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Partial abelianization of GL_n-local systems and non-commutative $\mathcal{A}$-coordinates

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Introduction en français

Une représentation du groupe fondamental d'une surface S de caractéristique d'Euler négative dans un groupe de Lie G peut être vue comme un G-fibré principal sur S, ce qui permet d'utiliser des outils topologiques et géométriques pour étudier ces représentations. Les fibrés obtenus à partir des représentations ont la propriété particulière d'avoir des changements de cartes localement constants. Ce type de fibrés est appelé un *G-système local* sur la surface S. La représentation correspondant à un G-système local est appelée *représentation de monodromie*. Habituellement, il est plus intéressant d'étudier l'espace de modules des représentations à conjugaison près (ou de manière équivalente les systèmes locaux à transformation de jauge près), que l'on appelle *variété de caractères*. Cependant, prendre le quotient par l'action de G par conjugaison résulte généralement en un espace non-Hausdorff. Il existe deux solutions à ce problème : prendre le quotient GIT qui nécessite des outils de géométrie algébrique, ou prendre le quotient Hausdorff à la place. Ces deux solutions coïncident lorsque G est un groupe réductif complexe, ce qui est l'un des cas couverts par le travail présenté ici. Un exemple classique de cette approche est l'espace de Teichmüller de la surface S qui peut être identifié à une composante connexe de l'espace de modules des $\mathrm{PSL}_2(\mathbb{R})$-systèmes locaux sur S. Il s'agit d'un exemple très particulier où une composante connexe entière de la variété de caractères est constituée uniquement de représentations fidèles et discrètes. Trouver des composantes connexes similaires dans la variété de caractères d'autres groupes de Lie est devenu un domaine très fructueux appelé *théorie des espaces de Teichmüller supérieurs*. Pour une exposition détaillée sur le sujet, voir [Wie18, FM22, GW18, BIW10].

Dans [FG06], Vladimir Fock et Alexander Goncharov introduisent les *variétés amassées* pour étudier les systèmes locaux en utilisant une structure algébrique appelée *algèbre amassée* introduite par Sergei Fomin et Andrei Zelevinsky dans [FZ02] quelques années auparavant. La théorie introduite par Fock et Goncharov décrit les G-systèmes locaux sur une surface ciliée, où G est un groupe de Lie semi-simple réel déployé. Les surfaces ciliées sont une classe très générique de surfaces hyperboliques contenant les surfaces épointées et des polygones idéaux, voir la Section 2.1 pour une définition précise. Les surfaces ciliées sont toujours non-compactes, et leurs groupes fondamentaux sont des groupes libres. Une propriété essentielle des surfaces ciliées est qu'elles admettent toutes des *triangulations idéales*. Une triangulation idéale Δ d'une surface ciliée S est un ensemble maximal de classes d'homotopie d'arcs sur S deux à deux sans intersections et non homotopes, ne s'auto-intersectant pas, dont les extrémités sont des pointes. De manière similaire aux coordonnées de Penner sur l'espace de Teichmüller décoré, les coordonnées de Fock-Goncharov nécessitent des données additionelles au système local : un cadrage ou une décoration (voir les sections 2.2.2 et 2.2.3 pour des définitions précises). Soit G un groupe de Lie semi-simple et réel déployé, B un sous-groupe de Borel et U le radical unipotent de B.

Définition. Un *cadrage* d'un G-système local $\mathcal{L}$ est la donnée pour chaque pointe de S d'une classe de G/B (appelée *drapeau*) stabilisée par la monodromie de $\mathcal{L}$ autour d'une petite boucle autour de la ppointe. Une *décoration* d'un G-système local $\mathcal{L}$ est la donnée pour chaque pointe d'une classe de G/U (appelée *drapeau décoré*) stabilisée par la monodromie de $\mathcal{L}$ autour d'une petite boucle autour de la pointe.

Les travaux de Fock et Goncharov reposent sur la définition d'un atlas spécial sur l'espace de modules des systèmes locaux encadrés (resp. décorés) sur une surface ciliée, appelé $\mathcal{X}$-coordonnées (resp. $\mathcal{A}$-coordonnées). Cet atlas spécial a une carte associée à chaque triangulation idéale de la surface ciliée et les changements de cartes peuvent être calculés avec une suite de *mutations* élémentaires, chacune étant une transformation birationnelle. Deux triangulations quelconques sont reliées par une suite finie de *flips* (changement de diagonale dans un quadrilatère), de sorte qu'il suffit de calculer les changements de cartes pour deux triangulations qui diffèrent d'un flip pour pouvoir calculer les changements de cartes de n'importe quelle triangulation vers n'importe quelle autre. La combinatoire de la suite de mutations conduisant à un flip de la triangulation est décrite par un carquois inscrit sur la surface, et les formules de mutations sont données par la structure d'algèbre amassée. En particulier, ces transformations birationnelles préservent la positivité des coordonnées. Une représentation est dite *positive* si toutes ses coordonnées dans n'importe quelle triangulation (donc dans toutes) sont positives. Cette notion de positivité étend la définition donnée par George Lusztig dans [Lus94]. Dans le cas $G = \mathrm{PGL}_n(\mathbb{R})$, les coordonnées $\mathcal{X}$ sont des *birapports* et des *triple rapports* de diverses droites construites en utilisant le cadrage du système local. Dans le cas $G = \mathrm{SL}_n(\mathbb{R})$, les coordonnées $\mathcal{A}$ sont des *volumes* de diverses bases construites en utilisant la décoration du système local. L'un des principaux résultats de leur théorie est l'identification entre la composante de Hitchin de la variétédes caractères et le sous-ensemble des représentations positives. Ils montrent également que les représentations positives sont fidèles et discrètes, et donc que les composantes de Hitchin sont des *espaces de Teichmüller supérieurs*.

La théorie des coordonnées de Fock-Goncharov s'applique aux groupes de Lie semisimples réels déployés G, et le cadrage doit être pris dans la variété des drapeaux *complets* G/B où B est le sous-groupe de Borel de G. Une autre classe connue d'espaces de Teichmüller supérieurs est l'espace des *représentations maximales*. Les représentations maximales ont été définies par Marc Burger, Alessandra Iozzi et Anna Wienhard dans [BIW10] pour les surfaces non fermées. Il s'agit de représentations dans un groupe de Lie *hermitien de type tubulaire* non compact pour lequel le nombre de Toledo, qui est une généralisation du nombre d'Euler pour les représentations dans $\mathrm{PSL}_2(\mathbb{R})$, prend sa valeur maximale. Dans [BIW10], Burger-Iozzi-Wienhard montrent que les représentations maximales sont fidèles et discrètes, et qu'elles satisfont une propriété de positivité, mais le cadrage dans ce cas est à valeurs dans une variété de drapeaux *partiels*, comme l'espace des sous-espaces lagrangiens lorsque $G = \mathrm{Sp}_{2n}(\mathbb{R})$. Contrairement aux composantes de Hitchin qui sont contractiles, l'espace des représentations maximales n'est en général ni connexe ni simplement connexe. Pour les groupes de Lie qui sont à la fois réels déployés et hermitiens (c'est-à-dire $\mathrm{Sp}_{2n}(\mathbb{R})$), l'ensemble des représentations maximales contient strictement la composante de Hitchin. Une grande classe de groupes de Lie hermitiens de type tubulaire peut être exprimée comme des groupes symplectiques à coefficients dans une algèbre involutive (voir [ABR+22] ou Section 2.7.2 pour plus de détails). Dans le cas symplectique, le nombre de Toledo peut être calculé en utilisant l'indice de Kashiwara-Maslov, voir [AGRW22]. Dans [AGRW22], Daniele Alessandrini, Olivier Guichard, Eugen Rogozinnikov et Anna Wienhard ont construit des généralisations non-commutatives des coordonnées de Fock-Goncharov $\mathcal{X}$ et $\mathcal{A}$ pour les représentations symplectiques, et ont montré que le lieu de positivité est exactement l'ensemble des représentations maximales. Ces coordonnées généralisées sont à valeurs dans des anneaux de matrices carrées, et sont donc génériquement non-commutatives. Cependant, elles admettent une forte structure de type "amassée", comme des cartes birationelles (non-commutatives) entre les coordonnées associées aux triangulations différant d'un flip.

Afin de disposer d'un cadre général pour étudier à la fois les représentations positives et les représentations maximales, Guichard-Wienhard ont introduit dans [GW18, GW22] la notion de Θ-positivité. Cette notion dépend de la donnée d'un sous-groupe parabolique P_Θ d'un groupe de Lie semi-simple G qui est déterminé par le choix d'un sous-ensemble Θ de racines restreintes de G. Toutes les paires (G, P) n'admettent pas une structure Θ-positive, et toutes les paires admettant une structure positive sont classifiées dans [GW18]. Il existe quatre familles de structures Θ-positives. La première est celle où G est réel déployé et $P = B$ est le sous-groupe de Borel de G (c'est-à-dire quand Θ est l'ensemble de toutes les racines), et les représentations Θ-positives sont exactement les représentations positives introduites par Fock-Goncharov. La seconde famille est celle où G est un groupe de Lie hermitien de type tubulaire et P est associé à la dernière racine du système de Dynkin de type B_n, et dans ce cas les représentations Θ-positives coïncident avec les représentations maximales. Les deux ”nouvelles” familles sont respectivement celle des groupes localement isomorphes à $SO(p, q)$ avec $p \neq q$ et celle qui contient une classe de groupes de Lie exceptionnels. Dans ces deux derniers cas, le sous-groupe parabolique est associé à un sous-ensemble strict de racines. Dans [GLW21], Guichard-Labourie-Wienhard ont montré que les représentations Θ-positives sont P_Θ-Anosov, donc fidèles et discrètes. La conjecture selon laquelle les représentations $Theta$-positives forment des espaces de Teichmüller supérieurs est encore ouverte dans le cas général au moment de l'écriture de ce manuscrit, mais elle a été démontrée pour une grande classe d'exemples, voir [GLW21, BP21].

L'idée principale du travail présenté ici est d'étudier l'espace de modules d'un système local encadré ou décoré sur une surface ciliée, mais avec un cadrage ou une décoration associé à *n'importe quel* sous-groupe parabolique d'un groupe de Lie semi-simple ou réductif. En particulier, nous cherchons des coordonnées qui partagent certaines propriétés calculatoires avec les coordonnées de Fock-Goncharov, à savoir une description combinatoire qui permet le calcul de formules de mutation de flips à partir d'une suite de mutations élémentaires. Dans ce travail, nous nous concentrons sur un groupe de Lie de type A, à savoir $\mathrm{SL}_n(\mathbb{R})$ (ou plutôt $\mathrm{GL}_n(\mathbb{R})$, comme nous l'expliquerons ci-dessous). Pour des raisons techniques, nous devrons nous limiter aux sous-groupes paraboliques de $\mathrm{GL}_n(\mathbb{R})$ dont le sous-groupe de Levi est de la forme $\mathrm{GL}_d(\mathbb{R})^k$, avec $n = kd$. Cela correspond à un sous-groupe parabolique conjugué au sous-groupe des matrices triangulaires supérieures où tous les blocs diagonaux sont des carrés de taille d. Pour ce faire, nous remplaçons le corps $\mathbb{R}$ par une $\mathbb{R}$-algèbre de dimension finie R. Dans tout ce travail, nous nous intéressons principalement au cas où $R = \mathcal{M}_n(\mathbb{R})$ est une algèbre de matrices carrées à coefficients réels. Ce faisant, nous obtiendrons des coordonnées qui sont à valeurs dans une certaine algèbre de matrices carrées, et qui sont donc génériquement non-commutatives. Ceci s'accompagne de plusieurs difficultés techniques. La première de ces difficultés est la définition même des coordonnées de type $\mathcal{X}$: l'une des définitions possibles du birapport (resp. du triple rapport) s'étend au cas non-commutatif, donnons d'abord ces définitions :

Définition. Soit L_1, L_2, L_3, L_4 quatre droites dans $\mathbb{R}^2$, transverses deux à deux. Soit $\pi_{i,j}$ la restriction de la projection canonique $\pi_{i,j} : L_j \to \mathbb{R}^2/L_i$ pour tous $i, j \in \{1, 2, 3, 4\}$ distincts. Le birapport des quatre droites est alors le suivant

$$[L_1 : L_2 : L_3 : L_4] = \pi_{4,1}^{-1} \circ \pi_{4,3} \circ \pi_{2,3}^{-1} \circ \pi_{2,1}.$$

Définition. Soit $(L_1, P_1), (L_2, P_2), (L_3, P_3)$ trois drapeaux complets dans $\mathbb{R}^3$ avec $\dim F_i = 1$ et $\dim P_i = 2$ pour $i = 1, 2, 3$, transverses deux à deux. Soit $\pi_{i,j}$ la restriction de la projection canonique $\pi_{i,j} : L_j \to \mathbb{R}^2/P_i$ pour tous ß, $j \in \{1, 2, 3\}$ distincts. Alors le triple rapport du triple des drapeaux est

$$r = \pi_{3,1}^{-1} \circ \pi_{3,2} \circ \pi_{1,2}^{-1} \circ \pi_{1,3} \circ \pi_{2,3}^{-1} \circ \pi_{2,1}.$$

Dans ces deux définitions, l'application $L_1 \to L_1$ est identifiée à un scalaire car sa matrice ne dépend pas du choix d'une base de L_1. Cependant, cette identification n'est plus possible si la dimension de L_1 est plus grande. La généralisation correcte de $\mathcal{X}$ serait alors la classe de conjugaison des applications définies ci-dessus, mais nous perdrions les formules de mutation, car on ne peut pas ajouter ou multiplier les classes de conjugaison. Cependant, ces "coordonnées $\mathcal{X}$ généralisées" conservent une partie du comportement de leur contrepartie commutative. En faisant attention à ces difficultés techniques, nous les étudions dans la Section 3.1.2.

Lorsque l'on considère les drapeaux décorés, la situation est toutefois très similaire dans le cadre commutatif et dans le cadre non-commutatif. Le besoin d'une base pour exprimer les applications ci-dessus est couvert par la décoration. Les définitions du birapport et du rapport triple indiquent également de bons candidats pour les $\mathcal{A}$-coordonnées non-commutatives. En effet, nous voulons que les $\mathcal{A}$-coordonnées déterminent les $\mathcal{X}$-coordonnées de la configuration de drapeaux, et dans le cas commutatif, les définitions ci-dessus donnent les formules habituelles concernant les $\mathcal{X}$-coordonnées et les $\mathcal{A}$-coordonnées, étant donné que les $\mathcal{A}$-coordonnées sont définies comme les matrices des cartes $\pi_{i,j}$ dans les bases données par la décoration des drapeaux. Cependant, cette affirmation n'est valable qu'au rang 2. Ceci nous amène à la difficulté technique suivante découlant de la non-commutativité : il n'y a pas de déterminant pour les matrices à valeurs dans un anneau générique, donc pas de groupe spécial linéaire SL_n. Puisque nous ne pouvons pas travailler avec SL_n, nous devons travailler dans le groupe général linéaire GL_n qui, même sur un anneau commutatif, n'est pas un groupe de Lie semi-simple, mais un groupe réductif. Par conséquent, les $\mathcal{A}$-coordonnées non-commutatives que nous définirons dans les Sections 3.3 et 3.2 ne se réduiront pas directement aux coordonnées $\mathcal{A}$ de Fock-Goncharov dans le cas commutatif. Au lieu de cela, nous obtenons une généralisation des coordonnées de Fock-Goncharov aux $\mathrm{GL}_n(\mathbb{R})$-systèmes locaux. Une autre difficulté technique rencontrée est la nécessité de travailler avec des systèmes locaux *tordus* pour définir les $\mathcal{A}$-coordonnées. Ces systèmes locaux tordus sont définis non pas sur S mais sur le fibré tangent unitaire $T'S$ de S, avec pour condition d'avoir la monodromie autour de la fibre de $T'S \to S$ égale à $-\mathrm{Id}$.

Maintenant que nous avons de bons candidats pour les $\mathcal{A}$-coordonnées généralisées, plusieurs questions se posent. Premièrement, existe-t-il des relations entre les coordonnées définies par rapport à une triangulation fixée ? Deuxièmement, comment calculer les formules de mutations reliant les coordonnées correspondant à deux triangulations différentes ? Pour répondre à ces deux questions, nous utilisons un outil introduit en 2014 par Davide Gaiotto, Gregory W. Moore et Andrew Neitzke dans [GMN13] appelé *réseaux spectraux*. Il s'agit d'un outil combinatoire utilisé pour étudier les systèmes locaux encadrés en permettant la définition d'une procédure appelée *abélianisation* (et son inverse appelée *non-abélianisation*). Cette procédure transforme un fibré vectoriel *plat* de rang n sur une surface ciliée S en un *fibré en droites* plat sur un certain revêtement *ramifié* à n feuilles Σ_n de S appelé *surface spectrale*. Il s'agit d'un outil très important en physique théorique pour étudier une certaine classe de théories quantiques des champs topologiques, mais nous n'utiliserons pas tout le potentiel des réseaux spectraux dans ce travail. Initialement, les réseaux spectraux étaient définis comme le graphe critique d'une différentielle quadratique holomorphe sur

S relevée à la surface spectrale. Cependant, une description géométrique et combinatoire existe et sera plus pratique à utiliser dans notre étude. Pour cela, nous utilisons une description combinatoire de la surface spectrale introduite par Alexander Goncharov et Maxim Kontsevich dans [GK22] que nous rappelons dans la Section 2.3.1, et nous ajoutons la définition d'un réseau spectral adapté à cette surface spectrale dans la section 2.3.3. Ensuite, nous devons étendre la définition de la procédure de non-abélianisation pour travailler avec des systèmes locaux *tordus*. Cette partie est un travail commun avec Eugen Rogozinnikov dans [KR22] dans le cas $n = 2$, et se généralise immédiatement à $n \geq 3$. Cette extension est décrite dans la section 2.4. Le processus de non-abélianisation lui-même fonctionne exactement de la même manière que le fibré sur la surface spectrale soit un fibré en droites ou un fibré de rang supérieur. Une fois cela fait, nous décrivons une construction inverse au processus de non-abélianisation appelée *abélianisation* (partielle), qui transforme un fibré vectoriel plat encadré de rang nd sur la surface ciliée S en un fibré vectoriel plat de rang d sur la surface spectrale. Nous la décrivons plus précisément dans la section 2.5. Cela nous permet de décrire la topologie d'un sous-ensemble (en raison des hypothèses de généricité) ouvert de l'espace de modules des $\mathrm{GL}_2(R)$-systèmes locaux encadré sur une surface ciliée S, où R est une $\mathbb{R}$-algèbre de dimension finie, par exemple l'anneau des matrices $n \times n$ à coefficients réels.

Théorème. *L'espace de modules des $\mathrm{GL}_2(R)$-systèmes locaux encadrés tordus sur S qui sont Δ-génériques par rapport à une triangulation Δ est homéomorphe à l'espace de modules des $R^\times$-systèmes locaux tordus sur Σ_2 qui est homéomorphe à $(R^\times)^{1-4\chi(\overline{S})+2p+\sum n_i}/R^\times$ où $R^\times$ agit diagonalement par conjugaison sur $(R^\times)^{1-4\chi(\overline{S})+2p+\sum n_i}$.*

Cette construction nous permet également d'étudier les systèmes locaux symplectiques dans la Section 2.7 car une matrice dans le groupe symplectique s'écrit naturellement comme une matrice par bloc. Il s'agit d'un travail commun avec Eugen Rogozinnikov dans [KR22]. Pour cela, nous devons traduire la donnée de la forme symplectique présente dans un système local symplectique sur S en une donnée supplémentaire dans le système local abélianisé. Comme corollaire, nous obtenons aussi une description topologique de l'espace de modules des systèmes locaux encadrés *maximaux*.

Théorème. *Si A est hermitien, alors l'espace de modules des $\mathrm{Sp}_2(A, \sigma)$-systèmes locaux encadrés tordus maximaux sur S est homéomorphe à :*

$$\left((A_+^\sigma)^{-2\chi(\overline{S})+p} \times (A^\times)^{-2\chi(\overline{S})+2p-1+\sum n_i}\right)/A^\times$$

où $A^\times$ agit par conjugaison sur $(A^\times)^{-2\chi(\overline{S})+2p-1+\sum n_i}$ et par congruence sur $(A_+^\sigma)^{-2\chi(\overline{S})+p}$.

Dans le chapitre 3, nous nous concentrons sur la définition et l'étude d'une généralisation non-commutative des $\mathcal{A}$-coordonnées de Fock-Goncharov. Nous commençons par étudier le cas particulier des $\mathrm{GL}_2(R)$-systèmes locaux, étant donné que de nombreuses difficultés techniques apparaissant au rang supérieur ne sont pas présentes en rang 2. Dans ce cas, nous montrons que l'algèbre non-commutative introduite par Arkady Berenstein et Vladimir Retakh dans [BR18] a une représentation par des fonctions rationnelles à valeurs matricielles sur l'espace de modules des $\mathrm{GL}_2(R)$-systèmes locaux tordus, que nous appelons $\mathcal{A}$-*coordonnées*. Comme pour les coordonnées commutatives de Fock-Goncharov, il existe une famille de fonctions associées à chaque triangulation de la surface, et il existe des formules de mutations qui expriment les fonctions associées à une triangulation comme des fonctions rationnelles non-commutatives de fonctions associées à une autre triangulation. Cependant, il y a une différence fondamentale avec les coordonnées commutatives de Fock-Goncharov : les fonctions associées à une triangulation fixée ont des relations entre elles, appelées *relations triangulaires*. Le terme "coordonnées" est donc un abus de langage.

Définition. Soit S une surface ciliée. Nous définissons la $\mathbb{R}$-algèbre unitaire $\mathcal{A}_S$ engendrée par les symboles x_γ et x_γ^{-1} pour tous les arcs idéaux γ sur S (avec la convention $x_\gamma = 1$ si γ est trivial) avec les relations :

- $\forall \gamma \in E(S), x_\gamma^{-1} x_\gamma = x_\gamma x_\gamma^{-1} = 1$
- $\forall f : P_3 \to S$ triangle,
$$x_{\gamma_{1,3}} x_{\gamma_{2,3}}^{-1} x_{\gamma_{2,1}} = x_{\gamma_{1,2}} x_{\gamma_{3,2}}^{-1} x_{\gamma_{3,1}}$$
- $\forall f : P_4 \to S$ quadrilatère,
$$x_{\gamma_{4,2}} = x_{\gamma_{4,3}} x_{\gamma_{1,3}}^{-1} x_{\gamma_{1,2}} + x_{\gamma_{4,1}} x_{\gamma_{3,1}}^{-1} x_{\gamma_{3,2}}$$
  et
$$x_{\gamma_{2,4}} = x_{\gamma_{2,3}} x_{\gamma_{1,3}}^{-1} x_{\gamma_{1,4}} + x_{\gamma_{2,1}} x_{\gamma_{3,1}}^{-1} x_{\gamma_{3,4}}$$

Dans le cas $R = \mathcal{M}_d(\mathbb{R})$, l'espace X des $\mathrm{GL}_2(R)$-systèmes locaux décorés tordus sur S est une variété algébrique et le sous-ensemble X_Δ des systèmes locaux Δ-génériques est un sous-ensemble ouvert et dense de X. Chaque $\mathcal{A}$-coordonnée associée à Δ peut être considérée comme une fonction rationnelle sur X_Δ à coefficients dans $\mathcal{M}_d(\mathbb{R})$.

Théorème. *Soit S une surface ciliée. L'application*

$$\psi : \begin{array}{ccc} \mathcal{A}_S & \to & \mathrm{Rat}(X, \mathcal{M}_d(\mathbb{R})) \\ x_\gamma & \mapsto & a_\gamma \end{array}$$

est un homomorphisme d'algèbre.

Pour le démontrer, nous devons calculer à la fois les relations triangulaires et les formules de mutations. En rang 2, nous pouvons calculer les deux en utilisant les réseaux spectraux. Les relations triangulaires apparaissent naturellement dans le processus d'abélianisation :

les $\mathcal{A}$-coordonnées sont des holonomies de certains chemins dans le système local abélianisé, et les relations triangulaires correspondent à des holonomies de chemins contractiles, et sont donc triviales. Pour les formules de mutation, nous profitons du fait que le système local sur la surface de base S ne dépend pas de la triangulation, alors que la surface spectrale et le système local abélianisé en dépendent. Ensuite, en effectuant des abélianisations correspondant à deux triangulations différentes et en identifiant les bons termes, nous pouvons exprimer les $\mathcal{A}$-coordonnées correspondant à une triangulation en termes de $\mathcal{A}$-coordonnées correspondant à l'autre triangulation. La même technique peut être utilisée pour montrer qu'étant donné une triangulation spécifique Δ, toutes les autres coordonnées $\mathcal{A}$ peuvent être écrites comme un polynôme de Laurent non-commutatif des coordonnées associées à Δ. Cette propriété est connue sous le nom de phénomène de Laurent (non-commutatif) et est la généralisation non-commutative d'une propriété bien connue des algèbres amassées commutatives.

Théorème. *Soit $n \geq 3$ et S le disque fermé avec n pointes sur le bord. Soit $i, j \in \{1, \dots, n\}$, $i \neq j$. Alors, pour toute triangulation Δ de S et tout $\mathrm{GL}_2(R)$-système local tordu décoré $\mathcal{L}$ qui est Δ-générique et tel que $(F^{(i)}, F^{(j)})$ est générique, la coordonnée $a_{\gamma_{i,j}}$ est un polynôme de Laurent non-commutatif dans les coordonnées $(a_\gamma)_{\gamma\in\Delta}$ associées à la triangulation Δ.*

Nous étudions également dans les sections 3.1.4 et 3.1.5 un type de coordonnées intermédiaire entre les $\mathcal{X}$-coordonnées et les $\mathcal{A}$-coordonnées, qui donnent une représentation d'une sous-algèbre de $\mathcal{A}_S$ introduite dans [BR18]. Les $\mathcal{A}$-coordonnées présentées ici se restreignent aux coordonnées symplectiques non-commutatives introduites dans [AGRW22] lorsqu'elles sont définies sur un système local symplectique avec une décoration symplectique.

Nous généralisons ensuite ces coordonnées à des configurations de drapeaux dans R^n dans la Section 3.2 et à des $\mathrm{GL}_n(R)$-systèmes locaux tordus dans la Section 3.3. La raison pour laquelle nous avons divisé ceci en deux sections différentes est une difficulté technique provenant des systèmes locaux tordus. En effet, la généralisation que nous décrivons nécessite des données supplémentaires aux drapeaux décorés habituels. Nous appelons ces données supplémentaires un *extra-décoration*.

Definition. Soit $(A^{(1)}, \dots, A^{(k)})$ un k-uplet de drapeaux en position générique. Une *extra-décoration* de l'ensemble k de drapeaux est constitué des données d'une décoration de chaque drapeau $A^{(i)}$ ainsi que des données pour chaque $i_1, \dots, i_k \in \mathbb{N}$ tel que $i_1 + \cdots + i_k = (k-1)n+1$ d'un élément $b_{i_1,\dots,i_k} \in R^n$ qui engendre librement $A^{(1)}_{i_1} \cap \cdots \cap A^{(k)}_{i_k}$.

Pour une configuration de drapeaux dans R^n, la définition ne pose pas de problème, mais pour un système local tordu, ces bases devraient être des sections plates du système local tordu le long de petites boucles situées à l'intérieur de la surface. Cependant, ces petites boucles ont une monodromie $-\mathrm{Id}$ car nous travaillons avec des systèmes locaux tordus, il n'est donc possible de définir que ces sections *qu'au signe près*. Nous définissons une version légèrement modifiée des carquois habituels associés aux algèbres amassées commutatives, pour prendre en compte le fait qu'une extra-décoration dépend de la triangulation choisie, et doit donc muter en même temps que les coordonnées elles-mêmes. Nous ne décrivons que les mutations nécessaires à un flip, car il nous manque une interprétation géométrique des autres mutations pour les décrire.

Cette définition des coordonnées $\mathcal{A}$ non-commutatives est légèrement différente de celle introduite par Goncharov-Kontsevich dans [GK22]. En effet, les coordonnées de Goncharov-Kontsevich peuvent être écrites comme des quotients de deux des coordonnées introduites dans ce travail. En d'autres termes, les $\mathcal{A}$-coordonnées présentées ici donnent une factorisation des coordonnées de Goncharov-Kontsevich. Une autre différence est que les coordonnées de Goncharov-Kontsevich ne nécessitent qu'une décoration (au sens usuel) des systèmes locaux, alors que les coordonnées présentées ici nécessitent cette *extra-décoration*. La combinatoire des coordonnées présentées dans ce travail est beaucoup plus proche de la combinatoire d'une algèbre amassée, avec des mutations dictées par un carquois sous-jacent. En tant que telles, les coordonnées de Goncharov-Kontsevich peuvent être considérées comme "intermédiaires" entre les $\mathcal{A}$-coordonnées présentées ici et les $\mathcal{X}$-coordonnées non-commutatives. Il est intéressant de noter que pour $n = 2$ (c'est-à-dire pour $G = \mathrm{GL}_2(R)$) les coordonnées de Goncharov-Kontsevich coïncident avec les $\mathcal{A}$-coordonnées introduites ici, les différences entre les deux types de coordonnées apparaissant pour $n \geq 3$.

Le travail présenté dans le chapitre 3 est un pas vers une définition de structures amassées (éventuellement non-commutatives) sur les espaces de modules des G-systèmes locaux encadrés dans les variétés de drapeaux partiels G/P pour des groupes de Lie plus généraux et des sous-groupes paraboliques plus généraux que ce qui est déjà connu. Bien qu'il n'y ait pas de réduction directe des coordonnées présentées ici aux coordonnées sur $SO(p, q)$ ou sur les groupes de Lie exceptionnels avec leur sous-groupe parabolique P_Θ (les deux classes restantes de structures Θ-positives), les constructions présentées dans ce manuscrit devraient aider à comprendre la combinatoire attendue des coordonnées amassées non-commutatives dans un cas général.

Introduction

A representation of the fundamental group of a surface S with negative Euler characteristic into a Lie group G can be seen as a principal bundle over S, allowing topological and geometrical tools to be used to study those representations. The bundles obtained from representations have the special property that the changes of charts are locally constant. This kind of bundle is called a *G-local system* on the surface S. The representation corresponding to a G-local system is called the *monodromy representation*. Usually we are more interested in the moduli space representations up to conjugation (or equivalently local systems up to gauge transformation), which is called the *character variety*. However, taking the quotient by the action of G by conjugation results usually in a non-Hausdorff space. There are two workarounds to this problem: taking a GIT quotient which require algebraic geometry tools, or taking the Hausdorff quotient instead. Both these solutions coincides when G is a complex reductive group which is one of the case covered in the work presented here. A classical example of this approach is the Teichmüller space of the surface S which can be identified with a connected component in the moduli space of $\mathrm{SL}_2(\mathbb{R})$-local systems on S. This is a very special example where an entire connected component of the character variety is constituted of only discrete and faithful representations. Finding similar connected components in the character variety of other Lie groups has become a very fruitful domain called *higher Teichmüller theory*. For a detailed account on the subject, see [Wie18, FM22, GW18, BIW10].

In [FG06], Vladimir Fock and Alexander Goncharov introduce *cluster varieties* to study local systems using an algebraic structure called *cluster algebras* introduced by Sergei Fomin and Andrei Zelevinsky in [FZ02] a few years before. The theory introduced by Fock and Goncharov describes G-local systems over a ciliated surface, where G is a split-real semisimple Lie group. Ciliated surfaces are a very generic class of hyperbolic surfaces containing punctured surfaces and ideal polygons, see Section 2.1 for a precise definition. Ciliated surfaces are always non-compact, and their fundamental groups are free groups. A key property of ciliated surfaces is that they all admit *ideal triangulations*. An ideal triangulation Δ of a ciliated surface S is a maximal set of (homotopy classes of) pairwise non-intersecting and non-homotopic, not self-intersecting arcs on S with endpoints on punctures. Similarly to Penner's coordinates on the decorated Teichmüller space, Fock-Goncharov coordinates require additional data to a local system: a framing or a decoration (see Section 2.2.2 and 2.2.3 for precise definitions). Let G be a semisimple split-real Lie group, B be a Borel subgroup and U be the unipotent radical of B.

Definition. A *framing* of a G-local system $\mathcal{L}$ is the data for each puncture of a coset in G/B (called a *flag*) stabilized by the monodromy of $\mathcal{L}$ around a small loop around the puncture. A *decoration* of a G-local system $\mathcal{L}$ is the data for each puncture of a coset in G/U (called a *decorated flag*) stabilized by the monodromy of $\mathcal{L}$ around a small loop around the puncture.

The work of Fock and Goncharov revolves around defining a special atlas on an open dense subset of the moduli space of framed (resp. decorated) local systems on a ciliated surface, called $\mathcal{X}$-coordinates (resp. $\mathcal{A}$-coordinates). This special atlas has a chart associated to each triangulation of the ciliated surface and the changes of charts can be computed with a sequence of elementary *mutations*, each being a birational map. Any two triangulation are related by a finite sequence of *flips*, so it is enough to compute the change of charts for two triangulations that differ from a flip to be able to compute change of charts from any triangulation to any other. The combinatorics of the sequence of mutations leading to a flip of the triangulation is described by a quiver embedded on the surface, and the mutations formulas are given by the cluster algebra structure. In particular, these birational transformations preserve the positivity of the coordinates. A representation is said to be *positive* if all of its coordinates in any (hence all) triangulation are positive. This notion of positivity extends the definition given by George Lusztig in [Lus94]. In the case $G = \mathrm{PGL}_n(\mathbb{R})$, the $\mathcal{X}$-coordinates are *cross ratios* and *triple ratios* of various lines constructed using the given framing of the local system. In the case $G = \mathrm{SL}_n(\mathbb{R})$, the $\mathcal{A}$-coordinates are *volumes* of various bases constructed using the given decoration of the local system. One of the main result of their theory is the identification between the Hitchin component and the subset of positive representations. They also show that positive representations are discrete and faithful, hence that Hitchin components are *higher Teichmüller spaces.*

The theory of Fock-Goncharov coordinates hold for split-real semisimple Lie groups G, and the framing has to be taken in the *complete* flag variety G/B where B is the Borel subgroup of G. Another known class of higher Teichmüller space is the space of *maximal representations.* Maximal representations were defined by Marc Burger, Alessandra Iozzi and Anna Wienhard in [BIW10] for non-closed surfaces. They are representations in a non-compact *hermitian* Lie group *of tube type* for which the Toledo number, which is a generalization of the Euler number for representations into $\mathrm{PSL}_2(\mathbb{R})$, takes its maximal value. In [BIW10], Burger-Iozzi-Wienhard show that maximal representations are discrete and faithful, and that they satisfy a positivity property, but the framing in this case takes value in a *partial* flag variety, like the space of lagrangian subspaces when $G = \mathrm{Sp}_{2n}(\mathbb{R})$.

Unlike Hitchin components which are contractible, the space of maximal representations is in general neither connected nor simply connected. For Lie groups that are both split-real and Hermitian (i.e $\mathrm{Sp}_{2n}(\mathbb{R})$), the set of maximal representations strictly contains the Hitchin component. A large class of hermitian Lie group of tube type can we expressed as symplectic groups over involutive algebras (see [ABR+22] or Section 2.7.2 for more details). In the symplectic case, the Toledo number can be computed using the Kashiwara-Maslov index, see [AGRW22]. In [AGRW22], Daniele Alessandrini, Olivier Guichard, Eugen Rogozinnikov and Anna Wienhard constructed non-commutative generalizations of Fock-Goncharov $\mathcal{X}$ and $\mathcal{A}$ coordinates for symplectic representations, and showed that the positivity locus is exactly the set of maximal representations. These generalized coordinates take values in square matrix rings, hence are non-commutative. However, they admits strong "cluster-like" structure, like (non-commutative) birational maps between coordinates associated to triangulations differing from a flip.

In order to have a general framework to study both positive representations and maximal representations, Guichard-Wienhard introduced in [GW18, GW22] the notion of Θ-positivity. This notions depend on the data of a parabolic subgroup P_Θ of a semisimple Lie group G which is determined by the choice of a subset Θ of restricted roots of G. Not every pair (G, P) admits a Θ-positive structure, and all the pairs admitting a positive structure are classified in [GW18]. There are four families of Θ-positive structure. The first one is when G is split-real and $P = B$ is the Borel subgroup of G (i.e. when Θ is the set of all roots), and Θ-positive representations are exactly the positive representations introduced by Fock-Goncharov. The second family is G hermitian Lie group of tube type and P is the last root of the Dynkin system of type B_n, and in this case the Θ-positive representations coincide with the maximal representations. The two "new" family are respectively for groups locally isomorphic to $SO(p, q)$ with $p \neq q$ and for some class of exceptional Lie groups. In both those last cases, the parabolic subgroup is associated to a strict subset of roots. In [GLW21], Guichard-Labourie-Wienhard showed that Θ-positive representations are P_Θ-Anosov, hence discrete and faithful. The conjecture that Θ-positive representations form higher Teichmüller spaces is still open in the general case at the time of writing, but has been shown for large class of examples, see [GLW21, BP21].

The main idea behind the work presented here is to study the moduli space of framed or decorated local system on a ciliated surface, but with a framing/decoration associated to *any* parabolic subgroup of a semisimple or reductive Lie group. In particular, we are interested in finding coordinates that share some computational properties with Fock-Goncharov coordinates, namely a combinatorial description that allows the computation of flip mutation formulas from a sequence of elementary mutations. In this work we focus on A-type Lie group, namely $\mathrm{SL}_n(\mathbb{R})$ (or rather $\mathrm{GL}_n(\mathbb{R})$, as we will explain below). For technical reasons, we will need to restrict to parabolic subgroups of $\mathrm{GL}_n(\mathbb{R})$ whose Levi subgroup is of the form $\mathrm{GL}_d(\mathbb{R})^k$, with $n = kd$. This corresponds to a parabolic subgroup conjugated to the subgroup of blockwise upper-triangular matrices where all the diagonal blocks are square of size d. The way we do this is by replacing the field $\mathbb{R}$ by a finite dimensional $\mathbb{R}$-algebra R. In all this work, we are mainly interested in the case $R = \mathcal{M}_n(\mathbb{R})$ of square matrices with real coefficients. Doing so will result in coordinates which take values in some square matrix algebra, thus being generically non-commutative. This comes with several technical difficulties. The first of these difficulties is the very definition of cluster $\mathcal{X}$-coordinates: one of the possible definition of the cross ratio (resp. the triple ratio) extend to the non-commutative case, let us first give those definitions.

