and c_0 is the reference acoustic velocity. If we now suppose a slow envelope of variation, the acoustic pressure can be factorized by a Hankel function,

$$p(r,z) = u(r,z) H_0^{(1)}(k_0 r)\,.$$

We then apply two successive approximations: the far-field approximation and the small aperture approximation. Finally we obtain the simplest parabolic equation,

$$2ik_0 \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} + k_0^2(n^2 - 1)u = 0 \qquad (2.1)$$

for the factorised pressure, u . The transmission losses are then given by

$$\mathrm{TL} = 20 \log_{10} |p|\,.$$

Details can be found in [Stu, Tap].

In order to obtain a well-posed initial boundary value problem, we add an "initial" condition $u(r = 0, z = z_s)$ and suitable boundary conditions on the surface $z = 0$ and on the sea-bottom $z = z_b$.

2.1. **Direct Numerical Simulations.** We show a few direct simulations of the parabolic equation (2.1). The numerical method used is an implicit Crank-Nicolson finite difference discretization - see [Stk]. The physical data were: $c_0 = 1520\,\mathrm{ms}^{-1}$, $c(z) = 1500\,\mathrm{ms}^{-1}$, $f = 250\,\mathrm{Hz}$, $z_s = 135\,\mathrm{m}$, $z_b = 135\,\mathrm{m}$, $R = 5500\,\mathrm{m}$. The Figure 2.2 compares the losses, $|u(r,z)|$, for a Dirichlet condition ($u = 0$) on the seabed $z = z_b$ with those of a Neumann condition ($\frac{\partial u}{\partial z} = 0$) and the Robin condition, $\frac{\partial u}{\partial z} + i\gamma u = 0$ with $\gamma = -1$, which describes partial transmission. This simulation corresponds quite well to actual signal logs recorded during measurement campaigns.

2.2. **Energy.** In the sequel we will need estimates of the energy of the solution to (2.1). As for the wave equation, we define the energy as

$$E(t) = \int_\Omega |u_t|^2 + |\nabla u|^2 \, dx\,.$$

We can readily show that this energy is conserved and thus

$$E(t) = E(0) = \mathrm{cst}.$$

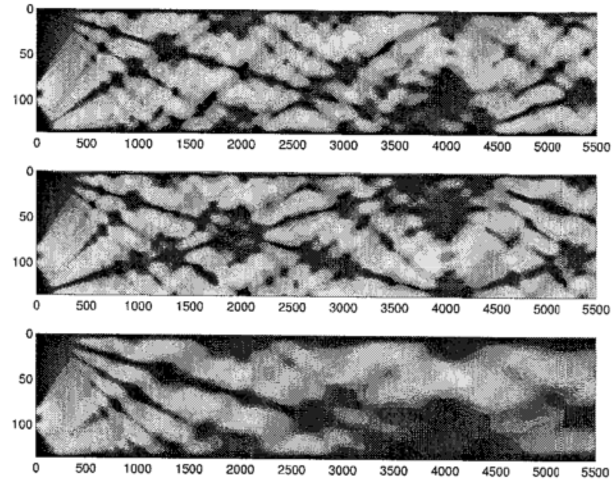

FIGURE 2.2. Direct simulations of the parabolic equation with a Dirichlet condition (top), a Neumann condition (middle) and a Robin condition (bottom) at $z = z_b$.

3. OPTIMAL CONTROL FOR THE INVERSE PROBLEM

In order to solve the inverse problem we propose to use an impedance boundary condition. We consider two boundary conditions for the equivalent seabottom. The first condition is a local one (LBC). This is the classical Robin or mixed condition. The second is a non local absorbing boundary condtion (NLBC) and was introduced by [Y], originally for the Fresnel equations and then for the paraxial wave equation [YJ]. The advantage of the second condition is its closeness to the physical reality and its ability to better model the problem at hand. However, its implementation is more complex.

