Elemental numerical methods

S1

Numerical integration

1. Recall the principles of the composit Gauss-Legendre numerical integration method and write a program that computes this algorithm
2. Apply your programm to the following integrals

$$\int_0^1 \exp(-t)dt,$$

$$\int_0^1 \ln(t)dt.$$

   How many intervals do you need for achieving a precision of $\epsilon = 10^{-8}$?

3. Explain why the method is much more precise for the first integral than for the second.

Numerical integration in higher dimensions

We consider a perfect incompressible fluid flowing from a source located at the point $(0, 0, 1)$ to a well located at $(0, 0, -1)$. The flow potential is given by

$$\phi(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + (z-1)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z+1)^2}}.$$

The velocity field is

$$u = -\nabla\phi.$$

We consider now the quadrilateral $Q = (A, B, C, D)$ in the plane $z = 0$ with $A = (0, 0, 0)$, $B = (1, 0, 0)$, $C = (1/2, 1, 0)$ and $D = (0, 1, 0)$. The normal vector to this surface, oriented in the direction of the flow, is

$$n = (0, 0, -1).$$

We want to compute numerically

$$I = \int_Q u \cdot n dxdy.$$

1. We define $f(x, y) = u(x, y, 0) \cdot n = 2\left(x^2 + y^2 + 1\right)^{-3/2}$. Check that

$$I = \int_Q f(x, y)dxdy.$$

2. Compute the exact value of I.
3. For the numerical integration, we split the big quadrilateral Q into small pieces Q_{ij}, $i = 1 \cdots N$, $j = 1 \cdots N$, where N is a given integer (the bigger N is the more precise will be the numerical integration). We define $h = 1/N$. The small quadrilateral Q_{ij} is $(A_{ij}, B_{ij}, C_{ij}, D_{ij})$ with $A_{ij} = ((i-1)h(1 - \frac{j-1}{2N}), (j-1)h, 0)$, $B_{ij} = (ih(1 - \frac{j-1}{2N}), (j-1)h, 0)$, $C_{ij} = (ih(1 - \frac{j}{2N}), jh, 0)$ and $D_{ij} = ((i-1)h(1 - \frac{j}{2N}), jh, 0)$. With consider a reference square $C = [0, 1] \times [0, 1]$. This square is transformed into Q_{ij} thanks to the transformation

$$(\xi, \eta) \to \tau_{ij}(\xi, \eta) = (1 - \xi)(1 - \eta)A_{ij} + \xi(1 - \eta)B_{ij} + \xi\eta C_{ij} + (1 - \xi)\eta D_{ij}.$$

The integral I can be written

$$I = \sum_{ij} I_{ij} = \sum_{ij} \int_{Q_{ij}} u \cdot n \tag{1}$$

and with a change of variables

$$I_{ij} = \int_{\xi=0}^{1} \int_{\eta=0}^{1} f(\tau_{ij}(\xi,\eta)) \left|\tau'_{ij}(\xi,\eta)\right| d\xi d\eta.$$

Verify that

$$\left|\tau'_{ij}(\xi,\eta)\right| = \frac{2N - j + 1 - \eta}{2N^3}$$

$$\tau_{ij}(\xi,\eta) = (\frac{i + j - 1 - 2N + \eta + \xi - \eta i + 2iN - \xi j + 2\xi N - ij - \xi\eta}{2N^2}, \frac{j - 1 + \eta}{N})$$

4. Write a program that computes an approximation of (1) with the two-dimensional Gauss-Legendre integration method. Verify that the numerical value I_N tends rapidly to the exact value I when N increases.