On combinatorial zeta functions

Christian Kassel

Institut de Recherche Mathématique Avancée
CNRS - Université de Strasbourg
Strasbourg, France

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Generating functions

• To a sequence $(a_n)_{n\geq 1}$ of numbers, it is customary to associate its generating function

$$g(t) = \sum_{n\geq 1} a_n t^n$$

• This is convenient because

∗ it puts an infinite number of data into a single expression

∗ very often finding equations for the generating function helps compute each individual number a_n

Another type of generating functions: Zeta functions

• To a sequence $(a_n)_{n\geq 1}$ of numbers, it is sometimes convenient to associate another generating function, a bit more involved, its zeta function:

$$Z(t) = \exp\left(\sum_{n\geq 1} a_n \frac{t^n}{n}\right)$$

• The ordinary generating function $g(t)$ of the sequence can be recovered from the zeta function as its logarithmic derivative:

$$g(t) = t\,\frac{\mathrm{dlog}\,Z(t)}{\mathrm{dt}} = t\,\frac{Z'\,t)}{Z(t)} \qquad (1)$$

where $Z'(t)$ is the derivative of $Z(t)$

Note that $Z(t)$ is the unique solution of (1) such that $Z(0) = 1$

• Let us give examples of zeta functions appearing in various situations

Zeta functions I. Geometric progressions

- Take the geometric progression $(a_n)_{n\geq 1}$ with $a_n = \lambda^n$ for some fixed scalar λ:

$$g(t) = \sum_{n\geq 1} \lambda^n t^n = \frac{\lambda t}{1-\lambda t}$$

We deduce

$$Z(t) = \frac{1}{1-\lambda t} \tag{2}$$

Observe that

∗ $Z(t)$ is a rational function; we shall see more examples of rational zeta functions

∗ $Z(t)$ is a "simpler" rational function than $g(t)$

∗ In the special case of the constant sequence $a_n = 1$ for all $n \geq 1$,

$$Z(t) = \frac{1}{1-t}$$

Proof of (2)**.** We have $Z(0) = 1$ and

$$t\,\frac{Z'\,t)}{Z(t)} = -t\,\frac{(1-\lambda t)'}{1-\lambda t} = -t\,\frac{-\lambda}{1-\lambda t} = \frac{\lambda t}{1-\lambda t} = g(t)$$

Zeta functions II. Algebraic varieties

• Let X be an algebraic variety defined as the set of zeroes of a system of polynomial equations with coefficients in the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$

• Recall that any finite field extension of $\mathbb{F}_p$ is of the form $\mathbb{F}_q$, where $\mathbb{F}_q$ is a finite field of cardinality $q = p^n$ for some integer $n \geq 1$

• Now X has a finite number of points $a_n = |X(\mathbb{F}_{p^n})|$ in all finite fields $\mathbb{F}_{p^n}$, so that it makes sense to consider the zeta function

$$Z_{X/\mathbb{F}_p}(t) = \exp\left(\sum_{n\geq 1} a_n \frac{t^n}{n}\right)$$

introduced by Emil Artin in his Leipzig thesis (1921)

• **Question.** *Why call this a zeta function?*
To answer this question, let us consider the case when X is a point

The connection with Riemann's zeta function

- **Example.** Let X be a point. Then $a_n = |X(\mathbb{F}_{p^n})| = 1$ for all $n \geq 1$ and

$$Z_{X/\mathbb{F}_p(t)} = \frac{1}{1-t}$$

- Now a point is defined over all finite fields. Putting all prime characteristics p together, we may form the global zeta function:

$$\zeta_X(s) = \prod_{p \text{ prime}} Z_{X/\mathbb{F}_p}(p^{-s})$$

- Let us compute the global zeta function when X is a point:

$$\begin{aligned} \zeta_X(s) &= \prod_{p \text{ prime}} \frac{1}{1-p^{-s}} \\ &\overset{\text{(Euler)}}{=} \sum_{n \geq 1} \frac{1}{n^s} = \zeta(s) \end{aligned}$$

