GEOMETRIC INEQUALITIES AND GENERALIZED RICCI BOUNDS IN THE HEISENBERG GROUP*

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ABSTRACT. We prove that no curvature-dimension bound $CD(K, N)$ holds in any Heisenberg group $\mathbb{H}_n$. On the contrary the measure contraction property $MCP(0, 2n+3)$ holds and is optimal for the dimension $2n+3$. For the non-existence of a curvature-dimension bound, we prove that the generalized "geodesic" Brunn-Minkowski inequality is false in $\mathbb{H}_n$. We also show in a new and direct way, (and for all $n \in \mathbb{N}\backslash\{0\}$) that the general "multiplicative" Brunn-Minkowski inequality with dimension $N > 2n+1$ is false.

INTRODUCTION

The Heisenberg group $\mathbb{H}^n$ turns up both in many parts of mathematics (see [25]) and in other scientific or technical domains. A reason for this is that it is the most basic and representative space of sub-Riemannian geometry, playing the role of $\mathbb{R}^N$ in Riemannian geometry. Many analytical tools have been developed in both settings. In particular $\mathbb{H}^n$ and $\mathbb{R}^N$ both have a doubling measure and satisfy a Poincaré inequality. This last setting has proved to be very efficient as a minimal framework permitting to generalize conformal geometry to some metric measure spaces (see [13] and the references therein). Recent developments tend to improve this analysis and introduce for some spaces analysis of second order. In particular it was very challenging to define metric spaces with a lower bound on the curvature. For sectional curvature, Alexandrov spaces were defined more than fifty years ago (see [6]). For Ricci curvature an amazing theory has been recently developed independently (but using essentially the same ideas) by Sturm (see [27], [28]) and by Lott and Villani (see [17], [18]) in terms of geometric curvature-dimension condition $CD(K, N)$ (The definition is different from the curvature-dimension of Bakry and Émery defined in [3]).

For many metric measure spaces, this very recent condition is stronger than the condition that there exists a Poincaré inequality and the measure is doubling. It uses an old probability tool: optimal transport of measure. This theory deals with the so-called Monge-Kantorovich problem, namely the problem of transporting one probability measure into another one whilst minimizing a transport cost (usually the square of the distance). For Riemannian manifolds and two absolutely continuous measures, McCann proved that there is a unique geodesic interpolation between them (see [19]) and studied the particular expression it takes. From this Cordero-Erausquin, McCann and Schmuckenschläger explained in [8] that the optimal transport of measure has a particular form if the Ricci curvature of the

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manifold is bounded below. In particular the transported measure has a relative entropy whose level of convexity along the transport depends on the lower bound of the Ricci curvature. Inspired by this observation Lott, Sturm and Villani defined the curvature-dimension condition $CD(K,N)$ and developed a theory of synthetic Ricci curvature in metric measure spaces. An other essential fact is that inspired by [19], Ambrosio an Rigot recently proved in [1] that the geodesic solutions of the Monge problem in the Heisenberg group $\mathbb{H}^n$ have a similar expression as the one for the Riemannian manifolds. We answer this question in this paper whether curvature-dimension condition also holds in this space.

The measure contraction property $MCP(K,N)$ is another geometrical property that involves curves in the space of measures on a given metric measure space. In a certain sense, the definition of the MCP involves curves in the space of measures one of whose extremities corresponds to a Dirac mass. The other measure is contracted onto this Dirac mass and this contraction reveals some geometrical aspects of the space. As CD, the measure contraction property can be seen as a generalization of the Ricci lower bound of a Riemannian manifold. As written in the appendix, under the hypothesis that there is almost surely a unique geodesic between two points (this is the case in $\mathbb{H}^n$) the curvature-dimension condition $CD(K,N)$ implies the measure contraction $MCP(K,N)$ and this last property implies that the metric measure space satisfies a Poincaré inequality and is doubling.

In this paper we prove the following theorem:

Main Theorem. *Let n be a non negative integer. We consider $(\mathbb{H}^n, d_{CC}, \mathcal{L}^{2n+1})$, the n-th Heisenberg group with its Carnot-Carathéodory distance and the Lebesgue measure of $\mathbb{R}^{2n+1}$. Then:*

- *For every $N \in [1,+\infty]$ and every $K \in \mathbb{R}$, the geometric curvature-dimension bound $CD(K,N)$ does* not *hold in $(\mathbb{H}^n, d_{CC}, \mathcal{L}^{2n+1})$.*
- *For $(N,K) \in [1,+\infty[\times\mathbb{R}$, the measure contraction property $MCP(K,N)$ holds in $(\mathbb{H}^n, d_{CC}, \mathcal{L}^{2n+1})$ if and only if $N \geq 2n+3$ and $K \leq 0$.*

A first surprise is that the geometric curvature-dimension and the measure contraction property behave differently. It is known that these properties are different but they are quite close: For a N-dimensional Riemannian manifold (M,g), the conditions $CD(K,N)$ and $MCP(K,N)$ are both equivalent to the property of having a Ricci curvature greater than Kg (Compare also with the Bakry and Émery curvature-dimension condition for the Laplace-Betrami operator in [2]). The second surprise is the dimension $2n+3$ that appears in the second item. As far as I know it has no classic significance and this may be the first time it arises in relation to the Heisenberg group.

In the first section, we give a short presentation of the Heisenberg group $\mathbb{H}_n$ ($n \in \mathbb{N}\backslash\{0\}$) and its geodesics. We also introduce two maps that will be helpful in the following sections: the geodesic-inversion map $\mathcal{I}$ and the intermediate points map $\mathcal{M}$. At the beginning of the second section we give the definition of $CD(K,N)$ and $MCP(K,N)$ for $K=0$ which is the only interesting case. We prove the second part of the Main Theorem in Theorem 2.3. The last section is devoted to a proof of the fact that there is no geodesic Brunn-Minkowski inequality in the Heisenberg group: it is the keystone of Theorem 3.3 which corresponds to the first part of the Main Theorem. Remark 3.4 of this section deals with MCP and CD for non-zero curvature parameters. This completes the Main Theorem. We also mention the multiplicative Brunn-Minkowski inequality and sketch the fact that this inequality does not hold in any dimension strictly greater than the topological dimension (i.e. $2n+1$). In the appendix, we sketch relations between the geodesic Brunn-Minkowski inequality, $CD(K,N)$, $MCP(K,N)$ and the Poincaré inequality plus doubling measure.

1. The Heisenberg Group and its Geodesics

1.1. **The Heisenberg group.** Let n be a non-negative integer. In this section we give a short presentation of the Heisenberg group $\mathbb{H}_n$ as a metric measure space equipped with the Lebesgue measure $\mathcal{L}^{2n+1}$ and Carnot-Carathéodory metric d_{CC}. As a set $\mathbb{H}_n$ can be written in the form $\mathbb{R}^{2n+1} \simeq \mathbb{C}^n \times \mathbb{R}$ and an element of $\mathbb{H}_1$ can also be written as $(z,t) = (z_1, \cdots, z_n, t)$ where $z_k := x_k + iy_k \in \mathbb{C}$ for $1 \leq k \leq n$ and $t \in \mathbb{R}$. The group structure of $\mathbb{H}_n$ is given by:

$$(z_1, \cdots, z_n, t) \cdot (z_1', \cdots, z_n', t') = \left(z_1 + z_1', \cdots, z_n + z_n', t + t' + 2 \sum_{k=1}^{n} \Im(z_k \overline{z_k'}) \right)$$

where $\Im$ denotes the imaginary part of a complex number. $\mathbb{H}_n$ is then a Lie group with neutral element $0_{\mathbb{H}_n} := (0,0)$ and inverse element $(-z,-t)$. The set $L = \{(z,t) \in \mathbb{H}_n \mid z = 0\}$ is the center of the group and will play an important role. Throughout this paper, $\tau_p : \mathbb{H}_n \to \mathbb{H}_n$ will be the left translation

$$\tau_p(q) = p \cdot q$$

where $p, q \in \mathbb{H}_n$. This map is affine and its vectorial part has the determinant 1. It follows that the Haar measure of $\mathbb{H}_n$ is the Lebesgue measure $\mathcal{L}^{2n+1}$ of $\mathbb{R}^{2n+1}$ which is left (and actually also right) invariant. For $\lambda > 0$, we denote by δ_λ the dilation

$$\delta_\lambda(z,t) = (\lambda z, \lambda^2 t).$$

The measures behaviour under dilation is also good:

$$\mathcal{L}^{2n+1}(\delta_\lambda(E)) = \lambda^{2n+2} \mathcal{L}^{2n+1}(E) \tag{1}$$

if $\lambda \geq 0$ and E is a measurable set.