3.1. **A local impedance boundary condition.** We suppose that the boundary condition on the seafloor $z = z_b$ is

$$\frac{\partial u}{\partial z} + i\gamma(r)u = 0 \qquad (3.1)$$

where γ is a real-valued function depending on the range r. The value of γ is not arbitrary, since the boundary condition must dissipate the energy of the system. In order to check this, we multiply the parabolic equation by $\overline{u}$ (the complex conjugate of u) and we take the real part:

$$\frac{1}{2}\frac{\partial |u|^2}{\partial r} + \Re\left(\frac{-i}{2k_0}\frac{\partial^2 u}{\partial z^2}\overline{u}\right) = 0.$$

We then integrate between $z = 0$ and $z = z_b$, and find

$$\frac{1}{2}\frac{d}{dr}\int_{z=0}^{z=z_b} |u|^2\, dz + \left[Re(\frac{-i}{2k_0}\frac{\partial u}{\partial z}\overline{u})\right]_{z=0}^{z=z_b} = 0,$$

that is

$$\frac{1}{2}\frac{d}{dr}\int_{z=0}^{z=Hz_b} |u|^2\, dz = \frac{1}{2k_0}u\overline{u}(r,H)Re(\gamma).$$

The energy will thus decrease when $Re(\gamma) \leq 0$.

3.2. **The adjoint method.** Let us suppose now that we measure $u_d(R,z)$ for a given range $r = R$. We would like to recover the boundary coefficient $\gamma(r)$ which plays the role of the control function. Let the cost function be (we can always add a penalisation term on γ)

$$J(\gamma) = \frac{1}{2}\int_{z=0}^{z=H} |u(\gamma;R,z) - u_d(R,z)|^2,$$

where $u(\gamma;r,z)$ is obtained from solving the parabolic equation with the boundary condition (3.1). The minimisation problem is thus to find

$$\inf_{\gamma\in\mathcal{G}_{ad}} J(\gamma).$$

where $\mathcal{G}_{ad}$ is a suitably defined space of admissible controls. The necessary condition for the existence of a minimum is given by the following basic theorem.

Theorem. *If J attains a (local) minimum at a point $\gamma_* \in \mathcal{G}$, then for any $\phi \in \mathcal{G}$*

$$\delta J(\gamma_*;\phi) = 0$$

where

$$\delta J(\gamma;\phi) = \lim_{\epsilon\to 0}\frac{J(\gamma+\epsilon\phi) - J(\gamma)}{\epsilon}$$

is the Gâteaux derivative of J at γ in the direction ϕ.

We must start by taking the variation of J

$$\begin{aligned}\lim_{t\to 0}\frac{J(\gamma+t\varphi) - J(\gamma)}{t} &= \frac{1}{2}\lim_{t\to 0}\int_{z=0}^{z=H}\frac{|u(\gamma+t\varphi;R,z) - u_d(R,z)|^2 - |u(\gamma;R,z) - u_d(R,z)|^2}{t} \\ &= J'(\gamma,\varphi) = g'(0),\end{aligned}$$

where g is the function defined by

$$g(t) = J(\gamma + t\varphi).$$

We introduce the real-valued scalar product

$$< u, v >= Re(u\overline{v}).$$

This scalar product satisfies

$$< u, u >= |u|^2, \quad < iu, v >= - < u, iv >,$$

$$< u, v >=< v, u >, \quad < \gamma u, v >=< u, \overline{\gamma} v >,$$

and the derivative formula

$$d|u|^2 = 2 < u, du > .$$

We then have

$$g'(t) = \int_{z,r=R} < u(\gamma+t\varphi) - u_d, \frac{du}{dt}(\gamma+t\varphi) > dz,$$

where $w = du/dt$ is the solution of the tangential problem (we suppose that $n^2 = 1$ for simplicity)

$$\begin{aligned}2ik_0 w_r + w_{zz} &= 0, \\ w(0,z) &= 0, \\ w(r,0) &= 0, \\ w_z(r,z_b) + i\gamma w(r,z_b) &= -i\varphi(r)u(\gamma+t\varphi;r,z_b).\end{aligned}$$