which is the famous Riemann zeta function

Artin's zeta functions and Weil conjectures

- **More examples of Artin's zeta functions.**

(a) Let $X = \mathbb{A}^1$ be an affine line. Then $a_n = |\mathbb{F}_{p^n}| = p^n$ and

$$Z_{\mathbb{A}^1/\mathbb{F}_p}(t) = \frac{1}{1-pt}$$

(b) Let $X = \mathbb{P}^1 = \mathbb{A}^1 \coprod \{\infty\}$ be a projective line. Then $a_n = p^n + 1$ and

$$Z_{\mathbb{P}^1/\mathbb{F}_p}(t) = \frac{1}{(1-t)(1-pt)}$$

- **Weil conjectures.** One of them is the following:

$*$ *If X is a quasi-projective variety (i.e. intersection of an open and of a closed subset of a projective space), then $Z_{X/\mathbb{F}_p}(t)$ is a rational function*

$*$ This conjecture was first proved by Dwork (Amer. J. Math. 82 (1960))

$*$ Later Deligne proved all Weil conjectures and expressed $Z_{X/\mathbb{F}_p}(t)$ in terms of étale cohomology

Zeta functions III. Graphs

• Let Γ be a finite connected graph (i.e. the set of vertices and the set of edges are finite, and one can pass from one vertex to another by a series of edges)

Assume Γ has no vertex of degree 1 (i.e. no vertex is related to only one other vertex)

• Let a_n be the number of closed paths of length n (without backtracking)

Y. Ihara (1966) formed the corresponding zeta function

$$Z_\Gamma(t) = \exp\left(\sum_{n\geq 1} a_n \frac{t^n}{n}\right)$$

• **Theorem.** *The zeta function of a graph is a rational function*

More precisely,...

Ihara's zeta function: rationality

• The zeta function $Z_\Gamma(t)$ of a graph Γ is the inverse of a polynomial:

$$Z_\Gamma(t) = \frac{1}{\det(I - tM)} \quad (3)$$

Here M is the edge adjacency matrix of Γ defined as follows:

• M is a matrix whose entries are indexed by all couples $(\vec{e}, \vec{f})$ of oriented edges of Γ (each edge has two orientations)

∗ By definition, $M_{\vec{e},\vec{f}} = 1$ if $\bullet \xrightarrow{\vec{e}} \bullet \xrightarrow{\vec{f}} \bullet$, i.e., if the terminal vertex of $\vec{e}$ is the initial vertex of $\vec{f}$ (provided $\vec{f}$ is not the edge $\vec{e}$ with reverse orientation)

∗ Otherwise, $M_{\vec{e},\vec{f}} = 0$

• To prove (3) one checks that

$$a_n = (\text{number of closed paths of length } n) = \mathrm{Tr}(M^n)$$

and one concludes with the following general fact

The zeta function of a matrix

- For any square matrix M with scalar entries (in a field, in $\mathbb{Z}$), define

$$Z_M(t) = \exp\left(\sum_{n \geq 1} \mathrm{Tr}(M^n)\,\frac{t^n}{n}\right)$$

Here $a_n = \mathrm{Tr}(M^n)$ is the trace of the n-th power of M

- **Proposition.** *We have Jacobi's formula*

$$Z_M(t) = \frac{1}{\det(I - tM)}$$

- **Proof.** M is conjugate to an upper triangular matrix N; we have $\mathrm{Tr}(M^n) = \mathrm{Tr}(N^n)$ and $\det(I - tM) = \det(I - tN)$

$*$ For Tr and det we need take care only of the diagonal elements

$*$ By multiplicativity we are reduced to a 1×1-matrix $M = (\lambda)$, hence to a geometric progression:

$$Z_M(t) = \frac{1}{1 - \lambda t} = \frac{1}{\det(I - tM)}$$

Group rings

We next consider matrices with entries in a group ring

- Let G be a group (finite or infinite). Any element of the group ring $\mathbb{Z}G$ is a finite linear combination of elements of G of the form

$$a = \sum_{g \in G} a_g g \qquad (a_g \in \mathbb{Z})$$

- Let $\tau_0 : \mathbb{Z}G \to \mathbb{Z}$ be the linear form defined by

$$\tau_0 \left(\sum_{g \in G} a_g g \right) = a_e \qquad (e \text{ is the identity element of } G)$$