In order to define the Carnot-Carathéodory metric, we consider the Lie algebra associated to $\mathbb{H}_n$. This is the vector space of left-invariant vector fields. A basis for this vector space is given by $\left(\overrightarrow{X}_1, \cdots, \overrightarrow{X}_n, \overrightarrow{Y}_1, \cdots, \overrightarrow{Y}_n, \overrightarrow{T}\right)$ where

$$\begin{aligned} \overrightarrow{X}_k &= \partial_{x_k} + 2y_k \partial_t \\ \overrightarrow{Y}_k &= \partial_{y_k} - 2x_k \partial_t \\ \overrightarrow{T} &= \partial_t. \end{aligned}$$

Roughly speaking, the Carnot-Carathéodory distance between two points p and q is the infimum of the lengths of the horizontal curves connecting p and q. By a horizontal curve we mean an absolutely continuous curve $\gamma : [0,r] \to \mathbb{H}_n$ whose derivative $\gamma'(s)$ is spanned by $\{\overrightarrow{X}_1(s), \cdots, \overrightarrow{X}_n(s), \overrightarrow{Y}_1(s), \cdots, \overrightarrow{Y}_n(s)\}$ in almost all points s. The length of this curve is then

$$\text{length}(\gamma) = \int_0^r \|\gamma'(s)\| \, ds$$

where $\| \sum_{k=1}^n (a_k \overrightarrow{X}_k + b_k \overrightarrow{Y}_k)\|^2 = \sum_{k=1}^n (a_k^2 + b_k^2)$. The value of the Carnot-Carathéodory distance between p and q is then

$$d_{CC}(p,q) := \inf \operatorname{length}(\gamma) \tag{2}$$

where the infimum is taken over all horizontal curves γ connecting p and q. The Chow Theorem (see for example [20]) ensures that this set is not empty. The Carnot-Carathéodory metric (like the Lebesgue measure) behaves well under translation τ_p and dilation δ_λ. It is left-invariant:

$$d_{CC}(\tau_p q, \tau_p q') = d_{CC}(q, q') \tag{3}$$

and

$$d_{CC}(\delta_\lambda(q), \delta_\lambda(q')) = \lambda d_{CC}(q, q') \tag{4}$$

for $\lambda > 0$. Because of (3), the Hausdorff measure (derived from d_{CC}) is a Haar measure. With the correct dimension, this measure is then proportional to the Lebesgue measure. By (1) and (4), it follows that the Hausdorff dimension of $(\mathbb{H}_n, d_{CC})$ is $2n+2$.

The metric space $(\mathbb{H}_n, d_{CC})$ is complete and separable. Another essential fact is that the topology given by d_{CC} is the usual topology on $\mathbb{R}^{2n+1}$. The Hausdorff dimension is then different from the topological dimension $(2n+2 \neq 2n+1)$, which is considered by some authors as the definition for a fractal set.

1.2. **A geodesic space.** Let us first give the terminology that we will use in this paper.

Definition 1.1. Let (X,d) be a metric space. Let m_0 and m_1 be two points of this set. An s-intermediate point between m_0 and m_1 is a point m_s such that

$$d(m_0, m_s) = s d(m_0, m_1) \quad \text{and}$$
$$d(m_s, m_1) = (1-s) d(m_0, m_1).$$

A *geodesic* from m_0 to m_1 is a continuous map γ defined on a segment $[a,b]$ (with $a < b$) such that for every $a', b', c \in [a,b]$ with $a' \leq c \leq b'$, the point $\gamma(c)$ is a $\frac{c-a'}{b'-a'}$-intermediate point between $\gamma(a')$ and $\gamma(b')$. A *normal geodesic* is a geodesic defined on $[0,1]$. A *local geodesic* is a curve γ defined on an interval I, such that for any point s in the interior of I there is an $\varepsilon > 0$ such that $[s-\varepsilon, s+\varepsilon] \subset I$ and $\gamma\mid_{[s-\varepsilon,s+\varepsilon]}$ is a geodesic. The metric space (X,d) is said to be a *geodesic space* if there is a geodesic between any two points of X.

We now come to the geodesics of $\mathbb{H}_n$. The infimum in (2) turns out to actually be a minimum. We have the following proposition, which can be found in [1] for example:

Proposition 1.2. *The metric space $(\mathbb{H}_n, d_{CC})$ is a geodesic space. Moreover every geodesic between two points p and q of $\mathbb{H}_n$ is horizontal and has length $d_{CC}(p,q)$.*

The equations of the local geodesics of $\mathbb{H}_n$ have been known since Gaveau's paper [11]. In [1] Ambrosio and Rigot give explicitly the cut locus of local geodesics passing through $0_{\mathbb{H}}$. See also [20] for the similarities with the Dido problem. In this paper, we will investigate how the measure is transported along the geodesics: to do this we need to know their equations. Because the Carnot-Carathéodory distance and hence the geodesics are left-invariant, it is enough to know the equations of the geodesics passing through $0_{\mathbb{H}}$. Let (χ, φ) be in $\mathbb{C}^n \times \mathbb{R}$. By a curve with parameters (χ, φ) we mean the curve $\gamma_{\chi,\varphi}$ defined on $\mathbb{R}$ by

$$\gamma_{\chi,\varphi}(s) = \begin{cases} \left(i\frac{e^{-i\varphi s}-1}{\varphi}\chi, 2|\chi|^2\frac{\varphi s - \sin(\varphi s)}{\varphi^2}\right) \in \mathbb{C}^n \times \mathbb{R} & \text{if } \varphi \neq 0 \\ (s\chi, 0) & \text{if } \varphi = 0. \end{cases} \tag{5}$$

Here $|\chi|$ is $\sqrt{|\chi_1|^2 + \cdots + |\chi_n|^2}$. Obviously, the map $(\chi, \varphi, s) \to \gamma_{\chi,\varphi}(s)$ is real analytic on $\mathbb{C}^n \times \mathbb{R} \times \mathbb{R}$ so all its partial derivatives are well defined and continuous. The curve $\gamma_{\chi,\varphi}$ is horizontal and its length between a and b is $|\chi|(b-a)$. In this paper, we denote by Γ_s the map

$$\Gamma_s(\chi, \varphi) := \gamma_{\chi,\varphi}(s).$$

In particular we will make use of it for $s = 1$. We note that $\Gamma_s(\chi, \varphi)$ is $\Gamma_1(s\chi, s\varphi)$.

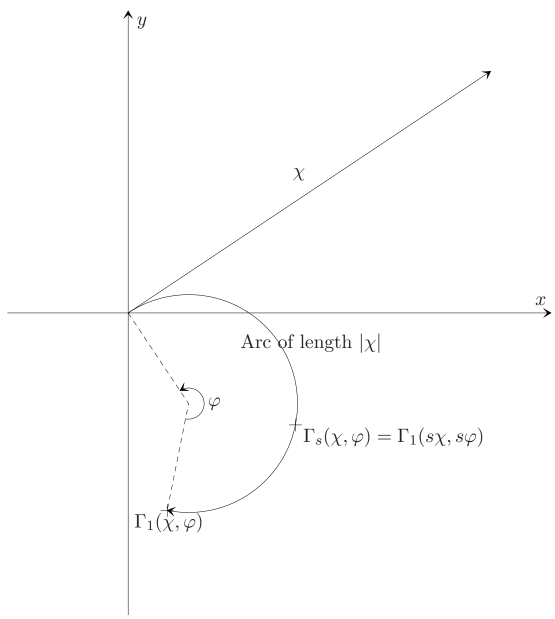

FIGURE 1. Projection of $\gamma_{\chi,\varphi}$ onto the $\mathbb{C}$-plane in $\mathbb{H}_1$.

The following proposition is proved in the paper [1] by Ambrosio and Rigot, and stated in almost the same formulation: we just adapted it to our notations. Let us recall that L is the set $\{(z, s) \in \mathbb{H}_n \mid z \neq 0\}$.

Proposition 1.3. *The normal geodesics starting from $0_{\mathbb{H}}$ are the restrictions to $[0,1]$ of curves with parameters (χ, φ) for $(\chi, \varphi) \in \mathbb{C}^n \times [-2\pi, 2\pi]$. In particular restrictions to $[0,1]$ of curves with parameters (χ, φ) with $|\varphi| > 2\pi$ are not normal geodesics. Conversely any restriction to $[0,1]$ of a curve with parameter $(\chi, \varphi) \in \mathbb{C}^n \times [-2\pi, 2\pi]$ is a normal geodesic starting from $0_{\mathbb{H}}$. Moreover we have the following more precise description:*

- *For any $p = (0,t) \in L^*$, normal geodesics from $0_{\mathbb{H}}$ to p are exactly the restrictions to $[0,1]$ of the curves with parameters $(\chi, \frac{t}{|t|}2\pi)$ where χ is an element of the sphere of vectors with norm $\sqrt{\pi|t|}$.*
- *For any $p \in \mathbb{H}_n \backslash L$ there exists a* unique *normal geodesic connecting $0_{\mathbb{H}}$ and p. This is the restriction to $[0,1]$ of a curve of parameters (χ, φ) with $|\varphi| < 2\pi$.*

Remark 1.4. The curves with parameters $(a+ib, v, r)$ from [1] (with $|a|^2+|b|^2 = 1$) have constant speed equal to one and are in fact curves with parameters $(a+ib, v)$ restricted to $[0,r]$. The restrictions to $[0,1]$ of the curves with parameters (χ, φ) have length $|\chi|$ and are simply the curves $s \in [0,1] \to \exp_{\mathbb{H}}(s\chi, s\varphi/4)$ where $\exp_{\mathbb{H}}$ is the Heisenberg-exponential map from the end of [1].

Remark 1.5. The map $s \to \delta_s(p)$ is not a geodesic.

We give a corollary of Proposition 1.3 for local geodesics.