The Gâteau derivative of J is then, with $t = 0$,

$$J'(\gamma,\varphi) = \int_{z,r=R} < u - u_d, w > dz.$$

The adjoint state p is written here as

$$\begin{aligned}2ik_0 p_r + p_{zz} &= 0, \\ p(R,z) &= u - u_d, \\ p(r,0) &= 0, \\ p_z(r,z_b) - i\overline{\gamma}p(r,z_b) &= 0.\end{aligned}$$

This equation is backwards in r, and the boundary condition is still dissipative in spite of the change of sign of γ! Integrating by parts, we obtain

$$\int_z\int_r < w_r - \frac{i}{2k_0}w_{zz}, p >= 0,$$

$$\int_z\int_r < w_r, p > -\frac{1}{2k_0} < iw_{zz}, p >= 0,$$

$$\int_z\int_r - < w, p_r > + \int_{z,r=R} < w, p > -\frac{1}{2k_0}\int_z\int_r < iw, p_{zz} > +$$
$$\frac{1}{2k_0}\int_{r,z=z_b} < iw, p_z > -\frac{1}{2k_0}\int_{r,z=z_b} < iw_z p >= 0,$$

$$\int_{z,r=R} < w, u-u_d > +\frac{1}{2k_0}\int_{r,z=z_b} < iw, p_z > -\frac{1}{2k_0}\int_{r,z=z_b} < i(-i\gamma w - iu\varphi), p >= 0,$$

$$\int_{z,r=R} < w, u-u_d > +\frac{1}{2k_0}\int_{r,z=z_b} < iw, p_z - i\gamma p > +\frac{1}{2k_0}\int_{r,z=z_b} < i(iu\varphi), p >= 0,$$

$$\int_{z,r=R} < u - u_d, w >= \frac{1}{2k_0}\int_{r,z=z_b} < u\varphi, p >,$$

whence the result

$$J'(\gamma,\varphi) = \frac{1}{2k_0}\int_{r,z=z_b} < \overline{u}p, \varphi >,$$

which can be rewritten as

(3.2) $$\nabla J = \frac{\overline{u}p}{2k_0}.$$

3.3. A non local impedance boundary condition. The nonlocal boundary condition provides us with a formula for the acoustic pressure field u at range $r + \Delta r$ in terms of the already calculated field between 0 and r at $z = z_b$. This is obtained by expanding the vertical wave number operator in powers of an exponential translation operator $R = \exp(-\Delta r\, \partial_r)$. Note that

$$R^j u(r,z) = u(r - j\Delta r, z).$$

In terms of the operator R, the Crank-Nicolson discretization of our paraxial equation becomes

$$\left[\frac{\partial^2}{\partial z^2} + \Gamma_0^2\right] u(r + \Delta r, z) = 0 \tag{3.3}$$

where the $z-$space vertical wave number operator is defined by

$$\Gamma_0^2 = k_0^2\left(n_b^2 - 1 + v^2\frac{1-R}{1+R}\right)$$

with $v^2 = 4i/k_0\Delta r$, the index b relates to the bottom parameters and the index w to those of the water at the water-sediment interface. Factoring the equation (3.3) into upgoing and downgoing components, we can identify the one-way radiation condition satisfied by the downgoing field at $z = z_b$ as

$$\left[\frac{\partial}{\partial z} - i\frac{\rho_w}{\rho_b}\Gamma_0\right] u(r + \Delta r, z_b) = 0. \tag{3.4}$$

This equation acounts for the *total* impedance jump (sound speed, attenuation and density) encountered by waves that cross the lower boundary of the waveguide. It is for this reason that we obtain an excellent model of the physical reality. From a numerical point of view, this nonlocal impedance condition is computed by expanding Γ_0 in terms of R, which gives

$$\left[\frac{\partial}{\partial z} - i\beta\right] u\left[(J+1)\Delta r, z_b\right] = i\beta \sum_{j=1}^{J+1} g_{0,j} u\left[(J+1-j)\Delta r, z_b\right]$$

where $\beta = (\rho_w/\rho_b)k_0\sqrt{n_b^2 - 1 + \nu^2}$ and the $g_{0,j}$ are the terms of the expansion of Γ_0. We note that the right-hand side depends only on the known values of the pressure field along the interface $z = z_b$. This is clearly convenient for a marching type scheme of numerical solution.