Exercise. Prove that τ_0 is a trace map, i.e., $\tau_0(ab) = \tau_0(ba)$ for all $a, b \in \mathbb{Z}G$

- **Example.** If $G = \mathbb{Z}$ is the group of integers, then $\mathbb{Z}G = \mathbb{Z}[X, X^{-1}]$ is the algebra of Laurent polynomials in one variable X and $\tau_0 \left(\sum_{k \in \mathbb{Z}} a_k X^k \right) = a_0$ is the constant coefficient of this Laurent polynomial

Matrices over group rings

- Let $M \in M_d(\mathbb{Z}G)$ be a $d \times d$-matrix with entries in the group ring $\mathbb{Z}G$. Set

$$\tau(M) = \tau_0\left(\mathrm{Tr}(M)\right) = \sum_i \tau_0(M_{i,i}) \in \mathbb{Z}$$

- **Definition.** *The zeta function of a matrix $M \in M_d(\mathbb{Z}G)$ is given by*

$$Z_M(t) = \exp\left(\sum_{n\geq 1} \tau(M^n)\,\frac{t^n}{n}\right)$$

- If G is the trivial group, then $\mathbb{Z}G = \mathbb{Z}$ and $\tau(M^n) = \mathrm{Tr}(M^n)$. Therefore,

$$Z_M(t) = \frac{1}{\det(I - tM)}$$

This again is a rational function

- **Question.** What can we say for a general group G?

Finite groups

- **Proposition.** *Let G be a finite group of order N. For any $M \in M_d(\mathbb{Z}G)$,*

$$Z_M(t) = \left(\frac{1}{\det(I - tM')}\right)^{1/N} \qquad (4)$$

for some $M' \in M_{dN}(\mathbb{Z})$

- **Proof.** It follows from a simple trick. We show how it works for $d = 1$.

  $*$ To $a = \sum_{g \in G} a_g g \in \mathbb{Z}G$ associate the matrix $M_a \in M_N(\mathbb{Z})$ of the multiplication by a in a basis $\{g_1, \dots, g_N\}$ of $\mathbb{Z}G$. It is easy to check that

$$\tau(a) = \tau_0(a) = a_e = \frac{1}{N}\operatorname{Tr}(M_a)$$

  $*$ Therefore,

$$\begin{aligned} Z_{(a)}(t) &= \exp\left(\sum_{n \geq 1} \tau(a^n)\,\frac{t^n}{n}\right) = \exp\left(\sum_{n \geq 1} \frac{1}{N}\operatorname{Tr}(M_a^n)\,\frac{t^n}{n}\right) \\ &= Z_{M_a}(t)^{1/N} \overset{\text{(Jacobi)}}{=} 1/\det(I - tM_a)^{1/N} \end{aligned}$$

Algebraic functions

So, if G is a non-trivial finite group, then $Z_M(t)$ is an algebraic function, not a rational function

• **Definition.** *A function $y = y(t)$ is algebraic if it satisfies an equation of the form*

$$a_r(t)y^r + a_{r-1}(t)y^{r-1} + \cdots + a_0(t) = 0$$

for some $r \geq 1$ and polynomials $a_0(t), a_1(t), \ldots, a_r(t)$ in t (not all of them 0)

• The zeta function $y = Z_M(t) = 1/\det(I - tM')^{1/N}$ of (4) satisfies the algebraic equation

$$\det(I - tM')y^N - 1 = 0$$

• Now we can state the main result...

The main result

• **Theorem**. *Let G be a virtually free group and $M \in M_d(\mathbb{Z}G)$. Then $Z_M(t)$ is an algebraic function.*

• **Remarks.** (a) A group G is virtually free if it contains a finite-index subgroup H which is free.