Corollary 1.6. *The curve with parameters (χ, φ) is a local geodesic. More precisely the restriction of $\gamma_{\chi,\varphi}$ to $[a,b]$ is a geodesic if and only if $(b-a)|\varphi| \leq 2\pi$. Moreover this is the unique geodesic defined on $[a,b]$ if and only if $(b-a)|\varphi| < 2\pi$.*

Proof. The case $a = 0$ and $\gamma_{\chi,\varphi}(a) = 0_{\mathbb{H}}$ is contained in Proposition 1.3. To complete the proof we compute the left translation of the curve mapping $\gamma_{,\varphi}(a)$ to $0_{\mathbb{H}}$ and obtain

$$\gamma_{\chi,\varphi}(a)^{-1} \cdot \gamma_{\chi,\varphi}(a+s) = \gamma_{\chi',\varphi}(s)$$

with $\chi' = e^{-i\varphi a}\chi$. The proposition follows from this equation and the fact that d_{CC} is left-invariant. □

We set $D_1 := (\mathbb{C}^n \backslash 0) \times]-2\pi, 2\pi[$ and similarly $D_s := (\mathbb{C}^n \backslash 0) \times]-2s\pi, 2s\pi[$ for $s \in [0,1]$. The following proposition is a second and important corollary of Proposition 1.3:

Proposition 1.7. *The map Γ_1 is a C^∞-diffeomorphism from D_1 to $\mathbb{H}_n \backslash L$. Similarly for $s \in [-1,0[\cup]0,1[$, the map Γ_s is a C^∞-diffeomorphism from D_1 to $\Gamma_1(D_{|s|})$.*

Proof. For general s the assertion is a direct consequence of the case $s = 1$ and the relation $\Gamma_s(\chi, \varphi) = \Gamma_1(s\chi, s\varphi)$. With Proposition 1.3, it is clear that Γ_1 is one-to-one on D_1 and it is C^∞-differentiable because it is real analytic. We postpone the proof that its Jacobian determinant does not vanish to Proposition 1.12 at the end of this section. □

We introduce two helpful maps: the intermediate-points map $\mathcal{M}$ and geodesic-inversion map $\mathcal{I}$. The left-invariance of the Carnot-Carathéodory metric tells us whether or not there is a unique normal geodesic between two given points. If $p = (z,t)$ and $q = (z',t')$, the isometry $\tau_{p^{-1}}$ maps p to $0_{\mathbb{H}}$ and q to $p^{-1} \cdot q = (z-z', t'')$ for some t'' in $\mathbb{R}$. It follows from Proposition 1.3 that there is a unique normal geodesic from p to q if and only if $z \neq z'$ or $p = q$. We will denote the open set $\{(p,q) \in (\mathbb{H}_n)^2 \mid z_p \neq z_q\} = \{(p,q) \in (\mathbb{H}_n)^2 \mid p^{-1} \cdot q \notin L\}$ by U. On this set we define our first map.

Definition 1.8. We define the *intermediate-points map* $\mathcal{M}$ from the set $U \times [0,1]$ to $\mathbb{H}_n$ by

$$\mathcal{M}(p,q,s) = \tau_p \circ \Gamma_s \circ \Gamma_1^{-1} \circ \tau_{p^{-1}}(q).$$

The point $\mathcal{M}(p,q,s)$ is actually the unique s-intermediate point between p and q. It is a s-intermediate point when $p=0_{\mathbb{H}}$ because $\Gamma_s \circ \Gamma_1^{-1}(\gamma_{\chi,\varphi}(1))$ is $\gamma_{\chi,\varphi}(s)$ for $(\chi,\varphi)\in D_1$. The general case follows from the left-invariance of the Carnot-Carathéodory metric. Moreover $\mathcal{M}(p,q,s)$ is the unique s-intermediate point between p and q because there is a unique normal geodesic from p to q (the pair (p,q) is in U) and because the s-intermediate points in a geodesic space lie on the geodesics connecting two points.

In the following sections, we will extend $\mathcal{M}$ in (two) different ways to $(\mathbb{H}_n)^2\times[0,1]$. Using the proposition 1.7 and recalling that τ_p is affine, we have the following regularity lemma.

Lemma 1.9. *The map $\mathcal{M}$ is measurable. It is C^∞ on $U\times]0,1[$. The curve $s\in[0,1]\to\mathcal{M}(p,q,s)$ is the unique normal geodesic from p to q.*

Let us now introduce the geodesic-inversion map $\mathcal{I}$.

Definition 1.10. We define the *geodesic-inversion map* $\mathcal{I}$ on $\mathbb{H}_n\backslash L$ by $\mathcal{I}(p)=\Gamma_{-1}\circ\Gamma_1^{-1}(p)$.

The name comes from the fact that for $(\chi,\varphi)\in D_1$ and $s\in[-1,1]$ we have by Proposition 1.7:

$$\begin{aligned}\mathcal{I}(\gamma_{\chi,\varphi}(s)) &= \mathcal{I}(\Gamma(s\chi,s\varphi))\\ &= \Gamma_{-1}\circ\Gamma_1^{-1}(\Gamma_1(s\chi,s\varphi))\\ &= \Gamma_{-1}(s\chi,s\varphi)\\ &= \gamma_{\chi,\varphi}(-s).\end{aligned}$$

It follows that $\mathcal{I}\circ\mathcal{I}$ is the identity on $\mathbb{H}_n\backslash L$. That is why for any $p\in\mathbb{H}_n$ we will call $(p,\mathcal{I}(p))$ a pair of $\mathcal{I}$-conjugate points. We now establish the connection between $\mathcal{M}$ and $\mathcal{I}$.

Lemma 1.11. *Let p be in $\mathbb{H}_n\backslash L$. Then $\mathcal{M}(\mathcal{I}(p),p,1/2)$ is well defined and is $0_{\mathbb{H}}$ if and only if the φ-coordinate of $\Gamma_1^{-1}(p)$ verifies $|\varphi|<\pi$, i.e when $p\in\Gamma_1(D_{1/2})$.*

Proof. Proposition 1.3 says that $p=\Gamma_1(\chi,\varphi)$ for some $|\varphi|<2\pi$. Moreover the definition of $\mathcal{I}$ implies that $\mathcal{I}(p)=\Gamma_{-1}(\chi,\varphi)$. Therefore we have to say when $\mathcal{M}(\gamma_{\chi,\varphi}(-1),\gamma_{\chi,\varphi}(1),1/2)$ exists and if it is $0_{\mathbb{H}}$.

It follows from equation (5) that the z-coordinates of $\gamma_{\chi,\varphi}(-1)$ and $\gamma_{\chi,\varphi}(1)$ are equal if and only if $|\varphi|=\pi$. Therefore $(\gamma_{\chi,\varphi}(-1),\gamma_{\chi,\varphi}(1))\in U$ if and only if $|\varphi|\neq\pi$. In this case there is a unique geodesic δ on $[-1,1]$ between the two points and we can define the midpoint

$$\delta(0)=\mathcal{M}(\delta(-1),\delta(1),1/2)=\mathcal{M}(\gamma_{\chi,\varphi}(-1),\gamma_{\chi,\varphi}(1),1/2).$$

If $|\varphi|<\pi$ then $2|\varphi|<2\pi$. In this case the curve δ is the restriction of $\gamma_{\chi,\varphi}$ to $[-1,1]$ because by Corollary 1.6 both maps are the unique geodesic defined on $[-1,1]$ that goes from $\mathcal{I}(p)$ to p. The midpoint is then $\delta(0)=\gamma_{\chi,\varphi}(0)=0_{\mathbb{H}}$.

If $\pi<|\varphi|<2\pi$ we argue by contradiction. Assume that $\delta(0)=0_{\mathbb{H}}$. Then by Proposition 1.3, the curve $\delta\mid_{[0,1]}$ is the unique normal geodesic from $0_{\mathbb{H}}$ to $p=\gamma_{\chi,\varphi}(1)$ and $s\in[0,1]\to\delta(-s)$ is the unique normal geodesic between $0_{\mathbb{H}}$ and $\mathcal{I}(p)=\gamma_{\chi,\varphi}(-1)$. It follows that δ is $\gamma_{\chi,\varphi}$ on $[0,1]$ and $[-1,0]$ contradicting the fact that $|\varphi|>\pi$. (For $2|\varphi|>2\pi$, Corollary 1.6 shows that the restriction to $[-1,1]$ of $\gamma_{\chi,\varphi}$ is not a geodesic and consequently can not be δ.) Hence $\mathcal{M}(p,\mathcal{I}(p),1/2)$ is not $0_{\mathbb{H}}$. $\square$

As mentioned above, we present the computation of the Jacobian determinant. To prove Proposition 1.7, we only need to prove that the Jacobian of Γ_1 does not vanish. This fact is mentioned in [1] where the authors state that Γ_1 is a diffeomorphism (in fact in this paper χ is given by its polar coordinates $(|\chi|, \frac{\chi}{|\chi|})$). The result of the calculation is given for $\mathbb{H}_1$ in the paper of Monti (see [21]). We now give all the details of this computation for every $n \in \mathbb{N}\backslash\{0\}$ because we do not only need the fact that the Jacobian determinant does not vanish, but also its exact value.

Proposition 1.12. *The Jacobian determinant of* Γ_1 *is given by*

$$\mathrm{Jac}(\Gamma_1)(\chi,\varphi) = \begin{cases} 2^{2n+2}|\chi|^2 \left(\frac{\sin(\varphi/2)}{\varphi}\right)^{2n-1} \frac{\sin(\varphi/2)-(\varphi/2)\cos(\varphi/2)}{\varphi^3} & \text{for } \varphi \neq 0, \\ |\chi|^2/3 & \text{otherwise.} \end{cases}$$

It does not vanish on D_1.

Proof. We start by writing exactly what Γ_1 is:

$$\Gamma_1(\chi,\varphi) = \begin{cases} \left(i\frac{e^{-i\varphi}-1}{\varphi}\chi_1, \cdots, i\frac{e^{-i\varphi}-1}{\varphi}\chi_n, 2|\chi|^2\frac{\varphi-\sin(\varphi)}{\varphi^2}\right) & \text{if } \varphi \neq 0, \\ (\chi, 0) & \text{otherwise} \end{cases}$$

where $|\chi|^2 = |\chi_1|^2 + \cdots + |\chi_n|^2$. We start by calculating $\mathrm{Jac}(\Gamma_1) = \det(D\Gamma_1)$ for $\varphi \neq 0$. The case $\varphi = 0$ is obtained as a limit.