Definition. Let L_1, L_2, L_3, L_4 be four lines in $\mathbb{R}^2$, pairwise transverse. Let $\pi_{i,j}$ denote the restriction of the canonical projection $\pi_{i,j} : L_j \to \mathbb{R}^2/L_i$ for all $i, j \in \{1, 2, 3, 4\}$ distinct. Then the cross-ratio of the four lines is

$$[L_1 : L_2 : L_3 : L_4] = \pi_{4,1}^{-1} \circ \pi_{4,3} \circ \pi_{2,3}^{-1} \circ \pi_{2,1}.$$

Definition. Let $(L_1, P_1), (L_2, P_2), (L_3, P_3)$ be three complete flags in $\mathbb{R}^3$ with $\dim F_i = 1$ and $\dim P_i = 2$, pairwise transverse. Let $\pi_{i,j}$ denote the restriction of the canonical projection $\pi_{i,j} : L_j \to \mathbb{R}^2/P_i$ for all $i, j \in \{1, 2, 3\}$ distinct. Then the triple-ratio of the triple of flags is

$$r = \pi_{3,1}^{-1} \circ \pi_{3,2} \circ \pi_{1,2}^{-1} \circ \pi_{1,3} \circ \pi_{2,3}^{-1} \circ \pi_{2,1}.$$

In both those definitions, the map $L_1 \to L_1$ is identified with a scalar because its (1 by 1) matrix does not depend on the choice of a basis of L_1. However, this identification is no longer possible if the dimension of L_1 is higher. The correct generalization of $\mathcal{X}$ would then be the conjugacy class of the maps defined above, however we would lose the mutation formulas, as one cannot add or multiply conjugacy classes. Still, those "generalized $\mathcal{X}$-coordinates" retain some of the behavior of their commutative counterpart. Being careful with those technical difficulties, we discuss those in Section 3.1.2.

When considering decorated flags however, the situation is very similar in the commutative setting and in the non-commutative setting. The need for a basis is covered by the decoration. The definitions of cross-ratio and triple-ratio also indicate good candidates for the non-commutative $\mathcal{A}$-coordinates. Indeed, we want the $\mathcal{A}$-coordinates to determine the $\mathcal{X}$-coordinates of the configuration of flags, and in the commutative case the definitions above give the usual formulas relating $\mathcal{X}$-coordinates and $\mathcal{A}$-coordinates, given that the $\mathcal{A}$-coordinates are defined as the matrices of the maps $\pi_{i,j}$ in the bases given by the decoration of the flags. However this statement only holds in rank 2. This lead us to the next technical difficulty arising from non-commutativity: there is no determinant for matrices with values in a general ring, hence no special linear group SL_n. Since we can't work with SL_n, we need do work in the general linear group GL_n which even over a commutative ring is not a semi-simple Lie group, but a reductive one. As a consequence, the non-commutative $\mathcal{A}$-coordinates we will define in Section 3.3 and 3.2 will not directly reduce to Fock-Goncharov's $\mathcal{A}$-coordinates in the commutative case. Instead, what we get is a generalization of Fock-Goncharov coordinates to $\mathrm{GL}_n(\mathbb{R})$-local systems. Another technical difficulty arising is the need to work with *twisted* local systems to define $\mathcal{A}$-coordinates. These twisted local systems are defined on the unit tangent bundle $T'S$ of S, with the extra condition of having monodromy around the fiber of $T'S \to S$ equal to $-\mathrm{Id}$.

Now that we have good candidates for generalized $\mathcal{A}$-coordinates, several questions arise. First, are there relations between the coordinates defined with respect to a fixed triangulation? Second, how to compute the mutations formulas relating the coordinates with respect to two different triangulations? To answer both questions, we make use of a tool introduced in 2014 by Davide Gaiotto, Gregory W. Moore and Andrew Neitzke in [GMN13] called *spectral networks*. This is a combinatorial tool used to study framed local systems by allowing the definition of a procedure called *abelianization*. This procedure transform a *flat* rank n vector bundle over a ciliated surface S into a *line* bundle over a certain *ramified* n-covering Σ_n of S called the *spectral surface*. This is a tool of great importance in theoretical physics to study a certain class of topological quantum field theories, but we won't make use of all the potential of spectral networks in this work. Initially, spectral network were defined as the critical graph of a holomorphic quadratic differential on S lifted to the spectral surface. However, a geometric and combinatorial description exists and will be more convenient to work with in our study. For this, we use a combinatorial description of the spectral surface introduced by Alexander Goncharov and Maxim Kontsevich in [GK22] which we recall in Section 2.3.1, and we add the definition to a spectral network adapted to this spectral surface in Section 2.3.3. Then, we need to extend the definition of the non-abelianization procedure to work with *twisted* local systems. This part is a joint work with Eugen Rogozinnikov in [KR22] in the case $n = 2$, and generalizes immediately to $n \geq 3$. This extension is described in Section 2.4. The non-abelianization process itself works exactly the same whether the the bundle over the spectral surface is a line bundle or a higher rank bundle. Once this is done, we describe an inverse construction to the non-abelianization process called (partial) *abelianization*, which transform a framed flat rank nd vector bundle over the ciliated surface S into a flat rank d vector bundle over the spectral surface. We describe it more precisely in Section 2.5. This allow us to describe the topology of a subset (due to genericity assumptions) of the moduli space of framed $\mathrm{GL}_2(R)$-local system over a ciliated surface S, where R is a finite dimensional $\mathbb{R}$-algebra, such as the ring of $n \times n$ matrices with real coefficients.

Theorem. *The moduli space of framed (twisted)* $\mathrm{GL}_2(R)$*-local systems on* S *that are* Δ*-generic with respect to a fixed triangulation* Δ *is homeomorphic to the moduli space of (twisted)* $R^\times$*-local systems on* Σ_2 *which is homeomorphic to* $(R^\times)^{1-4\chi(\overline{S})+2p+\sum n_i}/R^\times$ *where* $R^\times$ *acts diagonally by conjugation on* $(R^\times)^{1-4\chi(\overline{S})+2p+\sum n_i}$.

This construction also allow us to study symplectic local systems in Section 2.7 as a matrix in the symplectic group naturally writes as a two by two block matrix. This is a join work with Eugen Rogozinnikov in [KR22]. For this, we need to translate the data of the symplectic form present in a symplectic local system on S into an additional data in the abelianized local system. As a corollary we also get a topological description of the moduli space of *maximal* framed local systems.

Theorem. *If A is Hermitian, then the moduli space of framed (twisted) maximal $\mathrm{Sp}_2(A,\sigma)$-local systems on S is homeomorphic to:*

$$\left((A_+^\sigma)^{-2\chi(\overline{S})+p} \times (A^\times)^{-2\chi(\overline{S})+2p-1+\sum n_i}\right)/A^\times$$

where $A^\times$ acts componentwisely by conjugation on $(A^\times)^{-2\chi(\overline{S})+2p-1+\sum n_i}$ and by congruence on $(A_+^\sigma)^{-2\chi(\overline{S})+p}$.

In Chapter 3, we focus on defining and studying a non-commutative generalization of Fock-Goncharov's $\mathcal{A}$-coordinates. First we study the special case of $\mathrm{GL}_2(R)$-local systems, since a lot of technical difficulties appearing in higher rank are not present in rank 2. In this case, we show that the non-commutative algebra introduced by Arkady Berenstein and Vladimir Retakh in [BR18] have a representation as matrix valued rational function on the moduli space of decorated twisted $\mathrm{GL}_2(R)$-local systems, which we call *$\mathcal{A}$-coordinates*. Similarly to commutative Fock-Goncharov coordinates, there is a family of functions associated to each triangulation of the surface, and there are mutations formulas that express the functions associated to a triangulation as non-commutative rational functions of functions associated to another triangulation. However, there is a fundamental difference with commutative Fock-Goncharov coordinates: the functions associated to a fixed triangulations have relations between them, called *triangle relations*. The term "coordinates" is thus an an abuse of terminology.

Definition. Let S be a ciliated surface. We define the unitary $\mathbb{R}$-algebra $\mathcal{A}_S$ generated by the symbols x_γ and x_γ^{-1} for all homotopy class of arcs γ joining two punctures on S (with the convention $x_\gamma = 1$ if γ is trivial) with the relations:

- $\forall \gamma \in E(S), x_\gamma^{-1}x_\gamma = x_\gamma x_\gamma^{-1} = 1$
- $\forall f : P_3 \to S$ triangle,
$$x_{\gamma_{1,3}}x_{\gamma_{2,3}}^{-1}x_{\gamma_{2,1}} = x_{\gamma_{1,2}}x_{\gamma_{3,2}}^{-1}x_{\gamma_{3,1}}$$
- $\forall f : P_4 \to S$ quadrilateral,
$$x_{\gamma_{4,2}} = x_{\gamma_{4,3}}x_{\gamma_{1,3}}^{-1}x_{\gamma_{1,2}} + x_{\gamma_{4,1}}x_{\gamma_{3,1}}^{-1}x_{\gamma_{3,2}}$$
  and
$$x_{\gamma_{2,4}} = x_{\gamma_{2,3}}x_{\gamma_{1,3}}^{-1}x_{\gamma_{1,4}} + x_{\gamma_{2,1}}x_{\gamma_{3,1}}^{-1}x_{\gamma_{3,4}}$$

We define the non-commutative $\mathcal{A}$-coordinates associated to a decorated $\mathrm{GL}_2(R)$-local system $\mathcal{L}$ as follows: for each ideal arc γ on S (i.e. a homotopy class of arc joining two punctures) from p to q, let $\gamma^*\mathcal{L}$ be the pullback of $\mathcal{L}|_\gamma$. This is a trivial flat R^2-bundle over $[0,1]$ and the decoration of $\mathcal{L}$ induces two flat R-subbundles F_p and F_q of $\gamma^*\mathcal{L}$. We define

$$a_\gamma : F_p \to R^2/F_q$$

to be the restriction of the canonical projection to F_p, identified with its matrix in the bases induces by the decoration. In the case $R = \mathcal{M}_d(\mathbb{R})$, the space X of decorated twisted $\mathrm{GL}_2(R)$-local systems on S is an algebraic variety and the subset X_Δ of Δ-generic local systems is an open dense subset of X. Each $\mathcal{A}$-coordinate associated to Δ can be seen as a rational function on X_Δ with coefficient in $\mathcal{M}_d(\mathbb{R})$.

Theorem. *Let S be a ciliated surface. The map*

$$\psi : \begin{array}{ccc} \mathcal{A}_S & \to & \mathrm{Rat}(X, \mathcal{M}_d(\mathbb{R})) \\ x_\gamma & \mapsto & a_\gamma \end{array}$$

is an algebra homomorphism.

To show this, we need to compute both the triangle relations and the mutations relations associated with flips of the triangulation. In rank 2, we can compute both with the use of spectral networks. The triangle relations appear naturally in the abelianization process: the $\mathcal{A}$-coordinates are holonomies of certain paths on the abelianized local system, and the triangle relations correspond to holonomies of contractible paths, hence are trivial. For the mutation relation, we take advantage of the fact that the local system on the base surface S does not depend on the triangulation, while the spectral surface and the abelianized local system do. Then by doing abelianizations corresponding to two different triangulations and identifying the right terms, we can express the $\mathcal{A}$-coordinates corresponding to one triangulation in terms of the $\mathcal{A}$-coordinates corresponding to the other one. The same technique can be used to show that given a specific triangulation Δ, any other $\mathcal{A}$-coordinates can be written as a non-commutative Laurent polynomial of coordinates associated to Δ. This property is known as (non-commutative) Laurent phenomenon and is the non-commutative generalization of a well known property of commutative cluster algebras.

Theorem. *Let $n \geq 3$ and let S be the closed disk with n punctures on the boundary. Let $i, j \in \{1, \ldots, n\}$, $i \neq j$. Then for every triangulation Δ of S and every decorated twisted $\mathrm{GL}_2(R)$-local system $\mathcal{L}$ that is Δ-generic and such that $(F^{(i)}, F^{(j)})$ is generic, the $\mathcal{A}$-coordinate $a_{\gamma_{i,j}}$ is a non-commutative Laurent polynomial in the $\mathcal{A}$-coordinates $(a_\gamma)_{\gamma\in\Delta}$ associated to the triangulation Δ.*

We also study in Section 3.1.4 and 3.1.5 a type of coordinates in-between the $\mathcal{X}$-coordinates and the $\mathcal{A}$-coordinates, which yield a representation of a subalgebra of $\mathcal{A}_S$ introduced in [BR18]. The $\mathcal{A}$-coordinates presented here restrict to the non-commutative symplectic coordinates introduced in [AGRW22] when defined over a symplectic local system with a symplectic decoration.

We then generalize these coordinates to configurations of flags in R^n in Section 3.2 and to twisted $\mathrm{GL}_n(R)$-coordinates in Section 3.3. The reason we split this in two different sections is because of a technical difficulty arising from twisted local systems. Indeed, the generalization we describe require additional data to the usual decorated flags. We call that additional data an *extra-decoration*.

Definition. Let $(A^{(1)}, \dots, A^{(k)})$ be a k-tuple of flags in generic position (recall Section 1.2 for the definition of flags in generic position). An *extra-decoration* of the k-tuple of flags is the data of a decoration of each flag $A^{(i)}$ together with the data for every $i_1, \dots, i_k \in \mathbb{N}$ such that $i_1 + \cdots + i_k = (k-1)n + 1$ of an element $b_{i_1,\dots,i_k} \in R^n$ that freely span $A^{(1)}_{i_1} \cap \cdots \cap A^{(k)}_{i_k}$.

For a configuration of flags in R^n this is easy to define, but for a twisted local system, these bases should be sections of the local system along small loops sitting in the interior of the surface. However these small loops have monodromy $-\mathrm{Id}$ because we are working with twisted local systems, so this is only possible to define those section *up to sign*. We define a slightly altered version of the usual quivers associated to commutative cluster algebras, to take into account the fact that an extra-decoration depends on the triangulation chosen, so should mutate alongside the coordinates themselves. We only describe the mutations necessary to a flip, as we lack a geometric interpretation of a general mutation to describe it.

This definition of non-commutative $\mathcal{A}$-coordinates is slightly different than the one introduced by Goncharov-Kontsevich in [GK22]. Indeed, the Goncharov-Kontsevich coordinates can be written as quotients of two of the coordinates introduced in this work. In other words, the $\mathcal{A}$-coordinates presented here gives factorization of Goncharov-Kontsevich coordinates. Another difference is that Goncharov-Kontsevich coordinates only require a decoration of the local systems, whereas the coordinates presented here require this additional *extra-decoration*. The combinatorics of the coordinates presented in this work are much closer to the combinatorics of a cluster algebra, with mutations dictated by an underlying quiver. As such, Goncharov-Kontsevich coordinates can be thought as "in-between" the $\mathcal{A}$-coordinates presented here and non-commutative $\mathcal{X}$-coordinates. It is worth noting that for $n = 2$ (i.e. for $G = \mathrm{GL}_2(R)$) the Goncharov-Kontsevich coordinates coincide with the $\mathcal{A}$-coordinates introduced i this work, the differences between the two types of coordinates arising for $n \geq 3$.

The work presented in Chapter 3 is a step toward the definition of (possibly non-commutative) cluster structures on moduli spaces of G-local systems with framing in partial flag varieties G/P for more general Lie groups and more general parabolic subgroups than what is already known. While there is no direct reduction from the coordinates presented here to coordinates on $SO(p,q)$ or exceptional Lie groups with their parabolic subgroup P_Θ (the two remaining classes of Θ-positive structures), the constructions present in this manuscript should help to understand the expected combinatorics of non-commutative cluster coordinates a general case.

Chapter 1

Algebra and flags

The purpose of this first chapter is to collect basic definitions about flags in free modules over non-commutative algebras. The main example of such non-commutative algebras that we will be interested throughout this work is the algebra of square matrices over a commutative field. This chapter contains elementary linear algebra results that will serve as ground level results in the rest of this manuscript.

1.1 Algebra vocabulary

Let R be a finite dimensional unitary associative $\mathbb{R}$-algebra and $R^\times$ the group of invertible elements of R. For $n \geq 2$ we consider R^n as a right R-module. Let $\mathcal{M}_n(R)$ be the ring of all $n \times n$-matrices with entries in R, and $\mathrm{GL}_n(R)$ be the group of all invertible matrices of $\mathcal{M}_n(R)$. We see R^n as the set of column vectors with coefficients in R, so that $\mathrm{GL}_n(R)$ acts on R^n by left multiplication.

The work presented here could be generalized to a larger class of ring R, but we are formulating it here with R a finite dimensional $\mathbb{R}$-algebra so that R has a "nice" free module theory. In particular, under this assumption a free R-module M has a unique rank k, meaning that when there exists a non-negative integer k such that $M \simeq R^k$, then this integer k (called the *rank* of M) is unique. Moreover, if N is a free submodule of M, then the rank of N is necessarily lesser or equal to k, with equality if and only if $N = M$.

Definition 1.1.1. Let $0 \leq k \leq n$. A *k-dimensional subspace* of R^n is a R-submodule F isomorphic to R^k that is a direct factor of R^n, i.e. $R^n \simeq F \oplus R^{n-k}$. A 1-dimensional subspace of R^n will sometimes be called an *R-line*. We denote the space of R-lines of R^n by $\mathbb{P}(R^n)$.

Remark 1.1.2. If F is a i-dimensional subspace of R^n, then the quotient R^n/F is isomorphic to R^{n-i}. However a R-submodule F satisfying $F \simeq R^k$ and $R^n/F \simeq R^{n-k}$ for some $1 \leq k \leq n-1$ is not necessarily a k-dimensional subspace.

We denote by $x+F$ the image of an element $x \in R^n$ by the canonical projection $R^n \to R^n/F$.

Definition 1.1.3. We make the following definitions:

- An n-tuple $(x_1, \ldots, x_n)$ for $x_1, \ldots, x_n \in R^n$ is called *basis* of R^n if the map

$$\begin{array}{ccc} R^n & \to & R^n \\ (a_1, \ldots, a_n) & \mapsto & \sum_{i=1}^n x_i a_i \end{array}$$

  is an isomorphism of right R-modules.

- An element $x \in R^n$ is called *regular* if it freely spans an R-line of R^n.

- Regular elements $x_1, \ldots, x_k \in R^n$ for $k \leq n$ are called *linearly independent* if they freely span a k-dimensional subspace of R^n, i.e. there exists $x_{k+1}, \ldots, x_n$ such that $(x_i)_{1 \leq i \leq n}$ is a basis of R^n.

- A k-dimensional subspace F and a $(n-k)$-dimensional subspace G are called *transverse* if $F \oplus G = R^n$ as right R-modules.

Remark 1.1.4. Let $x \in R^n$. The map

$$m_x : \begin{array}{ccc} R & \to & R^n \\ a & \mapsto & xa \end{array}$$

is an isomorphism of right R-module if and only if x is regular, and if x is not regular then the kernel of m_x is a $\mathbb{R}$-subalgebra of R.

In this work, we are mainly interested in the case when R is the algebra $\mathcal{M}_d(K)$ of square matrices of size $d \geq 2$ over a commutative field K, or a subalgebra of it. The right R-module R^n is isomorphic to the right R-module $\mathcal{M}_{nd,d}(K)$ of matrices with nd rows and d columns with coefficients in K as a column of n square matrices of size d with coefficients in K is equivalent to a $nd \times d$ matrix with coefficients in K. We may consider an element $\mathbf{e}$ of R^n as a d-tuple $(e_1, \ldots, e_d)$ of vectors in K^{nd}. The group $R^\times$ is then $\mathrm{GL}_d(K)$ and the group $\mathrm{GL}_n(R)$ is identified with $\mathrm{GL}_{nd}(K)$. An R-submodule of R^n spanned by k elements $\mathbf{e}^1, \ldots, \mathbf{e}^k$ is a k-dimensional subspace if and only if the family of kd vectors $(e_1^1, \ldots, e_d^1, e_1^2, \ldots, e_d^k)$ is of rank kd because the vector space in K^{nd} spanned by $(e_1^1, \ldots, e_d^1, e_1^2, \ldots, e_d^k)$ is then kd-dimensional and admits a supplement that is $(n-k)d$-dimensional. The family $(e_1^1, \ldots, e_d^1, e_1^2, \ldots, e_d^k)$ is then a basis of the R-module it spans. A regular element is a family $\mathbf{e} = (e_1, \ldots, e_d)$ of rank d. Note that if $K = \mathbb{R}$ or $K = \mathbb{C}$ then $R^\times$ is dense in R and $\mathrm{GL}_n(R)$ is dense in $\mathcal{M}_n(R)$.

1.2 Configuration of flags

One of the reasons the notion of flags became central in the study of character varieties is because of its relations with Hitchin components. Indeed, studying the action of a split real Lie group on a space of (complete) flags proved to be useful to determine faithful and discreteness properties, see [FG06, Gui08]. The goal of this work is to work with partial flags instead of complete ones. A partial flag in $\mathbb{R}^{nd}$ with subspaces of dimensions $(0, d, 2d, \ldots, (n-1)d, nd)$ is the same as a complete flag in $\mathcal{M}_d(R)^n$, which lead to the following definitions.

Definition 1.2.1. A (complete) *flag* F in R^n is a sequence $F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n$ where every F_i is a i-dimensional subspace of R^n. A *decoration* of a flag F is the data of $(f_1, \ldots, f_n)$ such that $f_i \in F_i/F_{i-1}$ freely spans F_i/F_{i-1} for all $1 \leq i \leq n$.

Definition 1.2.2. A k-tuple of flags $(F^{(1)}, \ldots, F^{(k)})$ is said to be *in generic position* if :

- For every $i_1, \ldots, i_k \in \mathbb{N}$ such that $i_1 + \cdots + i_k = (k-1)n + r$ with $0 \leq r \leq n$, the R-submodule $F^{(1)}_{i_1} \cap \cdots \cap F^{(k)}_{i_k} \subset R^n$ is a r-dimensional subspace of R^n
- For every $i_1, \ldots, i_k \in \mathbb{N}$ such that $i_1 + \cdots + i_k = (k-1)n + 1$ and for all $1 \leq m \leq k$, the canonical projection $\pi_m : F^{(1)}_{i_1} \cap \cdots \cap F^{(k)}_{i_k} \to F^{(m)}_{i_m}/F^{(m)}_{i_m-1}$ is an isomorphism.

The group $\mathrm{GL}_n(R)$ acts naturally on the space of k-tuple of flags in R^n in generic position. We denote by $\mathrm{Conf}_{R,k}(n)$ the quotient of this space by the $\mathrm{GL}_n(R)$ action.

1.3 Kashiwara-Maslov map and the exchange relation

This section contains most of the generic linear algebra results that will play a fundamental role in the definition of both our non-commutative $\mathcal{A}$-coordinates (Section 3.1.2 and 3.2) and the abelianization procedure (Section 2.5).

Let (A, B, C) be a triple of flags in generic position. For all $i, j, k \in \mathbb{N}$ such that $i + j + k = 2n + 1$, we define the following maps by restriction of the canonical projections $A_i \to A_i/A_{i-1}, B_j \to B_j/B_{j-1}$ and $C_k \to C_k/C_{k-1}$:

$$a^A_{i,j,k} : A_i \cap B_j \cap C_k \to A_i/A_{i-1}$$

$$a^B_{i,j,k} : A_i \cap B_j \cap C_k \to B_j/B_{j-1}$$

$$a^C_{i,j,k} : A_i \cap B_j \cap C_k \to C_k/C_{k-1}$$

Remark 1.3.1. If $i = 1$, then $j = k = n$ and $A_i \cap B_j \cap C_k = A_1$. Then the map $a^A_{1,n,n} : A_1 \to A_1$ is the identity. Notice also that if $k = n$, then $i + j = n + 1$ and the maps $a^A_{i,j,n} : A_i \cap B_j \to A_i/A_{i-1}$ and $a^B_{i,j,n} : A_i \cap B_j \to B_j/B_{j-1}$ do not depend on the flag C.

For $i, j, k \in \mathbb{N}$ such that $i + j + k = 2n + 2$, we define the 2-dimensional subspace $P_{i,j,k} = A_i \cap B_j \cap C_k$. There is a map

$$\begin{array}{rcl} A_{i-1} \cap B_j \cap C_k \oplus A_i \cap B_{j-1} \cap C_k \oplus A_i \cap B_j \cap C_{k-1} & \to & P_{i,j,k} \\ (x, y, z) & \mapsto & x + y + z \end{array}$$

and we denote by $K_{i,j,k}$ its kernel. This space is endowed with three canonical projections

$$\begin{aligned} p^A_{i,j,k} &: K_{i,j,k} \to A_{i-1} \cap B_j \cap C_k, \\ p^B_{i,j,k} &: K_{i,j,k} \to A_i \cap B_{j-1} \cap C_k, \\ p^C_{i,j,k} &: K_{i,j,k} \to A_i \cap B_j \cap C_{k-1} \end{aligned}$$

Lemma 1.3.2. *The maps $p^A_{i,j,k}$, $p^A_{i,j,k}$ and $p^A_{i,j,k}$ defined above are isomorphisms. In particular $K_{i,j,k}$ is a R-line.*

Proof. We will show that $p^A_{i,j,k}$ is an isomorphism, the other case being similar. First, note that the R lines $A_{i-1} \cap B_j \cap C_k$, $A_i \cap B_{j-1} \cap C_k$ and $A_i \cap B_j \cap C_{k-1}$ have pairwise trivial intersection. Indeed, $(A_{i-1} \cap B_j \cap C_k) \cap (A_i \cap B_{j-1} \cap C_k) = A_{i-1} \cap B_{j-1} \cap C_k = 0$ because $(i-1) + (j-1) + k = 2n$ and the triple (A, B, C) is in generic position. Then we have

$$P_{i,j,k} = A_i \cap B_{j-1} \cap C_k \oplus A_i \cap B_j \cap C_{k-1}$$

because both terms are free R-module of rank 2 and the right-hand side is a submodule of the left-hand side. So given $x \in A_{i-1} \cap B_j \cap C_k$, there exists a unique pair $(y, z) \in A_i \cap B_{j-1} \cap C_k \oplus A_i \cap B_j \cap C_{k-1}$ such that $x = y + z$, i.e. $(x, -y, -z) \in K_{i,j,k}$. So $p^A_{i,j,k}(x, -y, -z) = x$ and $p^A_{i,j,k}$ is an isomorphism. $\square$

Proposition 1.3.3. *Let $i, j, k \in \mathbb{N}$ such that $i + j + k = 2n + 2$. The following diagram anti-commutes, i.e. $a^C_{i-1,j,k} \circ p^A_{i,j,k} = -a^C_{i,j-1,k} \circ p^B_{i,j,k}$:*

$$\begin{tikzcd}
& A_{i-1} \cap B_j \cap C_k \arrow[dr, "a^C_{i-1,j,k}"] & \\
K_{i,j,k} \arrow[ur, "p^A_{i,j,k}"] \arrow[dr, "p^B_{i,j,k}"] & & C_k/C_{k-1} \\
& A_i \cap B_{j-1} \cap C_k \arrow[ur, "a^C_{i,j-1,k}"] &
\end{tikzcd}$$

Proof. Let $(x, y, z) \in K_{i,j,k}$. Then

$$a^C_{i-1,j,k}(p^A_{i,j,k}(x, y, z)) = a^C_{i-1,j,k}(x) = x + C_{k-1}$$

and

$$a^C_{i,j-1,k}(p^B_{i,j,k}(x, y, z)) = a^C_{i-1,j,k}(y) = y + C_{k-1}.$$

Since $x + y + z = 0$ and $z \in A_i \cap B_j \cap C_{k-1} \subset C_{k-1}$, we have $x + C_{k-1} = -y + C_{k-1}$. $\square$

More generally for all $i, j, k \in \mathbb{N}$ such that $i + j + k = 2n + 2$, the following diagram anti-commutes, i.e. the monodromy around each cell is equal to $-\mathrm{Id}$:

$$\begin{tikzcd}
 & & A_{i-1} \cap B_j \cap C_k \arrow[dll, "a^C_{i-1,j,k}"'] \arrow[drr, "a^B_{i-1,j,k}"] & & \\
C_k/C_{k-1} & & & & B_j/B_{j-1} \\
 & & K_{i,j,k} \arrow[uu, "p^A_{i,j,k}"] \arrow[dll, "p^B_{i,j,k}"'] \arrow[drr, "p^C_{i,j,k}"] & & \\
A_i \cap B_{j-1} \cap C_k \arrow[uu, "a^C_{i,j-1,k}"] \arrow[drr, "a^A_{i,j-1,k}"'] & & & & A_i \cap B_j \cap C_{k-1} \arrow[uu, "a^B_{i,j,k-1}"'] \arrow[dll, "a^A_{i,j,k-1}"] \\
 & & A_i/A_{i-1} & &
\end{tikzcd}$$

In particular, we have

$$(a^B_{i-1,j,k})^{-1} \circ a^B_{i,j,k-1} \circ (a^A_{i,j,k-1})^{-1} \circ a^A_{i,j-1,k} \circ (a^C_{i,j-1,k})^{-1} \circ a^C_{i-1,j,k} = -\mathrm{Id}\,.$$

In the case $n = 2$ a flag is just a R-line and the map $\mu_A^{B,C} := a^A_{2,2,1}(a^B_{2,2,1})^{-1}a^B_{1,2,2} \colon C_1 \to R^2/C_1$ is called the *Kashiwara-Maslov map* of the triple of R-lines (A_1, B_1, C_1). Proposition 1.3.3 imply that $\mu_A^{B,C} = -\mu_A^{C,B}$.

When $n = 2$ we can also describe an additional relation between these isomorphisms for a configuration of 4 flags in generic position:

Proposition 1.3.4. *Let F_1, F_2, F_3, F_4 be four R-lines in R^2 in generic position. For all $i \neq j \in \{1, 2, 3, 4\}$ let $a_{j,i} : F_i \to R^2/F_j$ be the restriction of the canonical projection $R^2 \to R^2/F_j$ to F_i. These maps are isomorphisms and we have the following relation:*

$$a_{2,4} = a_{2,3}a_{1,3}^{-1}a_{1,4} + a_{2,1}a_{3,1}^{-1}a_{3,4}.$$

Proof. These maps are isomorphisms because of the genericity condition. Let $b \in F_4$ freely spanning F_4. Since $F_1 \oplus F_3 = R^2$ by genericity again, there exists $x_1 \in F_1$ and $x_3 \in F_3$ such that $b = x_1 + x_3$. The left-hand term is then $a_{2,4}(b) = b + F_2$. The first term of the right-hand side is

$$\begin{aligned}
a_{2,3}a_{1,3}^{-1}a_{1,4}(b) &= a_{2,3}a_{1,3}^{-1}(b + F_1) \\
&= a_{2,3}a_{1,3}^{-1}(x_1 + x_3 + F_1) \\
&= a_{2,3}a_{1,3}^{-1}(x_3 + F_1) \\
&= a_{2,3}(x_3) \\
&= x_3 + F_2
\end{aligned}$$

and similarly the second term is

$$a_{2,3}a_{1,3}^{-1}a_{1,4}(b) = x_1 + F_2$$

hence the relation. □

Lastly, we describe an additional relation in the case $n = 3$.

Proposition 1.3.5. *Let (A, B, C, D) be four flags in R^3 in generic position. The following diagram anti-commutes:*

$$\begin{tikzcd}
 & R^3/(A_2 \cap B_2 \oplus C_2 \cap D_2) \arrow[dr] & \\
B_2 \cap C_2 \arrow[ur] \arrow[dr] & & A_2 \cap D_2 \\
 & R^3/(A_2 \cap C_2 \oplus B_2 \cap D_2) \arrow[ur] &
\end{tikzcd}$$

where all the maps are restrictions of the canonical projections. All these maps are isomorphisms. We call the top map

$$T^+_{AD,BC} : B_2 \cap C_2 \to A_2 \cap D_2$$

and the bottom map

$$T^-_{AD,BC} : B_2 \cap C_2 \to A_2 \cap D_2$$

Proof. Let b freely spanning $B_2 \cap C_2$. By genericity we have

$$A_2 \cap B_2 \oplus C_2 \cap D_2 \oplus A_2 \cap D_2 = R^3$$

since $A_2 \cap B_2 \oplus A_2 \cap D_2 = A_2$ because $A_2 \cap B_2$ and $A_2 \cap D_2$ are two transverse R-lines in A_2 which is a 2-dimensional subspace, and $A_2 \cap (C_2 \cap D_2)$ is trivial by genericity. Similarly, we have

$$A_2 \cap C_2 \oplus B_2 \cap D_2 \oplus A_2 \cap D_2 = R^3$$

so there exists unique $x \in A_2 \cap C_2$, $y \in B_2 \cap D_2$, $x' \in A_2 \cap B_2$, $y' \in C_2 \cap D_2$ and $z, z' \in A_2 \cap D_2$ such that

$$\begin{aligned} b &= x + y + z \\ &= x' + y' + z' \end{aligned}$$

We want to show that $z + z' = 0$. We have $y + z \in D_2$ but since $y + z = b - x \in C_2$, we have $y + z \in C_2 \cap D_2$. Similarly $x' + z' \in A_2 \cap C_2$. So

$$b = x + (y + z) = (x' + z') + y'$$

are two decomposition of b in $A_2 \cap C_2 \oplus C_2 \cap D_2$, so they must coincide. This means that $x = x' + z'$, so $b = x' + z' + y + z$. So $z + z' = b - x' - y \in B_2$, and $z + z' \in A_2 \cap D_2$ which means that $z + z' = 0$ since $A_2 \cap D_2 \cap B_2 = 0$. $\square$

Chapter 2

Local systems and partial (non-)abelianization

In this chapter, we give all the definitions of framed and decorated local systems we need in this work. To understand the combinatorial properties of the non-commutative cluster coordinates we will define in Chapter 3, we start from a geometrical point of view. In 2014 Davide Gaiotto, Gregory W. Moore and Andrew Neitzke introduce in [GMN13] a combinatorial tool called *spectral networks* to describe a procedure called *abelianization* and its inverse called *non-abelianization*. The purpose of these constructions is to transform a flat vector bundle over a surface into a flat *line* bundle over a ramified covering of the surface. In this line bundle, many of the geometric invariant of the initial bundle can be seen. This is true in particular for the Fock-Goncharov coordinates of the initial bundle, which we find as monodromies of some special curves in the associated line bundle, see [GMN14, HN16]. The main idea behind this chapter is that the abelianization procedure can be carried "partially", i.e. transforming a vector bundle of high rank over a surface into a vector bundle of lower rank over a ramified covering but not necessarily a line bundle. In this lower rank bundle, looking at the monodromies of the curves representing the Fock-Goncharov coordinates in the abelian case should naturally give rise to non-commutative analogs of Fock-Goncharov coordinates. However since we want to define non-commutative cluster $\mathcal{A}$-coordinates, we need to work with twisted local system, as introduced in [FG06]. Thus, the first step is to extend the construction of Gaiotto-Moore-Neitzke to twisted local system. This is a joint work with Eugen Rogozinnikov in [KR22], together with a direct application of this construction to the study of symplectic local systems and maximal representations.

2.1 Ciliated surfaces, topological and combinatorial data

To study local system on surfaces, we first need to describe the kind of surfaces we want to consider. The construction of cluster coordinates on moduli spaces of local system in [FG06] uses a very general class of surfaces called *ciliated surfaces*. Since our goal is to extend this construction we will naturally deal with the same kind of surfaces.

Let $\overline{S}$ a smooth compact orientable surface with boundary. Let $\mathcal{P} \subset \overline{S}$ be a finite non-empty set such that every connected component of the boundary $\partial\overline{S}$ of $\overline{S}$ contains at least one point in $\mathcal{P}$. We call the elements of $\mathcal{P}$ *punctures*. We split the set of all punctures in two subsets: the set of *external punctures* $\mathcal{P}_{ext} = \mathcal{P} \cap \partial\overline{S}$ and the set of *internal punctures* $\mathcal{P}_{int} = \mathcal{P}\backslash\mathcal{P}_{ext}$. Let $S = \overline{S}\backslash\mathcal{P}$. We call a surface S obtained this way a *ciliated surface.* A ciliated surface will be called *hyperbolic* if it is not the sphere with one or two (internal) punctures nor the closed disk with one or two external punctures and no internal punctures. Every hyperbolic ciliated surface admits a hyperbolic metric of finite volume with totally geodesic boundary, but the choice of such a metric does not matter in this work. For every such hyperbolic structure, all the internal punctures are cusps and all boundary curves are (infinite) geodesics. Once equipped with a hyperbolic structure as above, the universal covering S' of S can be seen as a closed convex subset of the hyperbolic plane $\mathbb{H}^2$ with totally geodesic boundary, which is invariant under the natural action of $\pi_1(S)$ on $\mathbb{H}^2$ by the holonomy representation. Punctures of S are lifted to points of the ideal boundary of $\mathbb{H}^2$ which we call punctures of S' and denote their set by $\mathcal{P}' \subseteq \partial_\infty S' \subseteq \partial_\infty\mathbb{H}^2$. Notice, if $\overline{S}$ does not have boundary, then S' is the entire $\mathbb{H}^2$.

Example 2.1.1. An important example of a ciliated surface is the k-gon for any integer $k \geq 3$ which is a (closed) disk with k external punctures and no internal punctures. In the following, a 3-gon will be called a triangle and a 4-gon will ce called a quadrilateral.

Later on we will need to replace the base surface S with its unit tangent bundle (see Section 2.2.3). Let TS the tangent bundle of S and $TS\backslash\{0\}$ the tangent bundle without the zero section. The group $\mathbb{R}_+^*$ of positive real numbers acts on $TS\backslash\{0\}$ by scalar multiplication. With a slight abuse of terminology, we call *unit tangent bundle* the quotient $T'S = (TS\backslash\{0\})/\mathbb{R}_+^*$. This is a bundle over S of fiber $\mathbb{S}^1$. We will write an element of $T'S$ as an ordered pair (x, v) with $x \in S$ and v a non-zero vector in T_xS, identified with the half-line it spans. Every smooth path $\gamma : [0, 1] \to S$ lifts to a path

$$T'\gamma = (\gamma, \gamma') : [0, 1] \to T'S.$$

We fix an orientation on S. In the figures, the surface S will be oriented clockwise. For every internal puncture $p \in \mathcal{P}_{int}$ we choose a neighborhood V_p of p in S such that V_p is diffeomorphic to an open disk with one internal puncture, and such for $p \neq q$ two distinct punctures the neighborhoods V_p and V_q are disjoint. We fix a smooth oriented loop β_p

around p in V_p such that β_p is homotopic to p and the orientation of β_p is such that p is on its right. We call $T'\beta_p = (\beta_p, \beta'_p)$ the lift of β_p to $T'S$.