* A free group is virtually free: take $H = G$
* A finite group is virtually free: take $H = \{1\}$ (a free group of rank 0)

(b) The theorem is due to

* M. Kontsevich (*Arbeitstagung* Bonn 2011, arXiv:1109.2469) for $d = 1$,
* Christophe Reutenauer and me for $d \geq 1$ (Algebra Number Theory 2014)

• Using the finite group trick, one derives the theorem from the more precise following result:

Theorem 1. *Let $G = F_N$ be a free group and $M \in M_d(\mathbb{Z}G)$. Then the formal power series $Z_M(t)$ has integer coefficients and is algebraic.*

The proof: Kontsevich's three steps

Let us now outline the proof of Theorem 1 following Kontsevich

Starting from a matrix $M \in M_d(\mathbb{Z}F_N)$,

- **Step 1**. Prove that $Z_M(t)$ is a formal power series with integer coefficients
- **Step 2**. Prove that $g_M(t) = t \,\mathrm{dlog}(Z_M(t))/\,\mathrm{dt} = t\, Z'_M(t)/Z_M(t)$ is algebraic
- **Step 3**. Deduce from Steps 1–2 that $Z_M(t)$ is algebraic
- **Remark**.
  - Steps 1–2 use standard techniques of the theory of formal languages
  - Step 3 follows from a deep result in arithmetic geometry

A general setup

• Let A be a set and A^* the free monoid on the alphabet A:

$$A^* = \{\, \text{words from letters of the alphabet } A \,\}$$

• Let $\mathbb{Z}\langle\langle A \rangle\rangle$ be the ring of non-commutative formal power series on A with integer coefficients. For $S \in \mathbb{Z}\langle\langle A \rangle\rangle$ we have a unique expansion of the form

$$S = \sum_{w \in A^*} (S, w)\, w \qquad \text{with } (S, w) \in \mathbb{Z}$$

To such S we associate a generating function $g_S(t)$ and a zeta function $Z_S(t)$

• **Definition**. Set $a_n = \sum_{|w|=n} (S, w)$, where $|w|$ is the length of w. Then

$$g_S(t) = \sum_{n \geq 1} a_n t^n \in \mathbb{Z}[[t]] \quad \text{and} \quad Z_S(t) = \exp\left(\sum_{n \geq 1} a_n \frac{t^n}{n} \right) \in \mathbb{Q}[[t]]$$

As above, $g_S(t)$ and $Z_S(t)$ are related by $g_S(t) = t \, \mathrm{dlog}(Z_S(t))/\, \mathrm{dt} = t\, Z_S'(t)/Z_S(t)$

Cyclic non-commutative formal power series

- **Definition**. *An element $S \in \mathbb{Z}\langle\langle A \rangle\rangle$ is cyclic if*
  - $\forall u, v \in A^*, \ \ (S, uv) = (S, vu)$ *and*
  - $\forall w \in A^* - \{1\}, \forall r \geq 2, \ \ (S, w^r) = (S, w)^r$.

Definition. *(a) A word is primitive if it is not the power of a proper subword*
(b) Words w and w' are conjugate if $w = uv$ and $w' = vu$ for some u and v

- **Proposition**. *If $S \in \mathbb{Z}\langle\langle A \rangle\rangle$ is cyclic, then we have the Euler product*

$$Z_S(t) = \prod_{[\ell]} \frac{1}{1 - (S, \ell)\, t^{|\ell|}} = \prod_{[\ell]} \left(1 + (S, \ell)\, t^{|\ell|} + (S, \ell)^2\, t^{2|\ell|} + \cdots\right)$$

where the product is taken over all conjugacy classes of non-trivial primitive words ℓ

- For the proof, take $t\, \mathrm{dlog} / \mathrm{dt}$ of both sides and use the following two facts:
  - any word is the power of a unique primitive word
  - if w is of length n, then there are n words conjugate to w, all of them of length n

- **Corollary**. *If $S \in \mathbb{Z}\langle\langle A \rangle\rangle$ is cyclic, then $Z_S(t)$ has integer coefficients*

Algebraic non-commutative formal power series

• One can define the notion of an algebraic non-commutative power series S

Essentially, it means that S satisfies an algebraic system of equations

• Passing from $S \in \mathbb{Z}\langle\langle A \rangle\rangle$ to $g_S(t) \in \mathbb{Z}[[t]]$ consists in replacing in S each letter of the alphabet A by the variable t. Therefore,

if S is algebraic, then $g_S(t)$ is an algebraic function

• **Relation between algebraicity and virtually free groups.**

Let G be a group and $A \subset G$ be a subset generating G as a monoid. Consider

$$S_G = \sum w \quad \in \mathbb{Z}\langle\langle A \rangle\rangle$$

where the sum is taken over all words $w \in A^*$ representing the identity element of G (The series S_G incarnates the word problem for G)

