A simple entropy fix for the VFRoe schemes

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Outlines

Finite volumes

- Approximation of

$$\partial_t W + \partial_x F(W) = 0 \quad + \text{entropy condition}$$

- Mesh $x_i = i\Delta x,\ t_n = n\Delta t, W_i^n \simeq W(x_i, t_n)$
- Finite volume approach

$$\frac{W_i^{n+1} - W_i^n}{\Delta t} + \frac{F_{i+1/2}^n - F_{i-1/2}^n}{\Delta x} = 0$$

- Numerical flux

$$F_{i+1/2}^n = F(W_i^n, W_{i+1}^n)$$

- Conservative variables $W(Y)$ and primitive variables Y

$$\partial_t Y + A(Y)\partial_x Y = 0$$

Example: Rusanov scheme

The numerical flux of the Rusanov scheme is given by

$$F(W_L, W_R) = \frac{F(W_L) + F(W_R)}{2} - \frac{\lambda}{2}(W_R - W_L)$$

where

$$\lambda = \max(\rho(A(Y_L)), \rho(A(Y_R)))$$

The Rusanov scheme generally satisfies a numerical entropy dissipation principle. It is robust but very dissipative.

VFRoe approach

- solve the linearized Riemann problem

$$\partial_t Y + A(\overline{Y})\partial_x Y = 0$$

$$\overline{Y} = \frac{Y_L + Y_R}{2} \quad Y(x,0) = \begin{cases} Y_L \,\text{if}\, x < 0, \\ Y_R \,\text{if}\, x > 0. \end{cases}$$

The solution is noted

$$Y(x,t) = R(Y_L, Y_R, x/t)$$

- The numerical flux of the VFRoe scheme is then

$$F(W_L, W_R) = F(R(Y_L, Y_R, 0))$$

Linearized Riemann problem

The solution of the linearized Riemann problem is given by

$$Y(x,t) = R(Y_L, Y_R, x/t) = \frac{Y_L + Y_R}{2} - \frac{1}{2}\mathrm{sgn}(A(\overline{Y}) - \frac{x}{t} I)(Y_R - Y_L)$$

with

$$\mathrm{sgn(x)} = \begin{cases} 1 & \text{if} \quad x > 0, \\ 0 & \text{if} \quad x = 0, \\ -1 & \text{if} \quad x < 0. \end{cases}$$

The sgn function of the matrix A can be defined as follow.
Let $\lambda_1 < \lambda_2 < \cdots < \lambda_m$ be the ordered eigenvalues of A.
Let P be the interpolation polynomial of the sgn function on the eigenvalues of A:

$$d^\circ P \leq m-1$$

$$P(\lambda_i) = \mathrm{sgn}(\lambda_i) \quad i = 1 \cdots m$$

Then

$$\mathrm{sgn}(A) := P(A)$$

An efficient way to compute P is to use the Newton algorithm

$$P(A) = \mathrm{sgn}[\lambda_1] + \mathrm{sgn}[\lambda_1, \lambda_2](\mathrm{A} - \lambda_1 \mathrm{I}) + \cdots$$

$$+\mathrm{sgn}[\lambda_1 \cdots \lambda_m](A - \lambda_1 I) \cdots (A - \lambda_{m-1} I)$$

$$\mathrm{sgn}[\lambda_i] := \mathrm{sgn}(\lambda_i)$$

$$\mathrm{sgn}[\lambda_1 \cdots \lambda_{i+1}] = \frac{\mathrm{sgn}[\lambda_2 \cdots \lambda_{i+1}] - \mathrm{sgn}[\lambda_1 \cdots \lambda_i]}{\lambda_{i+1} - \lambda_1}$$

- easy to handle the case of multiple eigenvalues (away from 0)
- the computation of the eigenvectors is not necessary
- complexity equivalent to the Hörner algorithm ($\sim m-1$ matrix vector products)

A simple entropy fix

- The precision of the VFRoe scheme is equivalent to the precision of the Godunov or the Roe scheme.
- The choice of the primitive variables is important (and problem dependant) [2]
- The cost and simplicity are very interesting ($\sim$Rusanov + 15%)
- But an entropy fix is needed in sonic waves

We propose to follow the very simple idea: replace the VFRoe flux by the Rusanov flux if a sonic wave is present.
More precisely, if for a genuinely non-linear field we have

$$\lambda_i(W_L) < 0 < \lambda_i(W_R)$$

then replace the VFRoe flux by the Rusanov flux.

- No small parameter as for other entropy fix
- fast

It is not clear why it should work: numerical tests for the moment...