For every external puncture $p \in \mathcal{P}_{ext}$ we choose a neighborhood V_p of p in S such that V_p is diffeomorphic to $\left\{(x,y) \in \mathbb{R}^2 | x^2+y^2<1, x \geq 0, (x,y) \neq (0,0)\right\}$ (we call such a neighborhood a *punctured half-disk*). We choose the neighborhoods V_p to be pairwise disjoints. We also choose a smooth path β_p inside V_p going from one of the boundary components surrounding p to the other one, again oriented such that p is on its right. We also denote by $T'\beta_p$ its lift to $T'S$.

An *ideal triangulation* (which we will simply call *triangulation*) of S is a maximal set of (homotopy classes of) pairwise non-intersecting and non-homotopic, not self-intersecting arcs $\left\{\gamma_i : [0,1] \to \overline{S}\right\}_{i \in I}$ such that $\forall i \in I, \forall t \in [0,1], \gamma_i(t) \in \mathcal{P} \Leftrightarrow t \in \{0,1\}$. By maximality, the complement in S of a triangulation Δ is a disjoint union of open disks which we call *triangles* of Δ. Note that the boundary components of S (i.e. the connected components of $\partial S \backslash \mathcal{P}_{ext}$) are arcs present in every triangulation of S. We call those *external edges* of the triangulation, and we call the other arcs *internal edges*. Note that the endpoints of edges of Δ are punctures. For a triangle t of Δ with vertices $p,q,r \in \mathcal{P}$ we will sometimes write $t=(p,q,r)$ even though the data of the vertices does not in general determine the triangle t.

Remark 2.1.1. It is well known (see [Hat91]) that any two triangulations of S can be related by a finite sequence of *flips*. A flip is the change of triangulation obtained by changing the diagonal of a quadrilateral that is part of the triangulation.

2.2 Framed and decorated local systems

2.2.1 Local systems

Let X be a smooth manifold and let $n \geq 1$.

Definition 2.2.1. A $\mathrm{GL}_n(R)$-*local system* on X is a R^n-bundle $\mathcal{L}$ over X with locally constant transitions functions, i.e. there exists an open covering $X = \bigcup_{i \in I} U_i$ such that $\mathcal{L}|_{U_i} \overset{\alpha_i}{\simeq} U_i \times R^n$ and for every $i,j \in I$ such that $U_i \cap U_j \neq \varnothing$ (up to refining the covering we can assume $U_i \cap U_j$ to be connected), there exists $g_{i,j} \in \mathrm{GL}_n(R)$ such that the following diagram commutes:

$$\begin{tikzcd}
(U_i \cap U_j) \times R^n \arrow[r, "\mathrm{Id} \times g_{i,j}"] \arrow[d, "\alpha_i"'] & (U_i \cap U_j) \times R^n \arrow[d, "\alpha_j"'] \\
\mathcal{L}|_{U_i \cap U_j} \arrow[r, "\mathrm{Id}"] & \mathcal{L}|_{U_i \cap U_j}
\end{tikzcd}$$

Remark 2.2.2. This is the same as a locally constant sheaf on X of fiber R^n.

Given a local system $\mathcal{L}$ on X, we will denote by $\mathcal{L}|_Y$ its restriction to a subset $Y \subset X$ and by $\mathcal{L}_x$ its fiber above $x \in X$.

Definition 2.2.3. Let U be an open subset of X. A *regular R^k-subbundle* or *k-dimensional subbundle* F of a $\mathrm{GL}_n(R)$-local system $\mathcal{L}$ over U is a subbundle of $\mathcal{L}|_U$ such that for every $p \in U$ there exists a neighborhood U_p of p and a trivialization

$$\Phi_p \colon \mathcal{F}|_{U_p} \to U_p \times R^n$$

such that $\Phi_p(F|_{U_p}) = U_p \times G$ where G is an k-dimensional subspace in R^n.

A section $v \colon U \to \mathcal{F}$ is *regular* if its span vR is a regular R^1-subbundle of $\mathcal{F}|_U$.

Definition 2.2.4. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two $\mathrm{GL}_n(R)$-local systems over X. A morphism $\varphi : \mathcal{L}_1 \to \mathcal{L}_2$ of vector bundles is a *morphism of local systems* if there exists an open covering $X = \bigcup_{i\in I} U_i$ such that both $\mathcal{L}_1$ and $\mathcal{L}_2$ are trivial over U_i and such that the family of linear maps $(g(x) : R^n \to R^n)_{x\in U_i}$ induced by $U_i \times R^n \simeq \mathcal{L}_1|_{U_i} \xrightarrow{\varphi} \mathcal{L}_2|_{U_i} \simeq U_i \times R^n$ is locally constant.

The following results relate those geometrical objects to more algebraic objects, namely representations of groups. These are well-known results, the reader can find the proofs in [Del70].

Proposition 2.2.5. *A $\mathrm{GL}_n(R)$-local system on a simply connected manifold X is isomorphic to the trivial bundle $X \times \mathrm{GL}_n(R)$.*

Proposition 2.2.6 (Riemann-Hilbert correspondence)**.** *The set of $\mathrm{GL}_n(R)$-local systems on X up to isomorphism is in 1:1 correspondence with the set of representations $\rho : \pi_1(X) \to \mathrm{GL}_n(R)$ up to the action of $\mathrm{GL}_n(R)$ by conjugation.*

Definition 2.2.7. Let $\mathcal{L}$ be a $\mathrm{GL}_n(R)$-local system on X, and let $\gamma : [0,1] \to X$ be a path on X. Let $x = \gamma(0)$ and $y = \gamma(1)$. The bundle $\gamma^*\mathcal{L}$ is trivial since $[0,1]$ is simply connected, we fix an isomorphism $\gamma^*\mathcal{L} \simeq [0,1] \times R^n$. Hence we can define an isomorphism of R-module

$$\varphi_\gamma : \mathcal{L}_x \to \mathcal{L}_y$$

such that $\varphi_\gamma(v) = v'$ if and only if $\gamma^*(v) = (0, v'')$ and $\gamma^*(v') = (1, v'')$ for some $v'' \in R^n$. We call the map φ_γ the *parallel transport* of $\mathcal{L}$ along γ. This map only depends on the homotopy class of γ. The parallel transport along a path will sometimes be called the *holonomy* of $\mathcal{L}$ along γ, and in the special case when the path is a loop we will instead call the parallel transport the *monodromy* of $\mathcal{L}$ around γ.

2.2.2 Framed local systems

The framework of Fock-Goncharov requires to work with additional data to a local system on a surface. We will add to the local system the data of a flag at each puncture to get a *framed* local system. This impose a monodromy condition on the local system around internal puncture.

Let S be an hyperbolic ciliated surface.

Definition 2.2.8. Let $\mathcal{L}$ a $\mathrm{GL}_n(R)$-local system on S. A *framing* of $\mathcal{L}$ is the data of for each puncture p of $n+1$ subbundles $F_0^{(p)} \subset \cdots \subset F_n^{(p)}$ of $\mathcal{L}|_{\beta_p}$, with $F_i^{(p)}$ being a i-dimensional subbundle of $\mathcal{L}|_{\beta_p}$. The data of a $\mathrm{GL}_n(R)$-local system together with a framing is called a *framed* $\mathrm{GL}_n(R)$*-local system* on S.

Remark 2.2.9. For every puncture p, $F_0^{(p)}$ is the zero section of $\mathcal{L}|_{\beta_p}$ and $F_n^{(p)} = \mathcal{L}|_{\beta_p}$.

Remark 2.2.10. For a $\mathrm{GL}_n(R)$-local system on a hyperbolic ciliated surface S to admit a framing, the monodromy around β_p for every internal puncture p must be upper triangular in some basis.

Remark 2.2.11. Let $\mathcal{L}$ be a framed local system on S. Let Δ be a triangulation of S and let t be a triangle of Δ. Since $\mathcal{L}|_t$ is trivial we have in any fiber $\mathcal{L}_x$ over $x \in t$ a triple of flags given by the parallel transport of $F^{(p)}, F^{(q)}$ and $F^{(r)}$.

Definition 2.2.12. Let $\mathcal{L}$ be a framed $\mathrm{GL}_n(R)$-local system on a ciliated surface S, and let Δ be a triangulation of S. We say that the framing of $\mathcal{L}$ is Δ-*generic* if for every triangle $t = (p, q, r)$ of Δ and for any point $x \in t$, the triple of flags defined by parallel transport of $F^{(p)}, F^{(q)}$ and $F^{(r)}$ to $\mathcal{L}_x$ is in generic position.

Note that in a contractible subset of S like a triangle t in a triangulation Δ of S, for every $x, y \in t$ and $F \subset \mathcal{L}_x$ all the parallel transports of F from x to y in t define the same subspace $F' \subset \mathcal{L}_y$. Given a triangulation, a framing of $\mathcal{L}$ can thus be extended to a triple of flags in every fiber of $\mathcal{L}$.

2.2.3 Twisted and decorated local systems

The data of a framed local system is enough to carry the abelianization/non-abelianization constructions, but the moduli space of framed local system in the commutative setting carry $\mathcal{X}$-coordinates, which are less well behaved in the non-commutative setting. To define $\mathcal{A}$-coordinates, we need even more additional data, called a *decoration*: a basis of the graded of each flags of the framing. Very few local systems on a ciliated surface admit a decoration, hence we need to consider a slight modification of local systems called *twisted* local systems, introduced in [FG06] and used in numerous work in the field like [GRW22] or [GK22].

Definition 2.2.13. Let S be a ciliated surface and let $T'S$ be its unit tangent bundle. A *twisted* $\mathrm{GL}_n(R)$*-local system* on S is a $\mathrm{GL}_n(R)$-local system on $T'S$ such that the monodromy around the loop going once around the fiber of $T'S \to S$ is $-\mathrm{Id}$.

We define a framing of a twisted local system the same way as for not twisted ones, except the flag subbundles are now taken above the lifted peripheral paths $T'\beta_p$. More precisely, let $\mathcal{L}$ a twisted $\mathrm{GL}_n(R)$-local system on S. A *framing* of $\mathcal{L}$ is the data for each puncture p of $n+1$ subbundles $F_0^{(p)} \subset \cdots \subset F_n^{(p)}$ of $\mathcal{L}|_{T'\beta_p}$ such that for all $0 \leq i \leq n$, $F_i^{(p)}$ is a i-dimensional subbundle. The data of a $\mathrm{GL}_n(R)$-local system together with a framing is called a *framed twisted* $\mathrm{GL}_n(R)$*-local system* on S.

Note that since the monodromy of a twisted local system $\mathcal{L}$ around a fiber of $T'S \to S$ is $-\mathrm{Id}$, it preserves any subspace of a fiber of $\mathcal{L}$. Thus as in the non-twisted case, given a triangulation of S we can define a triple of flags in every fiber of $\mathcal{L}$.

Again, we define Δ-generic framing for twisted local system in a similar way to before. Let $\mathcal{L}$ be a framed twisted $\mathrm{GL}_n(R)$-local system on a ciliated surface S, and let Δ be a triangulation of S. We say that the framing of $\mathcal{L}$ is Δ*-generic* if for every triangle $t = (p, q, r)$ of Δ and for any point $x \in t$, the triple of flags defined by parallel transport of $F^{(p)}, F^{(q)}$ and $F^{(r)}$ to $\mathcal{L}_v$ is in generic position, where $v \in T'_x S$.

The fundamental group of $T'S$ is an extension of $\pi_1(S)$ by $\pi_1(\mathbb{S}^1) = \mathbb{Z}$:

$$0 \to \mathbb{Z} \to \pi_1(T'S) \to \pi_1(S) \to 0$$

and this sequence splits because $\pi_1(S)$ is a free group. We denote by $\pi_1^s(S)$ the quotient of $\pi_1(T'S)$ by the subgroup $2\mathbb{Z} \subset \mathbb{Z} = \pi_1(\mathbb{S}^1)$.

Proposition 2.2.14. *The set of twisted* $\mathrm{GL}_n(R)$*-local systems on* S *up to isomorphism is in one-to-one correspondence with the set of representations* $\rho : \pi_1^s(S) \to \mathrm{GL}_n(R)$ *such that* $\rho(\delta) = -\mathrm{Id}$ *up to the action of* $\mathrm{GL}_n(R)$ *by conjugation, where* δ *is the loop going once around the fiber of* $T'S \to S$ *in any direction.*

Remark 2.2.15. If the ciliated surface S does not have any internal puncture, S embeds into $T'S$ via the choice of any non-vanishing vector field agreeing with the paths $T'\beta_p$ for all internal puncture p. Thus, twisted $\mathrm{GL}_n(R)$-local system and $\mathrm{GL}_n(R)$-local system are equivalent on surfaces without internal punctures.

Definition 2.2.16. Let $\mathcal{L}$ be a framed twisted GL_n-local system on S. A *decoration* of $\mathcal{L}$ is the data for every puncture p and for every $1 \leq i \leq n$ of a flat section $b_i^{(p)}$ of $F_i^{(p)}/F_{i-1}^{(p)}$ along $T'\beta_p$ that freely spans $F_i^{(p)}/F_{i-1}^{(p)}$. The data of a framed twisted $\mathrm{GL}_n(R)$-local system together with a decoration is called a *decorated twisted* $\mathrm{GL}_n(R)$*-local system.*

Remark 2.2.17. For a twisted $\mathrm{GL}_n(R)$-local system on S to admit a decoration, the monodromy around every loop $T'\beta_p$ must be upper triangular unipotent in some basis.

2.3 Spectral data

In this section we introduce the necessary tools for the abelianization/non-abelianization construction, namely the spectral surface in Section 2.3.1 which is a type of ramified covering and the spectral network in Section 2.3.3. We also define a few special curves on the spectral surface that will be useful in Section 3.3 to define the cluster coordinates.

2.3.1 Ramified coverings

We first introduce some vocabulary about ramified coverings. We will only consider finite ramified covering.

Definition 2.3.1. Let $X, \tilde{X}$ be two smooth manifolds and let $n \geq 2$. A continuous surjective map $\pi\colon \tilde{X} \to X$ is called a *n-fold (topological) ramified covering* if there exists a nonempty finite set $B \subset X$ such that $\pi|_{\pi^{-1}(X \backslash B)} \colon \pi^{-1}(X\backslash B) \to X \backslash B$ is a n-fold covering and every $b \in B$ has strictly less than n preimages. The subset B above is called the *branch locus* of π, and a point $b \in B$ is called a *branch point*. If a branch point $b \in B$ has exactly $n-1$ preimages we call it a *simple* branch point, and if all the branch points are simple the ramified covering is called simple. Given a branch point $b \in B$, a preimage $\tilde{b} \in \pi^{-1}(b)$ is called a ramification point if in a neighborhood U of $\tilde{b}$ the map $\pi|_{U \backslash \tilde{b}} \colon U \backslash \tilde{b} \to \pi(U) \backslash b$ is a k-fold covering with $k \geq 2$.

In this section we will give a combinatorial construction of the spectral surface for bundles of rank $n \geq 2$ introduced in [GK22] by A. Goncharov and M. Kontsevich. We recall the construction of the spectral cover given in [GK22] and introduce a few notations, and we will use this construction in Section 2.3.3 to construct a spectral network associated to this simple ramified n-covering. Let S a ciliated surface, endowed with a triangulation Δ. For every integer $n \geq 2$ we define a bipartite graph Γ_n on S by gluing for all triangles t of S elementary pieces $\Gamma_n^{(t)}$ described as follows. We can fix for each triangle t of Δ a diffeomorphism φ_t between t and the euclidean triangle T in $\mathbb{R}^3$ that is the convex hull of the points of coordinates $(0, n+1, n+1)$, $(n+1, 0, n+1)$ and $(n+1, n+1, 0)$. This euclidean triangle T can be decomposed into smaller triangles as in Figure 3.1.

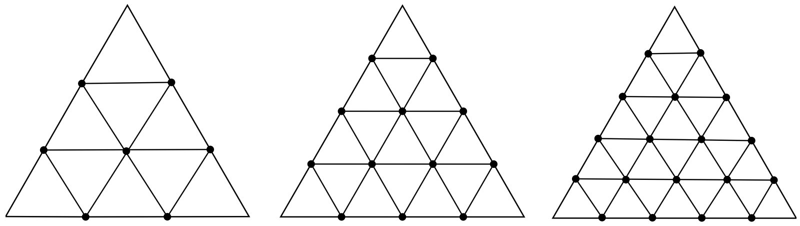


Figure 3.1: From left to right, a 3-subdivision, a 4-subdivision and a 5-subdivision of a triangle.

The vertices of those smaller triangles are the point of coordinates (a,b,c) with $a+b+c = 2n+2$ and a,b,c are non-negative integers, and each smaller triangle is one of two types: *upward* triangles have vertices $(a-1,b,c),(a,b-1,c),(a,b,c-1)$ with $a+b+c = 2n+3$ and *downward* triangles have vertices $(a+1,b,c),(a,b+1,c),(a,b,c+1)$ with $a+b+c = 2n+1$. We call this subdivision into smaller triangles a $n+1$-*subdivision* of T. On each side of T there are $n+1$ smaller upward triangles. There is a black vertex of $\Gamma_n^{(t)}$ on every point of coordinates (a,b,c) with $a+b+c = 2n+2$ and a,b,c positive integers (i.e. every vertex of smaller triangles except for the vertices of the big triangle T). There is a white vertex of $\Gamma_n^{(t)}$ inside every downward small triangle. Then for every downward small triangle, there is an edge in $\Gamma_n^{(t)}$ between the white vertex inside it and the three black vertices at the vertices of the downward small triangle, see Figure 3.2.

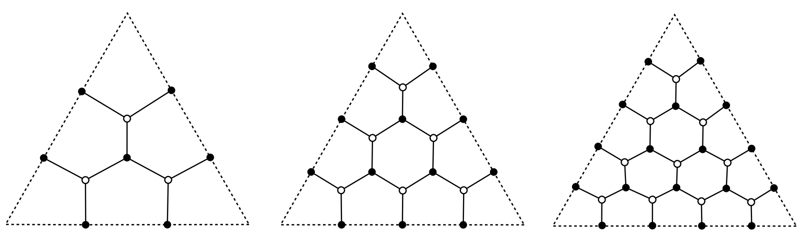


Figure 3.2: The graphs $\Gamma_2^{(t)}$, $\Gamma_3^{(t)}$ and $\Gamma_4^{(t)}$ restricted to a triangle t of Δ.

Then for every pair of triangles t and t' that share an edge e of the triangulation, we identify the corresponding black vertices of $\Gamma_n^{(t)}$ and $\Gamma_n^{(t')}$ on e, see Figure 3.3.

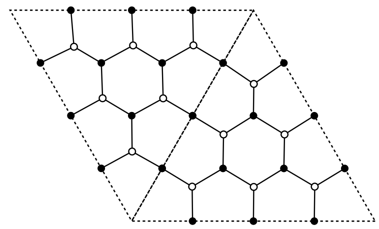


Figure 3.3: The gluing of $\Gamma_3^{(t)}$ and $\Gamma_3^{(t')}$ for two adjacent triangles t and t'.

The white vertices are trivalent and the black vertices are either univalent, bivalent or trivalent: the univalent and bivalent black vertices will be called *external* and the trivalent black vertices will be called *internal.* The graph Γ_n is embedded into S so every vertex inherits a cyclic order on the edges incident to it from the orientation of S. Let $\gamma = (v_1, \dots, v_r)$ be a path on Γ_n and $1 < i < r$. We say γ *turns left* (resp. *turns right*) at v_i if the edges (v_{i-1}, v_i) and (v_i, v_{i+1}) (resp. (v_i, v_{i+1}) and (v_{i-1}, v_i)) are consecutive in that order with respect to the cyclic ordering at v_i. Since the graph Γ_n is embedded into S, a path in Γ_n will be identified with its image in S.

A *zig-zag path* on Γ_n is a path $(v_1, \dots, v_r)$ that turns right at every white vertex and turns left at every black vertex. In particular, every oriented edge of Γ_n belong to exactly one maximal zig-zag path. Since the graph Γ_n is finite, a zig-zag path can be extended either to a cycle on Γ_n, every extensions of this cycle being periodic, or to a path connecting two boundary components of S. From the construction of Γ_n, we get the following proposition:

Proposition 2.3.2. *Let γ be a zig-zag path on Γ_n. Then the image of γ in S is either:*

- *homeomorphic to an oriented circle $\mathbb{S}^1$, and $S\backslash\gamma$ has two connected component, one being homeomorphic to an open disk with one puncture. We denote this connected component S_γ and the puncture inside it s_γ. The puncture s_γ is on the right of the path γ.*
- *a path without self-intersection, and $S\backslash\gamma$ has two connected component, one being homeomorphic to a punctured half-disk. We denote this connected component S_γ and the puncture inside it s_γ.*

Let γ a zig-zag path on Γ_n and $\overline{S_\gamma}$ the closure in S of S_γ. The boundary of S_γ is the image of γ in S, in particular it does not contain any punctures so $\overline{S_\gamma}$ is homeomorphic to either a closed disk with one puncture in its interior or a closed punctured half-disk.

Lemma 2.3.3. *Let $n \geq 2$. For every puncture s of S, there is exactly n zig-zag paths $\gamma_s^{(1)}, \dots, \gamma_s^{(n)}$ in Γ_n such that $s = s_{\gamma_s^{(i)}}$ for all $1 \leq i \leq n$. Moreover, up to reordering, for all $1 \leq i < j \leq k$ we have $\gamma_s^{(i)} \subset S_{\gamma_s^{(j)}}$.*

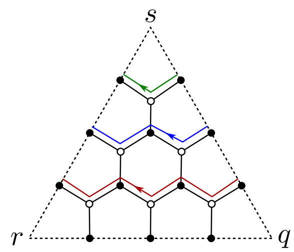


Figure 3.4: The three zig-zag paths $\gamma_s^{(1)}, \gamma_s^{(2)}$ and $\gamma_s^{(3)}$ in $\Gamma_3^{(t)}$ that circle around the puncture s. The subset of t above a path $\gamma_s^{(i)}$ is contained in $S_{\gamma_s^{(i)}}$.

We define Σ_n as the topological surface obtained by gluing punctured disks to Γ_n: along every zig-zag path γ in Γ_n we glue a copy of $\overline{S_\gamma}$. We define $\pi : \Sigma_n \to S$ as follows: the image of a point in Γ_n is the immersion in S, and every cell S_γ is endowed with a map $S_\gamma \to S$ given by the inclusion. For all puncture s of S, we denote by s_i the puncture inside $S_{\gamma_s^{(i)}}$ in Σ_n.

Proposition 2.3.4. *The map $\pi : \Sigma_n \to S$ is a is a surface, and a ramified n-fold covering with simple ramification points. The ramification points are the internal black vertices of Γ_n, we denote the set of all ramification points in Σ_n by $\mathcal{B}$, and the set of branch points of S by $B = \pi(\mathcal{B})$.*

Proof. Every edge of Γ_n is part of exactly two zig-zag paths so Σ_n is a topological surface in the neighborhood of a point in the interior of every edge of Γ_n and in the neighborhood of an external black vertex. Every white vertex is on the boundary of exactly three zig-zag paths, each cell S_γ being glued to a pair of adjacent edges (see Figure 3.5). The neighborhood of a white vertex is then a surface. In all these case, we also see that the covering π is regular. In the neighborhood of every internal black vertex Σ_n is also a surface, but the covering π is simply ramified at every internal black vertex (see Figure 3.5). $\square$

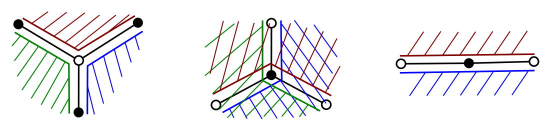


Figure 3.5: The neighborhood of a white vertex, an internal black vertex and an external black vertex, together with the different cells glued in the neighborhood of those vertices.

Note that the map $\pi|_{\Sigma_n \backslash \mathcal{B}} : \Sigma_n \backslash \mathcal{B} \to S \backslash \mathcal{B}$ is a local diffeomorphism. In the neighborhood

U_b of a ramification point $b \in \mathcal{B}$ the surface Σ_n is diffeomorphic to a disk and the map π is locally of the form $\begin{array}{ccc} \mathbb{D} & \to & \mathbb{D} \\ z & \mapsto & z^2 \end{array}$.

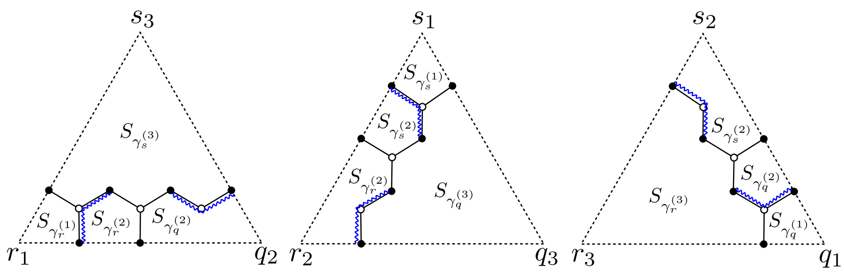


Figure 3.6: The 3-fold ramified covering of a triangle of S. The wavy blue lines are branch cuts.

Note that inside a triangle t of Δ with vertex s, q, r, each white vertex v of Γ_n is uniquely determined by the triple of integers (i_q, i_r, i_s) such that $v \in \gamma_q^{(i_q)} \cap \gamma_r^{(i_r)} \cap \gamma_s^{(i_s)}$. This triple satisfies $i_q + i_r + i_s = 2n + 1$.

In the case $n = 2$, we can describe more precisely the ramified covering. Let S be an hyperbolic ciliated surface and let Δ be a triangulation of S. We can endow $\overline{S}$ with a Euclidean structure with conical points by choosing for each triangle t of Δ an orientation preserving diffeomorphism $\varphi_t : \mathsf{T} \to t$ where T is the Euclidean triangle in $\mathbb{R}^2 = \mathbb{C}$ with vertices 1, $j = e^{\frac{2i\pi}{3}}$ and j^2. Then for each gluing of two triangles (not necessarily distinct) in S, glue the corresponding Euclidean triangles with the composition of a rotation and a translation. The conical points of this structure are exactly the points in $\mathcal{P}$, meaning that this structure once restricted to S is smooth. Let $B = \{\varphi_t(0) \mid t \text{ triangle of } \Delta\} \subset S$. There is one point of B in the interior of each triangle of $\mathcal{T}$. With this data, we can construct a two-fold branched covering $\pi \colon \overline{\Sigma}_2 \to \overline{S}$ such that the branched points are precisely elements of B and $\overline{\Sigma}_2$ has a Euclidean structure as follows. Let H be the Euclidean hexagon with vertices the sixth roots of unity in $\mathbb{C}$. Then the map $z \mapsto z^2$ is a ramified covering from H to T that has exactly one ramification of order 2 at the point 0. Then take as many copies of H as there are triangles in $\mathcal{T}$ and for each gluing of two triangles (not necessarily distinct) in S, glue the corresponding Euclidean hexagons on both edges that are mapped to the glued edge in S with rotation and a translation (see Figure 3.7). The surface obtained is Σ_2.

This defines a two-fold ramified covering $\pi : \overline{\Sigma_2} \to \overline{S}$ with ramification points at B, and the conical points of $\overline{\Sigma}_2$ are a subset of $\pi^{-1}(\mathcal{P})$. This means the map π restricted to $\Sigma_2 = \overline{\Sigma_2} \setminus \pi^{-1}(\mathcal{P})$ is a smooth two-fold branched covering from Σ_2 to S, with simple ramifications on points of B. The lift $\Delta^* := \pi^{-1}(\Delta)$ of Δ to Σ_2 induces a hexagonal tiling of Σ_2 such that in every hexagon there is exactly one element of $\pi^{-1}(B)$.

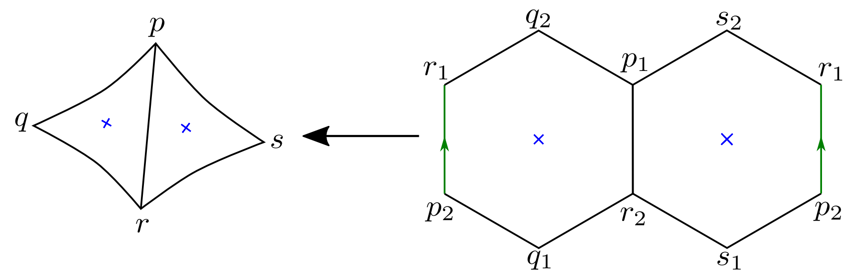


Figure 3.7: The ramified two-fold covering of two glued triangles. The preimages of p are p_1 and p_2, same for q, r, s. The branched points are the blue crosses. The two outer edges with an arrow are glued according to arrow orientation.

Remark 2.3.5. Since the map π is a local diffeomorphism on $\Sigma_2 \setminus \pi^{-1}(B)$, it induces the tangent (differential) map $d\pi\colon T(\Sigma_2 \setminus \pi^{-1}(B)) \to T(S \setminus B)$ that factorizes to unit tangent bundles $T'(\Sigma_2 \setminus \pi^{-1}(B)) \to T(S \setminus B)'$. In order to simplify the notation, we will sometime write $\pi_*\colon T\Sigma_2 \to TS$ and $\pi_*\colon T'\Sigma_2 \to T'S$ instead of $d\pi$.

Remark 2.3.6. The unit tangent bundle of H (resp. T) is canonically identified to $\mathsf{H} \times \mathbb{S}^1$ (resp. $\mathsf{T} \times \mathbb{S}^1$) as H (resp. T) is a subset of $\mathbb{R}^2$. With this identification, the preimages by $d\pi$ of $(x, v) \in T'S$ are of the form (x_1, v') and $(x_2, -v')$ where x_1 and x_2 are the preimages of x by π.

The following proposition describe the topology of the ramified covering Σ_2:

Proposition 2.3.7. *Let $\overline{S}$ be a compact orientable surface with $k \geq 0$ boundary components $C_1, \ldots, C_k$ and let $\mathcal{P}$ be a finite set of points (called punctures) of $\overline{S}$ such that for all $i \in \{1, \ldots, k\}$, $n_i = \#(C_i \cap \mathcal{P}) > 0$. Let k_e (resp. k_o) be the number of components of $\partial\overline{S}$ with an even (resp. odd) number of punctures, such that $k = k_e + k_o$. Let $p = \#(\mathcal{P} \setminus \partial\overline{S})$, let g be the genus of $\overline{S}$ and let $S = \overline{S} \setminus \mathcal{P}$. Then the two-fold ramified covering $\Sigma_2 = \overline{\Sigma}_2 \setminus \pi^{-1}(\mathcal{P})$ of S is a surface that verifies:*

- *$\overline{\Sigma}_2$ is a compact orientable surface of genus*

$$g' = \frac{1}{2}\left(2p + 2k_e + 3k_o + 8g - 6 + \sum_{i=1}^{k} n_i\right),$$

- *for each of the k_e boundary components C of $\overline{S}$ with aneven number n of punctures, $\pi^{-1}(C)$ is the union of two distinct boundary components in $\overline{\Sigma}_2$, each with n punctures,*

- *for each of the k_o boundary components C of $\overline{S}$ with an odd number n of punctures, $\pi^{-1}(C)$ is one boundary component in $\overline{\Sigma}_2$ with $2n$ punctures,*

- Σ_2 *has* $2p$ *internal punctures.*

Proof. First, note that the formula given above defines an integer because $3k_o + \sum n_i$ is always even. It is clear from the construction that $\overline{\Sigma_2}$ is compact and orientable, and that Σ_2 has $2p$ internal punctures. To compute the number of boundary components of $\overline{\Sigma_2}$, we will glue to each boundary component of $\overline{S}$ a disk with the corresponding number of puncture on the boundary to get a surface $\hat{S}$ with no boundary, only internal punctures. Since a disk with one (resp. two) puncture on the boundary does not admit an ideal triangulation, we glue a disk with one (resp. two) puncture on the boundary and one internal puncture instead. In the corresponding ramified covering $\hat{\Sigma_2}$ of $\hat{S}$, we then remove the lifts of the interior of the glued disks to obtain Σ_2. The result follows from the following lemma:

Lemma 2.3.8. *If S is a closed disk with $n \geq 3$ punctures on the boundary and no internal puncture, $\overline{\Sigma_2}$ has either one boundary component with $2n$ punctures if n is odd or two boundary components with n punctures each if n is even. If S is a disk with one internal puncture and one puncture on the boundary, $\overline{\Sigma_2}$ has one boundary component with two punctures. If S is a disk with one internal puncture and two punctures on the boundary, $\overline{\Sigma_2}$ has two boundary components with two punctures each.*

Proof. The two cases with an internal puncture can be computed individually. Let S be a disk with $n \geq 3$ punctures on the boundary. Let Δ be a triangulation of S and Σ_2 the corresponding ramified covering. Let γ be a loop homotopic to the boundary of the disk going around all the $n-2$ branched points in S. Let x be the base point of γ, and x_1, x_2 the lifts of x to Σ_2. Let $\tilde{\gamma}$ the lift of γ starting at x_1. If $\tilde{\gamma}$ is a loop then there are two lifts of the boundary of $\overline{S}$ to $\overline{\Sigma_2}$, and if $\tilde{\gamma}$ is a path from x_1 to x_2 then the lift of the boundary of $\overline{S}$ is connected in $\overline{\Sigma_2}$. The loop γ is homotopic to the concatenations of loops $\gamma_1, \ldots, \gamma_{\lfloor \frac{n-2}{2} \rfloor}, \gamma'$ based at x such that each γ_i goes around two branched points in S and γ' is either trivial if $n-2$ is even or goes around one branched point if $n-2$ is odd. Then $\tilde{\gamma}$ is the concatenation of the lifts $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_{\lfloor \frac{n-2}{2} \rfloor}, \tilde{\gamma}'$. Since the $\tilde{\gamma}_i$ are loops based at p_1 and $\tilde{\gamma}'$ is either trivial or a path from p_1 to p_2 (depending on the parity of n), we get the result. □

The Euler characteristic of $\overline{S}$ is

$$\chi(\overline{S}) = 2 - 2g - k = 2 - 2g - k_o - k_e$$

and the Euler characteristic of $\overline{\Sigma_2}$ is

$$\chi(\overline{\Sigma_2}) = 2 - 2g' - k_o - 2k_e.$$

The number of branched points is the same as the number of triangles in Δ, which is $-2\chi(\overline{S}) + 2p + \sum n_i$. Riemann-Hurwitz formula gives us:

$$\begin{aligned}\chi(\overline{\Sigma}_2) = 2 - 2g' - k_o - 2k_e &= 2\chi(\overline{S}) - \left(-2\chi(\overline{S}) + 2p + \sum_{i=1}^{k} n_i\right) \\ &= 4\chi(\overline{S}) - 2p - \sum_{i=1}^{k} n_i \\ &= 8 - 8g - 2p - 4k_o - 4k_e - \sum_{i=1}^{k} n_i\end{aligned}$$

We can then solve for g' to get the result. □

Remark 2.3.9. To precisely describe the topology of Σ_n for $n \geq 3$ we would need to understand first the topology of the ramified covering of a disk with external punctures. We haven't found a convenient description of these coverings yet.

We denote by $\theta\colon \Sigma_2 \to \Sigma_2$ the covering involution, i.e. the map swapping the two preimages of a regular point, and fixing ramification points. The following result is a direct consequence of the above proposition.

Corollary 2.3.10. *The fundamental group $\pi_1(\Sigma_2)$ is a free group of rank*

$$1 - \chi(\overline{\Sigma}_2) + 2p = 1 - 4\chi(\overline{S}) + 4p + \sum n_i.$$

Let $b \in \Sigma_2$ be a ramification point of the covering $\pi\colon \Sigma_2 \to S$. Let $\alpha_1, \ldots, \alpha_s\colon [0,1] \to S$ be free generators of the fundamental group $\pi_1(S, \pi(b))$ that do not pass through other branch points. The fundamental group $\pi_1(\Sigma_2, b)$ is the free group freely generated by the following collection of loops on Σ_2:

- *For every generator α_i, there are two closed lifts γ_i^1 and $\gamma_i^2 = \theta \circ \gamma_i^1$ on Σ_2 based at b (in total $2 - 2\chi(\overline{S}) + 2p$ curves);*

- *For every ramification point $b' \neq b$ in Σ_2, we fix a simple segment on S connecting $\pi(b)$ and $\pi(b')$ and take the lift of this segment on Σ_2. It is a closed loop ξ based at b (in total $-2\chi(\overline{S}) + 2p - 1 + \sum n_i$ curves).*

The fundamental group $\pi_1(T'\Sigma_2, \tilde{b})$ where $\tilde{b} \in T'\Sigma_2$ is a lift of b to $T'\Sigma_2$ is generated by lifts of curves described above and the curve going once around the fiber of $T'\Sigma_2 \to \Sigma_2$ at $\tilde{b}$.

2.3.2 Decorating loops and joining paths

We want to fix for every white vertex of Γ_n a small loop going twice around it, in both S and Σ_n. These loops will support sections of the local system which will be used to define the $\mathcal{A}$-coordinates of a local system on S in Section 3.3. However for consistency reasons, white vertices of Γ_n adjacent to a same external black vertex should be grouped together. This creates three different types of white vertices:

- White vertices that are not adjacent to any external black vertex, for which all three coordinates are at most $n-1$
- White vertices adjacent to exactly one external black vertex, for which exactly one of the three coordinates is equal to n. The white vertices adjacent to bivalent external black vertices are grouped two-by-two.
- White vertices adjacent to two external black vertices, for which two coordinates are equal to n and the last coordinate is 1. In this case the white vertex is on the path $\gamma_p^{(1)}$ for some puncture p. We call those *peripheral white vertices*.