Theorem (Muller & Schupp, 1983) *The non-commutative power series S_G is algebraic if and only if the group G is virtually free*

Steps 1–2

• To a matrix $M \in M_d(\mathbb{Z}F_N)$ there is a procedure to associate a formal power series $S \in \mathbb{Z}\langle\langle A \rangle\rangle$ satisfying:

- the generating functions coincide:

$$g_S(t) = g_M(t) \quad \text{and} \quad Z_S(t) = Z_M(t)$$

- S is cyclic
- S is algebraic

• **Consequence.** $Z_M(t)$ is a formal power series with integer coefficients and its logarithmic derivative $g_M(t)$ is algebraic

Step 3: An algebraicity theorem

To conclude we need the following

• **Lemma**. *If $f \in \mathbb{Z}[[t]]$ is a formal power series with integer coefficients and $t\,\mathrm{dlog}\,f/\mathrm{dt}$ is algebraic, then f is algebraic*

• **Remark.** The integrality condition ("with integer coefficients") is crucial: the transcendental formal power series

$$f(t) = \exp(t) = \sum_{n \geq 0} \frac{t^n}{n!}$$

has a logarithmic derivative $t\,\mathrm{dlog}\,f/\mathrm{dt} = t$ which is algebraic (even rational)

• This lemma belongs to a list of similar results, such as the 19th century result: *If $f \in \mathbb{Z}[[t]]$ is a formal power series with integer coefficients and its derivative is rational, then f is a rational function*

But passing from "rational" to "algebraic" is a more challenging problem, having received an answer only in the last 30 years

• **Problem.** Find an elementary proof of the lemma!

The Grothendieck-Katz conjecture

• The Grothendieck-Katz conjecture is a very general, mainly unproved, algebraicity criterion:

If $Y' = AY$ is a linear system of differential equations with $A \in M_r(\mathbb{Q}(t))$, then it has a basis of solutions which are algebraic over $\mathbb{Q}(t)$ if and only, for all large enough prime integers p, the reduction modulo p of the system has a basis of solutions that are algebraic over $\mathbb{F}_p(t)$

• Instances of the conjecture have been proved

- by Yves André (1989) following Diophantine approximation techniques of D. V. and G. V. Chudnovsky (1984),
- and by Jean-Benoît Bost (2001) using Arakelov geometry

• These instances cover the system consisting of the differential equation

$$y' = \frac{g_M}{t}\,y$$

which is of interest to us, and thus yield the desired lemma

(for an overview, see Bourbaki Seminar by Chambert-Loir, 2001)

An example by Kontsevich

- Let $G = F_2$ the free group with generators X and Y and

$$M = X + X^{-1} + Y + Y^{-1} \in \mathbb{Z}F_2 = M_1(\mathbb{Z}F_2)$$

- Easy to check that

$$\tau(M^n) = |\text{ words in the alphabet } \{X, X^{-1}, Y, Y^{-1}\} \text{ of length } n \text{ and representing the identity element of } F_2 \,|$$

- Kontsevich proves the following algebraic expression for $Z_M(t)$:

$$Z_M(t) = \frac{2}{3} \cdot \frac{1 + 2\sqrt{1 - 12t^2}}{1 - 6t^2 + \sqrt{1 - 12t^2}}$$

Expanding $Z_M(t)$ as a formal power series, we obtain

$$Z_M(t) = 1 + 2 \sum_{n \geq 1} 3^n \frac{(2n)!}{n!(n+2)!} t^{2n} \in \mathbb{Z}[[t]]$$