We first have to compute the real derivative of Γ_1, i.e. the derivative of Γ_1 as a map from $\mathbb{R}^{2n+1}$ to $\mathbb{R}^{2n+1}$. We write $D\Gamma_1$ as a matrix $\left(\begin{smallmatrix} P & C \\ R & q \end{smallmatrix}\right)$ where the block P is made of the $2n$ first rows and columns. If we identify complex numbers with 2×2 matrices ($a+ib$ is $\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$), we can write P as an $n \times n$ complex matrix $i\frac{e^{-i\varphi}-1}{\varphi}I_n$ where I_n is the identity matrix of $M_n(\mathbb{C})$. The column C is $(\frac{e^{-i\varphi}}{\varphi} - i\frac{e^{-i\varphi}-1}{\varphi^2})\chi$ seen as a $\mathbb{R}^{2n}$ vector, the row R is $(4x_1\frac{\varphi-\sin(\varphi)}{\varphi^2}, 4y_1\frac{\varphi-\sin(\varphi)}{\varphi^2}, \cdots, 4x_n\frac{\varphi-\sin(\varphi)}{\varphi^2}, 4y_n\frac{\varphi-\sin(\varphi)}{\varphi^2})$, and the real number q is $2|\chi|^2\left(\frac{2sin(\varphi)}{\varphi^3} - \frac{1+cos(\varphi)}{\varphi^2}\right)$.

It is difficult to compute directly the determinant of $\left(\begin{smallmatrix} P & C \\ R & q \end{smallmatrix}\right)$ in any point. Because of this we now prove that if $|\chi| = |\chi'|$, then $\mathrm{Jac}(\Gamma_1)(\chi,\varphi) = \mathrm{Jac}(\Gamma_1)(\chi',\varphi)$. Let T be a unitary $\mathbb{C}$-linear map so that $T(\chi) = \chi'$. Consider now T' defined by $T'(\chi,\varphi) = (T(\chi),\varphi)$. Then it is not difficult to see that $\Gamma_1 \circ T' = T' \circ \Gamma_1$. It follows that $(\mathrm{Jac}(\Gamma_1)\circ T')\cdot\det_{\mathbb{R}}(T') = \det_{\mathbb{R}}(T')\cdot\mathrm{Jac}(\Gamma_1)$ and hence we have $\mathrm{Jac}(\Gamma_1)(\chi,\varphi) = \mathrm{Jac}(\Gamma_1)(\chi',\varphi)$. We use this relation to simplify the computation by choosing $\chi' = (0,\cdots,0,|\chi|)$. With this new vector χ', most of the entries of C and R are equal to zero, so we can calculate the determinant of $D\Gamma_1 = \left(\begin{smallmatrix} P & C \\ R & q \end{smallmatrix}\right)$ by blocks. We get that $\mathrm{Jac}(\Gamma_1)(\chi,\varphi)$ is the product of

$$\begin{vmatrix} \sin(\varphi)/\varphi & (1-\cos(\varphi))/\varphi \\ (\cos(\varphi)-1)/\varphi & \sin(\varphi)/\varphi \end{vmatrix}^{n-1}$$

with

$$\begin{vmatrix} \sin(\varphi)/\varphi & (1-\cos(\varphi))/\varphi & |\chi|(\frac{\cos(\varphi)}{\varphi} - \frac{\sin(\varphi)}{\varphi^2}) \\ (\cos(\varphi)-1)/\varphi & \sin(\varphi)/\varphi & |\chi|(\frac{-\sin(\varphi)}{\varphi} - \frac{\cos(\varphi)-1}{\varphi^2}) \\ 4|\chi|\frac{\varphi-\sin(\varphi)}{\varphi^2} & 0 & 2|\chi|^2\left(\frac{2\sin(\varphi)}{\varphi^3} - \frac{1+\cos(\varphi)}{\varphi^2}\right) \end{vmatrix}.$$

This is just

$$2^{n-1}\left(\frac{2\sin^2(\varphi/2)}{\varphi^2}\right)^{n-1} 2|\chi|^2 \begin{vmatrix} \sin(\varphi)/\varphi & (1-\cos(\varphi))/\varphi & \frac{\cos(\varphi)}{\varphi} \\ (\cos(\varphi)-1)/\varphi & \sin(\varphi)/\varphi & \frac{-\sin(\varphi)}{\varphi} \\ 2\frac{\varphi-\sin(\varphi)}{\varphi^2} & 0 & \frac{1-\cos(\varphi)}{\varphi^2} \end{vmatrix}$$

which is

$$2^{2n+2}|\chi|^2\left(\frac{\sin(\varphi/2)}{\varphi}\right)^{2n-1}\frac{\sin(\varphi/2)-(\varphi/2)\cos(\varphi/2)}{\varphi^3}.$$

The continuous limit at $\varphi = 0$ is $|\chi|^2/3$.

It remains to show that $\mathrm{Jac}(\Gamma_1)$ does not vanish on D_1. This is clear for $\varphi = 0$. Otherwise we have to prove that the odd function $f(u) := \sin(u) - u\cos(u)$ does not vanish for $u \in]0, \pi[$. $f(0) = 0$. The first derivative of f is the map $f'(u) = u\sin(u)$ which is positive on $]0, \pi[$. On this interval f is non-decreasing and does not vanish. □

We recall that for $0 < |s| \leq 1$ we have $\Gamma_s(\chi, \varphi) = \Gamma_1(s\chi, s\varphi)$ and we get the following corollary.

Corollary 1.13. *Let $0 < |s| \leq 1$. The Jacobian determinant of Γ_s on D_1 is*

$$\mathrm{Jac}(\Gamma_s)(\chi,\varphi) = \begin{cases} 2^{2n+2}s|\chi|^2\left(\frac{\sin\frac{s\varphi}{2}}{\varphi}\right)^{2n-1}\frac{\sin\frac{s\varphi}{2}-\frac{s\varphi}{2}\cos\frac{s\varphi}{2}}{\varphi^3} & \text{for } \varphi \neq 0, \\ s^{2n+3}|\chi|^2/3 & \text{otherwise.} \end{cases}$$

2. The Measure Contraction Property in $\mathbb{H}_n$

In general metric measure spaces, there are two conditions which can be thought of as replacements for the Ricci curvature bounds of differential geometry: the geometric curvature-dimension $CD(K, N)$ and the measure contraction property $MCP(K, N)$. In our case where the geodesic between two points is almost surely unique, curvature-dimension $CD(K, N)$ is more restrictive than the measure contraction property $MCP(K, N)$, although it was not clear for a long time whether the two properties are equivalent. Moreover, in this situation (when there is almost surely a unique normal geodesic between two points), the measure contraction property implies a Poincaré inequality and the doubling property for metric measure spaces. This is shown in [32] and [18]. Metric measure spaces verifying a Poincaré inequality and the doubling property have proved to be a perfect setting for analysis with minimal hypotheses. A good reference on this theory is the book by Heinonen (see [13]). It is possible to define a differentiable structure on such space, as proved in the Cheeger's paper [7] or to define Sobolev spaces with interesting properties (see [7],[12] and [26]). Another area of application domain of the Poincaré inequality is conformal geometry where it enables to analyze the quasi-conformal maps between metric spaces (see the survey article [5]). Some of the more exotic examples of doubling metric measure spaces with a Poincaré inequality are the boundary of hyperbolic buildings (see [4]), some Cantor-like sets with worm-holes (see [14] and the erratum [15]) and the Heisenberg group (see [29]).

We now give the definition of the curvature-dimension $CD(0, N)$ and of the measure contraction property $MCP(0, N)$. We recall that our aim is to prove that the first property does not hold for any N whereas the Heisenberg group $\mathbb{H}_n$ verifies $MCP(0, 2n + 3)$ where the bound $2n + 3$ is sharp. The case where

$K \neq 0$ in not really interesting in the Heisenberg group. We will see why and which properties hold in Remark 3.4. Let (X, d, μ) be a metric measure space. The measure μ is assumed to be locally finite and defined on the Borel σ-algebra of (X, d). We assume moreover that this space is separable and complete (a Polish space). The curvature-dimension condition $CD(K, N)$ is a geometric condition on the optimal transport of measure between any pair of absolutely continuous probability measures on (X, d, μ). For $N \geq 1$, the curvature-dimension condition $CD(0, N)$ roughly speaking states that the functional $S_N(\cdot \mid \mu)$ is convex on the L^2-Wasserstein space $\mathcal{P}_2$. Here $S_N(\cdot \mid \mu)$ is the relative Rényi entropy functional defined for a measure m with density $\rho_m \in L^1(\mu)$ by:

$$S_N(m \mid \mu) = -\int_X \rho_m^{1-1/N} d\mu.$$

The basic invariant in optimal transport theory is the distance between two probability measures m_0 and m_1 known as the L_2-Wasserstein distance. This is defined by

$$d_W(m_0, m_1) = \sqrt{\inf_q \left(\int_{X \times X} d^2(x, y) dq(x, y) \right)}$$

where the infimum is taken over all couplings q of m_0 and m_1. In a Polish space such as X, there is a coupling that attains the infimum. It is said to be an optimal coupling. Let $\mathcal{P}_2(X)$ be the space of probability measures m on X with second moment (i.e. $\int_X d(x_0, x)^2 dm(x) < +\infty$ for some $x_0 \in X$). With the distance d_W the space $\mathcal{P}_2(X)$ is also a complete and separable metric space. Thus it is possible to speak about geodesics in $(\mathcal{P}_2(X), d_W)$ and actually if X is geodesic, $\mathcal{P}_2$ is geodesic as well. For a detailed presentation, and more about optimal transport, see [30] or [31]. We now give the definition of $CD(0, N)$. It is a specific case of the curvature-dimension condition introduced by Sturm in [28].