We fix the following paths on Σ_n, corresponding to each kind of white vertex in Γ_n:

- For every white vertex v that is not adjacent to any external black vertex, let β_v be a small loop going *twice* around v, with positive orientation.
- For every pair of white vertices v and v' adjacent to the same external black vertex b (and not adjacent to any other external black vertex), let $\beta_v = \beta_{v'}$ be a loop going *twice* around the three vertices v, v' and b, with positive orientation.
- For every puncture p of S, let β_p be a small smooth deformation of $\gamma_p^{(1)}$ in S. For every lift p_i of p to Σ_n, let β_{p_i} be the preimage of β_p by π going around p_i. For every white vertex v on $\gamma_p^{(1)}$, we set $\beta_v = \beta_{p_1}$.

We call all the paths $T'\beta_v$ and $T'\beta_{p_i}$ the *decorating curves* of Σ_n.

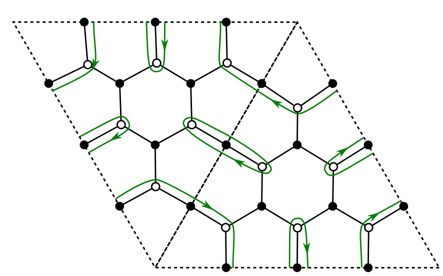


Figure 3.8: All projections on S of the decorating curves on a quadrilateral for $n=3$.

Remark 2.3.11. Notice that in the last case, the path β_p is not a loop if the puncture p is external, and if β_p is a loop, it goes only once around p and not twice as in the other cases. This is due to the fact we are working with twisted local systems on S and Σ_n, see Section 2.2.3.

Remark 2.3.12. All decorating curves β_v going twice around a vertex or a group of vertices are obtained by choosing a smooth loop $\tilde{\beta_v} : \mathbb{S}^1 \to \Sigma_n$ going once around, and $\beta_v = \tilde{\beta_v} \circ \tau$ where $\tau : \mathbb{S}^1 \to \mathbb{S}^1$ is the two-fold cover of the circle over itself. In particular, the image of every β_v in Σ_n is diffeomorphic to a circle.

This construction is equivalent to contracting each group of white vertices and external black vertices to a single white vertex, and taking a loop $\tilde{\beta_v}$ around that new white vertex v. Doing so will create a white vertex at every puncture. If this new white vertex v is a puncture we set $\beta_v = \tilde{\beta_v}$ and if not, we set $\beta_v = \tilde{\beta_v} \circ \tau$.

The cover $\Sigma_n \to S$ is regular in the neighborhood of punctures, white vertices and external black vertices, so by taking the images of the paths above by $\pi : \Sigma_n \to S$ we obtain similar paths $\pi(\beta_v)$ on S. Notice that for all puncture p and every $p_i \in \pi^{-1}(p)$, $\pi(\beta_{p_i}) = \beta_p$.

For every white vertex v of Γ_n and for every puncture p of S that is a vertex of the triangle containing v and such that $v \notin \gamma_p^{(1)}$, let p_i be the lift of p to Σ_n such that $v \in \gamma_p^{(i)}$. Fix a smooth path $\gamma_{p,v} : [0,1] \to \Sigma_n$ without self-intersection such that $T'\gamma_{p,v}(0) \in T'\beta_v$ and $T'\gamma_{p,v}(1) \in T'\beta_{p_i}$, as in Figure 3.9. If v is peripheral, $T'\gamma_{p,v}$ intersects $T'\beta_v$ only once. If v is not peripheral, then $T'\gamma_{p,v}$ intersects $T'\beta_v$ twice, so there exists $x_1, x_2 \in \mathbb{S}^1$ such that $T'\gamma_{p,v}(1) = T'\beta_v(x_1) = T'\beta_v(x_2)$, and x_1, x_2 satisfy $\tau^{-1}(x_1) = \tau^{-1}(x_2)$. We call those paths $\gamma_{p,v}$ *joining paths*.

For each white vertex v, we get a finite collection of points $E(v)$ on the circle $\mathbb{S}^1$ given by all $x \in \mathbb{S}^1$ such that $T'\beta_v(x) = T'\gamma_{p,v}(0)$ for some puncture p. This set $E(v)$ inherit the cyclic ordering of $\mathbb{S}^1$. In the next sections, it will be convenient to view this cyclic ordering as a bijection $\sigma : E(v) \to E(v)$ satisfying $\sigma^{\#E(v)} = \mathrm{Id}$, which sends an element $x \in E(v)$ to the next element in the cyclic order. If v is not peripheral, $\#E(v)$ is even and we have $T'\beta_v(x) = T'\beta_v(\sigma^{\#E(v)/2}(x))$ for all $x \in E(v)$ because β_v loops twice around v. We denote the union of all the images in Σ_n of the sets $\beta_v(E(v))$ for v a white vertex of Γ_n by $E_\Delta(\Sigma_n)$, and its projection on S by $E_\Delta(S) = \pi(E_\Delta(\Sigma_n))$.

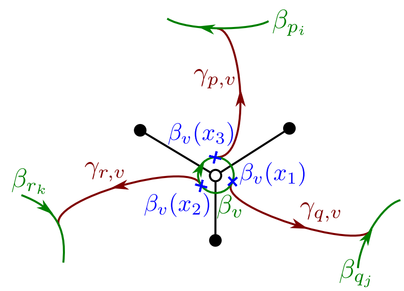


Figure 3.9: The decorating curve β_v around a non-peripheral white vertex $v \in \gamma_p^{(i)} \cap \gamma_q^{(j)} \cap \gamma_r^{(k)}$, together with the joining paths and the images of the points in $E(v)$ on β_v. The points in $E(v)$ are $x_1 < \cdots < x_6 < x_1$ and satisfy $\beta_v(x_i) = \beta_v(x_{i+3})$ for $i = 1, 2, 3$.

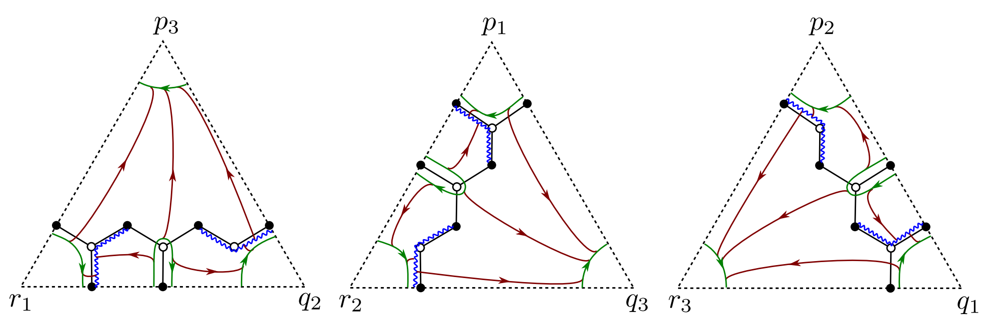


Figure 3.10: All decorating curves and the paths joining them in the lift of a triangle to Σ_3.

2.3.3 Spectral networks

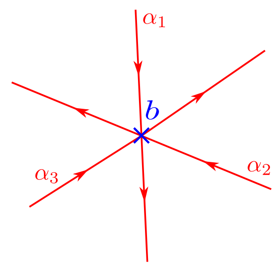
Spectral networks are tools introduced by Davide Gaiotto, Gregory W. Moore and Andrew Neitzke in [GMN13] to study local systems by breaking them down into a smaller rank local system on a ramified covering of the initial surface. In this section we give a combinatorial definition of finite spectral networks, which will be sufficient for our work. For a general account on spectral networks, see [GMN13, GMN14, HN16].

Let S be a ciliated surface, let $n \geq 2$ and let $\pi : \Sigma_n \to S$ be the ramified covering constructed in Section 2.3.1. We first recall the definition of a spectral network.

Definition 2.3.13. A *small spectral network* associated with the ramified covering $\pi : \Sigma_n \to S$ is a finite set $\mathcal{W}$ of paths $[-1, 1] \to \overline{\Sigma}_n$ (called *lines* of the spectral network) satisfying:

- $\forall \alpha \in \mathcal{W}$, $\alpha(-1), \alpha(1) \in \pi^{-1}(\mathcal{P})$, $\alpha(0) \in \mathcal{B}$ and if $t \notin \{-1, 0, 1\}$, then $\alpha(t) \notin \mathcal{P} \cup \mathcal{B}$

- $\forall \alpha \in \mathcal{W}$, $\forall t \in [-1, 1]$, $\pi(\alpha(t)) = \pi(\alpha(-t))$
- $\forall b \in \mathcal{B}$, there are exactly 3 lines $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{W}$ passing through b, and locally around b the lines look like so:



- $\forall \alpha, \alpha' \in \mathcal{W}$ with $\alpha \neq \alpha'$, $\alpha(]-1, 0[) \cap \alpha'(]0, 1[) \neq \varnothing$.

Remark 2.3.14. Each line α of a spectral network goes through exactly one ramification point b of Σ_n, and we say this ramification point b is *associated to* α, or that b is *the branch point of* α.

Remark 2.3.15. The general construction of a spectral network given in [GMN13] describe more general spectral networks. Generically they have infinitely many lines and are dense on the surface. However we will only need the very small class of spectral networks defined above.

We can now construct a small spectral network on Σ_n associated with the n-fold ramified covering $\pi : \Sigma_n \to S$. For this, we will describe the spectral network on the covering of a triangle of S, the complete spectral network on Σ_n being the union of all the spectral networks on the coverings of the triangles. First, notice that every puncture p of S is unramified, i.e. $|\pi^{-1}(p)| = n$. Those n lifts $p_1, \dots, p_n$ of p each lies in one of the punctured disks $S_{\gamma_p^{(i)}}$ defined in Lemma 2.3.3, and inherit the order given by the inclusion of these punctured disks $p_1 < \cdots < p_n$, where $p_i < p_j$ if $\gamma_p^{(i)} \subset S_{\gamma_p^{(j)}}$. Let t be a triangle of S and p a vertex of this triangle, and let $2 \leq i \leq n$. There are $i-1$ internal black vertices on the boundary of $S_{\gamma_p^{(i)}} \cap t$ and there are $i-1$ white vertices on the boundary of $S_{\gamma_p^{(i-1)}} \cap t$. Each of those black vertices is connected to exactly one of those white vertices via an edge of $\Gamma_n^{(t)}$, and those black vertices are ramification points. On Σ_n, we define for each of those black vertices a line of the spectral network as follows: let v_b be an internal black vertex on the boundary of $S_{\gamma_p^{(i)}} \cap t$ and v_w the white vertex on the boundary of $S_{\gamma_p^{(i-1)}} \cap t$ associated to v_b. Let c_1 a path from p_i to v_b in $S_{\gamma_p^{(i)}} \cap t$, let c_2 be the path from v_b to v_w obtained by following the edge of the graph $\Gamma_n^{(t)}$, and let c_3 be the path from v_w to p_{i-1} in $S_{\gamma_p^{(i-1)}} \cap t$. The line of the spectral network is the concatenation $\alpha = c_3.c_2.c_1 : [-1,1] \to \overline{\Sigma}_n$, parameterized such that $\alpha(0) = v_b$.

Remark 2.3.16. Notice that every line of this spectral network goes from the i-th preimage p_i of a puncture p to the $(i-1)$-th preimage p_{i-1} for some $2 \le i \le n$.

Example 2.3.1. Note that in the case $n = 2$, every puncture of Σ_2 is either at the end of every line of the spectral network containing it, in which case we call it a *sink*, or at the beginning of every line of the spectral network containing it, in which case it is called a *source*. This way, every point of $\mathcal{P}$ have two preimages, one source and one sink. For each hexagon H of $\pi^{-1}(\Delta)$, the three lines going through the ramification point in H are the three Euclidean segments going from the source to the sink for each of the three puncture in T.

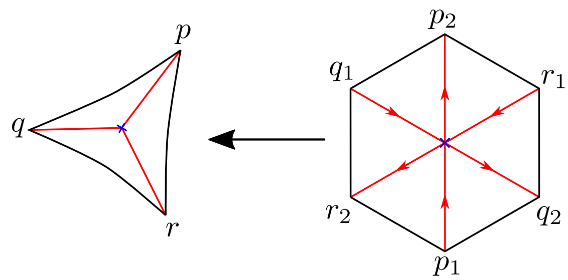


Figure 3.11: Picture of the spectral network on each triangle of the triangulation in the case $n = 2$. Here the sources are p_2, q_2, r_2 and the sinks are p_1, q_1, r_1.

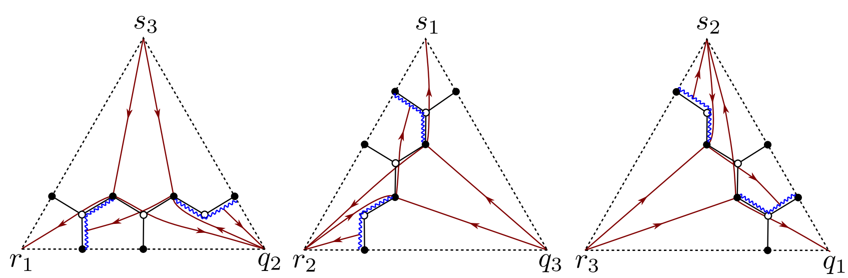


Figure 3.12: The small spectral network on the lift of a triangle to Σ_3.

Proposition 2.3.17. *The set $\mathcal{W}$ of all paths on Σ_n constructed as above is a small spectral network associated to the n-fold ramified covering $\pi : \Sigma_n \to S$.*

Proof. With the right coice of parametrization, every line $p : [-1,1] \to \overline{\Sigma}_n$ defined above satisfy $\pi(p(t)) = \pi(p(-t))$ for $t \in [-1,1]$, $\pi(p(-1)) = \pi(p(1))$ is a puncture in $\overline{S}$ and $p(0)$

is an internal black vertex, i.e. a ramification point. Every internal black vertex is part of three zig-zag paths, so it is on the boundary of three cells of the form $S_{\gamma_s^{(i)}}$. There is three line of $\mathcal{W}$ going through every black vertex, with alternating directions (see Figure 3.13).

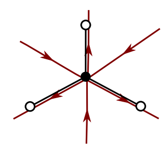


Figure 3.13: The intersection of three lines of the spectral network at an internal black vertex.

Each edge of Γ_n incident to an internal black vertex is followed by exactly one line of $\mathcal{W}$, and the edges incident to an external black vertex do not meet any lines of $\mathcal{W}$. There are no intersection between lines of $\mathcal{W}$ inside a punctured disk of the form $S_{\gamma_s^{(i)}}$, so the only intersections between lines of $\mathcal{W}$ except those at ramification points lies on white vertices of Γ_n. That means all those intersections happen between lines having already crossed their ramification points, thus the set $\mathcal{W}$ is a small spectral network. □

We can endow the neighborhood U_b of a ramification point $b \in B$ defined in Section 2.3.1 with an Euclidean structure such that the three lines of the spectral network going through b are straight lines with respect to this structure, so that locally the figure looks like the one in rank two (Figure 2.3.13). The Euclidean structure on U_b defines a flat connection ∇ on TU_b given by the restriction of the standard flat connection on $\mathbb{R}^2$. Since it is a bilinear map on the space of sections of TU_b (denoted $\Gamma(TU_b)$), this connection induces a flat connection (which we also call ∇) on the unit tangent bundle

$$\nabla : \Gamma(T'U_b) \times \Gamma(T'U_b) \to \Gamma(T'U_b).$$

Remark 2.3.18. A consequence of Remark 2.3.6 is that when $n = 2$, we can assume the neighborhood U_b of a ramification point $b \in B$ to be the whole hexagon tile containing it in Σ_2.

2.4 Partial non-abelianization

In this section we describe in detail the non-abelianization procedure introduced by Gaiotto-Moore-Neitzke in [GMN13]: we want to construct from a framed $\mathrm{GL}_n(R)$-local system on S a $R^\times$-local system on Σ_n. Since we will need to work with twisted local systems to define $\mathcal{A}$-coordinates, we first need to extend the spectral network lifting rule to work with paths on the unit tangent bundle of a surface. This construction is a joint work with Eugen Rogozinnikov in [KR22] in the case $n = 2$, and is presented here in the general case.

2.4.1 The spectral network map

Lifting paths to a ramified covering is not homotopy invariant: a contractible loop around a branch point $b \in B$ is lifted as two paths on Σ that are not loops, thus not homotopic to the lift of the trivial loop. The goal of this section is to construct a path-lifting map denoted by SN, which depends on the spectral network $\mathcal{W}$, from paths on $T'S$ to paths on $T'\Sigma$ such that SN is well-defined on homotopy classes.

We will use the symbol $\approx$ to represent homotopy (with fixed extremities) of paths.

Let S be a ciliated surface, Δ an ideal triangulation of S and $\pi : \Sigma_n \to S$ the n-fold branched covering constructed in Section 2.3.1. Let $\mathcal{W}$ be the spectral network adapted to this covering constructed in Section 2.3.3. Every path $\alpha :]-1,1[\to \Sigma_n$ of $\mathcal{W}$ is smooth. We can thus lift the paths of $\mathcal{W}$ to

$$T'\alpha : \begin{array}{rcl}]-1,1[& \to & T'\Sigma_n \\ t & \mapsto & (\alpha(t), \dot\alpha(t)) \end{array} .$$

We will call this set of paths in $T'\Sigma_n$ a tangent spectral network and denote it $T'\mathcal{W}$.

Let $U_b \subset \Sigma_n$ be the neighborhood of a ramification point b endowed with an Euclidean structure. Note that the Euclidean structure on U_b allows us to identify $T'U_b$ with $U_b \times \mathbb{S}^1$. In the following, a path γ on $T'U_b \simeq U_b \times \mathbb{S}^1$ will be written as a couple (x, v) where x is the projection of γ on $U_b \subset \Sigma_n$ and v is the projection of γ on $\mathbb{S}^1$. Note that $\mathbb{S}^1$ has a natural orientation given by the one on Σ_n. For all $\theta \in \mathbb{S}^1$, define s_θ^+ to be the (homotopy class of the) path in $\mathbb{S}^1$ going from θ to $-\theta$ following the orientation of $\mathbb{S}^1$, and s_θ^- going from θ to $-\theta$ in the opposite direction. For a path v on $\mathbb{S}^1$, we will denote $-v$ the image of v under the involution $\theta \mapsto -\theta$. The path $-v$ goes from $-v(0)$ to $-v(1)$. In particular, we have $s_\theta^- = -\overline{s_\theta^+}$ (where $\overline{s_\theta^+}$ denote the path obtained by reversing the direction of s_θ^+) and $(-s_\theta^+).s_\theta^+ = \delta_\theta^+$ (resp. $(-s_\theta^-).s_\theta^- = \delta_\theta^-$) where $\delta_\theta^+ : t \mapsto \theta + 2i\pi t$ (resp. $\delta_\theta^- : t \mapsto \theta - 2\pi t$) is the path going once around $\mathbb{S}^1$. When the context is clear, we will omit the subscript describing the starting point of the paths $s^\pm$ and $\delta^\pm$. The paths $\delta^\pm$ satisfy $\delta^- = \overline{\delta^+}$ and if v is a path on $\mathbb{S}^1$ from θ_1 to θ_2, we write $\delta^\pm.v \approx v.\delta^\pm$, meaning that $\delta_{\theta_2}^\pm.v \approx v.\delta_{\theta_1}^\pm$.

Definition 2.4.1. Let X be a topological space. The path algebra of X (denoted $\mathbb{Z}[\mathrm{Path}(X)]$) is the free abelian group generated by homotopy classes of paths $[0,1] \to X$, together with a product given by concatenation of paths: if $\gamma_1(0) \neq \gamma_2(1)$ then $\gamma_1.\gamma_2 = 0$ and if $\gamma_1(0) = \gamma_2(1)$ then $\gamma_1.\gamma_2$ is the path obtained by going through γ_2 then γ_1.

Remark 2.4.2. The path algebra of a topological space is in general a ring without a unit element.

Definition 2.4.3. Now let X be a smooth surface. Define the *twisted path algebra* of X as

$$\mathrm{TPA}(X) = \mathbb{Z}[\mathrm{Path}(T'X)]/\mathcal{I}$$

where $\mathcal{I}$ is the two-sided ideal generated by the elements $e_{x,\theta} + \delta_{x,\theta}$ for $(x,\theta) \in T'X$, with

$$e_{x,\theta} : \begin{array}{rcl} [0,1] & \to & T'X \\ t & \mapsto & (x,\theta) \end{array} \text{ and } \delta_{x,\theta} : \begin{array}{rcl} [0,1] & \to & T'X \\ t & \mapsto & (x,\theta+2\pi t) \end{array} .$$

Remark 2.4.4. Given any non-empty subset $E \subset T'X$, the subset

$$\{\gamma_1 + \cdots + \gamma_r + \mathcal{I} \mid \text{for all } 1 \leq i \leq r, \text{ endpoints of } \gamma_i \text{ are in } E\} \subset \mathrm{TPA}(X)$$

is a subring of $\mathrm{TPA}(X)$ because composition of paths preserves the set of endpoints. We will denote $\mathrm{TPA}_E(X)$ this subring.

We first describe the path lifting rule for a path on S, and then we will extend the construction to paths on $T'S$. Since there is a projection $T'S \to S$, we want the lift to $T'\Sigma_n$ of a path γ on $T'S$ to project onto the lift to Σ_n of the projection x of γ to S. Let x be a path on S intersecting only once (and not at its endpoints) the spectral network $\mathcal{W}$ and not going through a branch point. Let $\alpha \in \mathcal{W}$ be the path such that $\pi(\alpha)$ intersects x and let $b \in B$ be the ramification point associated to α. Up to homotopy, we can assume the intersection between x and α to be in U_b. Among the n standard lifts $x_1, \dots, x_n$ of x to Σ_n, two of them intersect once α in U_b, one of them intersecting $\alpha(]-1,0[)$ and the other intersecting $\alpha(]0,1[)$. Suppose x_1 is the one intersecting $\alpha(]-1,0[)$ and x_2 is the one intersecting $\alpha(]0,1[)$. We can then define a new path x' on Σ_n as the concatenation of 5 paths $x'_1, \dots, x'_5$ defined as follows:

- x'_1 is the part of x_1 from its starting point to the intersection point with α
- x'_2 is the part of α from the intersection with x_1 to the ramification point b
- x'_3 is a constant path at the ramification point b point (it will be useful in the next paragraph when we will consider the lifted spectral network $T'\mathcal{W}$)
- x'_4 is the part of α from b to the intersection with x_2
- x'_5 is the part of x_2 from the intersection with α to its endpoint.

The *spectral lifts* of x to Σ_n is the collection $x_1, \dots, x_n, x'$.

We now extend the path lifting rule to paths on $T'\Sigma_n$. Let $\gamma : t \mapsto (x(t), v(t))$ be a path on $T'S$ such that the path x on S intersects only once the spectral network on the projection $\pi(\alpha)$ of a line $\alpha \in \mathcal{W}$ at a time $t_0 \in\ ]0,1[$. Let b be the ramification point associated to α. Up to homotopy, we can assume that the intersection of x and α is in U_b. Let $\gamma_1 = (x_1, v_1), \dots, \gamma_n = (x_n, v_n)$ be the n standard lifts of γ to $T'\Sigma_n$, with γ_1 and γ_2 being the two lifts such that x_1 and x_2 intersect the spectral network. Note that γ_1 and γ_2 do not intersect $T'\mathcal{W}$ in general, but x_1 and x_2 intersect $\alpha \in \mathcal{W}$. Let x' be the path on Σ_n obtained

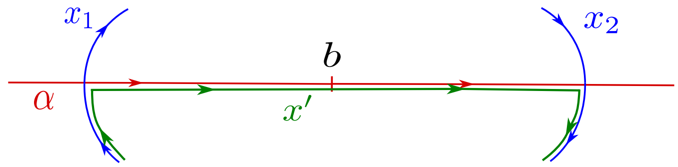


Figure 4.14: The path x' added by the intersection of the lifts x_1 and x_2 of x with α

with the construction above. We now want a continuous map $v' : [0,1] \to \mathbb{S}^1$ which coincide with the standard lifts v_1 and v_2 when x' coincides with either x_1 or x_2. Without loss of generality, suppose x is smooth at the intersection point with α and that the intersection is transverse. Then x_1 and x_2 are also smooth at their intersection points with α. We say the intersection of x_1 with α is *positively oriented* if $(\dot{x_1}(t_0), \dot{\alpha}(t_0))$ agrees with the orientation on Σ, *negatively oriented* if not.

Remark 2.4.5. The positivity of the intersection of a path (x, v) in $T'\Sigma$ with a line of the spectral network is determined using the derivative of the underlying path x, and does not depend on the vector field v on x.

Let v' be the concatenation of 5 paths $v'_1, \dots, v'_5$ defined as follows:

- v'_1 is the part of v_1 from its starting point to the intersection point with α
- v'_2 is obtained by parallel transport with respect to the flat connection ∇ on U_p from the vector $v_1(t_0)$ along the path x'_2
- v'_3 is the path $s^+_{v'_2(0)}$ in $T'_b\Sigma_n \simeq \mathbb{S}^1$ if the intersection of x_1 with α is positively oriented, and $s^-_{v'_2(0)}$ if the intersection is negatively oriented.
- v'_4 is obtained by parallel transport with respect to ∇ from the vector $v_2(t_0) = -v_1(t_0)$ along the path $\overline{x}'_4$
- v'_5 is the part of v_2 from the intersection with α to its endpoint

Remark 2.4.6. The resulting path v' on $\mathbb{S}^1$ is homotopic to $(-v_1^2).s^{\pm}_{v_1(t_0)}.v_1^1$ where $v_1^1 = v_1|_{[0,t_0]}$ and $v_1^2 = v_1|_{[t_0,1]}$. Note that for all path w on $\mathbb{S}^1$ from θ_0 to θ_1, we have

$$s^{\pm}_{\theta_1}.w \approx (-w).s^{\pm}_{\theta_0}$$

so the path v' is homotopic to $s^{\pm}_{v_1(1)}.v_1$.

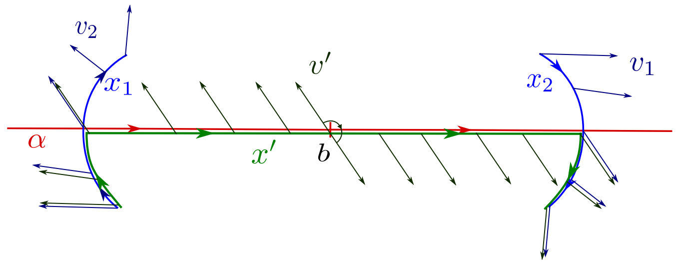


Figure 4.15: The path (x', v') added by the intersection with α. The intersection of x_1 with α is positively oriented if Σ_n is oriented clockwise.

Let $\gamma' = (x', v')$ and let $SN(\gamma)$ be the element $\gamma_1 + \cdots + \gamma_n + \gamma' \in \mathrm{TPA}(\Sigma_n)$. Let γ be a path in $T'S$. We can write γ as a concatenation of smaller paths $\gamma^1, \ldots, \gamma^r$, each having a projection to S intersecting at most once the spectral network and for each of these small paths, apply the construction above to obtain $SN(\gamma^1), \ldots, SN(\gamma^r)$ (if γ^i have a projection to S which does not intersect the spectral network, define $SN(\gamma^i)$ to be the sum of the n standard lifts of γ^i). Define the lift of γ with respect to the spectral network $\mathcal{W}$ to be the product $SN(\gamma) = SN(\gamma^1) \ldots SN(\gamma^r) \in \mathrm{TPA}(\Sigma)$.

Theorem 2.4.7. *Let γ_1 and γ_2 be two homotopic paths in $T'S$. Then $SN(\gamma_1) = SN(\gamma_2)$. In particular, the map*

$$SN : \begin{array}{ccc} \mathrm{TPA}(S) & \to & \mathrm{TPA}(\Sigma) \\ \gamma & \mapsto & SN(\gamma) \end{array}$$

is well-defined.

Remark 2.4.8. The map SN is not defined on the whole twisted path algebra of S as paths with endpoints on a line of the spectral network can not be lifted consistently, but we will never need to lift such paths. The subset of $\mathrm{TPA}(S)$ (resp. $\mathrm{TPA}(\Sigma_n)$) of elements where no term has an endpoint on $\mathcal{W}$ is a subring (see Remark 2.4.4), and with a slight abuse of notation we will still denote it $\mathrm{TPA}(S)$ (resp. $\mathrm{TPA}(\Sigma_n)$) instead of $\mathrm{TPA}(S)_{S\setminus\pi(\mathcal{W})}$ (resp. $\mathrm{TPA}(\Sigma_n)_{\Sigma_n\setminus\mathcal{W}}$).

The theorem is a consequence of the two following lemmas:

Lemma 2.4.9. *Let $\gamma = (x, v)$ be a path in $T'S$ that intersects exactly twice the same line α of the spectral network and no other line of $\mathcal{W}$, as in Figure 4.16. Then $SN(\gamma) = \gamma_1 + \cdots + \gamma_n$ where $\gamma_1, \ldots, \gamma_n$ are the n standard lifts of γ to $T'\Sigma_n$.*

Proof. Let $t_1 < t_2$ be the two elements of the interval $[0, 1]$ such that $x(t_1)$ and $x(t_2)$ are on α. Let $(x^{(1)}, v^{(1)}) = \gamma(t_1)$ and $(x^{(2)}, v^{(2)}) = \gamma(t_2)$, and let $\gamma_1 = (x_1, v_1)$ and $\gamma_2 = (x_2, v_2)$

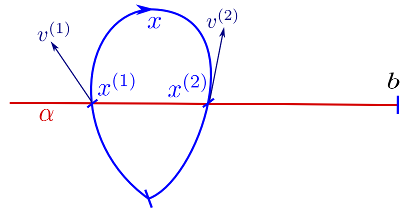


Figure 4.16: A loop intersecting twice the same line of $\mathcal{W}$

the two standard lifts of γ such that x_1 and x_2 intersect α, γ_1 being the lift such that x_1 intersects $\alpha(]-1,0[)$. Then $SN(\gamma) = \gamma_1 + \gamma_2 + \cdots + \gamma_n + \gamma' + \gamma''$ where $\gamma' = (x', v')$ is such that x' follow α from $x_1^{(1)}$ to $x_2^{(1)}$ and $\gamma'' = (x'', v'')$ is such that x'' follows α from $x_1^{(2)}$ to $x_2^{(2)}$.

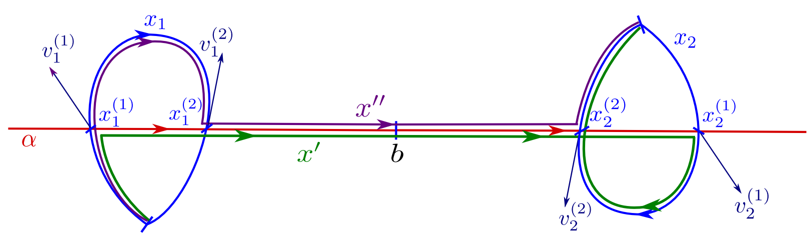


Figure 4.17: Spectral network lift of γ

In order to prove the lemma, we need to show that the two paths γ' and γ'' added by the intersections with the spectral network cancel each other in $\mathrm{TPA}(\Sigma)$, i.e. that $\gamma' + \gamma'' = 0$. For this, we need to show that $\overline{\gamma}''.\gamma'$ is homotopic to an odd power of $\delta_{x_1(0),v_1(0)}$. The paths x' and x'' are homotopic on Σ so the concatenation $\overline{x}''.x'$ is trivial. What is left is to show that $\overline{v}''.v'$ is homotopic to an odd power of δ^+.

Suppose the intersection of x_1 with α at $x_1^{(1)}$ is positive, the other case being symmetric. Then the intersection of x_1 with α at $x_1^{(2)}$ is negative. Then by remark 2.4.6, $v' \approx s^+.v_1$ and $v'' \approx s^-.v_1$, so we have

$$\begin{aligned} \overline{v}''.v' &\approx \overline{v}_1.\overline{s^-}.s^+.v_1 \\ &\approx \overline{v}_1.\delta^+.v_1 \\ &\approx \delta^+. \end{aligned}$$

□

Lemma 2.4.10. *Let m be a point in S in a small neighborhood of a branch point b but not on a line of $\mathcal{W}$ and $\theta \in T'_m S$. Let γ be a path homotopic to $e_{m,\theta}$ in $T'S$ that loops around the branch point b in S, intersecting exactly once each of the three lines of $\mathcal{W}$ going out of b,* *as in Figure 4.18. Then* $SN(\gamma) = e_{m_1,\theta_1} + \cdots + e_{m_n,\theta_n}$ *where* $(m_1, \theta_1), \ldots, (m_n, \theta_n)$ *are* n *lifts of* (m, θ) *in* $T'\Sigma_n$*.*

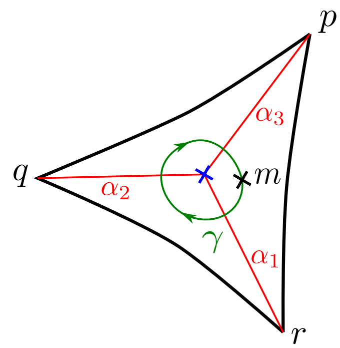


Figure 4.18: A small loop around a branch point.

Proof. Suppose the path γ is looping around b in the direction given by the orientation of Σ_n, the other case being symmetric. Then all the intersections of the standard lifts of γ with the spectral network in Σ_n are positive. By applying the spectral network lifting rule to γ, we get $n+6$ paths: the n standard lifts $\gamma_1 = (x_1, v_1), \ldots, \gamma_n = (x_n, v_n)$ (with x_1 and x_2 intersecting $\mathcal{W}$ as before), and 6 additional paths $\gamma'_1, \ldots, \gamma'_6$ shown in Figure 4.19. Let α_1, α_2 and α_3 be the three line of $\mathcal{W}$ intersected by γ, in that order. Let γ_1 be the standard lift of γ intersecting α_1 before the branch point, and let (m_1, θ_1) be its starting point and (m_2, θ_2) be its endpoint. We will label the spectral network lifts $\gamma'_i = (x'_i, v'_i)$ of γ as follows:

- γ'_1 follows γ_1 until the intersection with α_3, then α_3, then γ_2 until its end
- γ'_2 follows γ_2 until the intersection with α_2, then α_2, then γ_1 until its end
- γ'_3 follows γ_2 until the intersection with α_2, then α_2, then γ_1 until the intersection with α_3, then α_3, then γ_2 until its end
- γ'_4 follows γ_1 until the intersection with α_1, then α_1, then γ_2 until its end
- γ'_5 follows γ_1 until the intersection with α_1, then α_1, then γ_2 until the intersection with α_2, then α_2, then γ_1 until the intersection with α_3, then α_3, then γ_1 until its end
- γ'_6 follows γ_1 until the intersection with α_1, then α_1, then γ_2 until the intersection with α_2, then α_2, then γ_1 until its end.

The paths x'_1, x'_4 and x'_5 are homotopic to the trivial path e_{m_1}, x'_6 is homotopic to x_1, x'_2 is homotopic to e_{m_2} and x'_3 is homotopic to x_2. Since γ is homotopic to $e_{m,\theta}$ and x is looping

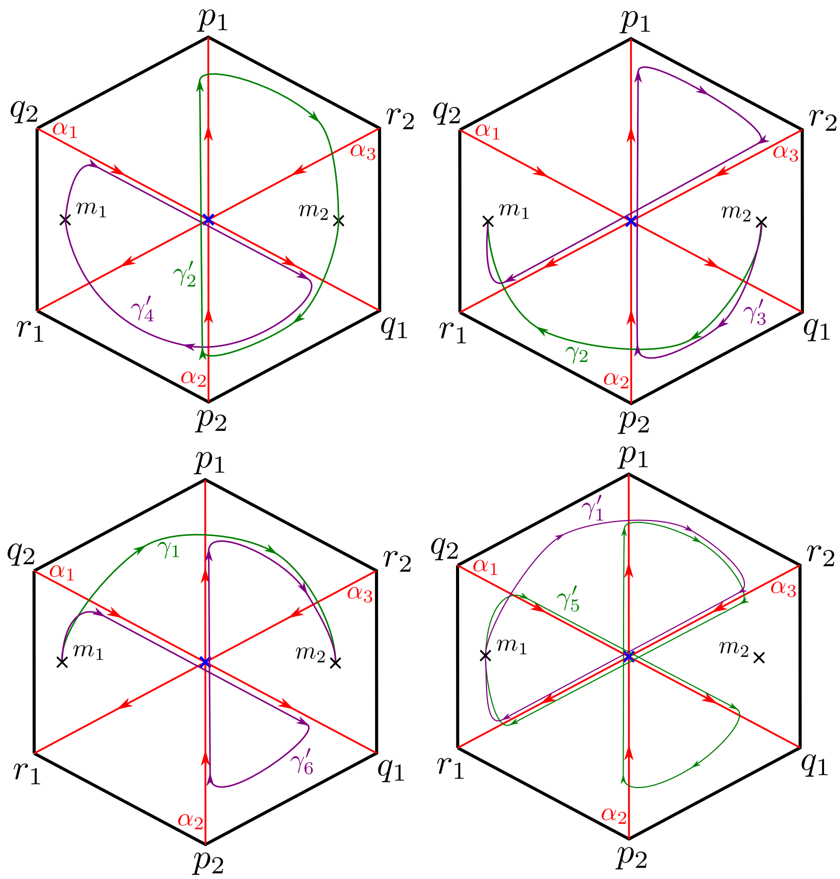


Figure 4.19: All 6 paths added by intersections with the spectral network, together with the standard lifts γ_1 and γ_2. On the upper left picture are the paths homotopic to trivial paths, and on the other are the remaining lifts, grouped as pairs of paths canceling each other in TPA(Σ).

around b in the direction given by the orientation of Σ, we have $v_1 \approx s^-_{\theta_1}$ and $v_2 \approx s^-_{\theta_2}$. Using the same reasoning, we get the following:

$$\begin{aligned} v'_1 &\approx s^+.v_1 \approx e_{\theta_1} \\ v'_2 &\approx s^+.v_2 \approx e_{\theta_2} \\ v'_3 &\approx s^+.s^+.v_2 \approx \delta^+.v_2 \\ v'_4 &\approx s^+.v_1 \approx e_{\theta_1} \\ v'_5 &\approx s^+.s^+.s^+.v_1 \approx \delta^+_{\theta_1} \\ v'_3 &\approx s^+.s^+.v_1 \approx \delta^+.v_1 \end{aligned}$$

So in TPA(Σ), we have:

$$\begin{aligned}
\gamma_2 + \gamma_3' &= 0 \\
\gamma_1 + \gamma_6' &= 0 \\
\gamma_1' + \gamma_5' &= 0 \\
\gamma_4' &= e_{m_1,\theta_1} \\
\gamma_2' &= e_{m_2,\theta_2}
\end{aligned}$$

so

$$SN(\gamma) = \gamma_1 + \gamma_2 + \gamma_1' + \gamma_2' + \gamma_3' + \gamma_4' + \gamma_5' + \gamma_6' = e_{m_2,\theta_2} + e_{m_2,\theta_2}.$$

□

2.4.2 Twisted path algebras and representations

Let X be an hyperbolic ciliated surface and $n \geq 1$. If $\mathcal{L}$ is a twisted $\mathrm{GL}_n(R)$-local system on X and γ is a path on $T'X$, the flat connection defines a holonomy map m_γ from $\mathcal{L}_{\gamma(0)}$ to $\mathcal{L}_{\gamma(1)}$. Moreover, by definition of a twisted local system the path $\delta_{x,\theta}$ induces the linear map $-\mathrm{Id}$ on $\mathcal{L}_{x,\theta}$. Thus, if $\gamma = \gamma_1 + \cdots + \gamma_r + \mathcal{I} \in \mathrm{TPA}(X)$ where all the $\gamma_i \in \mathrm{Path}(T'X)$ have the same extremities, the holonomy map $m_\gamma = m_{\gamma_1} + \cdots + m_{\gamma_r} : \mathcal{L}_{\gamma(0)} \to \mathcal{L}_{\gamma(1)}$ is well-defined (if there is more than one term in γ the holonomy map m_γ may not be an isomorphism). However, if γ_1 and γ_2 do not have the same extremities, it is not possible to associate an element of $\mathcal{M}_n(R)$ to $\gamma_1 + \gamma_2$, which is a problem we need to solve in order to consider representations of $\mathrm{TPA}(X)$. To make a link between twisted local systems and representations of $\mathrm{TPA}(S)$, we first need to modify the ring $\mathcal{M}_n(R)$ to solve this issue of endpoints. Since multiplication in $\mathrm{TPA}(X)$ is zero for paths whose extremities do not match, we need a ring with the same behavior.