See Sequence A000168 in Sloane's *On-Line Encyclopedia of Integer Sequences*

The zeta function is not always algebraic

- For $M = X + X^{-1} + Y + Y^{-1} \in \mathbb{Z}F_2 = M_1(\mathbb{Z}F_2)$ we observed that

$$\tau(M^n) = |\text{ words in the alphabet } \{X, X^{-1}, Y, Y^{-1}\} \text{ of length } n \text{ and representing the identity element of } F_2\,|$$

In particular, $\tau(XYX^{-1}Y^{-1}) = 0$

- Now, if $G = \mathbb{Z} \times \mathbb{Z}$ is the free abelian group with generators X and Y, then

$$XYX^{-1}Y^{-1} = 1$$

and so $\tau(XYX^{-1}Y^{-1}) = 1$

- Taking $M = X + X^{-1} + Y + Y^{-1}$ now in $\mathbb{Z}[\mathbb{Z} \times \mathbb{Z}] = \mathbb{Z}[X, X^{-1}, Y, Y^{-1}]$, one shows that

$$\tau(M^n) = \binom{2n}{n}^2 \sim \frac{1}{\pi}\,\frac{16^n}{n}$$

By a criterion due to Eisenstein (1852), the presence of $1/n$ in the previous asymptotics implies that the generating function $g_M(t)$, hence $Z_M(t)$, is not algebraic

The algebraic curve behind Kontsevich's example

• For $M = X + X^{-1} + Y + Y^{-1} \in \mathbb{Z}F_2$
the function $y = Z_M(t)$ satisfies the quadratic equation

$$27t^4y^2 - (18t^2 - 1)y + (16t^2 - 1) = 0 \tag{5}$$

This equation defines an algebraic curve C_M over $\mathbb{Z}$

What can we say about the curve C_M? about its genus?

• **How to compute the genus from an equation of the form $\sum_{i,j\geq 0} a_{i,j}\, t^i y^j = 0$?**

∗ Draw the Newton polygon which is the convex hull of the points $(i, j) \in \mathbb{R}^2$ for which $a_{i,j} \neq 0$

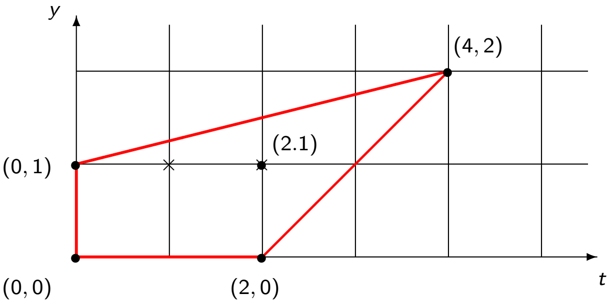
∗ The genus is the number of integral points of the interior of the Newton polygon

Reference. H. F. Baker, *Examples of applications of Newton's polygon to the theory of singular points of algebraic functions*, Trans. Cambridge Phil. Soc. 15 (1893), 403–450.

Newton polygon and genus

Equation of $y = Z_M(t)$ for $M = X + X^{-1} + Y + Y^{-1}$:

$$27t^4y^2 - (18t^2 - 1)y + (16t^2 - 1) = 0$$



The contour of the Newton polygon is in red

It contains 2 interior integral points marked $\times$

Therefore genus $= 2$

The algebraic curve behind a matrix

- For $M = X + X^{-1} + Y + Y^{-1}$, we have the formal power series expansion

$$Z_M(t) = 1 + 2t^2 + 9t^4 + 54t^6 + 378t^8 + 2916t^{10} + 24057t^{12} + 208494t^{14} + 1876446t^{16} + 17399772t^{18} + 165297834t^{20} + \cdots$$

The function $y = Z_M(t)$ satisfies the quadratic equation

$$27t^4y^2 - (18t^2 - 1)y + (16t^2 - 1) = 0$$

- For a general matrix $M \in M_d(\mathbb{Z}F_N)$, what are the connections between
  - ∗ the entries of M,
  - ∗ the integer coefficients of the formal power series $Z_M(t)$,
  - ∗ the integer coefficients of an algebraic equation satisfied by $Z_M(t)$,
  - ∗ the integral coordinates of the vertices of the Newton polygon of the equation?

Any idea?

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Ich danke für Ihre Aufmerksamkeit

Thank you for your attention!