Definition 2.1. Let $N \geq 1$. We say that the curvature-dimension condition $CD(0, N)$ holds in (X, d, μ) if for every pair (m_0, m_1) of absolutely continuous measure of $\mathcal{P}_2(X)$, there is a geodesic $(m_s)_{s \in [0,1]}$ connecting m_0 and m_1 and consisting of absolutely continuous measures m_s that verifies the following condition:

$$S_N(m_s \mid \mu) \leq (1 - s) S_N(m_0 \mid \mu) + s S_N(m_1 \mid \mu). \tag{6}$$

We will see in the next section (Theorem 3.3) that this property does not hold in the Heisenberg group.

Remark 2.2. In the paper by Ambrosio and Rigot (see [1]), the authors implicitly prove that there is a unique normal geodesic between two absolutely continuous measures of $\mathcal{P}_2(\mathbb{H}_n)$. As the interpolated measures are absolutely continuous too (see Remark 2.9) for $\mathbb{H}_n$ we can simply reformulate Definition 2.1 as the convexity of S_N along these (unique) geodesics.

The measure contraction property $MCP(0, N)$ (see [28], [18], [23]) is a condition on metric measure spaces (X, μ, d). Its formulation is much simpler if there exists a measurable map

$$\mathcal{N} : (x, y, s) \in X \times X \times [0, 1] \to X$$

such that for every $x \in X$ and μ-a.e $y \in X$, the curve $s \in [0, 1] \to \mathcal{N}(x, y, s)$ is the unique normal geodesic from x to y. Then the space (X, d, μ) satisfies $MCP(0, N)$

if and only if for almost every $x \in X$, every $s \in [0,1]$ and every μ-measurable set E

$$s^N \mu(\mathcal{N}_{x,s}^{-1}(E)) \leq \mu(E) \tag{7}$$

where $\mathcal{N}_{x,s}(y) := \mathcal{N}(x,y,s)$.

In Definition 1.8, we defined the map $\mathcal{M}$ on $U \times [0,1]$. We now extend it to $(\mathbb{H}_n)^2 \times [0,1]$ by $\mathcal{M}(p,q,s) = A$ if $(p,q) \notin U$ (We will use another extension in the next section). By Lemma 1.9, we see that $\mathcal{M}$ verifies the conditions of $\mathcal{N}$ on measurability and almost sure uniqueness of geodesics. We state in the next theorem that the property (7) also holds using $\mathcal{M}$ instead of $\mathcal{N}$.

Theorem 2.3. *The measure contraction property $MCP(0,N)$ holds in $\mathbb{H}_n$ if and only if $N \geq 2n+3$.*

We split the proof in two parts: in Proposition 2.4 we prove the easier part ($N < 2n+3$) and in Proposition 2.5 we prove the more difficult part which is based on a concavity statement (Lemma 2.6).

Proposition 2.4. *The measure contraction property $MCP(0,N)$ does not hold in $\mathbb{H}_n$ if $N < 2n+3$.*

Proof. Let N be strictly smaller than $2n+3$. As the Lebesgue measure $\mathcal{L}^{2n+1}$, the Carnot-Carathéodory distance d_{CC} and geodesics are left-invariant, it follows that $\mathcal{M}(\tau_p q, \tau_p q', s) = \tau_p \circ \mathcal{M}(q,q',s)$ for any p, q and q' in $\mathbb{H}_n$. If the relation (7) does not hold at some $B \in \mathbb{H}_n$, then it does not hold in any point. This is why it is enough to consider what happens when we contract from the point $0_{\mathbb{H}}$. This is why it is enough to consider what happens when we contract from the point $0_{\mathbb{H}}$. Let p be the point $((1,0,\cdots,0),0) = \Gamma_1((1,0,\cdots,0),0)$ and K_r the (Euclidian) ball $\mathcal{B}(p,r)$ with center p and radius $r < 1$. For a fixed s in $]0,1[$, we define the set E_r by $\mathcal{M}_{0_{\mathbb{H}},s}(K_r)$ where $\mathcal{M}_{0_{\mathbb{H}},s}$ is the map $\mathcal{M}(0_{\mathbb{H}},\cdot,s)$. As $\mathcal{B}(p,r)$ is contained in $\mathbb{H}_n \backslash L$ where $\mathcal{M}_{0_{\mathbb{H}},s}$ is one-to-one and differentiable, we have:

$$\mathcal{L}^{2n+1}(E_r) = \int_{K_r} \mathrm{Jac}\left(\mathcal{M}_{0_{\mathbb{H}},s}(q)\right) d\mathcal{L}^{2n+1}(q).$$

But from the definition of $\mathcal{M}$ in Definition 1.8 we deduce $\mathrm{Jac}(\mathcal{M}_{0_{\mathbb{H}},s})(p) = \frac{\mathrm{Jac}(\Gamma_s)}{\mathrm{Jac}(\Gamma_1)} \circ \Gamma_1^{-1}(p)$. This is s^{2n+3} by Proposition 1.12 and Corollary 1.13, plus the fact that the φ-coordinate of p is 0 (see equation (10)). It follows that $\mathrm{Jac}(\mathcal{M}_{0_{\mathbb{H}},s})(p) < s^N$. By continuity, we can find a small enough radius $r > 0$ such that $\mathrm{Jac}(\mathcal{M}_{0_{\mathbb{H}},s})(q) < s^N$ holds for every $q \in K_r$. For this choice of r we get that $s^N \mathcal{L}^{2n+1}(K_r) > \mathcal{L}^{2n+1}(E_r)$ which contradicts $MCP(0,N)$. □

Proposition 2.5. *The measure contraction property $MCP(0,N)$ holds in $\mathbb{H}_n$ if $N \geq 2n+3$.*

Proof. Let N be greater than $2n+3$. As in the proof of Proposition 2.4, we only need to prove (7) for $x = 0_{\mathbb{H}}$. Let E be a μ-measurable set with non-zero measure and $s \in]0,1[$. The map $\mathcal{M}_{0_{\mathbb{H}},s} := \mathcal{M}(0_{\mathbb{H}},\cdot,s)$ maps the line L on $0_{\mathbb{H}}$ (because of the definition of our extension) but is one-to-one on $\mathbb{H}_n \backslash L$ where it equals $\Gamma_s \circ \Gamma_1^{-1}$. If we denote $F := \mathcal{M}_{0_{\mathbb{H}},s}^{-1}(E)$, then we have:

$$q\mathcal{L}^{2n+1}(E) \geq \int_{F \backslash L} \mathrm{Jac}(\mathcal{M}_{0_{\mathbb{H}},s})(q) d\mathcal{L}^{2n+1}(q). \tag{8}$$

From our expression for $\mathcal{M}_{0_{\mathbb{H}},s}$ on $\mathbb{H}_n\backslash L$ we get that $\mathrm{Jac}(\mathcal{M}_{0_{\mathbb{H}},s}) = \frac{\mathrm{Jac}(\Gamma_s)}{\mathrm{Jac}(\Gamma_1)} \circ \Gamma_1^{-1}$. But we know the expression of these Jacobian determinants by Proposition 1.12 and Corollary 1.13. Hence to get (7), since $\mathcal{L}(L) = 0$, it is enough to prove that

$$\frac{\mathrm{Jac}(\Gamma_s)}{\mathrm{Jac}(\Gamma_1)}(\chi,\varphi) = s\left(\frac{\sin(s\varphi/2)}{\sin\varphi/2}\right)^{2n-1}\left(\frac{\sin(s\varphi/2)-(s\varphi/2)\cos(s\varphi/2)}{\sin(\varphi/2)-(\varphi/2)\cos(\varphi/2)}\right) \geq s^N \tag{9}$$

when $(\chi,\varphi) \in D_1$ (in the case $\varphi \neq 0$). For $\varphi = 0$ this relation must be changed to

$$\frac{\mathrm{Jac}(\Gamma_s)}{\mathrm{Jac}(\Gamma_1)}(\chi,0) = s^{2n+3} \geq s^N \tag{10}$$

which is obviously true. Both sides of (9) are 0 at 0 and 1 at 1. It is the same if we raise these expressions to the power of $1/N$. Hence, we want to prove that $s \to \left(\frac{\mathrm{Jac}(\Gamma_s)}{\mathrm{Jac}(\Gamma_1)}\right)^{1/N}(\chi,\varphi)$ lies above the diagonal between $(0,0)$ and $(1,1)$. That is in particular true if this function is concave in s for each $(\chi,\varphi) \in D_1$. This last assertion is equivalent to the $1/N$-concavity ($1/N$-concavity means positivity and concavity when raised to the power of $1/N$) on $]0,\pi[$ of the even function g_{2n-1} defined for $k \in \mathbb{N}$ by $g_k(u) = u\sin^k(u)(\sin(u) - u\cos(u))$. In the next lemma, we will prove a stronger statement: g_k is $1/(k+4)$-concave. It follows that g_{2n-1} is $1/(2n+3)$-concave which implies that it is $1/N$-concave because $N \geq 2n+3$.