Definition 2.4.11. Let A be a unital ring and $E \subset X$ any non-empty subset. Let A_E be the ring $A^{(E\times E)}$ of finite formal sums of elements of the form $a_{(p,q)}, a \in A, p, q \in E$, endowed with the multiplication defined as follows:

- $\forall a, b \in A, \forall x, y, z \in E, a_{(x,y)}.b_{(y,z)} = (a.b)_{(x,z)}$
- $\forall a, b \in A, \forall x, y, z, t \in E, y \neq z, a_{(x,y)}.b_{(z,t)} = 0$

The elements of this ring are copies of elements of A indexed by pairs of points in E, thought as "endpoints" of these elements. The sum of two elements is a formal sum except when the indices match, it then agrees with the sum in A. The multiplication of two elements is made so it agrees with the composition of paths: multiplication of two elements with "non-composable" indices is zero and multiplication with "composable" indices agrees with the one on A, and the index of the result is the composition of the indices. Given two rings A and B, a ring homomorphism $\varphi : A_E \to B_E$ is called a *graded ring homomorphism* if for all $a \in A$ and for all $x, y \in E$, $\varphi(a_{x,y}) = b_{x,y}$ for some $b \in B$.

The ring A_E contains many isomorphic copies of A as subrings: for all $x \in E$,

$$A_x := \left\{ a_{(x,x)} \mid a \in A \right\}$$

is a subring of A_E isomorphic to A. Note however that the ring A_E is not unital if E is infinite, but contains many idempotent elements.

Let $\mathrm{TPA}_\Delta(S) = \mathrm{TPA}_{E_\Delta(S)}(S)$ be the subring of $\mathrm{TPA}(S)$ of paths with endpoints in $E_\Delta(S)$ described in Remark 2.4.4, where $E_\Delta(S)$ is defined in Section 2.3.2. Similarly, let $\mathrm{TPA}_\Delta(\Sigma_n) = \mathrm{TPA}_{E_\Delta(\Sigma_n)}(\Sigma_n)$. For any unital ring A, let $A_{\Delta,S} = A_{E_\Delta(S)}$ and $A_{\Delta,\Sigma_n} = A_{E_\Delta(\Sigma_n)}$. In the following, X will denote either the surface S or its ramified covering Σ_n. Since $E_\Delta(S)$ and $E_\Delta(\Sigma)$ are finite, $\mathrm{TPA}_\Delta(X)$ and $A_{\Delta,X}$ are unital, the units elements being respectively

$$\sum_{x \in E_\Delta(X)} e_x$$

and

$$\sum_{x \in E_\Delta(X)} 1_{(x,x)}.$$

There is then a diagonal embedding

$$\begin{array}{rcl} A^{E_\Delta(X)} & \to & A_{\Delta,X} \\ (a_x)_{x \in E_\Delta(X)} & \mapsto & \sum_{x \in E_\Delta(X)} (a_x)_{(x,x)} \end{array}.$$

Remark 2.4.12. For every $x \in E_\Delta(X)$, there is an injective group homomorphism

$$\pi_1^s(X, x) \to \mathrm{TPA}_x(X)^\times \subset \mathrm{TPA}_\Delta(X)$$

where $\mathrm{TPA}_x(X)^\times$ denotes the group of invertible elements of $\mathrm{TPA}_x(X)$.

Two elements $a, b \in A_{\Delta,X}$ are said *conjugated* if there exists an invertible element u in $A^{E_\Delta(X)}$ such that $b = u.a.u^{-1}$. This is an equivalence relation.

Proposition 2.4.13. *Let R be a finite dimensional $\mathbb{R}$-algebra. Let Δ be a triangulation of a hyperbolic ciliated surface S and let X be either the surface S or its ramified covering Σ_n. There is a 1:1 correspondence between the set of twisted $\mathrm{GL}_n(R)$-local systems on X up to isomorphism and the set of graded ring homomorphisms $\mathrm{TPA}_\Delta(X) \to \mathcal{M}_n(R)_{\Delta,X}$ up to the action of $\mathrm{GL}_n(R)^{E_\Delta(X)}$ by conjugation.*

Proof. Given a twisted $\mathrm{GL}_n(R)$-local system $\mathcal{L}$ on X, for all $x \in E_\Delta(X)$ choose a basis of the fiber of $\mathcal{L}$ over x. The map

$$\varphi : \begin{array}{ccc} \mathrm{TPA}_\Delta(X) & \to & \mathcal{M}_n(R)_{\Delta,X} \\ \sum \gamma & \mapsto & \sum (Hol_{\mathcal{L}}(\gamma))_{(t(\gamma),s(\gamma))} \end{array}$$

is a ring homomorphism, where $s(\gamma)$ (resp. $t(\gamma)$) is the source (resp. the target) of γ (which are in $E_\Delta(X)$), and $Hol_{\mathcal{L}}(\gamma)$ is the holonomy of γ in $\mathcal{L}$ in the corresponding bases. The conjugacy class of φ does not depend on the choices of the bases.

Conversely, let φ be the conjugacy class of a representation $\mathrm{TPA}_\Delta(X) \to \mathcal{M}_n(R)_{\Delta,X}$, and let $x \in E_\Delta(X)$. Then $\mathrm{TPA}_x(X)$ contains an isomorphic copy of $\pi_1^s(X,x)$ and the restriction of φ to $\pi_1^s(X,x)$ yield a representation $\pi_1^s(X,x) \to \mathrm{GL}_n(A)$ mapping $\delta_x^{\pm}$ to $-\mathrm{Id}$, which define a unique isomorphism class of twisted $\mathrm{GL}_n(A)$-local system by Proposition 2.2.14, having holonomies described by φ. By construction, those are inverse. $\square$

2.4.3 Partial non-abelianization of twisted local systems

In Section 2.4.1, we constructed an algebra homomorphism $SN : \mathrm{TPA}(S) \to \mathrm{TPA}(\Sigma_n)$. This homomorphism restricts to a graded ring homomorphism

$$SN : \mathrm{TPA}_\Delta(S) \to \mathrm{TPA}_\Delta(\Sigma_n)$$

as mentioned in Remark 2.4.4. Let $\gamma \in \mathrm{TPA}_\Delta(S)$ be a path from p to q, $p, q \in E_\Delta(S)$, and let $p_1, \ldots, p_n$ be the lifts of p to Σ_n labeled as in Section 2.3.1, and $q_1, \ldots, q_n$ the lifts of q. Then

$$SN(\gamma) = \sum_{1 \le i,j \le n} \gamma_{j,i}$$

where $\gamma_{j,i}$ is the sum of all terms of $SN(\gamma)$ from p_i to q_j ($\gamma_{j,i}$ may be 0). Instead of a formal sum, it will be more convenient to see $SN(\gamma)$ as a n by n matrix with coefficients in $\mathrm{TPA}_\Delta(\ \Sigma_n)$. The definition of the multiplication on $\mathrm{TPA}_\Delta(\Sigma_n)$ makes it so the map:

$$SN_n : \begin{array}{ccc} \mathrm{TPA}_\Delta(S) & \to & \mathcal{M}_n(\mathrm{TPA}_\Delta(\Sigma_n)) \\ \gamma & \mapsto & \begin{pmatrix} \gamma_{1,1} & \cdots & \gamma_{1,n} \\ \vdots & \ddots & \vdots \\ \gamma_{n,1} & \cdots & \gamma_{n,n} \end{pmatrix} \end{array}$$

is a ring homomorphism. We also have a ring homomorphism $\pi_R : \mathcal{M}_n(R_{\Delta,\Sigma_n}) \to \mathcal{M}_n(R)_{\Delta,S}$ such that for all $p, q \in \mathcal{P}$ and for all $(a^{(i,j)})_{1 \le i,j \le n} \in R^{n^2}$,

$$\pi_R \begin{pmatrix} a^{(1,1)}_{q_1,p_1} & \cdots & a^{(1,n)}_{q_1,p_n} \\ \vdots & \ddots & \vdots \\ a^{(n,1)}_{q_n,p_1} & \cdots & a^{(n,n)}_{q_n,p_n} \end{pmatrix} = \begin{pmatrix} a^{(1,1)} & \cdots & a^{(1,n)} \\ \vdots & \ddots & \vdots \\ a^{(n,1)} & \cdots & a^{(n,n)} \end{pmatrix}_{q,p} .$$

Note that we can always write an element of $\mathcal{M}_n(R_{\Delta,\Sigma_n})$ as the sum of elements of the form

$$\begin{pmatrix} a^{(1,1)}_{q_1,p_1} & \dots & a^{(1,n)}_{q_1,p_n} \\ \vdots & \ddots & \vdots \\ a^{(n,1)}_{q_n,p_1} & \dots & a^{(n,n)}_{q_n,p_n} \end{pmatrix}$$

(possibly with some coefficients equal to 0), with $a^{(i,j)} \in R$ for all $1 \leq i, j \leq n$. Lastly given any ring homomorphism $\varphi : R_1 \to R_2$ between two rings R_1 and R_2, we will denote by $\mathcal{M}_n(\varphi) : \mathcal{M}_n(R_1) \to \mathcal{M}_n(R_2)$ the morphism obtained by applying φ to each entry of the matrix.

Proposition 2.4.14. *Let $\mathcal{E}$ be a twisted $R^\times$-local system over Σ_n and let $\varphi : \mathrm{TPA}_\Delta(\Sigma_n) \to R_{\Delta,\Sigma_n}$ the corresponding ring homomorphism given by Proposition 2.4.13. Then the ring homomorphism*

$$\psi = \pi_R \circ \mathcal{M}_n(\varphi) \circ SN_n : \mathrm{TPA}_\Delta(S) \to \mathcal{M}_n(R)_{\Delta,S}$$

corresponds to a twisted $\mathrm{GL}_n(R)$-local system on S, together with a Δ-generic framing.

Proof. Let $\mathcal{E}$ be a twisted $R^\times$-local system on Σ_n, and let $\mathcal{L}$ be the twisted $\mathrm{GL}_n(R)$-local system obtained on S. We need to show that $\mathcal{L}$ admits a flat section on any peripheral curve β_p on S corresponding to an internal puncture, i.e. that the monodromy along β_p is upper triangular in some basis. Let $p \in \mathcal{P}_{int}$ and $p_1, \dots, p_n$ the lifts of p to Σ_n, labeled as in Section 2.3.1. Let $q \in E_\Delta(S) \cap \beta_p$ and $q_1, \dots, q_n$ the lifts of q to Σ_n, $q_i \in \beta_{p_i}$. We will assume β_p is a loop based on q. The fiber $\mathcal{L}_q$ of $\mathcal{L}$ over q can be identified with the direct sum $\mathcal{E}_{q_1} \oplus \cdots \oplus \mathcal{E}_{q_n}$ of the fibers of $\mathcal{E}$ over q_1 to q_n. Every line of the spectral network $\mathcal{W}$ crossed by β_p on S lifts to a line from p_i to p_{i-1} for some $2 \leq i \leq n$ on Σ_n. This means that the lifts added by the spectral network all go from q_i to q_j with $j < i$, so the image of β_p via $SN : \mathrm{TPA}_\Delta(S) \to \mathcal{M}_n(\mathrm{TPA}_\Delta(\Sigma))$ is upper triangular. Then $\psi(\beta_p) \in \mathcal{M}_n(R)_{\Delta,S}$ which is the monodromy of β_p is also upper triangular. For each $1 \leq k \leq n$, the k-dimensional subspace $F^{(p)}_k = \mathcal{E}_{q_1} \oplus \cdots \oplus \mathcal{E}_{q_k} \subset \mathcal{L}_q$ is preserved by the peripheral monodromy which means that the parallel transport of $F^{(p)}$ along β_p defines a framing $F^{(p)} \subset \mathcal{L}_{\beta_p}$ around p. This framing is Δ-generic because for every triangle $t = (p, q, r)$ of Δ and every triple $i, j, k \in \mathbb{N}$ such that $i + j + k = 2n + 1$, the map

$$F^{(p)}_i \cap F^{(q)}_j \cap F^{(r)} \to F^{(p)}_i / F^{(p)}_{i-1}$$

is the holonomy of the path $\gamma_{p,v}$ on Σ_n defined in Section 2.3.2, thus is an isomorphism. $\square$

The twisted $\mathrm{GL}_n(R)$-local system $\mathcal{L}$ on S obtained from a twisted $R^\times$-local system $\mathcal{E}$ on Σ_n via this construction is called the *partial non-abelianization* of $\mathcal{E}$. In the next part, we define an inverse construction.

2.5 Partial abelianization

2.5.1 Partial abelianization of generic framed local systems

Let S be a ciliated surface and let Δ be an ideal triangulation of S. Let $n \geq 2$. Let $\pi : \Sigma_n \to S$ be the ramified n-fold covering constructed in section 2.3.1. Let $\mathcal{L}$ be a Δ-generic framed twisted $\mathrm{GL}_n(R)$-local system.

We construct the line bundle $\mathcal{E}$ on Σ_n as follows. Let p be a puncture of S and let $1 \leq i \leq n$. Let $S_{\gamma_p^{(i)}}$ be the punctured disk defined in Lemma 2.3.3. On $S_{\gamma_p^{(i)}}$ the line bundle $\mathcal{E}$ is the pullback of the bundle F_i^p/F_{i-1}^p by π. Let $t = (p, q, r)$ be a triangle of Δ. Let v be a white vertex of $\Gamma_n^{(t)}$ of coordinates (i, j, k), and let $U_{i,j,k}$ be a small contractible neighborhood of v together with the three edges incident to v in Σ_n that does not contain any black vertex of Γ_n (see Figure 5.20).

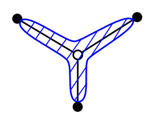


Figure 5.20: The neighborhood $U_{i,j,k}$ of a white vertex of coordinates (i, j, k) in Σ_n.

The line bundle $\mathcal{E}$ on $T'U_{i,j,k}$ is the pullback of the subbundle $F_i^{(p)} \cap F_j^{(q)} \cap F_k^{(r)}$ via π. On the intersection of $T'U_{i,j,k}$ with $T'S_{\gamma_p^{(i)}}$, the transition function is given by the isomorphism

$$F_i^{(p)} \cap F_j^{(q)} \cap F_k^{(r)} \to F_i^p/F_{i-1}^p$$

which is well defined in every point of the intersection $T'U_{i,j,k} \cap T'S_{\gamma_p^{(i)}}$.

This defines a line bundle $\mathcal{E}$ on Σ_n minus the set of all black vertex of Γ_n. To extend $\mathcal{E}$ to Σ_n, we need to check that the monodromy of $\mathcal{E}$ along a smooth loop without self-intersection going around a black vertex of $\Gamma_n \subset \Sigma_n$ is $-\mathrm{Id}$. When the black vertex v is external this is immediate because the line bundle $\mathcal{E}$ is trivial around v. When the black vertex v is internal, the monodromy of such a loop around v is $-\mathrm{Id}$ as computed in Proposition 3.2.4.

This construction of $\mathcal{E}$ has locally constant transition function so it defines a twisted $R^\times$ local-system $\mathcal{E}$ on Σ_n, which we call the *partial abelianization* of $\mathcal{L}$. By construction, the partial abelianization and the partial non-abelianization processes are inverse.

Proposition 2.5.1. *Let $\mathcal{L}$ be a Δ-generic framed twisted $\mathrm{GL}_n(R)$-local system, let $\mathcal{E}$ be the partial abelianization of $\mathcal{L}$. Let $\mathcal{L}'$ be the partial non-abelianization of $\mathcal{E}$ and let $\mathcal{E}'$ be the partial abelianization of $\mathcal{L}'$. Then $\mathcal{L} \simeq \mathcal{L}'$ and $\mathcal{E} \simeq \mathcal{E}'$.*

2.5.2 Partial abelianization of generic decorated local systems

Applying the construction of Section 2.5.1 to a decorated twisted $\mathrm{GL}_n(R)$-local system, we get an $R^\times$-local system $\mathcal{E} \to T'\Sigma_n$. Moreover, the existence of sections $b_i^{(p)}$ over $T'\beta_p$ for all $p \in \mathcal{P}$ and for all $1 \leq i \leq n$ induces that the holonomies of $\mathcal{E}$ around internal punctures of Σ_n are all trivial, i.e. the bundle $\mathcal{E}$ is trivial over the punctured disk around p bordered by β_p. That means that the local system $\mathcal{E} \to T'\Sigma_n$ can be uniquely extended to the local system over $T'\overline{\Sigma}_n$. For simplicity, slightly abusing the notation, we will write $\mathcal{E} \to T'\overline{\Sigma}_n$ this new local-system.

A decoration of $\mathcal{L}$ provides additionally a parallel section of $\mathcal{E}|_{T'\beta_p}$. We call the set of all those parallel sections a *decoration* of the twisted $R^\times$-local system $\mathcal{E}$.

Theorem 2.5.2. *The partial abelianization and partial non-abelianization processes define a bijection between the set of decorated twisted $R^\times$-local systems on Σ_n with trivial monodromy around punctures up to isomorphism and the set of decorated Δ-generic twisted local systems on S up to isomorphism.*

2.6 Topology of the moduli space of framed $\mathrm{GL}_2(R)$-twisted local systems

In this section, we focus on the special case $n = 2$. We describe the topology of the moduli space of framed twisted $\mathrm{GL}_2(R)$-local systems on S that are Δ-generic with respect to a fixed triangulation Δ.

As we have seen, framed twisted $\mathrm{GL}_2(R)$-local systems on S that are Δ-generic with respect to a fixed triangulation Δ are in 1:1-correspondence with twisted $R^\times$-local systems on Σ_2. Since Σ_2 has punctures, the space of twisted and non-twisted $R^\times$-local systems are homeomorphic. So we obtain the following theorem, using the same notations as in Proposition 2.3.7:

Theorem 2.6.1. *The moduli space of framed (twisted) $\mathrm{GL}_2(R)$-local systems on S that are Δ-generic with respect to a fixed triangulation Δ is homeomorphic to the moduli space of (twisted) $R^\times$-local systems on Σ_2 which is homeomorphic to $(R^\times)^{1-4\chi(\overline{S})+2p+\sum n_i}/R^\times$ where $R^\times$ acts diagonally by conjugation on $(R^\times)^{1-4\chi(\overline{S})+2p+\sum n_i}$.*

Remark 2.6.2. In [GRW22] the authors prove the same result using different techniques. They define local systems on some appropriate graphs over S and parametrize them using coordinates that are similar to Fock-Goncharov's GL_n-cluster $\mathcal{X}$-coordinates [FG06].

2.7 Symplectic local systems

Involutive algebras are an important class of non-commutative algebras. Over involutive algebras, generalizations of many classical groups can be constructed (e.g. orthogonal groups, symplectic groups). In this chapter which is a joint work with Eugen Rogozinnikov, we define algebras with anti-involutions and symplectic groups over such algebras that were introduced and studied in [ABR+22]. Further, we introduce framed twisted symplectic local system and characterize them in terms of partial abelianization introduced before. As a result of this construction, we describe the topology of the moduli space of framed or decorated twisted maximal symplectic local systems.

2.7.1 Involutive algebras

Let A be a unital associative, possibly non-commutative finite-dimensional $\mathbb{R}$-algebra.

Definition 2.7.1. An *anti-involution* on A is a $\mathbb{R}$-linear map $\sigma\colon A \to A$ such that

- $\sigma(ab) = \sigma(b)\sigma(a)$;
- $\sigma^2 = \mathrm{Id}$.

An *involutive* $\mathbb{R}$*-algebra* is a pair (A, σ), where A is a $\mathbb{R}$-algebra and σ is an anti-involution on A.

Example 2.7.1. The set of $n \times n$ matrices with real or complex coefficients endowed with the transposition is an involutive algebra.

Definition 2.7.2. Two elements $a, a' \in A$ are called *congruent*, if there exists $b \in A^\times$ such that $a' = \sigma(b)ab$. We call the action of $A^\times$ on A by $b \mapsto (a \mapsto \sigma(b)ab)$ *the action by congruence.*

Definition 2.7.3. An element $a \in A$ is called σ*-symmetric* if $\sigma(a) = a$. An element $a \in A$ is called σ*-anti-symmetric* if $\sigma(a) = -a$. We denote

$$A^\sigma := \mathrm{Fix}_A(\sigma) = \{a \in A \mid \sigma(a) = a\},$$

$$A^{-\sigma} := \mathrm{Fix}_A(-\sigma) = \{a \in A \mid \sigma(a) = -a\}.$$

Definition 2.7.4. The closed subgroup

$$U_{(A,\sigma)} = \{a \in A^\times \mid \sigma(a)a = 1\}$$

of $A^\times$ is called the *unitary group* of A. It is a Lie group whose Lie algebra agrees with $A^{-\sigma}$.

Definition 2.7.5. Let (A,σ) be an $\mathbb{R}$-algebra with an anti-involution. We define two set of squares:

$$A^\sigma_+ := \left\{a^2 \mid a \in (A^\sigma)^\times\right\}, \; A^\sigma_{\geq 0} := \left\{a^2 \mid a \in A^\sigma\right\}.$$

Remark 2.7.6. Since the algebra A is unital, we always have the canonical copy of $\mathbb{R}$ in A, namely $\mathbb{R}\cdot 1$ where 1 is the unit of A. We will always identify $\mathbb{R}\cdot 1$ with $\mathbb{R}$. Moreover, since σ is linear, for all $k \in \mathbb{R}$, $\sigma(k\cdot 1) = k\sigma(1) = k\cdot 1$, i.e. $\mathbb{R}\cdot 1 \subseteq A^\sigma$ and $\mathbb{R}_{>0}\cdot 1 \subseteq A^\sigma_+$.

Definition 2.7.7. A unital associative finite dimensional $\mathbb{R}$-algebra with an anti-involution (A,σ) is called *hermitian* if for all $x, y \in A^\sigma$, $x^2 + y^2 = 0$ implies $x = y = 0$.

Remark 2.7.8. In [ABR+22] it is shown that, if (A,σ) is a Hermitian algebra, then A^σ_+ is an open proper convex cone in A^σ, where proper means that the set does not contain (affine) lines.

If (A,σ) is Hermitian, for an element $a \in A^\sigma$ the *signature* can be defined, which is a bounded function $\operatorname{sgn}\colon A^\sigma \to \mathbb{Z}$ that is invariant under congruence by elements of $A^\times$. The elements of maximal signature are precisely the elements of A^σ_+. For more details about the signature see [ABR+22].

When $(A,\sigma) = (\mathcal{M}_n(\mathbb{R}), \cdot^T)$, the set A^σ is the set of symmetric matrices, $A^\sigma_{\geq 0}$ is the set of positive symmetric matrices and A^σ_+ is the set of positive definite symmetric matrices. The signature of a symmetric matrix with p positive eigenvalues and q negative eigenvalues (and any multiplicity of 0 as eigenvalue) is $p - q$.

2.7.2 Symplectic groups over non-commutative algebras

Let A be a unital associative finite dimensional $\mathbb{R}$-algebra with an anti-involution σ. We consider A^2 as a right A-module over A.

Definition 2.7.9. Let $\omega(x,y) := \sigma(x)^T\Omega y$ with $\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The group

$$\mathrm{Sp}_2(A,\sigma) := \mathrm{Aut}(\omega) = \{g \in \mathcal{M}_2(A) \mid \sigma(g)^T\Omega g = \Omega\}$$

is the *symplectic group* Sp_2 over (A,σ). The form ω is called the *standard symplectic form* on A^2.

We have

$$\mathrm{Sp}_2(A,\sigma) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \sigma(a)c,\, \sigma(b)d \in A^\sigma,\, \sigma(a)d - \sigma(c)b = 1\right\} \subseteq \mathrm{GL}_2(A)$$

We can also determine the Lie algebra $\mathfrak{sp}_2(A,\sigma)$ of $\mathrm{Sp}_2(A,\sigma)$:

$$\mathfrak{sp}_2(A,\sigma) = \left\{\begin{pmatrix} x & z \\ y & -\sigma(x) \end{pmatrix} \mid x \in A,\; y, z \in A^\sigma\right\} \subseteq \mathcal{M}_2(A).$$

Remark 2.7.10. In [ABR+22] is shown that, if A is a Hermitian algebra, then $\mathrm{Sp}_2(A,\sigma)$ is a Hermitian Lie group of tube type.

Let (x,y) be a basis of A^2. We say that this basis is *isotropic* if $\omega(x,x)=\omega(y,y)=0$. We say that this basis is *symplectic* if furthermore $\omega(x,y)=1$.

Let $x\in A^2$ be a regular isotropic element (recall the definition of a regular element in Section 1.1). We call the set $xA:=\{xa\mid a\in A\}$ an *isotropic A-line*. The space of all isotropic A-lines is denoted by $\mathrm{Is}(\omega)$.

2.7.3 Symplectic local systems

We consider a twisted $\mathrm{GL}_2(A)$-local system $\mathcal{L}\to T'S$. We say that $\mathcal{L}$ is a *twisted* $\mathrm{Sp}_2(A,\sigma)$-*local system* (or just *twisted symplectic local system*) if the transition functions between trivializations of $\mathcal{L}$ preserve the standard symplectic 2-form on A^2, i.e. belong to $\mathrm{Sp}_2(A,\sigma)$. We then get a well defined symplectic form $\omega\colon \mathcal{L}\times\mathcal{L}\to A$ on $T'S$.

A framing of a parabolic twisted symplectic local system is called *isotropic* if the A-line defining the framing in a neighborhood of every puncture is isotropic with respect to the field of the form ω. A decoration $((b_1^{(p)})_{p\in\mathcal{P}},(b_2^{(p)})_{p\in\mathcal{P}})$ of a twisted symplectic local system is called *symplectic* if $\omega(b_1^{(p)},b_1^{(p)})=0$ and $\omega(b_1^{(p)},b_2^{(p)})=1$.

Remark 2.7.11. Since $b_2^{(p)}\in\mathcal{L}/b_1^{(p)}A$, we need to check that symplectic decorations are well defined. Notice that if $\omega(b_1^{(p)},b_1^{(p)})=0$, then the expression $\omega(b_1^{(p)},b_2^{(p)})$ is well-defined. Indeed, let $\tilde b_2^{(p)}$ and $(\tilde b_2^{(p)})'$ be two lifts of $b_2^{(p)}$ to A^2. Then $(\tilde b_2^{(p)})'=\tilde b_2^{(p)}+b_1^{(p)}a$ for some $a\in A$. Further,

$$\omega(b_1^{(p)},(\tilde b_2^{(p)})')=\omega(b_1^{(p)},\tilde b_2^{(p)}+b_1^{(p)}a)=\omega(b_1^{(p)},\tilde b_1^{(p)})=:\omega(b_1^{(p)},b_2^{(p)}).$$

It is always enough to choose $b_1^{(p)}$ for every $p\in\mathcal{P}$. Then $b_2^{(p)}$ becomes uniquely defined.

A *framed twisted symplectic local system* is a twisted symplectic local system with an isotropic framing. A *decorated twisted symplectic local system* is a twisted symplectic local system with a symplectic decoration.

Remark 2.7.12. For a twisted symplectic local system to admit a framing (resp. a decoration), the monodromy around every internal puncture must stabilize (resp. fix) an isotropic A-line.

Remark 2.7.13. Notice, that since ω is a parallel form of even degree, the parallel transport of ω around the fiber of $T'S$ is trivial.

Let $\pi\colon\Sigma_2\to S$ be the ramified two-fold covering defined in Section 2.3.1. Let $\mathcal{E}\to T'\Sigma_2$ be an $A^\times$-local system over the spectral covering Σ_2 of S that is obtained by the partial abelianization procedure.

Let $\theta\colon\Sigma_2\to\Sigma_2$ be the covering involution. Slightly abusing the notation, we also denote $\theta=\theta_*\colon T'\Sigma_2\to T'\Sigma_2$.

Remark 2.7.14. Notice that θ does not have fixed points in $T'\Sigma_2$, even though it has fixed points in Σ_2.

We consider the pull-back of $\mathcal{E}$ with respect to θ and denote it by $\mathcal{E}' := \theta^*\mathcal{E}$. To simplify the notation, we will identify $\mathcal{E}'_p$ and $\mathcal{E}_{\theta(p)}$ for all $p \in \Sigma_2$. We denote by $P_\gamma \colon \mathcal{E}_{\gamma(0)} \to \mathcal{E}_{\gamma(1)}$, $P'_\gamma = P_{\theta\circ\gamma} \colon \mathcal{E}_{\theta(\gamma(0))} \to \mathcal{E}_{\theta(\gamma(1))}$ the parallel transport along $\gamma \colon [0,1] \to \Sigma_2$ in $\mathcal{E}$ and $\mathcal{E}'$. We denote by $P^S_\alpha \colon \mathcal{L}_{\alpha(0)} \to \mathcal{L}_{\alpha(1)}$ the parallel transport along $\alpha \colon [0,1] \to S$ in $\mathcal{L}$.

Definition 2.7.15. Let V and V' be two right A-modules. A map $b \colon V \times V' \to A$ is called an *A-sesquilinear pairing* between V and V' if it is additive in each argument and if for all $v \in V$, $v' \in V'$, and for all $a, a' \in A$, $b(va, v'a') = \sigma(a)b(v, v')a'$. An A-sesquilinear paring b is *non-degenerate* if for every regular $v \in V$ there exists $v' \in V'$ such that $b(v, v') \in A^\times$ and for every regular $v' \in V'$ there exists $v \in V$ such that $b(v, v') \in A^\times$.

We denote by $B(\mathcal{E}, \mathcal{E}') \to T'\Sigma_2$ the vector bundle of all A-sesquilinear parings between $\mathcal{E}$ and $\mathcal{E}'$. A section $\beta \in \Gamma(T'\Sigma_2, B(\mathcal{E}, \mathcal{E}'))$ is called *parallel* if

$$\beta_{\gamma(0)}(x, y) = \beta_{\gamma(1)}(P_\gamma(x), P'_\gamma(y)) = \beta_{\gamma(1)}(P_\gamma(x), P_{\theta\circ\gamma}(y))$$

for every $\gamma \colon [0,1] \to T'\Sigma_2$ and for every $x \in \mathcal{E}_{\gamma(0)}$, $y \in \mathcal{E}'_{\gamma(0)} = \mathcal{E}_{\theta(\gamma(0))}$.

Remark 2.7.16. Notice that if $\beta \in \Gamma(T'\Sigma_2, B(\mathcal{E}, \mathcal{E}'))$ is parallel and β_p is non-degenerate for one $p \in T'\Sigma_2$, then β_p is non-degenerate for all $p \in T'\Sigma_2$.

Theorem 2.7.17. *Let $\mathcal{L}$ be a framed twisted $\mathrm{GL}_2(A)$-local system on S and let $\mathcal{E}$ be the partial abelianization of $\mathcal{L}$ over Σ_2. Then $\mathcal{L}$ is an $\mathrm{Sp}_2(A, \sigma)$-local system if and only if there exists a non-degenerate parallel section $\beta \in \Gamma(T'\Sigma_2, B(\mathcal{E}, \mathcal{E}'))$ such that $\beta_p(x, y) = -\sigma(\beta_{\theta(p)}(y, x))$ for every $p \in T'\Sigma_2$, for every $x \in \mathcal{E}_p$ and for every $y \in \mathcal{E}_{\theta(p)}$.*

Proof. ($\Rightarrow$) Assume, $\mathcal{L}$ is an $\mathrm{Sp}_2(A, \sigma)$-local system. That means, there exists a field of standard symplectic forms ω on $\mathcal{L} \to T'S$, such that for every $\alpha \colon [0,1] \to T'S$ and for every $v, w \in \mathcal{L}_{\alpha(0)}$,

$$\omega_{\alpha(0)}(v, w) = \omega_{\alpha(1)}(P^S_\alpha(v), P^S_\alpha(w)).$$

Let $\gamma \colon [0,1] \to T'\Sigma_2$ be a smooth path such that $\gamma(0), \gamma(1)$ do not project to points on lines of the spectral network on Σ_2, and $x \in \mathcal{E}_{\gamma(0)}$ and $y \in \mathcal{E}_{\theta(\gamma(0))}$ regular elements. We consider $\gamma' = \theta \circ \gamma$ and $\alpha = \pi \circ \gamma = \pi \circ \gamma'$. Moreover, $(\pi_*(x), \pi_*(y))$ is an isotropic basis of $\mathcal{L}_{\alpha(0)}$. We can define

$$\beta_{\gamma(0)}(x, y) := \omega_{\alpha(0)}(\pi_*(x), \pi_*(y)).$$

Since ω is non-degenerate and skew-Hermitian, β is non-degenerate and sesquilinear pairing. Moreover, we have $\beta_{\gamma(0)}(x, y) = -\sigma(\beta_{\theta(\gamma(0))}(y, x))$ because $\omega_{\alpha(0)}(\pi_*(x), \pi_*(y)) = -\sigma(\omega_{\alpha(0)}(\pi_*(y), \pi_*(x)))$.

If γ does not intersect lines of the spectral network, then β along γ is parallel because in this case $P^S_\alpha = P_\gamma \oplus P_{\sigma\circ\gamma}$

If γ is a small segment intersecting a line of spectral network, then

$$\omega_{\alpha(1)}(P^S_\alpha(\pi_*(x)), P^S_\alpha(\pi_*(y))) = \omega_{\alpha(1)}(\pi_*(P_\gamma(x)) + \pi_*(P_{\tilde\gamma}(x)), \pi_*(P_{\theta\circ\gamma}(y)))$$

where $\tilde\gamma$ is a lift of α going along a line of spectral network from $\gamma(0)$ to $\theta(\gamma(1))$. But elements $P_{\tilde\gamma}(x), P_{\theta\circ\gamma}(y) \in \mathcal{E}_{\theta(\gamma(1))}$, therefore, $\omega(\pi_*(P_{\tilde\gamma}(x)), \pi_*(P_{\theta\circ\gamma}(y))) = 0$. So

$$\begin{aligned}\beta_{\gamma(0)}(x,y) &= \omega_{\alpha(0)}(\pi_*(x), \pi_*(y))\\ &= \omega_{\alpha(1)}(P^S_\alpha(\pi_*(x)), P^S_\alpha(\pi_*(y)))\\ &= \omega_{\alpha(1)}(\pi_*(P_\gamma(x)), \pi_*(P_{\theta\circ\gamma}(y)))\\ &= \beta_{\alpha(1)}(P_\gamma(x), P_{\theta\circ\gamma}(y)),\end{aligned}$$

i.e. β is parallel and extends also along lines of the spectral network on Σ_2.

Finally, let $p \in T'\Sigma_2$. Let $x \in \mathcal{E}_p$ and $y \in \mathcal{E}_{\theta(p)}$ regular elements. Then $(\pi_*(x), \pi_*(y))$ is an isotropic basis of $\mathcal{L}_{\pi(p)}$, i.e. $\beta(x,y) = \omega(\pi_*(x), \pi_*(y)) \in A^\times$. So the pairing β is non-degenerate.

($\Leftarrow$) Assume, there exists a non-degenerate parallel sesquilinear pairing β. Let $p \in T'\Sigma_2$ that does not project to a point on a line of the spectral network on Σ_2. We define for every $x \in \mathcal{E}_p, y \in \mathcal{E}_{\theta(p)}$:

$$\omega_{\pi(p)}(\pi_*(x), \pi_*(y)) := \beta_p(x,y).$$

Because $(\pi_*(x), \pi_*(y))$ is a basis of $V_{\pi(p)}$, ω extends by sesquilinearity on $V_{\pi(p)}$ if we assume

$$\omega_{\pi(p)}(\pi_*(x), \pi_*(x')) = \omega_{\pi(p)}(\pi_*(y), \pi_*(y')) = 0$$

for all $x, x' \in \mathcal{E}_p$ and $y, y' \in \mathcal{E}_{\theta(p)}$. Since β is non-degenerate, ω is non-degenerate as well.