Our approach is to generalize these equations in order to be able to treat any type of sea bottom. To this end we rewrite the nonlocal impedance condition as

$$\left[\frac{\partial}{\partial z} - i\beta(r)\right] u(r, z_b) = F(r)$$

where β an F are complex functions of r. This yields a vector function for the control which is now defined as

$$\psi = \begin{bmatrix} \beta \\ F \end{bmatrix}.$$

We do not go through the detailed derivation of the gradient (see [JCLth]) but only present the final result. It suffices to say that the steps are identical. Finally

we obtain

$$\nabla J = \begin{bmatrix} -\frac{\overline{u(r,z_b)}p(r,z_b)\big|_{r\in[0,R]}}{2k_0} \\ -\frac{ip(r,z_b)\big|_{r\in[0,R]}}{2k_0} \end{bmatrix}. \tag{3.5}$$

3.4. The conjugate gradient method. Once the gradient of the cost fonction ∇J is known, we can seek a local minimum of $J(\gamma)$. The simplest method for doing this is by steepest descent which uses the update

$$\gamma^{(n+1)} = \gamma^{(n)} - \alpha\nabla J(\gamma^{(n)}),\ \alpha > 0,$$

for $n = 0, 1, \ldots$until convergence. In order to accelerate the convergence we will use a conjugate gradient method of Fletcher-Reeves or Polak-Ribière type (see [NR]). Here the update is given by

$$\gamma^{(n+1)} = \gamma^{(n)} + \alpha_n p_n,$$

where α_n is the step-length that minimises J in the direction p_n, and this direction is computed in two steps:

$$\beta_n = \frac{\nabla J_{n+1}^T \nabla J_{n+1}}{\nabla J_n^T \nabla J_n}$$

$$p_{n+1} = -\nabla J_{n+1} + \beta_{n+1} p_n.$$

We point out that our gradients are complex valued functionals. This implies a very careful implementation of the minimization algortihm - see below.

3.5. Summary: formulation of the inverse problem. We recapitulate. The inverse problem for a local (respectively nonlocal) impedance boundary condition is: "For measurements of an acoustic field $u_d(R, z)$ at $r = R$ and $0 \le z \le z_b$, calculate the impedance boundary control $\gamma(r)$ (respectively $\psi(r) = [\beta(r)\quad F(r)]^T$) that minimizes the cost function

$$J(\gamma) = \frac{1}{2}\int_{z=0}^{z=z_b} |u(\gamma; R, z) - u_d(R, z)|^2\, dz, \tag{3.6}$$

where $u(r, z)$ is the solution of the paraxial wave equation

$$2ik_0\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} + k_0^2(n^2 - 1)u = 0 \tag{3.7}$$

in the rectangular domain $[0 \le r \le R] \times [0 \le z \le z_b]$, with known initial condition

$$u(0, z) = u_0(z), \tag{3.8}$$

known surface boundary condition

$$u(r, 0) = 0 \tag{3.9}$$

and unknown bottom boundary condition

$$\frac{\partial u}{\partial z} + i\gamma(r)u\bigg|_{z=z_b} = 0 \tag{3.10}$$

(respectively

$$\left[\frac{\partial}{\partial z} - i\beta(r)\right] u(r, z_b) = F(r) \tag{3.11}$$

)."

We assume that the initial boundary value problem (3.7, 3.8, 3.9, 3.10) (respectively (3.7, 3.8, 3.9, 3.11)) is well-posed and we seek the control $\gamma(r)$ (respectively $\psi(r)$) in a suitable space of admissible functions.

4. EXISTENCE FOR THE INVERSE PROBLEM

The question of existence of a solution to this inverse problem can be approached in two ways:

(1) Through the use of control theory by proving controllability (or observability.)
(2) By showing that the algorithm made up of the direct problem, the adjoint problem, the gradient calculation and the mimimization converges to a critical point of the cost functional.