Lemma 2.6. *For all $k \in \mathbb{N}$ the function g_k is $(k+4)^{-1}$-concave on $]0,\pi[$.*

Proof. We will prove this lemma by induction. We begin by proving that g_0 is 1/4-concave. For simplicity we will denote $g = g_0$. This function is positive because it is the product of $\mathrm{Id} : u \to u$ with the function f that we met in the proof of Proposition 1.12. Its first derivative is $g'(u) = (1+u^2)\sin(u) - u\cos(u)$ and its second derivative is $g''(u) = 3u\sin(u) + u^2\cos(u)$. After differentiating one more time it follows that g is concave on $[\alpha,\pi]$ where α can be calculated to be smaller than 2.46. It is true that 1/4-concavity is a weaker statement than concavity but we want it on all $[0,\pi]$. It is equivalent to the negativity of $(g''g - g'^2) + \frac{1}{4}g'^2$. A first step is to prove the weaker relation $g''g - g'^2 \leq 0$ which is the differential version of log-concavity (g positive and $\log(g)$ concave). Both factors of g are log-concave : Id is concave and

$$f''f - f'^2 = (\sin u + u\cos u)(\sin u - u\cos u) - (u\sin u)^2 = \sin^2 u - u^2 \leq 0.$$

It follows that g is log-concave. Alternatively we can write

$$g''g - g'^2 = (\mathrm{Id})^2(f''f - f'^2) + (\mathrm{Id}''\,\mathrm{Id} - \mathrm{Id}'^2)f^2$$

where both terms of the sum are negative on $]0,\pi[$. For 1/4-concavity, we have to prove the negativity of $(g''g - g'^2) + \frac{1}{4}g'^2$, which is

$$\begin{aligned} &u^2\left[\sin^2(u) - u^2\right] + [0-1]\,(\sin(u) - u\cos(u))^2 \\ &+\frac{1}{4}\left[(1+u^2)\sin(u) - u\cos(u)\right]^2 \end{aligned} \tag{11}$$

for $u \in [0,\pi]$. It is quite difficult to prove that this expression is negative. We replace the last expression by a pointwise greater polynomial. To do this, we replace cos and sin in each term by the beginning of their Taylor series. We start with $\frac{1}{4}g'^2(u)$. It is constructed from g' which is positive for $u \in [0,\pi]$. On this interval, we have:

$$0 \le (1+u^2)\sin(u) - u\cos(u) \le (1+u^2)(u - u^3/6 + u^5/120) - u(1-u^2/2).$$

For $0 \le u \le 2\sqrt{2}$, we have

$$\sin(u) - u\cos(u) \ge (u - u^3/6) - u(1 - u^2/2 + u^4/24) = u^3/3 - u^5/24 \ge 0$$

and finally, for $u \in [0,\pi]$ we have

$$0 \le \sin(u) \le u - u^3/6 + u^5/120.$$

We can then estimate (11) for $u \le 2\sqrt{2}$:

$$\begin{aligned}
&u^2\left[\sin^2(u) - u^2\right] - (\sin(u) - u\cos(u))^2 \\
&+ \frac{1}{4}\left[(1+u^2)\sin(u) - u\cos(u)\right]^2 \\
=&u^2\left[(u - u^3/6 + u^5/120)^2 - u^2\right] - (u^3/3 - u^5/24)^2 \\
&+ \frac{1}{4}((1+u^2)(u - u^3/6 + u^5/120) - u(1-u^2/2))^2 \\
=&-\frac{1}{30}u^8 + \frac{421}{57600}u^{10} - \frac{17}{28800}u^{12} + \frac{1}{57600}u^{14} \\
\le&u^8\left(\left(\frac{8}{57600} - \frac{17}{28800}\right)(u^2)^2 + \frac{421}{57600}u^2 - \frac{1}{30}\right) \le 0.
\end{aligned}$$

So we have $1/4$-concavity of g on $[0, 2\sqrt{2}]$. But we already proved that g is concave on $[2.46, \pi]$. Thus g is $1/4$-concave on $[0,\pi]$ which is the reunion of the two intervals.

Let us now prove by induction that g_{k+1} is $1/(k+5)$-concave. For this let us assume that g_k is $1/(k+4)$-concave for some integer k. Then $g_{k+1} = g_k \cdot \sin$. We have now to prove the negativity of

$$\begin{aligned}
&\left((g_k\sin)''(g_k\sin) - (g_k\sin)'^2\right) + \frac{1}{k+5}(g_k\sin)'^2 \\
=&(g_k''g_k - g_k'^2)\sin^2 + (-\sin\sin - \cos^2)g_k^2 + \frac{1}{k+5}(g_k\sin)'^2 \\
=&(g_k''g_k - g_k'^2)\sin^2 - g_k^2 + \frac{g_k'^2\sin^2 + 2g_kg_k'\sin\cos + g_k^2\cos^2}{k+5} \\
=&(g_k''g_k - g_k'^2 + \frac{g_k'^2}{k+4})\sin^2 - \frac{g_k'^2\sin^2}{k+4} - g_k^2 + \frac{g_k'^2\sin^2 + 2g_kg_k'\sin\cos + g_k^2\cos^2}{k+5} \\
=&(g_k''g_k - g_k'^2 + \frac{g_k'^2}{k+4})\sin^2 + \frac{-g_k'^2\sin^2}{(k+4)(k+5)} + g_k^2\left(\frac{\cos^2}{k+5} - 1\right) + \frac{2g_kg_k'\sin\cos}{k+5}.
\end{aligned}$$

The first term T_1 in the last sum is negative because of the $1/(k+4)$-concavity of g_k. The second term T_2 is clearly negative. The third term T_3 is also negative. It remains to prove that $|T_4| \le |T_2| + |T_3|$ where T_4 is the last term. We compare

$|T_4|^2$ and $(2\sqrt{|T_2||T_3|})^2 \leq (|T_2| + |T_3|)^2$:

$$\begin{aligned}
&4|T_2||T_3| - T_4^2 \\
=&4\left[\frac{g_k'^2 \sin^2}{(k+4)(k+5)}\right]\left[g_k^2\left(1 - \frac{\cos^2}{k+5}\right)\right] - \left[\frac{2g_k g_k' \sin\cos}{k+5}\right]^2 \\
=&4g_k^2 g_k'^2 \sin^2\left[\frac{\frac{k+5-\cos^2}{k+4} - \cos^2}{(k+5)^2}\right] \geq 0.
\end{aligned}$$

□

□

Remark 2.7. The exponent $2n+3$ in Theorem 2.3 can appear surprising because we should have expected the topological dimension $(2n+1)$ or the Hausdorff dimension $(2n+2)$ instead of $2n+3$. We now illustrate how this exponent arises for the unit ball $B_1^{\mathbb{H}}$, of $\mathbb{H}_1$. For $0 < s < 1$, the contraction $\mathcal{M}_{0_{\mathbb{H}},s}(B_1^{\mathbb{H}})$ is certainly contained in the Heisenberg ball $B_s^{\mathbb{H}}$ with center $0_{\mathbb{H}}$ and radius s. This ball is the dilation $\delta_s(B_1^{\mathbb{H}})$ of the unit ball and its volume is $s^4\mathcal{L}(B_1^{\mathbb{H}})$. Nevertheless, $MCP(0,5)$ (the best relation in $\mathbb{H}_1$) says that $\mathcal{L}(\mathcal{M}_{0_{\mathbb{H}},s}(B_1^{\mathbb{H}})) \geq s^5\mathcal{L}(B_1^{\mathbb{H}})$. Rescaling, we get $\mathcal{L}(\delta_{1/s}(\mathcal{M}_{0_{\mathbb{H}},s}(B_1^{\mathbb{H}}))) > s\mathcal{L}(B_1^{\mathbb{H}})$ where $\delta_{1/s}(\mathcal{M}_{0_{\mathbb{H}},s}(B_1^{\mathbb{H}}))$ is a subset of $B_1^{\mathbb{H}}$. It is possible to interpret the factor s appearing in this expression on writing down an explicit expression for this subset. It is actually the subset of points whose angle φ in the (χ,φ)-coordinate is between $-s2\pi$ and $s2\pi$. Indeed in equation (5), we see that φ is linearly increasing on geodesic paths starting from $0_{\mathbb{H}}$ and moreover the dilation $\delta_{1/s}$ does not change the value of φ. It is possible to calculate that the Lebesgue measure of $\mathcal{L}(\delta_{1/s}(\mathcal{M}_{0_{\mathbb{H}},s}(B_1^{\mathbb{H}}))$ is equivalent to $s\frac{\pi^2}{3}$ for s close to 0, which justifies the factor s. See the figure 2 which shows the set $\{y = 0\}$. The sets $B_1^{\mathbb{H}}$ and $\delta_{1/s}(\mathcal{M}_{0_{\mathbb{H}},s}(B_1^{\mathbb{H}}))$ are then obtained by rotating this figure around the ax $L = \{(0,0)\} \times \mathbb{R}$.

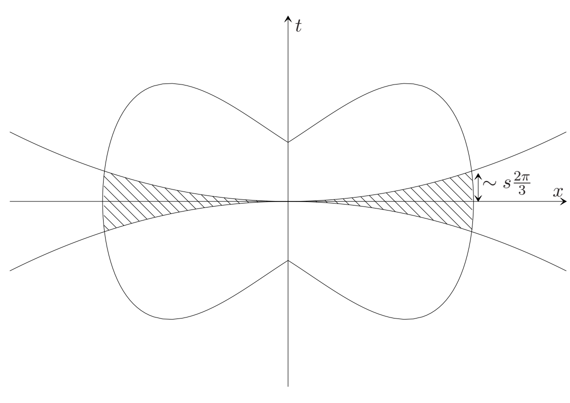

FIGURE 2. The sets $B_1^{\mathbb{H}}$ and $\delta_{1/s}(\mathcal{M}_{0_{\mathbb{H}},s}(B_1^{\mathbb{H}}))$.

Remark 2.8. The measure contraction property $MCP(0, 2n+3)$ can be directly applied to the Heisenberg group $\mathbb{H}_n$ in to prove a $(1,1)$-Poincaré inequality. To do this we follow the plan given at the end of the book by Saloff-Coste (see [24, 5.6.3]) for manifolds with a lower Ricci bound. This can be easily adapted: we obtain a constant $2^{2n+3}/n$. For every $p \in \mathbb{H}_n$, $r > 0$ and smooth function f we have:

$$\int_{B^{\mathbb{H}}(p,r)} |f(q) - f_B| \, d\mathcal{L}(q) \le \frac{2^{2n+3}}{n} r \int_{B^{\mathbb{H}}(p,2r)} |\nabla_{\mathbb{H}} f(q)|_{\mathbb{H}} \, d\mathcal{L}(q)$$

where $B_{\mathbb{H}}(p,r)$ is the d_{CC}-ball with center p and radius r, where

$$f_B = \frac{1}{\mathcal{L}(B)} \int_{B^{\mathbb{H}}(p,r)} f(q) \, d\mathcal{L}(q)$$

and $\nabla_{\mathbb{H}} f$ is the Heisenberg gradient defined by

$$\nabla_{\mathbb{H}} f = \sum_{k=1}^{n} (\overrightarrow{X}_k \cdot f)\overrightarrow{X}_k + (\overrightarrow{Y}_k \cdot f)\overrightarrow{Y}_k.$$

A Poincaré inequality on $\mathbb{H}_n$ was first proved by Varopoulos in [29].