Since $\beta_p(x,y) = -\sigma(\beta_{\theta(p)}(y,x))$, we get

$$\omega_{\pi(p)}(\pi_*(y), \pi_*(x)) = \beta_{\theta(p)}(y,x) = -\sigma(\beta_p(x,y)) = -\sigma(\omega_{\pi(p)}(\pi_*(x), \pi_*(y))).$$

Further, ω is parallel. Indeed, let $\alpha\colon [0,1] \to T'S$ be a path such that the projections of $\alpha(0)$ and $\alpha(1)$ to S are not on the lines of the spectral network. Let $x, y \in \mathcal{L}_{\alpha(0)}$. Let α_1, $\alpha_2 := \theta \circ \alpha_1$ are two standard lifts of α to $T'\Sigma_2$. Then $x = \pi_*(x_1) + \pi_*(x_2)$ and $y = \pi_*(y_1) + \pi_*(y_2)$ where $x_1, y_1 \in \mathcal{E}_{\alpha_1(0)}$ and $x_2, y_2 \in \mathcal{E}_{\alpha_2(0)}$. If the projection of α to Σ_2 does not intersect the spectral network, then the projection $T'\Sigma_2 \to T'S$ and the parallel transport along α and α_1, α_2 commute. So ω is parallel because β is parallel.

Assume now that the projection of α intersects the spectral network once. We denote by α_3 the additional lift of α along the spectral network. Without loss of generality, assume

$\alpha_3(0) = \alpha_1(0)$ and $\alpha_3(1) = \alpha_2(1)$. Notice that the path $\theta \circ (\alpha_3.\overline{\alpha}_1).\alpha_3.\overline{\alpha}_1$ is homotopic to the fiber of $T'\Sigma_2 \to \Sigma_2$. Therefore, $P_{\theta\circ\overline{\alpha}_1.\alpha_3} = -P_{\theta\circ\overline{\alpha}_3.\alpha_1}$. Therefore,

$$\begin{aligned}
\omega_{\alpha(1)}(P^S_\alpha(x), P^S_\alpha(y)) &= \omega_{\alpha(1)}(P^S_\alpha(x), P^S_\alpha(y)) \\
&= \omega_{\alpha(1)}(P^S_\alpha(\pi_*(x_1)) + P^S_\alpha(\pi_*(x_2)), P^S_\alpha(\pi_*(y_1)) + P^S_\alpha(\pi_*(y_2))) \\
&= \omega_{\alpha(1)}(\pi_*(P_{\alpha_1}(x_1) + P_{\alpha_3}(x_1) + P_{\alpha_2}(x_2)), \\
&\pi_*(P_{\alpha_1}(y_1) + P_{\alpha_3}(y_1) + P_{\alpha_2}(y_2))) \\
&= \omega_{\alpha(1)}(\pi_*(P_{\alpha_1}(x_1), \pi_*(P_{\alpha_3}(y_1) + P_{\alpha_2}(y_2)))) \\
&+ \omega_{\alpha(1)}(\pi_*(P_{\alpha_3}(x_1) + P_{\alpha_2}(x_2)), \pi_*(P_{\alpha_1}(y_1))) \\
&= \beta_{\alpha_1(1)}(P_{\alpha_1}(x_1), P_{\alpha_3}(y_1) + P_{\alpha_2}(y_2)) + \beta_{\alpha_2(1)}(P_{\alpha_3}(x_1) \\
&+ P_{\alpha_2}(x_2), P_{\alpha_1}(y_1)) \\
&= \beta_{\alpha_1(1)}(P_{\alpha_1}(x_1), P_{\alpha_3}(y_1)) + \beta_{\alpha_1(1)}(P_{\alpha_1}(x_1), P_{\alpha_2}(y_2)) \\
&+ \beta_{\alpha_2(1)}(P_{\alpha_3}(x_1), P_{\alpha_1}(y_1)) + \beta_{\alpha_2(1)}(P_{\alpha_2}(x_2), P_{\alpha_1}(y_1)) \\
&= \beta_{\alpha_1(1)}(P_{\alpha_1}(x_1), P_{\alpha_2}(y_2)) + \beta_{\alpha_2(1)}(P_{\alpha_2}(x_2), P_{\alpha_1}(y_1)) \\
&+ \beta_{\alpha_1(0)}(x_1, P_{(\theta\circ\overline{\alpha}_3).\alpha_1}(y_1) + P_{(\theta\circ\overline{\alpha}_1).\alpha_3}(y_1)) \\
&= \beta_{\alpha_1(1)}(P_{\alpha_1}(x_1), P_{\alpha_2}(y_2)) + \beta_{\alpha_2(1)}(P_{\alpha_2}(x_2), P_{\alpha_1}(y_1)) \\
&+ \beta_{\alpha_1(0)}(x_1, P_{\theta\circ\overline{\alpha}_3.\alpha_1}(y_1) + P_{\theta\circ\overline{\alpha}_1.\alpha_3}(y_1)) \\
&= \beta_{\alpha_1(1)}(P_{\alpha_1}(x_1), P_{\alpha_2}(y_2)) + \beta_{\alpha_2(1)}(P_{\alpha_2}(x_2), P_{\alpha_1}(y_1)) \\
&= \beta_{\alpha_1(0)}(x_1, y_2) + \beta_{\alpha_2(0)}(x_2, y_1) \\
&= \omega_{\alpha(0)}(x, y).
\end{aligned}$$

So ω is parallel and extends also along lines of the spectral network on S.

Finally, let $p \in \Sigma_2$ and $x \in \mathcal{E}_p$, $y \in \mathcal{E}_{\theta(p)}$ such that $\beta_p(x, y) = 1$, then $\omega(\pi_*(x), \pi_*(y)) = 1$. So ω is a field of standard symplectic forms. □

2.7.4 Topology of the moduli space of framed twisted symplectic local systems

We keep the same notation as in Proposition 2.3.7. Our goal in this section is to prove the following theorem:

Theorem 2.7.18. *The moduli space of framed (twisted)* $\mathrm{Sp}_2(A, \sigma)$*-local systems on* S *that are* Δ*-generic with respect to a fixed triangulation* Δ *is homeomorphic to:*

$$\left(((A^\sigma)^\times)^{-2\chi(\overline{S})+2p-1+\sum n_i} \times (A^\times)^{1-\chi(\overline{S})+p} \right) / A^\times$$

where the group $A^\times$ *acts componentwisely by conjugation on* $(A^\times)^{1-\chi(\overline{S})+p}$ *and by congruence on* $((A^\sigma)^\times)^{-2\chi(\overline{S})+2p-1+\sum n_i}$.

Proof. We use the 1:1-correspondence between framed twisted $\mathrm{Sp}_2(A,\sigma)$-local systems on S that are transverse to a fixed triangulation Δ and twisted $A^\times$-local systems on Σ_2 equipped with a non-degenerate parallel pairing β as in Theorem 2.7.17.

Let $\tilde{b} \in T'\Sigma_2$ such that it projects to a ramification point $b \in \Sigma_2$. Let $\alpha_1, \dots, \alpha_s \colon [0,1] \to S$ are free generators of the fundamental group $\pi_1(S, \pi(b))$. Let γ_i^1, γ_i^2 are closed lifts of α_i to $T'\Sigma_2$ such that $\theta \circ \gamma_i^1 = \gamma_i^2$ and γ_i^1 is based at $\tilde{b}$. Notice, that then γ_i^2 is based at $\theta(\tilde{b})$.

Let $s^+_{\tilde{b}}$ be as before the path from $\tilde{b}$ to $\theta(\tilde{b})$ going along the fiber at b in the positive direction and $s^-_{\theta(\tilde{b})} := \overline{s^+_{\tilde{b}}}$ the path from $\theta(\tilde{b})$ to $\tilde{b}$ going along the fiber at b in the negative direction. If the context is clear, we just write s^+ or s^- to simplify the notation.

Let $x \in \mathcal{E}_{\tilde{b}}$. Then on one hand: $\beta_{\tilde{b}}(x, P_{s^+}(x)) = -\sigma(\beta_{\theta(\tilde{b})}(P_{s^+}(x), x))$. On the other hand, since β is parallel:

$$\begin{aligned}\beta_{\tilde{b}}(x, P_{s^+}(x)) &= \beta_{\theta(\tilde{b})}(P_{s^+}(x), P_{s^+}(P_{s^+}(x)))\\ &= \beta_{\theta(\tilde{b})}(P_{s^+}(x), -x)\\ &= -\,\beta_{\theta(\tilde{b})}(P_{s^+}(x), x).\end{aligned}$$

So we obtain:

$$\beta_{\tilde{b}}(x, P_{s^+}(x)) = -\beta_{\theta(\tilde{b})}(P_{s^+}(x), x) = -\sigma(\beta_{\theta(\tilde{b})}(P_{s^+}(x), x)) =: a_0 \in A^\sigma.$$

Let now γ be a loop based at $\tilde{b}$ and

$$a_0 = \beta_{\tilde{b}}(x, P_{s^+}(x)) = \beta_{\tilde{b}}(P_\gamma(x), P_{\theta\circ\gamma} P_{s^+}(x)).$$

For every $x \in \mathcal{E}_{\tilde{b}}$, $P_\gamma(x) = x a_\gamma$ where $a_\gamma \in A^\times$. Let $P_{\theta\circ\gamma} P_{s^+}(x) = P_{s^+}(x) a'_\gamma$ for $a'_\gamma \in A^\times$. Then

$$a_0 = \sigma(a_\gamma)\beta_{\tilde{b}}(x, P_{s^+}(x)) a'_\gamma = \sigma(a_\gamma) a_0 a'_\gamma,$$

$$a'_\gamma = a_0^{-1}\sigma(a_\gamma^{-1}) a_0.$$

Let γ and $s^-.(\theta\circ\overline{\gamma}).s^+$ are different generators of $\pi_1(T'\Sigma_2, \tilde{b})$ (this corresponds to curves γ_i^1 and γ_i^2 of Lemma 2.3.10 case (1) lifted to $T'\Sigma_2$). In particular, they are not homotopic. Then a_γ and a_0 determine uniquely a'_γ.

Let $\gamma \colon [0,1] \to T'\Sigma_2$ and $\theta\circ\gamma \colon [0,1] \to T'\Sigma_2$ are two lifts to $T'\Sigma_2$ of a segment in S connecting $\pi(b)$ and $\pi(b')$ where b' is another ramification point on Σ_2. Let $\tilde{b} := \gamma(0)$ and $\tilde{b}' := \gamma(1)$. In this case, $\xi_{\tilde{b}'} := \xi := s^-_{\theta(\tilde{b})}.\theta(\overline{\gamma}).s^+_{\tilde{b}'}.\gamma$ and $s^-.(\theta\circ\overline{\xi}).s^+$ are homotopic in $T'\Sigma_2$. Therefore, $a_\xi = a_0^{-1}\sigma(a_\xi)a_0$, i.e. $a_0 a_\xi \in A^\sigma$. Moreover, an easy calculation shows that $a_0 a_\xi = \beta_{\tilde{b}'}(y, P_{s^+_{\tilde{b}'}} y)$ where $y = P_\gamma(x)$.

So the symplectic local system provides us elements $a_i \in A^\times$ corresponding to $P_{\gamma_i^1}$, $a_0 \in A^\sigma$ and $a_0 a_\xi \in A^\sigma$ for every ξ as in (2) of Lemma 2.3.10 (lifted to $T'\Sigma_2$). These elements are well-defined up to a common conjugation of all a_i and common congruence of all a_0 and $a_0 a_\xi$ by an element of $A^\times$.

Conversely, if elements a_i, a_0, a_ξ as above are given, then a twisted $A^\times$-local systems on Σ_2 equipped with a non-degenerate parallel pairing β can be reconstructed uniquely. Equivalent local system correspond to a common conjugation of all a_i and common congruence of all a_0 and $a_0 a_\xi$ by an element of $A^\times$. $\square$

2.7.5 Symplectic local system over Hermitian algebras

Let A be a Hermitian algebra. Let A, B, C be a triple of generic isotropic A-lines. The Kashiwara-Maslov index of the triple (A, B, C) is the signature of the element $\omega(x, \mu_A^{B,C}(x)) \in (A^\sigma)^\times$ for a regular $x \in A$ where $\mu_A^{B,C}$ is the Kashiwara-Maslov map defined in Section 1.3. In fact, this signature does not depend on $x \in A$, and it is invariant under cyclic permutations of the triple (A, B, C) and it changes the sign by transposition of the elements of the triple.

Let $\mathcal{L}$ be a framed symplectic twisted local system on S. Let $t \subset S$ be a triangle of the triangulation Δ that is incident to punctures p, q, r and the orientation of the triangle agrees with the orientation of the triple (p, q, r). Let $H = \pi^{-1}(t) \subset \Sigma_2$ be the hexagon that covers t. Let b be the ramification point in H, let $\tilde{b}$ be a lift of b in $T'H$ and let s^+ be a path in $T'H$ going from $\tilde{b}$ to $\theta(\tilde{b})$ along the fiber in the positive direction.

The following proposition is immediate:

Proposition 2.7.19. *Let $z \in T't$. The Kashiwara-Maslov index of $(F_1^{(p)}(z), F_1^{(q)}(z), F_1^{(r)}(z))$ agrees with the signature of the element $\beta_{\tilde{b}}(x, P_{s^+}(x)) \in A^\sigma$ for a regular $x \in \mathcal{E}_{\tilde{b}}$.*

Definition 2.7.20. Let $\mathcal{L}$ be a framed twisted $\mathrm{Sp}_2(A, \sigma)$-local systems on S. We say that $\mathcal{L}$ is *maximal* if for some triangulation Δ of S, for all triangle t of Δ the Kashiwara-Maslov index of the triple of isotropic A-lines associated to t is maximal.

The following proposition is proven in [BIW10]:

Proposition 2.7.21. *Twisted maximal $\mathrm{Sp}_2(A, \sigma)$-local systems on S are Δ-generic with respect to any triangulation Δ of S.*

Theorem 2.7.22. *If A is Hermitian, then the moduli space of framed (twisted) maximal $\mathrm{Sp}_2(A, \sigma)$-local systems on S is homeomorphic to:*

$$\left((A_+^\sigma)^{-2\chi(\overline{S})+p} \times (A^\times)^{-2\chi(\overline{S})+2p-1+\sum n_i}\right)/A^\times$$

where $A^\times$ acts componentwisely by conjugation on $(A^\times)^{-2\chi(\overline{S})+2p-1+\sum n_i}$ and by congruence on $(A_+^\sigma)^{-2\chi(\overline{S})+p}$.

Proof. Following the notation of the proof of Theorem 2.7.18, notice that the signature of $a_0 \in A^\sigma$ agrees with the Kashiwara-Maslov index of the oriented triangle where the ramification point $\pi(b) \in S$ lies, and the signature of $a_0 a_{\xi_{\tilde{b}'}} \in A^\sigma$ agrees with the Kashiwara-Maslov index of the oriented triangle where the ramification point $\pi(b') \in S$ lies. A twisted symplectic local system is maximal if and only if Kashiwara-Maslov indices of all oriented triangles are maximal. So we obtain the statement of the theorem. $\square$

Remark 2.7.23. The results of this and previous sections agree with the results from [GRW22] obtained using different techniques (see also Remark 2.6.2).

Chapter 3

Cluster $\mathcal{A}$-coordinates

In this chapter we present an attempt at the construction of a non-commutative generalization of the Fock-Goncharov cluster $\mathcal{A}$-coordinates introduced in [FG06] in the type A, i.e. for $\mathrm{SL}_n(\mathbb{R})$-local systems. Note however that many Lie groups does not admit a definition over a non-commutative algebra: this is the case in particular of SL_n for which there is no analog over a non-commutative ring, thus forcing us to deal with GL_n instead. We cannot yet express the mutation formulas for any mutation, we only describe those which are necessary to describe a flip in the triangulation chosen. We lack a geometric interpretation of a general mutation to extend the formula to any mutation in a given quiver. Still, the coordinates presented here have some convenient characteristics to justify the name of cluster $\mathcal{A}$-coordinates. First, they admit a nice description using (slightly altered) quivers which make their computation easy when changing the underlying triangulation. Second, they coincide with the non-commutative algebra introduced by Arkady Berenstein and Vladimir Retakh in [BR18] in the case $n = 2$, while also providing a quiver combinatorial description of the mutation formulas.

The coordinates introduced here are closely related to the cluster coordinates introduced by Alexander Goncharov and Maxim Kontsevich in [GK22]. The Goncharov-Kontsevich coordinates are products of two of the coordinates presented here. Another way to put it is to say that the coordinates presented here describe a factorization of each of the Goncharov-Kontsevich. It is worth noting that both coordinates systems coincide with the algebra defined by Berenstein-Retakh in the case $n = 2$. These two different definition of "cluster $\mathcal{A}$-coordinates" both have their advantages and drawbacks. The Goncharov-Kontsevich coordinates describe the moduli space that is the most natural generalization of the $\mathcal{A}$-moduli space introduced by Fock-Goncharov in [FG06], namely the moduli space of twisted decorated $\mathrm{GL}_n(R)$-local systems on an hyperbolic ciliated surface. However, they present some differences with the commutative Fock-Goncharov coordinates: the combinatorial description of the mutations sequences uses a bipartite graph instead of a quiver, and the mutation formulas do not coincide with the usual commutative ones when the ring R is commutative. A goal of the work presented here is to give a solution to both of those problems, and to see what is the trade-off to have coordinates that fits on a quiver with mutations formulas as close to the commutative ones as possible.

3.1 Rank 2

In rank 2 most of the technical difficulties arising from our definition of non-commutative cluster coordinates do not appear. We show in Section 3.1.2 that the non-commutative cluster algebra $\mathcal{A}_S$ introduced in [BR18] can be realized as the ring of non-commutative rational functions on the moduli space of decorated twisted $\mathrm{GL}_2(R)$-local systems over an hyperbolic ciliated surface S. Furthermore, the mutation formulas can be explicitly computed using spectral networks, which allow us to give a geometric and topological proof of an algebraic property of this algebra: the non-commutative Laurent phenomenon. We also give in Section 3.1.5 a representation of a subalgebra $\mathcal{Q}_S$ of $\mathcal{A}_S$ introduced in [BR18], and give a presentation of the subalgebra $\mathcal{Q}_S$.

3.1.1 The rank 2 cluster algebra

Arkady Berenstein and Vladimir Retakh introduced in [BR18] a non-commutative algebra associated to any hyperbolic ciliated surface that have some properties that can be thought as generalizations of properties of (commutative) cluster algebras. This algebra arose from the study of non-commutative Plücker coordinates constructed with a non-commutative algebra tool called *quasideterminants* introduced in [GGRW05]. In this section we recall some definitions and properties of these algebras introduced in [BR18], and in Section 3.1.2 we will give a geometric representation of these non-commutative algebras.

Let S be an hyperbolic ciliated surface, endowed with a hyperbolic metric with totally geodesic boundary. In this section, an *arc* of S is a (homotopy class) of path $\gamma : [0,1] \to \overline{S}$ such that $\gamma(0), \gamma(1) \in \mathcal{P}$ and $\gamma(]0,1[) \subset S$. We will denote by $E(S)$ the set of all arcs of S. Two arcs γ_1 and γ_2 are *composable* if $\gamma_1(1) = \gamma_2(0)$.

Given two hyperbolic ciliated surfaces S_1 and S_2 with puncture set $\mathcal{P}_1$ and $\mathcal{P}_2$ respectively, a map $f : S_1 \to S_2$ is a *morphism of ciliated surface* if it extends to an orientation-preserving local homeomorphism $f : \overline{S}_1 \to \overline{S}_2$ such that $f^{-1}(\mathcal{P}_2) = \mathcal{P}_1$.

For $n \geq 3$, we will call a n-gon a closed disk with n external punctures, and will denote it by P_n. The punctures of a n-gon are cyclically ordered from 1 to n. A *triangle* of S is a morphism $P_3 \to S$ of ciliated surface and a quadrilateral of S is a morphism $P_4 \to S$ of ciliated surface. Given a morphism $f : P_n \to S$, we will denote by $\gamma_{i,j}$ the image of the arc from the puncture j to the puncture i in P_n by f.

Definition 3.1.1. We define the unitary $\mathbb{R}$-algebra $\mathcal{A}_S$ generated by the symbols x_γ and x_γ^{-1} for all arcs $\gamma \in E(S)$ (with the convention $x_\gamma = 1$ if γ is trivial) with the relations:

- $\forall \gamma \in E(S), x_\gamma^{-1} x_\gamma = x_\gamma x_\gamma^{-1} = 1$
- $\forall f : P_3 \to S$ triangle,
$$x_{\gamma_{1,3}} x_{\gamma_{2,3}}^{-1} x_{\gamma_{2,1}} = x_{\gamma_{1,2}} x_{\gamma_{3,2}}^{-1} x_{\gamma_{3,1}}$$
- $\forall f : P_4 \to S$ quadrilateral,
$$x_{\gamma_{4,2}} = x_{\gamma_{4,3}} x_{\gamma_{1,3}}^{-1} x_{\gamma_{1,2}} + x_{\gamma_{4,1}} x_{\gamma_{3,1}}^{-1} x_{\gamma_{3,2}}$$
  and
$$x_{\gamma_{2,4}} = x_{\gamma_{2,3}} x_{\gamma_{1,3}}^{-1} x_{\gamma_{1,4}} + x_{\gamma_{2,1}} x_{\gamma_{3,1}}^{-1} x_{\gamma_{3,4}}$$

For Δ a triangulation of S, we define $\mathcal{A}_\Delta$ the subalgebra of $\mathcal{A}_S$ generated by the set of all x_γ for $\gamma \in E(S)$ and x_γ^{-1} for $\gamma \in \Delta$. When S is the n-gon, we will sometimes write $\mathcal{A}_n$ instead of $\mathcal{A}_{P_n}$.

The study of the subalgebra $\mathcal{A}_\Delta$ for a fixed triangulation Δ of S is considerably easier than the study of $\mathcal{A}_S$. Given a group G and a commutative field $\mathbb{K}$, the *group algebra* of G denoted by $\mathbb{K}G$ is the vector space freely spanned by the elements of G, together with the multiplication obtained by extending the multiplication on G by linearity. Any element $x \in \mathbb{K}G$ can be written as
$$x = \sum_{w \in G} \lambda_w w$$
where all $\lambda_w \in \mathbb{K}$ but a finite number are zero, and $\mathrm{Supp}(x)$ is the finite set of $w \in G$ such that $\lambda_w \neq 0$. The following definition is used in [BFR19] and will be used later in Theorem 3.1.12.

Definition 3.1.2. A *cyclically pinched 1-relator group* is a finitely generated group G that writes as an amalgamated product
$$G = F_1 \underset{w_1 = w_2}{*} F_2$$
where F_1 and F_2 are finitely generated free groups and $w_i \in F_i$ is a non-trivial element for $i = 1, 2$.

Theorem 3.1.3. *Let S be an hyperbolic ciliated surface and let Δ be a triangulation of S. Then the algebra $\mathcal{A}_\Delta$ is isomorphic to the group algebra $\mathbb{R}G$ of G, where G is :*

- *A finitely generated free group if S has at least one boundary component or if S is a sphere with 3 internal punctures*
- *A cyclically pinched 1-relator torsion free group otherwise.*

Proof. See [BR18], Theorem 3.26. □

3.1.2 Coordinates on decorated GL_2-local systems

In this section we show that the algebra introduced in 3.1.1 can be realized as non-commutative rational functions on the moduli space of decorated twisted $\mathrm{GL}_2(R)$-local systems on an hyperbolic ciliated surface S. Additionally, many algebraic computations can be done with purely topological/geometrical tools, namely spectral networks.

Let S be an hyperbolic ciliated surface and let Δ be a triangulation of S. Let $\Sigma = \Sigma_2$ and $\pi : \Sigma \to S$ be the ramified covering constructed in Section 2.3.1. Then for every arc γ of Δ from $p \in \mathcal{P}$ to $q \in \mathcal{P}$ we can bend it so that $T'\gamma$ intersect $T'\beta_p$ and $T'\beta_q$ as in Figure 1.1, i.e. β_p and γ are tangent at their intersection point.

Let $E_\Delta(S) \subset T'S$ be the set of intersection points between Δ and the lifted decoration curves $T'\beta_p$. This means that now each edge of the triangulation Δ is endowed with two special points (one for each extremity) lying on the peripheral curves associated to its endpoints. For every edge $\gamma \in \Delta$ of the triangulation, let τ_γ be the path in $T'S$ with extremities in $E_\Delta(S)$ obtained by restricting γ to the part in between the two special points on it. Note that since this is applied to all the oriented arcs of the triangulation, the chosen representative for τ_γ and $\tau_{\overline{\gamma}}$ are such that $T'(\tau_\gamma.\tau_{\overline{\gamma}})$ is homotopic to a fiber $T'S \to S$ (see Figure 1.1).

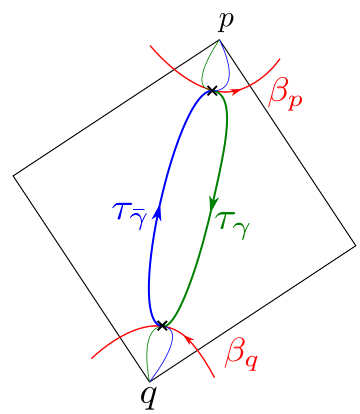


Figure 1.1: The bending of an edge of the triangulation. In red are the peripheral decoration, in blue and green are the oriented edges of the triangulation, the crosses are the points in $E_\Delta(S)$ and the thicker part of the edges in between the peripheral curves are the paths τ_γ and $\tau_{\overline{\gamma}}$

We also apply the same construction in Σ to equip each edge of the hexagonal tiling $\pi^{-1}(\Delta)$ with two special points, and denote $E_\Delta(\Sigma)$ the set of all special points in $T'\Sigma$. Note that both $E_\Delta(\Sigma)$ and $E_\Delta(S)$ are finite sets, and that the restriction of $d\pi : T'\Sigma \to T'S$ is 2:1 from $E_\Delta(\Sigma)$ to $E_\Delta(S)$.

Since the points in $E_\Delta(\Sigma)$ lie on peripheral curves associated with punctures, they inherit the source/sink naming from the puncture.

Let $\mathcal{L}$ be a decorated twisted $\mathrm{GL}_2(R)$-local system on the surface S, and assume $\mathcal{L}$ is Δ-generic. Let $\gamma \in \Delta$ be an edge from $p \in \mathcal{P}$ to $q \in \mathcal{P}$. We can trivialize the $\mathrm{GL}_2(R)$-local system $\mathcal{L}$ over $T'\tau_\gamma$ and the 1-dimensional subbundles $F_1^{(p)}$ and $F_1^{(q)}$ are transverse. The natural projection

$$a_\gamma : F_1^{(p)} \to \mathcal{L}/F_1^{(q)}$$

is an isomorphism, and we can identify it with its (1 by 1) matrix in the bases $b_1^{(p)}$ and $b_2^{(q)}$. We thus obtain a family $(a_\gamma)_{\gamma\in\Delta}$ of elements of $R^\times$ which we call *non-commutative $\mathcal{A}$-coordinates* of $\mathcal{L}$. The name "coordinates" is a slight abuse, since they are not independent.

Proposition 3.1.4. *For every oriented triangle $(\gamma_1, \gamma_2, \gamma_3)$ of Δ, we have*

$$a_{\gamma_3} a_{\overline{\gamma}_2}^{-1} a_{\gamma_1} = a_{\overline{\gamma}_1} a_{\gamma_2}^{-1} a_{\overline{\gamma}_3} \tag{1.1}$$

Proof. This is a direct corollary of Proposition 1.3.3. The minus sign appearing in the relation from Proposition 1.3.3 is canceled out by the minus sign appearing from the fact the local system is twisted. Indeed, the path $\tau_{\gamma_3}\overline{\tau}_{\overline{\gamma}_2}\tau_{\gamma_1}\overline{\tau}_{\overline{\gamma}_1}\tau_{\gamma_2}\overline{\tau}_{\overline{\gamma}_3}$ is homotopic to a loop going once around the fiber of $T'S \to S$. □

The coordinates of a decorated twisted $\mathrm{GL}_2(R)$-local system are the holonomies of its abelianized system $\mathcal{E}$ along the lifts of the arcs τ_γ, $\gamma \in \Delta$. Indeed, for each puncture $p \in \mathcal{P}$, the two lifts of p to Σ are a sink p_1 and a source p_2. In the neighborhood of p_1, the bundle $\mathcal{E}$ is the pullback of $F_1^{(p)}$ and in the neighborhood of p_2 the bundle is the pullback of $\mathcal{L}/F_1^{(p)}$. Now for an arc $\gamma \in \Delta$ from $p \in \mathcal{P}$ to $q \in \mathcal{P}$, the lifts of τ_γ to Σ join a sink and a source. Denote γ_1 the lift from p_1 to q_2 and γ_2 the lift from p_2 to q_1. Then a_γ is the holonomy of $\mathcal{E}$ along $T'\tau_{\gamma_1}$ and $a_{\overline{\gamma}}$ is thus the holonomy of $\mathcal{E}$ along $T'\overline{\tau_{\gamma_2}}$.

Let G be the graph embedded in $T'\Sigma$ with vertices the points of $E_\Delta(\Sigma)$ and edges the arcs τ_γ, $\gamma \in \pi^{-1}(\Delta)$ oriented from sink to source and portions of lifts of peripheral curves connecting two different points of $E_\Delta(\Sigma)$. To each oriented edge of this graph a coordinate is associated, and we assign to the edges with reversed orientation the inverse of this coordinate. The monodromy of $\mathcal{E}$ around a peripheral curve is trivial because $\mathcal{L}$ is decorated, so given a path on G, its holonomy in $\mathcal{E}$ is well-defined and only depends on the edges of the form τ_γ of the path. Then the triangle relation (1.1) imply that the monodromy of the abelianized system $\mathcal{E}$ restricted to the graph G is trivial around every hexagonal tile. Note that since Σ is obtained by gluing the hexagons back to G, the data of a family $(a_\gamma)_{\gamma\in\Delta}$ of invertible elements of R satisfying the relations 1.1 give rise to a twisted local system on Σ.

We now want to compute the "change of charts" induced by a flip in the triangulation. Let Δ_1 and Δ_2 be two triangulations differing only by one flip. Let p_1, p_2, p_3, p_4 be the four (not necessarily distinct) punctures at the vertices of the quadrilateral supporting the flip, in the cyclic order such that $\Delta_1 \setminus \Delta_2 = \{\gamma_{1,3}, \gamma_{3,1}\}$ and $\Delta_2 \setminus \Delta_1 = \{\gamma_{2,4}, \gamma_{4,2}\}$ where $\gamma_{i,j}$ is the arc of the quadrilateral going from p_j to p_i. Using the path-lifting map constructed in Section 2.4.1, we can compute the relations between the $\mathcal{A}$-coordinates associated to Δ_1 and the $\mathcal{A}$-coordinates associated to Δ_2.

Proposition 3.1.5. *Let $\mathcal{L}$ a decorated twisted* $\mathrm{GL}_2(R)$*-local system that is both Δ_1-generic and Δ_2-generic. Then its $\mathcal{A}$-coordinates with respect to Δ_1 and Δ_2 satisfy the following exchange relations:*

$$a_{\gamma_{2,4}} = a_{\gamma_{2,1}} a_{\gamma_{3,1}}^{-1} a_{\gamma_{3,4}} + a_{\gamma_{2,3}} a_{\gamma_{1,3}}^{-1} a_{\gamma_{1,4}}$$
$$a_{\gamma_{4,2}} = a_{\gamma_{4,1}} a_{\gamma_{3,1}}^{-1} a_{\gamma_{3,2}} + a_{\gamma_{4,3}} a_{\gamma_{1,3}}^{-1} a_{\gamma_{1,2}}$$

Remark 3.1.6. This way we retrieve with a geometric argument the relation given in Proposition 1.3.4.

Proof. For $i \in \{1,2,3,4\}$, let p_i', p_i'' be the two lifts of p_i to Σ where p_i' is the sink and p_i'' is the source. Let $s \in E_{\Delta_1}(S) \cap E_{\Delta_2}(S)$ be the intersection of $T'\beta_{p_2}$ and $\gamma_{2,1}$ and let $t \in E_{\Delta_1}(S) \cap E_{\Delta_2}(S)$ be the intersection of $T'\beta_{p_4}$ and $\gamma_{3,4}$. Let δ be a path in $T'S$ from s to t as in Figure 1.2.

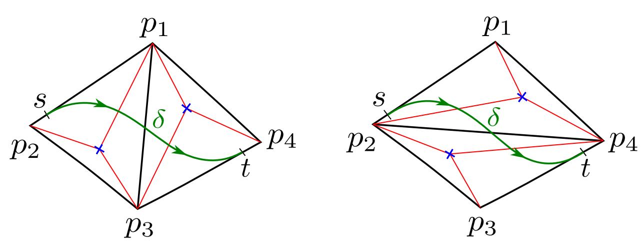


Figure 1.2: The path δ on S with triangulation Δ_1 on the left and with triangulation Δ_2 on the right.

The holonomy of $\mathcal{L}$ along γ does not depend on the triangulation.

Let Σ_1 and $\mathcal{W}_1$ be the ramified covering and the spectral network associated to the triangulation Δ_1 and $\Sigma_2, \mathcal{W}_2$ the ones associated to Δ_2. The corresponding path-lifting maps will be denoted SN_1 and SN_2, and the corresponding partial abelianizations of $\mathcal{L}$ will be denoted $\mathcal{E}_1$ and $\mathcal{E}_2$.

First, let's lift δ to Σ_2 using SN_2. Let s_1, s_2 be the lifts of s to Σ_2, s_1 being the sink and s_2 the source. Similarly, let t_1, t_2 the lifts of t, t_1 being the sink and t_2 the source. We get

$$SN_2(\delta) = \delta_1 + \delta_2 + \delta_1' + \delta_2' + \delta_3'$$

where δ_1 is a standard lift from s_1 to t_2, δ_2 is a standard lift from s_2 to t_1, δ_1' is a lift from s_2 to t_2 added by the spectral network, δ_2' is a lift from s_2 to t_1 added by the spectral network and δ_3' is a lift from s_1 to t_1 added by the spectral network (see Figure 1.3). The path δ_1 is the only lift going from s_1 to t_2, and its holonomy in $\mathcal{E}_2$ in the corresponding bases is $a_{\gamma_{4,2}}$ since it is homotopic to $\tau_{\gamma_{4,2}}$ precomposed with a piece of β_{p_2} and postcomposed with a piece of β_{p_4}, both of which have trivial holonomies. Since $\mathcal{L}$ is the partial non-abelianization of $\mathcal{E}_2$, this means that the map $F_1^{(p_2)} \to \mathcal{L}/F_1^{(p_4)}$ obtained by trivializing $\mathcal{L}$ along δ is exactly $a_{\gamma_{4,2}}$.

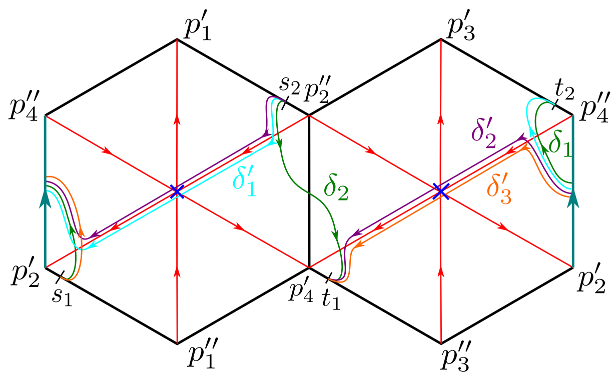


Figure 1.3: All the lifts of δ to Σ_2 using SN_2.

Now we will lift δ to Σ_1 using SN_1. Let s_1, s_2 be the lifts of s to Σ_1, s_1 being the sink and s_2 the source. Similarly, let t_1, t_2 the lifts of t, t_1 being the sink and t_2 the source. We get

$$SN_1(\delta) = \delta_1 + \delta_2 + \delta_1' + \delta_2' + \delta_3'$$

where δ_1 is a standard lift from s_1 to t_2, δ_2 is a standard lift from s_2 to t_1, δ_1' is a lift from s_1 to t_1 added by the spectral network, δ_2' is a lift from s_1 to t_2 added by the spectral network and δ_3' is a lift from s_2 to t_2 added by the spectral network (see Figure 1.4). The paths going from s_1 to t_2 are δ_1 and δ_2', and their holonomies in $\mathcal{E}_1$ in the corresponding bases are respectively $a_{\gamma_{4,3}} a_{\gamma_{1,3}}^{-1} a_{\gamma_{1,2}}$ and $a_{\gamma_{4,1}} a_{\gamma_{3,1}}^{-1} a_{\gamma_{3,2}}$. These are obtained by retracting the paths on the graph Γ, as the oriented edges of Γ have holonomies given by the $\mathcal{A}$-coordinates. Since $\mathcal{L}$ is also the partial non-abelianization of $\mathcal{E}_1$, this means that the map $F_1^{(p_2)} \to \mathcal{L}/F_1^{(p_4)}$ obtained by trivializing $\mathcal{L}$ along δ must be equal to the holonomy of $\delta_1 + \delta_2'$, which give the formula:

$$a_{\gamma_{4,2}} = a_{\gamma_{4,1}} a_{\gamma_{3,1}}^{-1} a_{\gamma_{3,2}} + a_{\gamma_{4,3}} a_{\gamma_{1,3}}^{-1} a_{\gamma_{1,2}}$$

The formula for $a_{\gamma_{2,4}}$ is obtained similarly. □

The proposition above yield a geometric realization of the non-commutative algebra $\mathcal{A}_S$ introduced in [BR18]. Let S be an hyperbolic ciliated surface and let Δ be a triangulation of S. We now focus on the case $R = \mathcal{M}_d(\mathbb{R})$. In this case, the space X of decorated twisted $\mathrm{GL}_2(R)$-local systems on S is an algebraic variety and the subset X_Δ of Δ-generic local

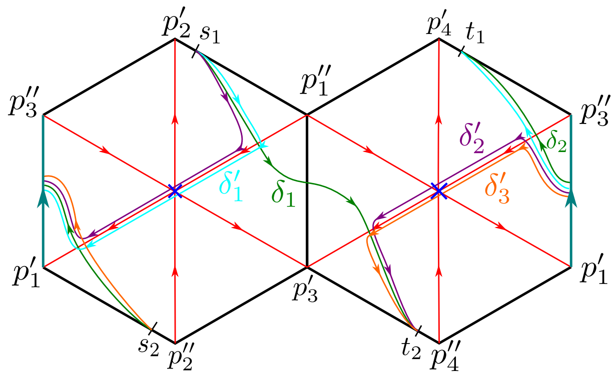


Figure 1.4: All the lifts of δ to Σ_1 using SN_1.

systems is an open dense subset of X. Each $\mathcal{A}$-coordinate associated to Δ can be seen as a rational function on X_Δ with coefficient in $\mathcal{M}_d(\mathbb{R})$.