We will consider these two possibilities separately, but first of all we consider the equation itself with the (local) impedance condition....

4.1. Control theory and existence. Let us recall what is known about the controllability of the Schrödinger equation.

4.2. Convergence theory and existence. This approach attempts to avoid the theoretical problems of control theory for non linear problems but, as we will see, cannot avoid some rather technical estimations.

5. NUMERICAL RESULTS

5.1. Discretization and precision.

5.2. Minimization of a complex functional.

5.3. Results for the local impedance condition.

5.4. Results for the nonlocal impedance condition. In the Figure 5.1 we show the initial acoustic field (top), the true field (middle) and the field obtained after inversion (bottom). It should be noted that the initial field is a propagating field with very little reflection, whereas the field we seek is a highly reflective one. The inversion succeeds admirably in both the reconstruction of the initial condition at $r = 0$ as well as the principal features of the field. Since the observation is at $r = 5500$ m we have more precision for large ranges than for ranges close to the source. The relative errors in a 2-norm, range from 2% at $r = 5500$ to 20% around $r = 0$. These errors can be seen in Figure 5.2.

REFERENCES

[Stu] Sturm, F. *Doctoral Thesis.* Université de Toulon et du Var (1999).

[Tap] Tappert F., Smith K. *UMPE Version1.1.* MPL Technical Memorandum 432, University of Miami (1994).

[Stk] Strikwerda, *Finite difference schemes and partial differential equations,* Wadsworth and Brooks Cole, 1989.

[NR] Press, Flannery, Teukolsky, Vetterling, *Numerical Recipes*, Cambridge University Press, 1986.

[ALG] Asch, Le Gac, *Adjoint method for geoacoustical inversions with local and nonlocal impedance boundary conditions,* in preparation, 2002.

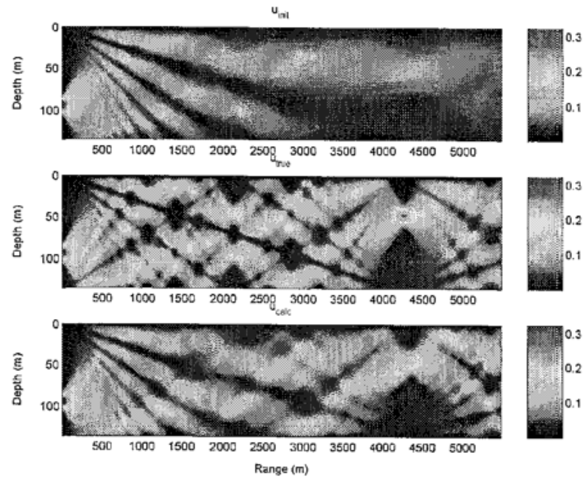

FIGURE 5.1. Initial (top), true (middle) and inverted (bottom) fields for the nonlocal boundary condition.

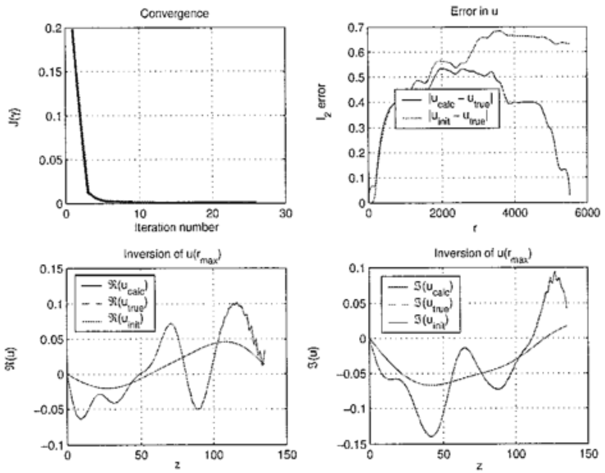

FIGURE 5.2. Convergence and errors for the inversion with nonlocal boundary condition.