Remark 2.9. In [10], Figalli and the author use $MCP(0, 2n+3)$ to answer positively an open question of Ambrosio and Rigot [1, section 7(c)]. In $\mathbb{H}_n$, the measures interpolated by optimal transport between an absolutely continuous measure and an other measure are absolutely continuous as well. As a consequence $\mathcal{P}_2^{ac} \subset \mathcal{P}_2$, the subspace of absolutely continuous measure is geodesic.

3. The Brunn-Minkowski Inequalities in $\mathbb{H}_n$

The classical Brunn-Minkowski inequality in $\mathbb{R}^N$ (see [9, 3.2.41] for instance) is a very useful geometric lower bound on the measure of the Minkowski sum (i.e the usual sum of two sets in $\mathbb{R}^N$) of two compact sets in $\mathbb{R}^N$. This inequality is equivalent to the following statement: given two compact sets K_0 and K_1, in $\mathbb{R}^N$ and $s \in [0,1]$ then

$$(\mathcal{L}^N)^{1/N}(sK_1 + (1-s)K_0) \ge s(\mathcal{L}^N)^{1/N}(K_1) + (1-s)(\mathcal{L}^N)^{1/N}(K_0)$$

with $sK_1 + (1-s)K_0 = \{sk_1 + (1-s)k_0 \in \mathbb{R}^N \mid k_1 \in K_1 \quad k_0 \in K_0\}$. We want to give a meaning to $sK_1 + (1-s)K_0$ in a geodesic metric space. For this we consider the set of the s-intermediate points from a point k_0 in K_0 to a point k_1 in K_1. We call this set the s-intermediate set and denote it by "$sK_1 + (1-s)K_0$".

Let (X, d, μ) be a metric measure space and N be greater than 1. We say that the *geodesic* Brunn-Minkowski inequality $BM(0, N)$ holds in (X, d, μ) if the inequality

$$\mu^{1/N}(\text{“}sK_1 + (1-s)K_0\text{”}) \ge s\mu^{1/N}(K_1) + (1-s)\mu^{1/N}(K_0) \tag{12}$$

is true for every pair compact sets K_0 and K_1 of non-zero measure. Here $\mu(\text{“}sK_1 + (1-s)K_0\text{”})$ will denote the outer measure of "$sK_1 + (1-s)K_0$" if the latter is not measurable. There is also a "multiplicative" Brunn-Minkowski inequality that has been introduced in the Heisenberg group by Monti in [22] (see also [16]). We deal with this inequality in Remark 3.5.

Remark 3.1. Let K be a real number and $N \ge 1$. The general definition of $CD(K, N)$ (see [28]) involves a modification of the geometric inequality (6) by factors roughly depending on the Wasserstein distance between the measures m_0

and m_1. These factors also appear in $MCP(K,N)$ and $CD(K,N)$ in the generalization of the inequalities (7) and (12). These three geometric properties have a common hierarchy when K and N vary: the property for (K,N) implies the property for (K',N) for all $K'<K$. Similarly for a fixed curvature K, the property (K,N) implies the property (K,N') for all $N'>N$. Nevertheless a priori there is no optimal pair (K,N) when the curvature and the dimension both vary (see [23],[28]).

It is proved in [28] that the curvature-dimension property $CD(0,N)$ implies $BM(0,N)$. In order to prove that $CD(0,N)$ does not hold in $\mathbb{H}_n$, we will prove that no geodesic Brunn-Minkowski inequality holds in this space.

In $\mathbb{H}_n$ it will be useful to interpret the s-intermediate set using the intermediate-points map $\mathcal{M}$. To do this we extend $\mathcal{M}$ in a way different from that used in the last section. Here $\mathcal{M}$ is no longer a map but a multi-valued map defined on $(\mathbb{H}_n)^2\times]0,1[$ by

$$\mathcal{M}(p,q,s)=\{m_s\in\mathbb{H}_n \mid d_{CC}(p,q)=\frac{1}{s}d_{CC}(p,m_s)=\frac{1}{1-s}d_{CC}(m_s,q)\}.$$

If (p,q) is in U, we identify the single-valued set $\mathcal{M}(p,q,s)$ with its unique element, which is coherent with Definition 1.8. To get more information on the values taken by $\mathcal{M}$ on $(\mathbb{H}_n)^2\backslash U\times]0,1[$, it is enough to use Proposition 1.3 and left translations. We will now prove the following lemma:

Lemma 3.2. *There are two compact sets K and K' such that*

$$\mathcal{L}^{2n+1}(K)=\mathcal{L}^{2n+1}(K')>\mathcal{L}^{2n+1}(\mathcal{M}^{1/2}(K,K'))$$

where $\mathcal{M}^{1/2}(K,K')=\{\mathcal{M}(k,k',1/2)\in\mathbb{H}_n \mid k\in K$ and $k'\in K'\}$.

Let N be a dimension greater than 1. We can raise the inequality in Lemma 3.2 to the power $1/N$ and using (12) we obtain as a corollary the following theorem.

Theorem 3.3. *The geodesic Brunn-Minkowski inequality $BM(0,N)$ and the geometric curvature-dimension $CD(0,N)$ do not hold for any N.*

We now give a proof of Lemma 3.2.

Proof. Let us consider a simple geodesic: the curve of parameter $((1,\cdots,0),0)$ on the interval $[-1,1]$. As $2\cdot 0<2\pi$ Corollary 1.6 says that this is the unique geodesic defined on $[-1,1]$ from $p'=(-1,0,\cdots,0)$ to $p=(1,0,\cdots,0)$: the points p and p' are $\mathcal{I}$-conjugate and have midpoint $0_{\mathbb{H}}$. Actually $\mathcal{M}^{1/2}:=\mathcal{M}(\cdot,\cdot,1/2)$ is simply the midpoint map. On U this map is single and is directly defined by setting $s=1/2$ in Definition 1.8:

$$\mathcal{M}^{1/2}(q',q)=\tau_{q'}\circ\Gamma_{1/2}\circ\Gamma_1^{-1}\circ\tau_{q'^{-1}}(q). \tag{13}$$

We will now use the geodesic-inversion map introduced in the first section. We recall that Lemma 1.11 exactly tells us exactly when the midpoint of two $\mathcal{I}$-conjugate points in U is $0_{\mathbb{H}}$. For p and p' this is the case so p and p' are in the open set $\Gamma_1(D_{1/2})$. Our counterexample consists of a small compact ball $K_r:=\mathcal{B}(p,r)$ with center p and (Euclidian) radius r and $K'_r=\mathcal{I}(K_r)$: we then consider the set of midpoints between K_r and K'_r. By continuity we can choose r small enough such that $K_r\subset\Gamma_1(D_{1/2})$ and $K_r\times K'_r\subset U$.

We have to show that K'_r has the same measure as K_r and this measure is greater than the measure of $\mathcal{M}^{1/2}(K_r,K'_r)$. The first claim is actually straightforward: Γ_1

and Γ_{-1} are diffeomorphisms between the same sets (Proposition 1.7) and have the same Jacobian determinant up to sign (Corollary 1.13). Hence

$$\mathcal{L}^{2n+1}(K_r') = \mathcal{L}^{2n+1}(\Gamma_{-1}(\Gamma_1^{-1}(K_r))) = \mathcal{L}^{2n+1}(\Gamma_1(\Gamma_1^{-1}(K_r))) = \mathcal{L}^{2n+1}(K_r).$$

The key to the second claim is the fact that

$$\mathcal{M}^{1/2}(K_r', K_r) = \bigcup_{a,b\in K_r} \mathcal{M}^{1/2}(\mathcal{I}(a), b) = \bigcup_{a,b\in K_r} \mathcal{M}^{1/2}(\mathcal{I}(a), a + (b-a)). \tag{14}$$

As $K_r \subset \Gamma_1(D_{1/2})$, Lemma 1.11 shows that if $a \in K_r$, then $\mathcal{M}^{1/2}(\mathcal{I}(a), a) = 0_{\mathbb{H}}$. Therefore the mid-set $\mathcal{M}^{1/2}(K_r', K_r)$ has very small measure. We will use differentiation tools to quantify this idea. By Lemma 1.9, $\mathcal{M}^{1/2}$ is C^∞-differentiable on U. For any $q \in \mathbb{H}_n \backslash L$ let $\mathcal{M}_q^{1/2}$ be the map $\mathcal{M}(q, \cdot, 1/2)$. We now write

$$\begin{aligned}
&\mathcal{M}^{1/2}(\mathcal{I}(a), a + (b-a)) \\
=&0 + D\mathcal{M}^{1/2}_{\mathcal{I}(a)}(a).(b-a) \\
&+ \left[\mathcal{M}^{1/2}\left(\mathcal{I}(a), a + (b-a)\right) - D\mathcal{M}^{1/2}_{\mathcal{I}(a)}(a).(b-a)\right] \\
=&D\mathcal{M}^{1/2}_{p'}(p).(b-a) + \left[\left(D\mathcal{M}^{1/2}_{\mathcal{I}(a)}(a) - D\mathcal{M}^{1/2}_{p'}(p)\right).(b-a)\right] \\
&+ \left[\mathcal{M}^{1/2}\left(\mathcal{I}(a), a + (b-a)\right) - D\mathcal{M}^{1/2}_{\mathcal{I}(a)}(a).(b-a)\right].
\end{aligned} \tag{15}$$