Corollary 3.1.7. *Let S be an hyperbolic ciliated surface. The map*

$$\psi : \begin{array}{ccc} \mathcal{A}_S & \to & \mathrm{Rat}(X, \mathcal{M}_d(\mathbb{R})) \\ x_\gamma & \mapsto & a_\gamma \end{array}$$

is an algebra homomorphism.

Using the same type of arguments as above, we can give a topological/geometrical proof of the Laurent phenomenon for the cluster algebra of a polygon:

Theorem 3.1.8. *Let $n \geq 3$ and let S be the closed disk with n punctures on the boundary. Let $i, j \in \{1, \dots, n\}$, $i \neq j$. Then for every triangulation Δ of S and every decorated twisted $\mathrm{GL}_2(R)$-local system $\mathcal{L}$ that is Δ-generic and such that $(F^{(i)}, F^{(j)})$ is generic, the $\mathcal{A}$-coordinate $a_{\gamma_{i,j}}$ is a non-commutative Laurent polynomial in the $\mathcal{A}$-coordinates $(a_\gamma)_{\gamma \in \Delta}$ associated to the triangulation Δ.*

Proof. All the edges of the form $\gamma_{i,i+1}$, with $i \in \mathcal{P}$ ordered cyclically, belong to every triangulation of S so the result is immediate if $j \in \{i-1, i+1\}$. Now let $i, j \in \{1, \dots, n\}$, $i \neq j \pm 1$. Let Δ_0 be a triangulation of S containing the edges $\gamma_{i,j}$, $\gamma_{i,j-1}$ and $\gamma_{i-1,j}$. Such a triangulation always exists when $i \neq j \pm 1$. Let $s \in E_{\Delta_0}(S) \cap E_\Delta(S)$ be the intersection of β_j and $\gamma_{j-1,j}$ and let $t \in E_{\Delta_0}(S) \cap E_\Delta(S)$ be the intersection of β_i and $\gamma_{i-1,i}$. Let δ be the path from s to t drawn in Figure 1.5.

As we have seen in the proof of the flip relation (Proposition 3.1.5) and keeping the same notations, in the spectral network lift of δ with respect to the triangulation Δ_0 the only term from s_1 to t_2 has the holonomy $a_{\gamma_{i,j}}$ in the partial abelianization of $\mathcal{L}$ with respect to Δ_0. This means that the map $F^{(j)} \to \mathcal{L}/F^{(i)}$ obtained by trivializing $\mathcal{L}$ on δ is $a_{\gamma_{i,j}}$.

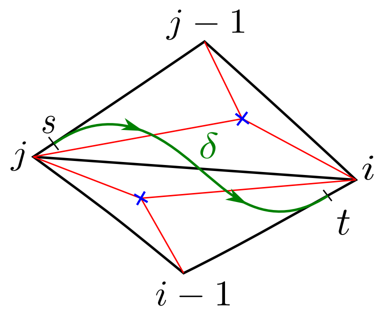


Figure 1.5: The path δ in the triangulation Δ_0. Only the quadrilateral $(i-1, i, j-1, j)$ is drawn.

Let $\mathcal{E}$ be the partial abelianization of $\mathcal{L}$ with respect to Δ. In spectral network lift of δ with respect to the triangulation Δ, let $\delta' = \delta'_1 + \cdots + \delta'_r$ be the sum of all paths from s_1 to t_2. Each δ'_k has a holonomy in $\mathcal{E}$ that is a monomial in the coordinates $(a_\gamma^{\pm 1})_{\gamma \in \Delta}$ as it retracts on the graph G. Since $\mathcal{L}$ is the partial non-abelianization of $\mathcal{E}$, the map $F^{(j)} \to \mathcal{L}/F^{(i)}$ obtained by trivializing $\mathcal{L}$ on δ is equal to the sum of the holonomies of the δ'_k in $\mathcal{E}$, so it is a Laurent polynomial in the $\mathcal{A}$-coordinates $(a_\gamma)_{\gamma \in \Delta}$. $\square$

Using these $\mathcal{A}$-coordinates, we can describe precisely the changes on the abelianized $R^\times$-local system on $\Sigma = \Sigma_2$ induced by a flip in the triangulation. We use the same notations as in Proposition 3.1.5. Let $\mathcal{L}$ be a framed twisted $\mathrm{GL}_2(R)$-local system on S that is transverse with respect to two triangulations Δ_1 and Δ_2. Let $\mathcal{E}_1$ (resp. $\mathcal{E}_2$) be the $R^\times$-local system on Σ obtained by abelianizing $\mathcal{L}$ with respect to Δ_1 (resp. Δ_2). These changes on the abelianized local system are supported in the lift C_Q of the quadrilateral Q surrounding the flip, which is homeomorphic to a cylinder with four punctures on each boundary components in Σ. Let γ be a loop on $T'\Sigma$. If γ crosses only one of the two boundary component of $\overline{C}_Q$ then the monodromies of γ in $\mathcal{E}_1$ and $\mathcal{E}_2$ are equal. Suppose γ crosses exactly once each of the two boundary components of $\overline{C}_Q$. Let γ_Q be the loop going around C_Q with the same orientation as the boundary of C_Q containing the sinks lifts of p_2 and p_4 (we refer to this boundary as the *positive* one, and the other one as *negative*).

Remark 3.1.9. We think of the holonomy of γ_Q in $\mathcal{E}$ as a generalization in the non-commutative setting of Fock-Goncharov's $\mathcal{X}$-coordinate of the quadrilateral Q. If $R = \mathbb{R}$, the holonomy of γ_Q is the cross-ratio of the four lines in $\mathbb{R}^2$ given by the framing of $\mathcal{L}$.

Up to homotopy, we can assume γ is going through at least one point $x_0 \in E_{\Delta_1}(\Sigma) \cap E_{\Delta_2}(\Sigma)$ on one of the eight external edges of the hexagon tiling of Q. We also choose a representative of γ_Q based at x_0. Let b be a basis of the fiber of $\mathcal{E}_1$ over x_0. Since x_0 is not in the interior of the cylinder supporting the flip in Σ, the fibers of $\mathcal{E}_1$ and $\mathcal{E}_2$ over x_0 are the same. Let $Y_1 \in R^\times$ (resp. Y_2) be the holonomy of γ in $\mathcal{E}_1$ (resp. $\mathcal{E}_2$), and let X be the holonomy of γ_Q in both $\mathcal{E}_1$ and $\mathcal{E}_2$.

Proposition 3.1.10. *If the part of γ inside C_Q goes from the positive boundary to the negative boundary, then*

$$Y_2 = Y_1(1+X).$$

If the part of γ inside C_Q goes from the negative boundary to the positive boundary, then

$$Y_2 = Y_1(1+X^{-1})^{-1}$$

Remark 3.1.11. The element $1+X^{-1} \in R$ is invertible because of the transversality of $\mathcal{L}$ with respect to Δ_2.

We have shown that a representation $\mathcal{A}_\Delta \to R$ give rise to a Δ-generic decorated twisted $\mathrm{GL}_2(R)$-local system on S. We now want to show a partial converse result: that there is no additional relations satisfied by the $\mathcal{A}$-coordinates. For this, we will focus on $R = \mathcal{M}_d(\mathbb{R})$.

A group G is called *residually finite* if for all $g \in G \setminus \{1\}$, there exists a finite index normal subgroup H not containing g, or equivalently if there exists a finite group G_g and a morphism $\varphi_g : G \to G_g$ such that the image of g by φ_g is non-trivial. All the finitely generated free groups are residually finite. Indeed, a rank n free group $\mathbb{F}_n = \langle x_1, \ldots, x_n \rangle$ embeds into $\mathbb{F}_2 = \langle a, b \rangle$ via the map

$$\begin{array}{ccc} \mathbb{F}_n & \to & \mathbb{F}_2 \\ x_i & \mapsto & ab^i \end{array}.$$

In turn, $\mathbb{F}_2$ embeds into $\mathrm{SL}_n(\mathbb{Z}) \subset \mathrm{Isom}^+(\mathbb{H}^2)$ via the group homomorphism defined by

$$a \mapsto \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \text{ and } b \mapsto \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

Given any non-trivial $M \in \mathrm{SL}_2(\mathbb{Z})$, there exists an integer $N \geq 2$ such that the image of M in the group $\mathrm{SL}_n(\mathbb{Z}/N\mathbb{Z})$ is non-trivial. This hows that $\mathrm{SL}_n(\mathbb{Z})$ is residually finite, hence $\mathbb{F}_n$ is too.

The following theorem is due to Baumslag in [BFR19].

Theorem 3.1.12. *A cyclically pinched 1-relator group is residually finite.*

Let S be an hyperbolic ciliated surface and let Δ be a triangulation of S. We now focus on the case $R = \mathcal{M}_d(\mathbb{R})$. In this case, the space X of decorated twisted $\mathrm{GL}_2(R)$-local systems on S is an algebraic variety and the subset X_Δ of Δ-generic local systems is an open Zariski-dense subset of X. Each $\mathcal{A}$-coordinate associated to Δ can be seen as a rational function on X_Δ with coefficient in $\mathcal{M}_d(\mathbb{R})$. Let $\psi_d : \mathcal{A}_\Delta \to \mathrm{Rat}(X_\Delta, \mathcal{M}_d(\mathbb{R}))$ be the representation defined in Proposition 3.1.7.

Theorem 3.1.13. *The family of representations $(\psi_d : \mathcal{A}_\Delta \to \mathrm{Rat}(X_\Delta, \mathcal{M}_d(\mathbb{R})))_{d\geq 2}$ is asymptotically injective, i.e.*

$$\bigcap_{d\geq 2} \mathrm{Ker}(\psi_d) = \{0\}.$$

Proof. Combining the Theorem 3.1.3 and 3.1.12, the algebra $\mathcal{A}_\Delta$ is isomorphic to $\mathbb{R}G$ where G is a residually finite group. Let $x \in \mathcal{A}_\Delta \setminus \{0\}$. We can write

$$x = \sum_{w \in \mathrm{Supp(x)}} \lambda_w w.$$

Since G is residually finite, for all $w, w' \in \mathrm{Supp}(x)$ distinct there exists a finite group $G_{ww'^{-1}}$ and a morphism $\varphi_{ww'^{-1}} : G \to G_{ww'^{-1}}$ such that $\varphi_{ww'^{-1}}(ww'^{-1}) \neq 1$. Let

$$\widetilde{G} = \prod_{w,w' \in \mathrm{Supp}(x), w \neq w'} G_{ww'^{-1}}$$

and let

$$\widetilde{\varphi} = \prod_{w,w' \in \mathrm{Supp}(x), w \neq w'} \varphi_{ww'^{-1}}$$

be the morphism associated. It satisfies $\widetilde{\varphi}|_{\mathrm{Supp}(x)}$ is injective because for any $w, w' \in \mathrm{Supp}(x)$ distinct, $\widetilde{\varphi}(ww'^{-1}) \neq 1$ so $\widetilde{\varphi}(w) \neq \widetilde{\varphi}(w')$. The map $\varphi : \mathbb{R}G = \mathcal{A}_\Delta \to \mathbb{R}\widetilde{G}$ satisfies $\varphi(x) \neq 0$. The algebra $\mathbb{R}\widetilde{G}$ acts on itself by left multiplication, providing an embedding $\mathbb{R}\widetilde{G} \to \mathrm{End}(\mathbb{R}\widetilde{G})$. Since $\widetilde{G}$ is finite, the algebra $\mathbb{R}\widetilde{G}$ is finite dimensional so $\mathrm{End}(\mathbb{R}\widetilde{G})$ is isomorphic to $\mathcal{M}_d(\mathbb{R})$ where $d = |G|$. Since $\widetilde{\varphi}(x) \neq 0$, $\widetilde{\varphi}(x)$ acts non trivially on $\mathbb{R}\widetilde{G}$, hence is mapped to an non-zero element of $\mathcal{M}_d(\mathbb{R})$. This yield a representation $f : \mathcal{A}_\Delta \to \mathcal{M}_d(\mathbb{R})$ such that $f(x) \neq 0$, so $x \notin \mathrm{Ker}(\varphi_d)$. □

3.1.3 $\mathcal{A}$-coordinates for symplectic local systems

Recall the definition of symplectic local systems given in Section 2.7. Since $\mathrm{Sp}_2(A, \sigma)$ is a subgroup of $\mathrm{GL}_2(A)$, the $\mathcal{A}$-coordinates defined in section 3.1.2, a twisted symplectic local system have well-defined $\mathcal{A}$-coordinates, and because of the additional structure of symplectic local systems, they satisfy additional relations. The following proposition is immediate:

Proposition 3.1.14. *Let S be an hyperbolic ciliated surface and let Δ be a triangulation of S. Let $\mathcal{L} \to S$ be a Δ-transverse decorated symplectic local system. Let γ be an arc of the triangulation Δ from $p \in \mathcal{P}$ to $q \in \mathcal{P}$. Then $a_\gamma = \omega(b_1^{(q)}, b_1^{(p)})$. In particular $a_{\overline{\gamma}} = -\sigma(a_\gamma)$.*

Proof. By definition of non-commutative $\mathcal{A}$-coordinates, $b_1^{(p)} \in F_1^{(p)}$ projects to $b_2^{(q)} a_\gamma \in \mathcal{L}/F_1^{(q)}$, i.e. for some lift $\hat{b}_2^{(q)} \in A^2$ of $b_2^{(q)}$, $b_1^{(p)} = \hat{b}_2^{(q)} a_\gamma + b_1^{(q)} r$ for some $r \in R$. Therefore, $\omega(b_1^{(q)}, b_1^{(p)}) = \omega(b_1^{(q)}, \hat{b}_2^{(q)} a_\gamma + b_1^{(q)} r) = \omega(b_1^{(q)}, \hat{b}_2^{(q)}) a_\gamma = a_\gamma$. □

From Proposition 1.1 follows:

Corollary 3.1.15. *For each oriented triangle $t = (\gamma_1, \gamma_2, \gamma_3)$ of Δ, we have*

$$\beta_t := a_{\gamma_3} a_{\overline{\gamma}_2}^{-1} a_{\gamma_1} \in A^\sigma.$$

If A is Hermitian, the signature of β_t agrees with the Kashiwara-Maslov index of t. Furthermore if $\beta_t \in A_+^\sigma$ for all oriented triangles t of Δ, then the decorated local system is maximal.

3.1.4 A subalgebra of $\mathcal{A}_S$

In [BR18], Berenstein and Retakh define a subalgebra $\mathcal{Q}_n$ of $\mathcal{A}_n$ and give a representation of both $\mathcal{A}_n$ and $\mathcal{Q}_n$ in terms of quasideterminants. Our goal in this section is to define a subalgebra $\mathcal{Q}_S$ of $\mathcal{A}_S$ that coincides with $\mathcal{Q}_n$ when $S = P_n$, and to give a presentation of $\mathcal{Q}_S$ similar to the presentation of $\mathcal{Q}_n$ given in [BR18], Theorem 2.14. We follow a similar approach to the original proof of Berenstein and Retakh. For this, we write the algebra $\mathcal{A}_S$ as a quotient of an algebra defined similarly to $\mathcal{A}_n$, except over an "infinite polygon", namely the universal cover of S in $\mathbb{H}^2$.

Let S be an hyperbolic ciliated surface, endowed with a hyperbolic metric with totally geodesic boundary. Let $\pi : \tilde{S} \to S$ be the universal cover of S, seen as a closed convex totally geodesic subspace of $\mathbb{H}$. We have $S = \Gamma \backslash \tilde{S}$ where $\pi_1(\Sigma) \simeq \Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ acts properly discontinuously on $\tilde{S} \subset \mathbb{H}$. We endow $\mathbb{P}^1\mathbb{R}$ with the canonical cyclic order, and we see $\mathbb{P}^1\mathbb{R} = \partial\mathbb{H}$ as the boundary of the hyperbolic plane $\mathbb{H}$. Let $\Lambda = \pi^{-1}(\mathcal{P}) \subset \mathbb{P}^1\mathbb{R}$ be the Farey set of $\tilde{S}$, i.e. the preimage of all punctures of S. A pair $(p, q) \in \Lambda^2$ with $p \neq q$ is identified with the oriented geodesic in $\mathbb{H}$ from p to q. The surface $\tilde{S}$ admits (infinite) ideal triangulations, in particular any pullback of a triangulation of S.

Definition 3.1.16. Let $\mathcal{Q}_S$ be the subalgebra of $\mathcal{A}_S$ spanned by the elements $y_{\alpha,\alpha'} = x_{\overline{\alpha}}^{-1} x_{\alpha'}$ with α and α' composable arcs on S. Given a triangulation Δ of S, we denote by $\mathcal{Q}_\Delta$ the subalgebra of $\mathcal{Q}_S$ spanned by the elements $y_{\alpha,\alpha'} = x_{\overline{\alpha}}^{-1} x_{\alpha'}$ with α and α' composable arcs in Δ.

Definition 3.1.17. We define the unitary $\mathbb{R}$-algebra $\mathcal{A}_\Lambda$ generated by the symbols $x_{p,q}$ and $x_{p,q}^{-1}$ for $p, q \in \Lambda$ (with the convention $x_{p,p} = 1$ for all $p \in \Lambda$) with the relations:

- $\forall p, q \in \Lambda, x_{p,q}^{-1} x_{p,q} = x_{p,q} x_{p,q}^{-1} = 1$
- $\forall p, q, r \in \Lambda, x_{p,q} x_{r,q}^{-1} x_{r,p} = x_{p,r} x_{q,r}^{-1} x_{q,p}$
- $\forall (p, q, r, s)$ cyclically ordered, $x_{p,r} = x_{p,q} x_{s,q}^{-1} x_{s,r} + x_{p,s} x_{q,s}^{-1} x_{q,r}$

For Δ a triangulation of $\tilde{S}$, we define $\mathcal{A}_\Delta$ the subalgebra of $\mathcal{A}_\Lambda$ generated by the set of all $x_{p,q}$ for $p, q \in \Lambda$ and $x_{p,q}^{-1}$ for $(p, q) \in \Delta$. We define $\mathcal{Q}_\Lambda$ the subalgebra of $\mathcal{A}_\Lambda$ generated by the elements $y_{p,q}^r = x_{r,p}^{-1} x_{r,q}$ for $p, q, r \in \Lambda$, $r \neq p, q$.

Definition 3.1.18. A *filtered set* is a nonempty set I together with a (partial) order $\leqslant$, with the additional property that every pair of elements has an upper bound.

Let $(I, \leqslant)$ a filtered set. Let $(A_i)_{i\in I}$ be a family of unitary algebras and let $(f_{i,j})_{i,j\in I}$ with $f_{i,j} : A_i \to A_j$ algebra homomorphisms such that $f_{ii} = \mathrm{Id}_{A_i}$ and $f_{i,j} \circ f_{k,i} = f_{k,j}$. We call $((A_i)_{i\in I}, (f_{ij})_{i,j\in I})$ a *direct system*. We denote by $(\varinjlim A_i, (f_i)_{i\in I})$ the direct limit of the system, which is defined by the following universal property: for all algebra B and for all families of algebra homomorphisms $(g_i)_{i\in I}$ such that for all $i \leqslant j \in I$, $g_i = g_j \circ f_{ij}$, there exists a unique homomorphism $\varphi : \varinjlim A_i \to B$ such that the following diagram commutes:

$$\begin{tikzcd}
A_i \arrow[rr, "f_{ij}"] \arrow[dr, "f_i"'] \arrow[ddr, bend right, "g_i"'] & & A_j \arrow[dl, "f_j"] \arrow[ddl, bend left, "g_j"] \\
& \varinjlim A_i \arrow[d, dotted, "\varphi"'] & \\
& B &
\end{tikzcd}$$

This direct limit is isomorphic to

$$\left(\bigsqcup_{i\in I} A_i\right) / \sim,$$

where the equivalence relation $\sim$ is defined by $(i, a_i) \sim (j, a_j) \Leftrightarrow \exists k \in I, i \leqslant k, j \leqslant k, f_{i,k}(a_i) = f_{j,k}(a_j)$. In particular, for all $a \in \varinjlim A_i$, there exists $i \in I$ and $a_i \in A_i$ such that $a = f_i(a_i)$.

Let $\mathcal{P}$ be the set of all finite subsets of Λ. Endowed with the inclusion, it is a filtered set. Let $\varinjlim \mathcal{A}_P$ be the direct limit of the system $((\mathcal{A}_{\mathcal{P}})_{P\in\mathcal{P}}, (j_{P,P'})_{P\subset P'\in\mathcal{P}})$, where $j_{P,P'} : \mathcal{A}_P \to \mathcal{A}_{P'}$ is the algebra homomorphism induced by the inclusion $P \subset P'$ (i.e. $\forall p, q \in P, j_{P,P'}(x_{p,q}) = x_{p,q}$).

Theorem 3.1.19. *The inclusions $\mathcal{A}_P \to \mathcal{A}_\Lambda$ for $P \in \mathcal{P}$ induce an algebra isomorphism*

$$\varinjlim \mathcal{A}_P \simeq \mathcal{A}_\Lambda.$$

Proof. Let $P \in \mathcal{P}$. Let $j_P : \mathcal{A}_P \to \varinjlim \mathcal{A}_P$ be the direct limit map associated to P, and let $f_P : \mathcal{A}_P \to \mathcal{A}_\Lambda$ be the algebra homomorphism induced by the inclusion $P \subset \Lambda$. This algebra homomorphism is well defined because $\tilde{\Sigma}$ contains all the triangles and the quadrilaterals of P. The morphisms f_P then satisfy $f_P = f_{P'} \circ j_{P,P'}$ for all $P \subset P' \in \mathcal{P}$. By universal property, there exists an algebra homomorphism $\varphi : \varinjlim \mathcal{A}_P \to \mathcal{A}_\Lambda$ such that $\varphi \circ j_P = f_P$ for all $P \in \mathcal{P}$. For $p, q \in \Lambda$, we define $\psi(x_{p,q}) = j_P(x_{p,q}) \in \varinjlim \mathcal{A}_P$, where $P \in \mathcal{P}$ contains

p and q. This definition does not depends on the choice of P by the property of direct limits. Moreover, for all triangle or quadrilateral of $\widetilde{S}$, there exists $P \in \mathcal{P}$ containing it. So $\psi : \mathcal{A}_\Lambda \to \varinjlim \mathcal{A}_P$ is a well defined algebra homomorphism. By construction, the morphisms φ and ψ are inverse one of each other.

$$\begin{tikzcd}
\mathcal{A}_P \arrow[rr, "j_{P,P'}"] \arrow[dr, "j_P"'] \arrow[ddr, bend right, "f_P"'] & & \mathcal{A}_{P'} \arrow[dl, "j_{P'}"] \arrow[ddl, bend left, "f_{P'}"] \\
& \varinjlim \mathcal{A}_P \arrow[d, shift right, "\varphi"'] & \\
& \mathcal{A}_\Lambda \arrow[u, shift right, "\psi"'] &
\end{tikzcd}$$

□

The following lemmas recall a few elementary properties of direct limits, they are stated here without proof.

Lemma 3.1.20. *Let $(I, \leqslant)$ be a filtered set. Let $((A_i)_{i\in I}, (f_{ij})_{i,j\in I})$ be a direct system and let $(B_i)_{i\in I}$ a family of subalgebras $B_i \subset A_i$ such that $f_{ij}(B_i) \subset B_j$. Then the maps $B_i \to \varinjlim A_j$ induce an identification of the limit $\varinjlim B_i$ of the system $((B_i)_{i\in I}, (f_{ij}|_{B_i})_{i,j\in I})$ with a subalgebra of $\varinjlim A_i$.*

Corollary 3.1.21. *For all triangulations Δ of $\tilde{S}$, the subalgebra $\mathcal{A}_\Delta$ of $\mathcal{A}_\Lambda$ is identified with the direct limit $\varinjlim \mathcal{A}_{\Delta_P}$ with $P \in \mathcal{P}$ which admits a triangulation Δ_P such that $\Delta_P \subset \Delta$. We then have an isomorphism $\mathcal{A}_\Delta \simeq \mathbb{R}\mathbb{F}_\infty$, where $\mathbb{F}_\infty$ is the free group freely spanned by a countable infinite set of generators.*
The subalgebra $\mathcal{Q}_\Lambda$ of $\mathcal{A}_\Lambda$ is identified with the direct limit $\varinjlim \mathcal{Q}_P$ with $P \in \mathcal{P}$.

Lemma 3.1.22. *Let P be a polygon. The subalgebra $\mathcal{Q}_P$ of $\mathcal{A}_P$ is generated by the elements $y^r_{p,q}$ with $p, q, r \in P$ distinct, with the relations:*

1. *$\forall p, q, r \in P$ pairwise distinct, $y^r_{p,q} y^r_{q,p} = 1$*
2. *$\forall p, q, r \in P$ pairwise distinct, $y^r_{p,q} y^p_{q,r} y^q_{r,p} = 1$*
3. *$\forall p, q, r, s \in P$ pairwise distinct, $y^s_{p,q} y^s_{q,r} y^s_{r,p} = 1$*
4. *$\forall (p, q, r, s) \in P$ pairwise distinct cyclically ordered, $y^q_{p,s} = y^r_{p,q} y^p_{q,s} + y^r_{p,s}$*

Proof. See [BR18], Theorem 2.14. □

Lemma 3.1.23. *Let $0 \longrightarrow I_i \xrightarrow{m_i} A_i \xrightarrow{\pi_i} Q_i \longrightarrow 0$ be a family of short exact sequences indexed by $i \in I$, with $((A_i)_{i\in I}, (f_{ij})_{i,j\in I})$ a direct system such that $f_{ij} \circ m_i(I_i) \subset m_j(I_j)$. Let $g_{ij} = f_{ij}|_{I_i} : I_i \to I_j$ and $h_{ij} : Q_i \to Q_j$ the map induced by $\pi_j \circ f_{ij}$. Then we have a short exact sequence $0 \longrightarrow \varinjlim I_i \longrightarrow \varinjlim A_i \longrightarrow \varinjlim Q_i \longrightarrow 0$.*

Corollary 3.1.24. *The subalgebra $\mathcal{Q}_\Lambda$ of $\mathcal{A}_\Lambda$ is generated by the elements $y^r_{p,q}$ with $p, q, r \in \Lambda$ distinct, with the relations:*

1. *$\forall p, q, r \in \Lambda$ distinct, $y^r_{p,q} y^r_{q,p} = 1$*
2. *$\forall p, q, r \in \Lambda$ distinct, $y^r_{p,q} y^p_{q,r} y^q_{r,p} = 1$*
3. *$\forall p, q, r, s \in \Lambda$ distinct, $y^s_{p,q} y^s_{q,r} y^s_{r,p} = 1$*
4. *$\forall (p, q, r, s) \in \Lambda$ distinct cyclically ordered, $y^q_{p,s} = y^r_{p,q} y^p_{q,s} + y^r_{p,s}$*

Given two algebras A and B, we denote by $A * B$ their free product.

Proposition 3.1.25. *Let $f : \Lambda \to \Lambda$ be a map without fixed points. Let $\mathbb{F}_\Lambda$ be the free group generated by $c_i, i \in \Lambda$ and let $\mathbb{R}\mathbb{F}_\Lambda$ be its group algebra. Then the map:*

$$\begin{array}{ccc} \mathcal{A}_\Lambda & \to & (\mathbb{R}\mathbb{F}_\Lambda) * \mathcal{Q}_\Lambda \\ x_{ij} & \mapsto & c_i * y^i_{f(i),j} \end{array}$$

is an algebra isomorphism.

Proof. Let $\varphi : \begin{array}{ccc} \mathcal{A}_\Lambda & \to & (\mathbb{R}\mathbb{F}_\Lambda) * \mathcal{Q}_\Lambda \\ x_{ij} & \mapsto & c_i * y^i_{f(i),j} \end{array}$. Firs we show that φ is well defined. Let $i, j, k \in \Lambda$. We have

$$\begin{aligned} \varphi(x_{ij} x_{kj}^{-1} x_{ki}) &= c_i * y^i_{f(i),j} * (c_k * y^k_{f(k),j})^{-1} * c_k * y^k_{f(k),i} \\ &= c_i * y^i_{f(i),j} * (y^k_{f(k),j})^{-1} * c_k^{-1} * c_k * y^k_{f(k),i} \\ &= c_i * y^i_{f(i),j} * y^k_{j,f(k)} * y^k_{f(k),i} \\ &= c_i * y^i_{f(i),j} * y^k_{j,i} \\ &= c_i * y^i_{f(i),j} * y^i_{j,k} * y^j_{k,i} \\ &= c_i * y^i_{f(i),k} * y^j_{k,i} \end{aligned}$$

$$\begin{aligned} \varphi(x_{ik} x_{jk}^{-1} x_{ji}) &= c_i * y^i_{f(i),k} * (c_j * y^j_{f(j),k})^{-1} * c_j * y^j_{f(j),i} \\ &= c_i * y^i_{f(i),k} * (y^j_{f(j),k})^{-1} * c_j^{-1} * c_j * y^j_{f(j),i} \\ &= c_i * y^i_{f(i),k} * y^j_{k,f(j)} * y^j_{f(j),i} \\ &= c_i * y^i_{f(i),k} * y^j_{k,i}. \end{aligned}$$

Let $(i,j,k,\ell) \in \Lambda$ cyclically ordered.

$$\begin{aligned}
\varphi(x_{jk}x_{ik}^{-1}x_{i\ell} + x_{ji}x_{ki}^{-1}x_{k\ell}) &= c_j * y^j_{f(j),k} * (c_i * y^i_{f(i),k})^{-1} * c_i * y^i_{f(i),\ell} \\
&\quad + c_j * y^j_{f(j),i} * (c_k * y^k_{f(k),i})^{-1} * c_k * y^k_{f(k),\ell} \\
&= c_j * y^j_{f(j),k} * y^i_{k,f(i)} * y^i_{f(i),\ell} + c_j * y^j_{f(j),i} * y^k_{i,f(k)} * y^k_{f(k),\ell} \\
&= c_j * y^j_{f(j),k} * y^i_{k,\ell} + c_j * y^j_{f(j),i} * y^k_{i,\ell} \\
&= c_j * y^j_{f(j),k} * (y^i_{k,\ell} + y^j_{k,i} * y^k_{i,\ell}) \\
&= c_j * y^j_{f(j),k} * y^j_{k,\ell} \\
&= c_j * y^j_{f(j),\ell} \\
&= \varphi(x_{j\ell})
\end{aligned}$$

So φ is well defined. Let $\psi_1 : \begin{array}{ccc} \mathbb{R}\mathbb{F}_\Lambda & \to & \mathcal{A}_\Lambda \\ c_i & \mapsto & x_{i,f(i)} \end{array}$ and let $\psi_2 : \mathcal{Q}_\Lambda \to \mathcal{A}_\Lambda$ be the canonical inclusion. Let

$$\psi = \psi_1 * \psi_2 : (\mathbb{Q}\mathbb{F}_\Lambda) * \mathcal{Q}_\Lambda \to \mathcal{A}_\Lambda.$$

We now show that φ and ψ are reciprocal inverse. We have $(\psi \circ \varphi)(x_{ij}) = \psi(c_i * y^i_{f(i),j}) = x_{i,f(i)} y^i_{f(i),j} = x_{ij}$. Moreover, $(\varphi \circ \psi)(c_i) = \varphi(x_{i,f(i)}) = c_i * y^i_{f(i),f(i)} = c_i$ and $(\varphi \circ \psi)(y^k_{ij}) = \varphi(y^k_{ij}) = (c_k * y^k_{f(i),i})^{-1} * c_k * y^k_{f(i),j} = y^k_{i,f(i)} * y^k_{f(i),j} = y^k_{ij}$. So φ is an isomorphism with inverse ψ. $\square$

Corollary 3.1.26. *With the same notation as above,*

$$\begin{array}{ccc} \mathcal{A}_\Lambda & \to & \mathcal{Q}_\Lambda \\ x_{ij} & \mapsto & y^i_{f(i),j} \end{array}$$

is a surjective algebra homomorphism, and its kernel is spanned by the elements $x_{f(i),i} - 1$ with $i \in \Lambda$.

Let $\mathcal{A}_S$ and $\mathcal{Q}_S$ be the algebras defined in Section 3.1.1. For all $p, q \in \Lambda$ distinct, we denote by (p,q) the oriented geodesic of $\tilde{S}$ from p to q and $\pi(p,q)$ the image of this geodesic in S. Let Δ be a triangulation of S and let $\tilde{\Delta}$ be the lift of Δ to $\tilde{S}$. Since π maps a polygon of $\tilde{S}$ to a polygon of S, π induces a surjective algebra homomorphism $\pi_* : \mathcal{A}_\Lambda \to \mathcal{A}_S$ defined by $\pi_*(x_{p,q}) = x_{\pi(p,q)}$.

The action of $\Gamma = \pi_1(S)$ is orientation-preserving, so it preserves the cyclic order on Λ. An element $\gamma \in \Gamma$ induces an algebra automorphism $\mathcal{A}_\Lambda$ given by $\gamma \cdot x_{p,q} = x_{\gamma(p),\gamma(q)}$.

Proposition 3.1.27. *The kernel of $\pi_* : \mathcal{A}_\Lambda \to \mathcal{A}_S$ is generated by the elements $(\gamma \cdot x) - x$ with $\gamma \in \Gamma$ and $x \in \mathcal{A}_\Lambda$. In particular, we have an isomorphism*

$$\mathcal{A}_S \simeq \mathcal{A}_\Lambda / ((\gamma \cdot x) - x)$$

Proof. For all $x \in \mathcal{A}_\Lambda$ and $\gamma \in \Gamma$, we have $(\gamma \cdot x) - x \in \ker \pi_*$. Let $\varphi : \mathcal{A}_\Lambda / ((\gamma \cdot x) - x) \to \mathcal{A}_\Sigma$ be the morphism induced by π_*. Let

$$\psi : \begin{array}{ccc} \mathcal{A}_\Lambda & \to & \mathcal{A}_\Lambda / ((\gamma \cdot x) - x) \\ x_\alpha & \mapsto & \overline{x_{\widetilde{\alpha}}} \end{array}, \text{ where } \alpha = \pi(\widetilde{\alpha}) \text{ is an arc of } S.$$

This defines an algebra homomorphism: on one hand if $\widetilde{\alpha}'$ is another lift of α by π, then there exists $\gamma \in \Gamma$ such that $\widetilde{\alpha}' = \gamma \cdot \widetilde{\alpha}$ so $\overline{x_{\widetilde{\alpha}}} = \overline{x_{\widetilde{\alpha}'}}$ and the definition of $\psi(x_\alpha)$ does not depend on the choice of the lift of α. On the other hand, if P is a n-gon of S, then there exists a n-gon $\widetilde{P}$ of $\tilde{S}$ such that $\pi(\widetilde{P}) = P$. This allows to deduce that ψ preserves the relations of $\mathcal{A}_\Lambda$, hence defines an algebra homomorphism. By construction, φ and ψ are inverse, so φ is an isomorphism. □

Theorem 3.1.28. *Let $I \subset \mathcal{Q}_\Lambda$ be the ideal spanned by the elements $\gamma \cdot y - y, \gamma \in \Gamma, y \in \mathcal{Q}_\Lambda$ and $y^r_{\gamma(p),q} - y^r_{p,q}, p, q, r \in \Lambda$ distinct with $\gamma \in \mathrm{Stab}_\Gamma(r)$. Then $\mathcal{Q}_S \simeq \mathcal{Q}_\Lambda / I$.*

Proof. The kernel of the morphism

$$\varphi : \begin{array}{ccc} \mathcal{Q}_\Lambda & \to & \mathcal{Q}_S \\ y^r_{p,q} & \mapsto & y_{\pi(p,r),\pi(r,q)} \end{array}$$

satisfies $I \subset \ker \varphi$, so we get a morphism

$$\overline{\varphi} : \begin{array}{ccc} \mathcal{Q}_\Lambda / I & \to & \mathcal{Q}_S \\ \overline{y^r_{p,q}} & \mapsto & y_{\pi(p,r),\pi(r,q)} \end{array}.$$

Let $f : \Lambda \to \Lambda$ without fixed point such that for all $\gamma \in \Gamma$ and for all $p \in \Lambda$, $f(\gamma(p)) \in \gamma.\mathrm{Stab}_\Gamma(p) \cdot f(p)$. One can construct such a function by choosing for each puncture p of S a lift $p_0 \in \Lambda$, then choosing $f(p_0) \in \Lambda$ and extending f to the orbit of p by Γ with the choice of an element of $\mathrm{Stab}_\Gamma(p)$ for each point in the orbit of p. Indeed, if $\gamma(p) = \gamma'(p)$, then $\gamma'^{-1}\gamma \in \mathrm{Stab}_\Gamma(p)$ so $\gamma'.\mathrm{Stab}_\Gamma(p) = \gamma.\mathrm{Stab}_\Gamma(p)$.