Numerical results

We first consider a Riemann problem for the Euler system with a strong rarefaction wave

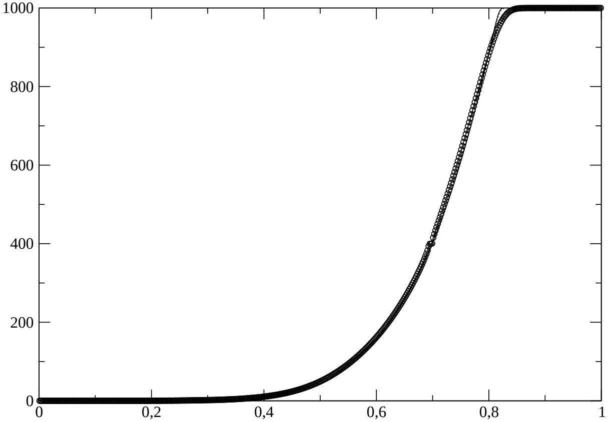
$$W = (\rho, \rho u, \frac{p}{\gamma - 1} + \frac{\rho u^2}{2})$$

$$F(W) = (\rho u, \rho u^2 + p, \frac{\gamma p u}{\gamma - 1} + \frac{\rho u^3}{2})$$

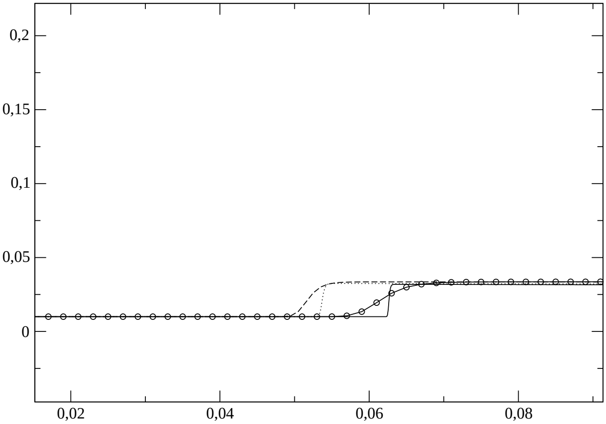
$$Y = (\rho, u, s = \frac{p}{\rho^\gamma})$$

$\gamma = 1.4,\ CFL = 1/2,\ \rho_L = 0.01,\ u_L = 0,\ p_L = 5,\ \rho_R = 1000,\ u_R = 0,$
$p_R = 10^5$
The initial jump is at $x = 1/2$

Density



Density (zoom)



Density profiles obtained by using 500 cells (circles), 1000 cells (dashes), 5000 cells (dotted), 10000$ cells (plain).

Magnetohydrodynamics

The MHD equations with divergence cleaning [3] read

$$W = \left(\rho, \rho u^T, \frac{p}{\gamma - 1} + \frac{\rho u \cdot u + B \cdot B}{2}, B^T, \psi\right)^T$$

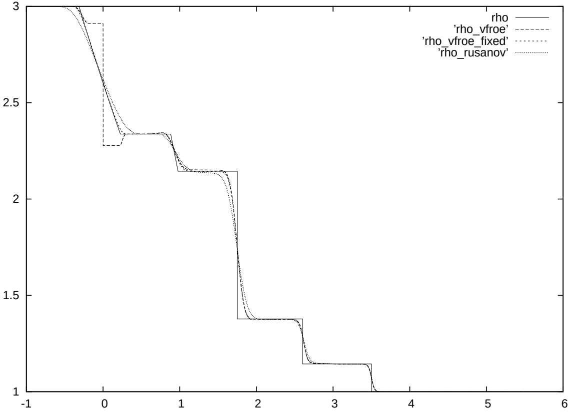
$$u = (u_1, u_2, u_3)^T, \quad B = (B_1, B_2, B_3)^T, \quad n = (1, 0, 0)^T$$

$$F(W) = \begin{pmatrix} \rho u \cdot n \\ \rho(u \cdot n)u + (p + \frac{B \cdot B}{2})n - (B \cdot n)B \\ (\frac{\gamma p}{\gamma - 1} + \frac{\rho u \cdot u}{2} + B \cdot B)u \cdot n - (B \cdot u)(B \cdot n) \\ (u \cdot n)B - (B \cdot n)u + \psi n \\ c_h^2 B \cdot n \end{pmatrix}$$

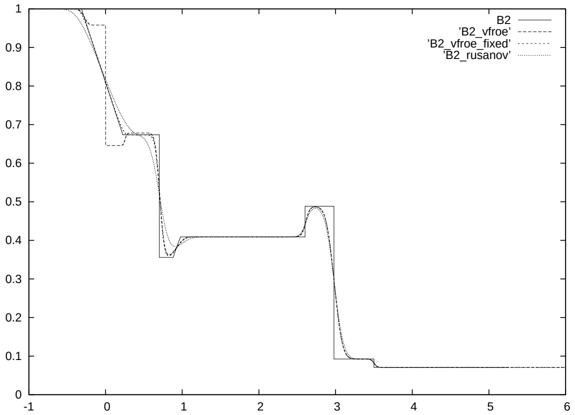
$$Y = (\rho, u^T, p, B^T, \psi)^T$$

$\rho_L = 3,\ u_L = (1.3, 0, 0)^T,\ p_L = 3,\ B_L = (1.5, 1, 1)^T,\ \psi_L = 0$
$\rho_R = 1,\ u_R = (1.3, 0, 0)^T,\ p_R = 1,\ B_R = (1.5, \cos(1.5), \sin(1.5))^T,$
$\psi_R = 0$
$CFL = 0.8,\ x \in [-1; 6].$
$c_h = 3.8$
$\gamma = 5/3$
The initial jump is at $x = 0$
We take 2000 cells

Density



Magnetic field B_2



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