For a and b close to p, the two last terms of the last sum are small and can be bounded using the smoothness of $\mathcal{M}$ in the differential calculus of $\mathbb{R}^{2n+1}$ (see Lemma 1.9): when r tends to zero,

$$\begin{aligned}
\sup_{a,b\in K_r} \Big| &\left(D\mathcal{M}^{1/2}_{\mathcal{I}(a)}(a) - D\mathcal{M}^{1/2}_{p'}(p)\right).(b-a) \\
&+ \mathcal{M}^{1/2}(\mathcal{I}(a), a + (b-a)) - D\mathcal{M}^{1/2}_{\mathcal{I}(a)}(a).(b-a)\Big| = o(r).
\end{aligned}$$

Therefore, as $K_r - K_r = \{q \in \mathbb{R}^{2n+1} \mid q = b - a \quad a, b \in \mathcal{B}(p, r)\} = \mathcal{B}(0, 2r)$, the relations (14) and (15) give the following set inclusion

$$\mathcal{M}^{1/2}(K_r', K_r) \subset D\mathcal{M}^{1/2}_{p'}(p).(\mathcal{B}(0, 2r)) + \mathcal{B}(0, \varepsilon(r)r) \tag{16}$$

where $\varepsilon(r)$ is a non-negative function which tends to zero when r tends to zero. We observe now that the measure of the right-hand set is equivalent to the measure of $D\mathcal{M}^{1/2}_{p'}(p).(\mathcal{B}(0, 2r))$. Considering relation (13) and recalling that the left-invariant affine maps $\tau_{p'}$ and $\tau_{p'^{-1}} = \tau_{p'}^{-1}$ have derivative equal to their linear part, we get that $\mathrm{Jac}(\mathcal{M}^{1/2}_{p'})(p)$ has the same value as $\mathrm{Jac}(\Gamma_{1/2} \circ \Gamma_1^{-1})$ taken at the point $p'^{-1} \cdot p$ which is $((2, 0, \cdots, 0), 0) = \Gamma_1((2, 0, \cdots, 0), 0)$. This Jacobian determinant was calculated in the second section (see equations (9) and (10)). In our case as the φ-coordinate of $\Gamma_1^{-1}(p'^{-1} \cdot p)$ is 0, we have to use equation (10) for $s = 1/2$. The value of the Jacobian determinant is then $\frac{1}{2^{2n+3}}$. It follows that

$$\mathcal{L}^{2n+1}(D\mathcal{M}^{1/2}_{p'}(p).(\mathcal{B}(0, 2r))) = \frac{2^{2n+1}}{2^{2n+3}}\mathcal{L}^{2n+1}(\mathcal{B}(p, r)) = \frac{1}{4}\mathcal{L}^{2n+1}(K_r).$$

Hence by (16) and the remark that follows it, we get that

$$\mathcal{L}^{2n+1}(\mathcal{M}^{1/2}(K_r', K_r)) \leq \frac{1}{4}\mathcal{L}^{2n+1}(K_r)(1 + o(r))$$

when r tends to zero. We now choose a small enough r and the lemma is proved. $\square$

Remark 3.4.
(i) The previous result does not only yields that $CD(0,N)$ does not hold. This also implies that $CD(K,N)$ does not hold for any $K>0$ because this condition is less demanding than $CD(0,N)$. Alternatively, spaces verifying $CD(K,N)$ with $K>0$ are bounded.
(ii) Also for any $K<0$, the curvature-dimension bound $CD(K,N)$ does not hold. We argue by contradiction. Assume that $CD(K,N)$ holds in the space $(\mathbb{H}_n, d_{CC}, \mathcal{L}^{2n+1})$ for $K<0$. Then the "scaled space" property from [28] tells us that $(\mathbb{H}_n, \lambda^{-1}d_{CC}, \lambda^{-(2n+2)}\mathcal{L}^{2n+1})$ verifies $CD(\lambda^2 K, N)$ for all $\lambda>0$. But this last space is exactly isomorphic to our metric measure space via the dilation δ_λ. Hence $CD(K',N)$ would hold in $(\mathbb{H}_n, d_{CC}, \mathcal{L}^{2n+1})$ for every non-positive K'. It is proved in [1] that the optimal transport between two measures is unique, so inequality (6) defining $CD(0,N)$ is obtained as limit of the corresponding inequalities for $CD(K',N)$, which contradicts Theorem 3.3. It follows that $CD(K,N)$ does not hold in $\mathbb{H}_n$.
(iii) In the same way, we could have proved directly that $BM(K,N)$ is false for any $K\in\mathbb{R}$ using the dilations of $\mathbb{H}_n$. It follows that $CD(K,N)$ does not hold because $CD(K,N)$ implies $BM(K,N)$.
(iv) The property $CD(K,+\infty)$ is defined in [27]. With the same argument as (ii) it implies $CD(0,+\infty)$. This property implies the Brunn-Minkowski inequality $(1-s)\ln(\mathcal{L}(K_0)) + s\ln(\mathcal{L}(K_1)) \le \ln(\mathcal{L}(\mathcal{M}^s(K_0,K_1)))$. It is also false because of Lemma 3.2.
(iv) For every N, the measure contraction property $MCP(K,N)$ is false for $K>0$. As for CD, the spaces verifying this condition are bounded (see [28]).
(vi) The property $MCP(K,N)$ also does not hold for $N>2n+3$ and $K<0$. This case is similar to (ii): using dilations we can show that $MCP(K,N)$ implies $MCP(0,N)$, which contradicts Theorem 2.3.

Remark 3.5. In [22], Monti compares the measure of two compact sets F and F' to the measure of $F\cdot F' = \{a\cdot b\in\mathbb{H}_n \mid a\in F \quad b\in F'\}$. He proves that

$$\mathcal{L}^3(F\cdot F')^{1/4} \ge \mathcal{L}^3(F)^{1/4} + \mathcal{L}^3(F')^{1/4}$$

does not hold in $\mathbb{H}_1$ (4 is the Hausdorff dimension of $\mathbb{H}_1$) using an argument based on the non-optimality of the unit ball in the isoperimetric inequality for $\mathbb{H}_1$.

Another proof for $\mathbb{H}_n$ of Hausdorff dimension $2n+2$ is the following: Take F to be the set K_r defined above and denote by F' the set $\{b\in\mathbb{H}_n \mid \exists c\in F,\, c\cdot b = 0_{\mathbb{H}}\}$ of inverse elements (it is simply $-F$ because $(z,t)^{-1} = (-z,-t)$). Using the methods of this section we get that $F\cdot F'$ is very close to $D\tau_{p'}(p).(\mathcal{B}(0,2r))$. The measure of this last set is $2^{2n+1}\mathcal{L}(F)$ because, as we said in the first section, $\mathrm{Jac}(\tau_{p'})=1$ in every point. As $\mathcal{L}^{2n+1}(F) = \mathcal{L}^{2n+1}(F')$ it follows that for r small enough

$$\mathcal{L}^{2n+1}(F\cdot F')^{\frac{1}{2n+2}} < \mathcal{L}^{2n+1}(F)^{\frac{1}{2n+2}} + \mathcal{L}^{2n+1}(F')^{\frac{1}{2n+2}} \tag{17}$$

and the multiplicative Brunn-Minkowski inequality is false for Hausdorff dimension (i.e. $2n+2$). In the paper by Leonardi and Masnou (see [16]), the authors show that the multiplicative Brunn-Minkowski inequality is true with topological dimension (i.e. $2n+1$). They explain that there could be in principle an $N\in]2n+1,2n+2[$ such that the multiplicative Brunn-Minkowski inequality holds in $\mathbb{H}_n$: in fact if this equality holds for N, then it holds for $N'<N$. On the other hand, as mentioned in Remark 3.4, $BM(K,N')$ is a consequence of $BM(K,N)$ if $N'>N$.

We proved in (17) that the sets F and F' defined in this remark are a counterexample to the multiplicative Brunn-Minkowski inequality with dimension $N = 2n+2$. They are actually also counterexamples for any $N > 2n+1$. It follows that $2n+1$ is the largest dimension for which the multiplicative Brunn-Minkowski inequality is true.

Implication Graph

Let (X, d, μ) be a metric measure space, K a real curvature parameter and N a dimension parameter greater than 1. The measure μ is assumed to be locally finite and defined on the Borel σ-algebra of (X, d). We have the following implication graph

$$\begin{tikzcd}
 & CD(K,N) \arrow[r, Rightarrow, "3"] \arrow[d, Rightarrow, "1{,}2"] & BM(K,N) \arrow[dl, Rightarrow, "1"] \\
\mathrm{Ricc} \geq K \arrow[ur, Rightarrow, "2"] & MCP(K,N) \arrow[l, Rightarrow, "2"] \arrow[d, Rightarrow, "1'"] & \\
 & \text{Poincaré+doubling} &
\end{tikzcd}$$

where the number on the arrow indicates that the implication is subject to certain conditions. Index 1 indicates that the implication is valid if there exists $\mu \otimes \mu$-almost surely a unique geodesic from x to y (see Theorem 5.4. in [28] for 1 and [32] or [18] with Remark 5.3 of [28] for $1'$) which is true in $\mathbb{H}_n$ (see section 1). Index 2 implications holds if X is a N-dimensional complete Riemannian manifold, d the geodesic distance and μ the Riemannian volume (see for example Corollary 5.5 and Theorem 1.7 in [28]). Index 3 means that the implication always holds (see [28, Proposition 2.1]).

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