Moreover as shown in Corollary 3.1.26, the map

$$\psi' : \begin{array}{ccc} \mathcal{A}_\Lambda & \to & \mathcal{Q}_\Lambda / I \\ x_{p,q} & \mapsto & \overline{y^p_{f(p),q}} \end{array}$$

is an algebra homomorphism. We now show that the ideal $((\gamma \cdot x) - x)$ is contained in its kernel. Let $p, q \in \Lambda$ and let $\gamma \in \Gamma$. We have $y^{\gamma(p)}_{f(\gamma(p)),\gamma(q)} = \gamma \cdot y^p_{\gamma^{-1}(f(\gamma(p))),q}$ and $\gamma^{-1}(f(\gamma(p))) \in \mathrm{Stab}_\Gamma(p) \cdot f(p)$, so there exists $\gamma' \in \mathrm{Stab}_\Gamma(p)$ such that $\gamma^{-1}(f(\gamma(p))) = \gamma'(f(p))$. Then $\gamma \cdot y^p_{\gamma^{-1}(f(\gamma(p))),q} = \gamma \cdot y^p_{\gamma'(f(p)),q}$, and since $\overline{\gamma \cdot y^p_{\gamma'(f(p)),q}} = \overleftarrow{y}^r_{p,q}$ so $\psi'(x_{p,q}) = \psi'(\gamma \cdot x_{p,q})$. So ψ' is an algebra homomorphism $\mathcal{A}_\Lambda/((\gamma \cdot x) - x) \to \mathcal{Q}_\Lambda / I$, whose composition with the isomorphism $\mathcal{A}_S \to \mathcal{A}_\Lambda/((\gamma \cdot x) - x)$ from Proposition 3.1.27 yield a morphism

$$\psi : \begin{array}{ccc} \mathcal{A}_S & \to & \mathcal{Q}_\Lambda / I \\ x_\alpha & \mapsto & \overline{y^p_{f(p),q}} \end{array} \text{ with } \alpha = \pi(p,q).$$

We now show that $\psi|_{\mathcal{Q}_\Sigma} = \overline{\varphi}^{-1}$.

$$\begin{aligned}(\psi \circ \overline{\varphi})(\overline{y^r_{p,q}}) &= \psi(y_{\pi(p,r),\pi(r,q)}) \\ &= \psi(x^{-1}_{\pi(r,p)} x_{\pi(r,q)}) \\ &= \overline{y^r_{f(r),p}}^{-1} \overline{y^r_{f(r),q}} \\ &= \overline{y^r_{p,f(r)} y^r_{f(r),q}} \\ &= \overline{y^r_{p,q}}\end{aligned}$$

Let α, α' be two arcs of S with $\alpha(1) = \alpha'(0)$, and let (p,r) and (r,q) be lifts of α and α' respectively, sharing an endpoint $r \in \Lambda$. We have

$$\begin{aligned}(\overline{\varphi} \circ \psi)(y_{\alpha,\alpha'}) &= \overline{\varphi}(\psi(x^{-1}_{\overline{\alpha}} x_{\alpha'})) \\ &= \overline{\varphi}(\overline{y^r_{f(r),p}}^{-1} \overline{y^r_{f(r),q}}) \\ &= \overline{\varphi}(\overline{y^r_{p,f(r)} y^r_{f(r),q}}) \\ &= \overline{\varphi}(\overline{y^r_{p,q}}) \\ &= y_{\alpha,\alpha'}\end{aligned}$$

So $\overline{\varphi} : \mathcal{Q}_\Lambda / I \to \mathcal{Q}_S$ is an isomorphism. $\square$

Corollary 3.1.29. *The algebra $\mathcal{Q}_S$ is generated by the elements $y_{\alpha,\alpha'}$ with α, α' composable arcs of S with the following relations:*

1. *$y_{\alpha,\overline{\alpha}} = 1$ for all α arc of S*
2. *$y_{\alpha,\alpha'} y_{\alpha',\alpha} = 1$ for all α, α' composable arcs of S*
3. *For all triangle $(\alpha, \alpha', \alpha'')$ of S:*
$$y_{\alpha,\alpha'} y_{\alpha'',\alpha} y_{\alpha',\alpha''} = 1$$
4. *For all $\alpha, \alpha', \alpha''$ such that $\alpha(1) = \alpha'(1) = \alpha''(1)$:*
$$y_{\alpha,\overline{\alpha}'} y_{\alpha',\overline{\alpha}''} y_{\alpha'',\overline{\alpha}} = 1$$
5. *$\forall f : P_4 \to S$ quadrilateral,*
$$y_{\alpha_{12},\alpha_{24}} = y_{\alpha_{13},\alpha_{32}} y_{\alpha_{21},\alpha_{14}} + y_{\alpha_{13},\alpha_{34}}$$

Proof. Let Q be the algebra defined with the presentation above. We have a morphism

$$\varphi : \begin{array}{ccc} \mathcal{Q}_\Lambda & \to & Q \\ y^r_{p,q} & \mapsto & y_{\pi(p,r),\pi(r,q)} \end{array}$$

because the relations of $\mathcal{Q}_\Lambda$ given by Corollary 3.1.24 are satisfied by φ. This morphism induces $\overline{\varphi} : \mathcal{Q}_\Lambda/I \to Q$. By composing it with the isomorphism of Theorem 3.1.28, we get a morphism $\varphi' : \mathcal{Q}_\Sigma \to Q$. The relations of Q are satisfied in $\mathcal{Q}_\Sigma$, so we have a morphism

$$\psi : \begin{array}{ccc} Q & \to & \mathcal{Q}_\Sigma \\ y_{\alpha,\alpha'} & \mapsto & y_{\alpha,\alpha'} \end{array}$$

and ψ and φ' are inverse. □

3.1.5 Coordinates on semi-decorated GL_2-local systems

We now give a representation of the subalgebra $\mathcal{Q}_S$ as coordinates on GL_2-local systems on S with a new type of additional data in between a decoration and a framing.

Definition 3.1.30. Let S be an hyperbolic ciliated surface and let $\mathcal{L}$ be a twisted $\mathrm{GL}_2(R)$-local system on S. A *semi-decoration* of $\mathcal{L}$ is the data of a framing $(F^{(p)})_{p\in\mathcal{P}}$ together with a regular section $b_1^{(p)}$ freely spanning $F_1^{(p)}$ for each puncture $p \in \mathcal{P}$.

Remark 3.1.31. For a twisted $\mathrm{GL}_2(R)$-local system on S to admit a semi-decoration, its monodromies around each internal punctures must fix a R-line in R^2. One can obtain a semi-decorated local system from a decorated one by forgetting about the sections $(b_2^{(p)})_{p\in\mathcal{P}}$ spanning the quotients $\mathcal{L}/F_1^{(p)}$.

Let $\mathcal{L}$ be a Δ-generic semi-decorated $\mathrm{GL}_2(R)$-local system on S. Let $\alpha, \alpha' \in \Delta$ be composable edges. Let $p \in \mathcal{P}$ be the start of α' and $q \in \mathcal{P}$ be the end of α. The map

$$a_{\overline{\alpha}}^{-1} \circ a_{\alpha'} : F_1^{(p)} \to F_1^{(q)}$$

is well defined. We denote by $q_{\alpha,\alpha'}$ its (1 by 1) matrix in the respective bases of $F_1^{(p)}$ and $F_1^{(q)}$ given by the semi-decoration. We call those elements the $\mathcal{Q}$-coordinates of $\mathcal{L}$.

Let $R = \mathcal{M}_d(\mathbb{R})$. Let S be an hyperbolic ciliated surface and let Δ be a triangulation of S. The space Q of semi-decorated twisted $\mathrm{GL}_2(R)$-local systems on S is an algebraic variety and the subset Q_Δ of Δ-generic local systems is an open Zariski-dense subset of Q. Each $\mathcal{Q}$-coordinate associated to Δ can be seen as a rational function on Q_Δ with coefficient in $\mathcal{M}_d(\mathbb{R})$. As a corollary of 3.1.7, we get:

Corollary 3.1.32. *Let S be an hyperbolic ciliated surface and let Δ be a triangulation of S. The map*

$$\psi_Q : \begin{array}{ccc} \mathcal{Q}_\Delta & \to & \mathrm{Rat}(Q_\Delta, \mathcal{M}_d(\mathbb{R})) \\ y_{\alpha,\alpha'} & \mapsto & q_{\alpha,\alpha'} \end{array}$$

is an algebra homomorphism.

3.2 Coordinates on extra-decorated configurations of flags

We want extend the $\mathcal{A}$-coordinates defined in Section 3.1 for $\mathrm{GL}_2(R)$-local systems to $\mathrm{GL}_n(R)$-local systems. Because of some technical difficulties related to twisted local system, it will be more convenient to introduce first those $\mathcal{A}$-coordinates in the case of a configuration of flags where those technical difficulties do not arise. The mutations formulas of the coordinates defined in this section are meant to be the closest possible to both the commutative case, with a quiver to break down the mutation of a flip into smaller mutations, and the non-commutative rank 2 case described above. For this, it is natural to introduce a new kind of decoration on a configuration of flags, which we call an *extra-decoration* since it contains a decoration in the sense introduced in Section 1.2 plus additional data. This new kind of decoration depends on the choice of a triangulation, so the mutation rules must include a way to "mutate" an extra-decoration associated to a triangulation to one associated to another triangulation. The coordinates introduced can be expressed in terms of quasideterminants (see Section 3.2.7). It should be noted however that these coordinates fail to satisfy the non-commutative Laurent phenomenon in general, however other situations in which non-commutative Laurent phenomenon arise in small rank also fail to satisfy it in higher rank. These coordinates also fail to satisfy the pentagon relation because the extra-decoration itself does not satisfy it, so the coordinates that depends on this extra-decoration cannot satisfy it either.

3.2.1 Extra-decorated configurations of flags

Definition 3.2.1. Let $(A^{(1)},\dots,A^{(k)})$ be a k-tuple of flags and let Δ be a triangulation of the k-gon. If for all triangle (i,j,k) of Δ, $(A^{(i)},A^{(j)},A^{(k)})$ is a triple of flag in generic position, the k-tuple $(A^{(1)},\dots,A^{(k)})$ is called Δ-*generic*. The group $\mathrm{GL}_n(R)$ acts naturally on the space of Δ-generic k-tuple of flags in R^n. We denote by $\mathrm{Conf}^{\Delta}_{R,k}(n)$ the quotient of this space by the $\mathrm{GL}_n(R)$ action.

Definition 3.2.2. Let $(A^{(1)},\dots,A^{(k)})$ be a k-tuple of flags in generic position (recall Section 1.2 for the definition of flags in generic position). An *extra-decoration* of the k-tuple of flags is the data of a decoration of each flag $A^{(i)}$ together with the data for every $i_1,\dots,i_k\in\mathbb{N}$ such that $i_1+\dots+i_k=(k-1)n+1$ of an element $b_{i_1,\dots,i_k}\in R^n$ that freely span $A^{(1)}_{i_1}\cap\dots\cap A^{(k)}_{i_k}$.

Definition 3.2.3. Let Δ be a triangulation of the k-gon. Let $(A^{(1)},\dots,A^{(k)})$ be a Δ-generic k-tuple of flags. A Δ-*extra-decoration* of $(A^{(1)},\dots,A^{(k)})$ is the data for each triangle (i,j,k) of Δ of an extra-decoration of the triple $(A^{(i)},A^{(j)},A^{(k)})$. The data of a Δ-generic k-tuple of flags together with a Δ-extra-decoration is called a Δ-*extra-decorated k-tuple of flags*. The group $\mathrm{GL}_n(R)$ acts naturally on the space of Δ-extra-decorated k-tuple of flags in R^n. We denote by $\mathrm{DecConf}^{\Delta}_{k}(n)$ the quotient of this space by the $\mathrm{GL}_n(R)$ action.

3.2.2 Coordinates on the configuration space of extra-decorated triple or quadruple of flags

Let (A, B, C) be a triple of flags in generic position. Recall from Section 1.3 that we defined for all $i, j, k \in \mathbb{N}$ such that $i + j + k = 2n + 1$ the following maps by restriction of the canonical projections $R^n \to A_i/A_{i-1}$, $R^n \to B_j/B_{j-1}$ and $R^n \to C_k/C_{k-1}$:

$$a^A_{i,j,k} : A_i \cap B_j \cap C_k \to A_i/A_{i-1}$$

$$a^B_{i,j,k} : A_i \cap B_j \cap C_k \to B_j/B_{j-1}$$

$$a^C_{i,j,k} : A_i \cap B_j \cap C_k \to C_k/C_{k-1}$$

Given an extra-decoration of the triple of flags (A, B, C), we identify these linear maps with their matrices in the corresponding bases. We call these elements of $R^\times$ the (non-commutative) *$\mathcal{A}$-coordinates* of the extra-decorated triple of flags.

A reformulation of Proposition 1.3.3 is the following:

Proposition 3.2.4. *The $\mathcal{A}$-coordinates satisfy the following relations, called* triangle relations*:*

$$\forall i,j,k \in \mathbb{N} \ s.t. \ i+j+k = 2n+2, a^A_{i,j,k-1}.\left(a^B_{i,j,k-1}\right)^{-1}.a^B_{i-1,j,k} = -a^A_{i,j-1,k}.\left(a^C_{i,j-1,k}\right)^{-1}.a^C_{i-1,j,k}$$

3.2.3 Quiver and mutations

Let P be a polygon and Δ a triangulation of P. Let Q be the A_{n-1}-type quiver on P associated to the triangulation Δ, see Figure 2.6. More precisely, we see any triangle t as the Euclidean triangle in $\mathbb{R}^3$ with vertices of coordinates $(n, 0, 0)$, $(0, n, 0)$ and $(0, 0, n)$. The A_{n-1}-type quiver of the triangle is the quiver with vertex the points of t with integer coordinates except for the vertices of t itself, and a vertex of coordinates (a, b, c) has outgoing arrows to $(a+1, b-1, c)$, $(a, b+1, c-1)$ and $(a-1, b, c+1)$ and ongoing arrows from vertex $(a+1, b, c-1)$, $(a-1, b+1, c)$ and $(a, b-1, c+1)$, provided those vertex belong to t and that the edge is not contained in an edge of t. The quiver Q is planar, and every vertex has the same number of edges going in and going out of it. Note that these two properties are preserved during the mutation sequence that leads to a flip in the triangulation. To handle the fact we are working with GL_n and not SL_n or PGL_n –and our coordinates being non-commutative– we need to make a few adjustments to the usual mutation rules.

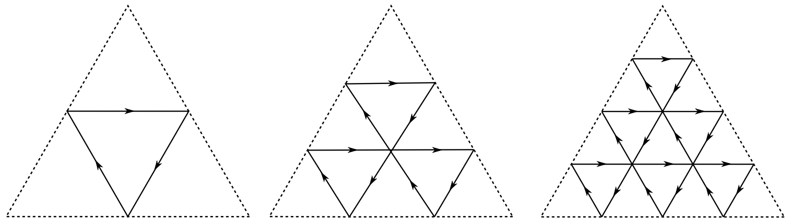


Figure 2.6: From left to right, the quivers of type A_1, A_2, A_3 on a triangle.

First, we want to allow pairs of vertices v and v' with arrows both from v to v' and from v' to v, although only in some precise situations.

Definition 3.2.5. Let $Q = (Q_0, Q_1)$ be a cyclic quiver, i.e. a quiver with n vertices $Q_0 = \{v_1, \ldots, v_n\}$ and $n-1$ arrows $Q_1 = \{(v_i, v_{i+1}) | 1 \leq i \leq n\}$ forming a cycle, with $v_{n+1} = v_1$. An *quiver!internal oriented triangulation* of Q is a set of arrows Q_1' such that $(Q_0, Q_1 \cup \overline{Q_1} \cup Q_1')$ is the graph of a polygon with an oriented triangulation, where $\overline{Q_1}$ denote the set of reversed edges of Q_1.

See Figure 2.7 for an example of a cyclic quiver with an internal oriented triangulation.

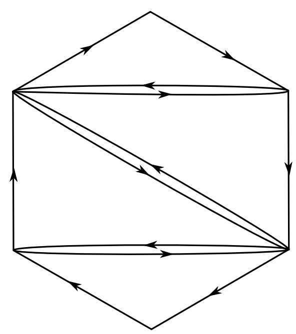


Figure 2.7: A cyclic quiver with 6 vertices with an internal oriented triangulation.

Definition 3.2.6. A *triangulated quiver* is a planar quiver Q embedded in an oriented polygon P such that:

- Q has no trivial loop
- every edge of P has at least one vertex of Q
- The subset of vertices of Q that are on an edge of P are called frozen

- Each vertex of Q has as many arrows going in and going out of it. We define the *valency* of a vertex v, denoted by $\mathrm{val}(v)$, to be the number of arrows going to v if v is not frozen and this number plus one if v is frozen
- There are two types of arrows in Q, we call them either *plain* or *dashed*
- At each vertex of Q, there exists a pairing between arrows going in and arrows going out, such that each pair of paired arrows is part of an oriented triangle for which the orientation agrees with the orientation on P (in the following, an *oriented cycle* in a quiver will always suppose matching orientation with P), and there are no edges inside such oriented triangles
- If an arrow $v \to v'$ is dashed, then there is another dashed arrow $v' \to v$ and those are the only arrows between v and v'
- Each oriented cycle of plain arrows of Q has an internal oriented triangulation of dashed arrows, and every dashed arrow belongs to one such triangulation
- If there is a plain arrow $v \to v'$, then there is no arrow $v' \to v$.
- There is at most one arrow from a vertex v to another vertex v'

When we say "arrow" without specifying if its dashed or plain it means any arrow of the quiver regardless of their nature, same for triangles and oriented cycles.

A notable difference with the usual quiver of cluster algebras is that we allow pairs of opposite arrows between two vertices, but only inside the oriented plain cycles. The set of all plain arrows in a triangulated quiver is a quiver in the usual sense. The dashed arrows are here to carry an additional information: a triangulation of the oriented cycles of plain arrows.

Proposition 3.2.7. *Let P be a polygon, let Δ be a triangulation of P and let $n \geq 1$. The A_n-type quiver associated with (P, Δ) is a triangulated quiver (without any dashed arrow).*

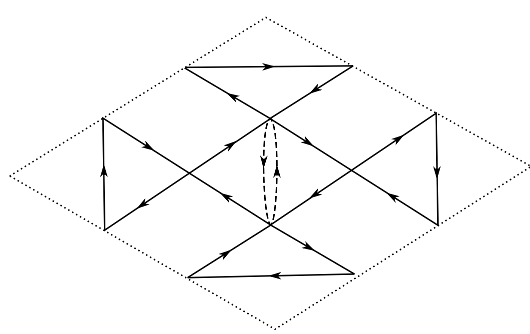


Figure 2.8: A triangulated quiver with one oriented plain cycle of length 4.

At every vertex v of a triangulated quiver, we will assign r variables $a_1(v), \dots, a_r(v) \in R^\times$ where r is the valency of v. Notice there is exactly r oriented triangles $t_1, \dots, t_r$ containing v in Q if v is not frozen. If v is frozen, then there is $r-1$ oriented triangles $t_1, \dots, t_{r-1}$ containing v in Q, and we set t_r to be the boundary containing v. The labeling on $t_1, \dots, t_r$ is set to agree with the cyclic ordering induced by the orientation of P. To get consistent mutation formulas, we will think of those r variables $a_1(v), \dots, a_r(v)$ as "in between" the oriented triangles containing v. More precisely, we choose a cyclic order on the set of those r variables together with the oriented triangles containing v such that $a_1(v) < t_1 < a_2(v) < \cdots < t_{r-1} < a_r(v) < t_r < a_1(v)$. We call the choice of those variable together with the cyclic order an *affectation* of Q, and we will denote by $\mathcal{V}$ the set of all the variables.

Around every oriented triangle t, the situation looks like the one in Figure 2.10.

Notice that in the case when one (or more) of the arrow of the oriented triangle t is dashed, then there are variables located "in between" the two opposite dashed arrows, see Figure 2.9.

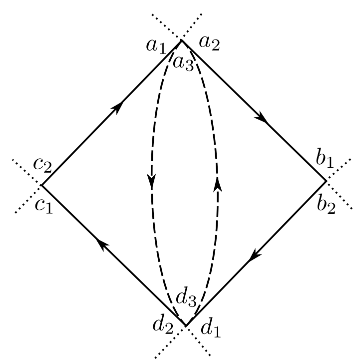


Figure 2.9: An affectation on a quiver containing a triangulated oriented quadrilateral.

Definition 3.2.8. Let Q be a triangulated quiver, together with an affectation $\mathcal{V}$. Let t be an oriented triangle of Q. The affectation around t gives 6 variables $a_1, a_2, b_1, b_2, c_1, c_2$ as in Figure 2.10. We define two elements which we call *triangle elements* at a:

$$R^+_{t,a} = b_1 a_2^{-1} c_1 \text{ and } R^-_{t,a} = c_2 a_1^{-1} b_2.$$

We define similarly $R^\pm_{t,b}$ and $R^\pm_{t,c}$. An affectation of a triangulated quiver Q is said to satisfy the triangle relations at an oriented triangle t if the variables satisfy the following relation, which we call *triangle relation* at t:

$$R^+_{t,x} = -R^-_{t,x}$$

where x is any of the three vertices of t. If the affectation satisfies the triangle relation for every oriented triangle of Q, it is said to satisfy the triangle relations.

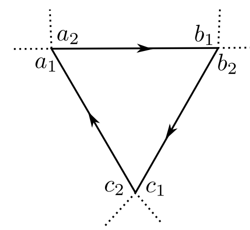


Figure 2.10: An affectation around an oriented triangle of Q.

Remark 3.2.9. At every oriented triangle of Q there is three equivalent triangles relations, one for each vertex. Also note that the triangle elements are always invertible elements of R.

Definition 3.2.10. A *non-commutative seed* is the data of:

- A triangulated quiver Q
- An affectation $\mathcal{V}$ of Q satisfying the triangle relations. We call those elements of R the $\mathcal{A}$-coordinates of the seed.

To alleviate the notations, we will just refer to non-commutative seeds as *seeds*.

Definition 3.2.11. Let $(Q, \mathcal{V})$ be a seed. Let x be a vertex of Q of valency 2. We say that x is *mutable* (or that the $\bowtie$-mutation at x is *admissible*) if none of the arrows incident to x are dashed and if the elements

$$x_1' = d_2 x_1^{-1} a_2 + c_1 x_2^{-1} b_1$$

$$x_2' = b_2 x_1^{-1} c_2 + a_1 x_2^{-1} d_1$$

are invertible in R, where the variables a_i, b_i, y_i, z_i are defined as in Figure 2.11. If x is mutable, the $\bowtie$-*mutation* of $(Q, \mathcal{V})$ at x is a new seed $(Q', \mathcal{V}') = \mu_{\bowtie,x}(Q, \mathcal{V})$ defined as follows:

- The new quiver Q' is obtained by reversing the direction of the arrows incident to x and adding the four plain arrows needed to complete the triangles created. If this process creates a pair of plain arrows between two vertex with opposite directions, then if this pair lies inside an oriented plain cycle of the quiver we change the pair of arrows to be dashed, and if not we remove the pair of arrows.
- The two $\mathcal{A}$-coordinates x_1, x_2 at the vertex x are replaced by the new variables x_1' and x_2' defined above. The new cyclic ordering at the vertex x is described on Figure 2.11.

The result of an admissible $\bowtie$-mutation is still a seed, i.e. satisfies the triangle relations: only two triangles are modified by the mutation, and using the same notation as in Figure 2.11 the two new triangle relations are:

$$\begin{aligned}
R^+_{t_1',a} &= x_1' a_2^{-1} c_2 \\
&= (d_2 x_1^{-1} a_2 + c_1 x_2^{-1} b_1) a_2^{-1} c_2 \\
&= d_2 x_1^{-1} c_2 + c_1 x_2^{-1} b_1 a_2^{-1} c_2 \\
&= R^+_{t_2,x} + c_1 (R^-_{t_1,b})^{-1} c_2 \\
&= -R^-_{t_2,x} - c_1 (R^+_{t_1,b})^{-1} c_2 \\
&= -c_1 x_2^{-1} d_1 - c_1 a_1^{-1} b_2 x_1^{-1} c_2 \\
&= -c_1 a_1^{-1} (b_2 x_1^{-1} c_2 + a_1 x_2^{-1} d_1) \\
&= -c_1 a_1^{-1} x_2' \\
&= -R^-_{t_1',a}
\end{aligned}$$

and a similar computation shows

$$R^+_{t_2',b} = -R^-_{t_2',b}.$$

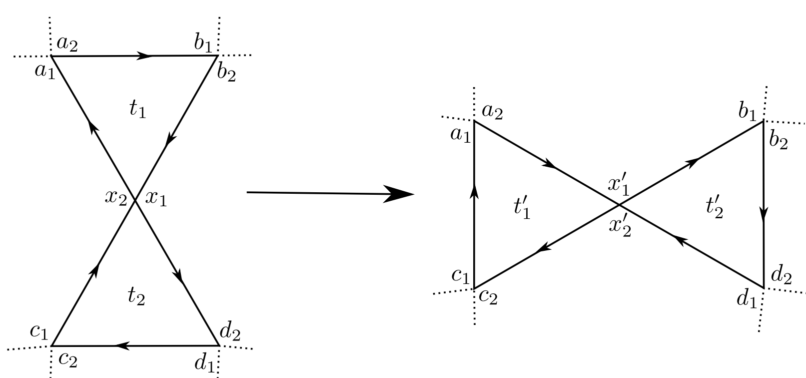


Figure 2.11: The situation before and after the $\bowtie$-mutation at the vertex x.

Remark 3.2.12. We can express the new triangle elements in term of the previous triangle elements and the unchanged $\mathcal{A}$-coordinates. Using the same notations as in Figure 2.11, we have:

$$\begin{aligned}
(R^{+}_{t_1',x})^{-1} &= (c_2 x_2'^{-1} a_1)^{-1} \\
&= a_1^{-1} x_2' c_2^{-1} \\
&= a_1^{-1}(b_2 x_1^{-1} c_2 + a_1 x_2^{-1} d_1) c_2^{-1} \\
&= a_1^{-1} b_2 x_1^{-1} + x_2^{-1} d_1 c_2^{-1} \\
&= (R^{+}_{t_1,b})^{-1} + (R^{+}_{t_2,d})^{-1}
\end{aligned}$$

and

$$\begin{aligned}
R^{+}_{t_1',a} &= x_1' a_2^{-1} c_2 \\
&= d_2 x_1^{-1} c_2 + c_2 x_2^{-1} b_1 a_2^{-1} c_2 \\
&= R^{+}_{t_2,x} + c_1 (R^{-}_{t_1,x})^{-1} c_2
\end{aligned}$$

The other triangle elements are obtained using similar formulas.

Proposition 3.2.13. *Let $(Q,\mathcal{V})$ be a seed, let x be a mutable vertex of Q and let $(Q',\mathcal{V}') = \mu_{\bowtie,x}(Q,\mathcal{V})$. Then x is mutable in $(Q',\mathcal{V}')$ and $\mu_{\bowtie,x}(Q',\mathcal{V}') = (Q,\mathcal{V})$.*

Proof. Let x_1'' be the variable of $\mu_{\bowtie,x}(Q',\mathcal{V}')$ at the location of x_1. We have:

$$\begin{aligned}
x_1'' &= a_2 x_1'^{-1} d_2 + c_2 x_2'^{-1} b_2 \\
&= a_2 (d_2 x_1^{-1} a_2 + c_1 x_2^{-1} b_1)^{-1} d_2 + c_2 (b_2 x_1^{-1} c_2 + a_1 x_2^{-1} d_1)^{-1} b_2 \\
&= (x_1^{-1} + d_2^{-1} c_1 x_2^{-1} b_1 a_2^{-1})^{-1} + (x_1^{-1} + b_2^{-1} a_1 x_2^{-1} d_1 c_2^{-1})^{-1} \\
&= (x_1^{-1} - x_1^{-1} c_2 d_1^{-1} b_1 a_2^{-1})^{-1} + (x_1^{-1} - x_1^{-1} a_2 b_1^{-1} d_1 c_2^{-1})^{-1} \\
&= \left((1 - c_2 d_1^{-1} b_1 a_2^{-1})^{-1} + (1 - a_2 b_1^{-1} d_1 c_2^{-1})^{-1}\right) x_1
\end{aligned}$$

To conclude, we need a general identity on non-commutative algebras:

Lemma 3.2.14. *Let $x \in R^{\times}$ such that $1 + x \in R^{\times}$. Then we have:*

$$(1 + x^{-1})^{-1} + (1 + x)^{-1} = 1.$$

Proof.

$$\begin{aligned}
(1 + x^{-1})^{-1} + (1 + x)^{-1} &= x(1 + x)^{-1} + (1 + x)^{-1} \\
&= (1 + x)(1 + x)^{-1} \\
&= 1
\end{aligned}$$

□

□

Since the number of coordinates associated to a vertex depends on its valency and the valency can change during the sequence of mutation of a flip, we need a second type of mutation that changes the valency of a vertex.

Definition 3.2.15. Let $(Q, \mathcal{V})$ be a seed. Let a and d two vertex of Q with a pair of dashed arrows between them. Let $b, c \in Q_0$ such that the oriented triangle containing $a \to d$ (resp. $d \to a$) is $t_1 = c \to a \to d \to c$ (resp. $t_2 = a \to b \to d \to a$), see Figure 2.12. The $\boxslash$-*mutation* of $(Q, \mathcal{V})$ at the pair of arrows $a \leftrightarrow d$ is a new seed $(Q', \mathcal{V}') = \mu_{\boxslash, a \leftrightarrow d}(Q, \mathcal{V})$ defined as follows:

- The pair of dashed arrow is part of an internal triangulation of a plain oriented cycle. The new quiver Q' is obtained by doing a flip of this edge in the internal triangulation:

$$Q'_1 = (Q_1 \setminus \{a \to d, d \to a\}) \cup \{b \to c, c \to b\}.$$

  The new arrows are also dashed.

- The valency of a and d is decreased by 1 so the variables a_3 and d_3 in between the removed dashed arrows are also removed. We then need to add two new variables in between the new pair of dashed arrows. The new added variable are:

$$b_3 = R^+_{t_1,c} = a_1 c_2^{-1} d_3 \text{ and } c_3 = R^+_{t_2,b} = d_1 b_2^{-1} a_3.$$

  See Figure 2.12.

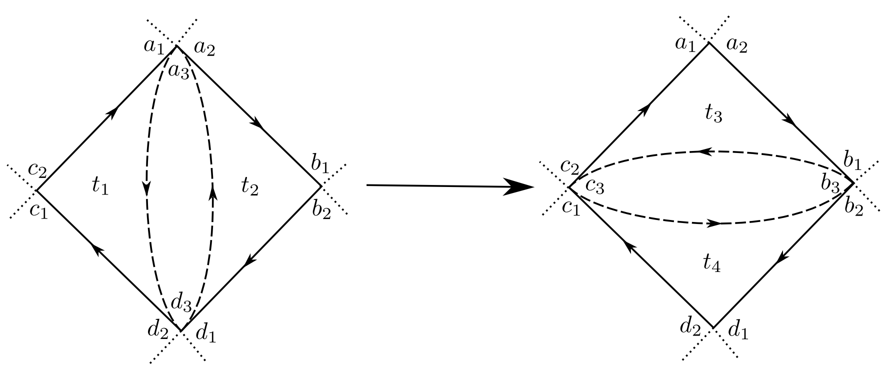


Figure 2.12: The situation before and after the $\boxslash$-mutation in an oriented quadrilateral of Q.

Again, the result of a $\boxslash$-mutation is a seed, i.e. satisfies the triangle relations. The new triangle relations are:

$$\begin{aligned}
R^+_{t_3,a} &= b_1 a_2^{-1} c_3 \\
&= b_1 a_2^{-1} d_1 b_2^{-1} a_3 \\
&= b_1 (R^-_{t_2,d})^{-1} a_3 \\
&= -b_1 (R^+_{t_2,d})^{-1} a_3 \\
&= -b_1 b_1^{-1} d_3 a_3^{-1} a_3 \\
&= -d_3 \\
&= -c_2 a_1^{-1} a_1 c_2^{-1} d_3 \\
&= -c_2 a_1^{-1} b_3 \\
&= -R^-_{t_3,a}
\end{aligned}$$

and similarly we have

$$R^+_{t_4,d} = -R^-_{t_4,d}.$$

Proposition 3.2.16. *Let $(Q, \mathcal{V})$ be a seed, let a, d be a pair of vertices of Q with a pair of dashed arrows between them and let $(Q', \mathcal{V}') = \mu_{\boxslash, a \leftrightarrow d}(Q, \mathcal{V})$. We use the same notations as in Figure 2.12. Then $\mu_{\boxslash, b \leftrightarrow c}(Q', \mathcal{V}') = (Q, \tilde{\mathcal{V}})$, where $\tilde{\mathcal{V}}$ is the same affectation as $\mathcal{V}$ except for $\tilde{a}_3 = -a_3$ and $\tilde{d}_3 = -d_3$.*

Proof. The coordinate $\tilde{a}_3$ is by definition equal to $R^+_{t_3,d} = c_1 d_2^{-1} b_3$. By replacing b_3 by $R^+_{t_1,c}$, we have:

$$\begin{aligned}
\tilde{a}_3 &= R^+_{t_2,d} \\
&= c_1 d_2^{-1} b_3 \\
&= c_1 d_2^{-1} a_1 c_2^{-1} d_3 \\
&= -a_3 d_3^{-1} c_2 c_2^{-1} d_3 \\
&= -a_3
\end{aligned}$$

$\square$

Remark 3.2.17. The $\boxslash$-mutation not being involutive is a consequence of the arbitrary choice of R^+ instead of R^- in the definition of the new variables. There is no canonical choice that would solve this problem, however when we will deal with twisted local systems (see Section 2.2.3) these coordinates will be defined up to sign anyway.

3.2.4 Coordinates on configurations of flags

Let (A, B, C) be an extra-decorated triple of flags in R^n. Let Q be the standard A_{n-1}-type quiver in a triangle (see Figure 2.6). The quiver Q is a triangulated quiver on a disk with 3 external punctures (with clockwise orientation). We label those three punctures p_A, p_B, p_C (in that order) and think of the flags A, B, C as associated to their corresponding puncture. Every vertex of Q is determined by a triple of non-negative integers (i, j, k) such that $i + j + k = 2n$, where i (resp. j, k) is the distance of the vertex to the puncture p_A (resp. p_B, p_C). This labeling is different from the one used in Section 3.2.3, a vertex of coordinates (a, b, c) with Section 3.2.3 notations will have coordinates $(i, j, k) = (n - a, n - b, n - c)$ in this new notation. We denote the vertex with coordinates (i, j, k) by $v_{i,j,k}$. The two vertices closest to p_A are $v_{1,n-1,n}$ and $v_{1,n,n-1}$. Frozen vertices of Q are the vertices for which one of the three coordinates is n.

Every non-frozen vertex $v_{i,j,k}$ of Q has valency three, and the three oriented triangles containing $v_{i,j,k}$ are:

$$t_{i,j,k,A} = v_{i,j,k} \to v_{i-1,j+1,k} \to v_{i-1,j,k+1} \to v_{i,j,k}$$

$$t_{i,j,k,B} = v_{i,j,k} \to v_{i,j-1,k+1} \to v_{i+1,j-1,k} \to v_{i,j,k}$$

$$t_{i,j,k,C} = v_{i,j,k} \to v_{i+1,j,k-1} \to v_{i,j+1,k-1} \to v_{i,j,k}$$

We assign to $v_{i,j,k}$ the coordinates $a^A_{i+1,j,k}, a^B_{i,j+1,k}, a^C_{i,j,k+1}$ constructed in Section 3.2.2 with the following cyclic order:

$$a^A_{i+1,j,k} < t_{i,j,k,C} < a^B_{i,j+1,k} < t_{i,j,k,A} < a^C_{i,j,k+1} < t_{i,j,k,B} < a^A_{i+1,j,k}$$

To remember this, $t_{i,j,k,A}$ is the triangle the closest to p_A and $a^A_{i,j,k+1}$ is the coordinate the furthest from p_A, with a similar statement for the other triangles/coordinates.

Every frozen vertex $v_{i,j,k}$ of Q has valency two and is part of one oriented triangle t of Q. Suppose $v_{i,j,k}$ is on the boundary between p_A and p_B (which means that $k = n$ and $i + j = n$), the other cases being symmetric. Let $b_{A,B}$ be the side of t between p_A and p_B. We assign to $v_{i,j,n}$ the variables $a^A_{i+1,j,n}$ and $a^B_{i,j+1,n}$ with the following cyclic ordering:

$$a^A_{i+1,j,n} < t < a^B_{i,j+1,n} < b_{A,B} < a^A_{i+1,j,n}.$$

As an immediate corollary from Proposition 3.2.4, we have:

Proposition 3.2.18. *The set of variables described above is an affectation of Q and satisfy the triangle relations.*

Let P be a k-gon ($k \geq 3$) and Δ be a triangulation of P. Let Q be the quiver obtained by gluing together the A_{n-1}-type quiver of each triangle of Δ along internal edges of Δ. The vertices on the internal edges of Δ are no longer frozen. Let $(A^{(1)}, \ldots, A^{(k)}) \in \mathrm{DecConf}_k^{\Delta}(n)$. Inside each triangle (i, j, k) of Δ, the triple $(A^{(i)}, A^{(j)}, A^{(k)})$ is extra-decorated. The affectation described above is compatible with the gluing of the quivers along edges of Δ. Indeed, for every pair of triangles (t_1, t_2) sharing an edge e, the variables affected to the vertices on e are identical in Q_{t_1} and Q_{t_2}, where Q_{t_1} (resp. Q_{t_2}) is the quiver inside t_1 (resp. t_2). For every vertex v on e, the oriented triangle in Q_{t_1} (resp. Q_{t_2}) replaces the boundary of the triangle t_1 (resp. t_2) regarding the cyclic order of the variables and oriented triangles at v.

Remark 3.2.19. Let $(Q, \mathcal{V})$ be the seed described above. The complement of Q in P has two types of connected components:

- interior of oriented triangles, which carry triangle relations
- interior of cycles with opposite orientation to P (including the regions containing an edge of P) which carries variables, all corresponding to maps with the same starting space. In this case we will say this common starting space is associated to this region. The four regions surrounding the two oriented triangle containing a vertex of valency 2 (as in Figure 2.11) are associated to four R-lines spanning a 2-dimensional subspace. The five regions surrounding a pair of vertices with a pair of dashed arrows are associated to five R-lines obtained from intersections of four 2-dimensional subspaces spanning a 3-dimensional subspace.

As a corollary to Proposition 1.3.4, doing a $\bowtie$-mutation at a mutable vertex yield a new set of coordinates on the same extra-decorated configuration of flags. However, doing a $\boxslash$-mutation replaces an anti-oriented region with another one. This means that for two triangulations Δ_1 and Δ_2 differing from a flip, we need a map from one Δ_1-extra-decoration of a configuration of flags to a Δ_2-extra-decoration of the same configuration of flags. For this, we use the isomorphism $T^{+}_{AD,BC}$ defined in 1.3.5. This isomorphism transports the vector spanning the R-line associated to the region in-between a pair of dashed arrows to a vector spanning the R-line associated to the region in between the new pair of dashed arrows after the $\boxslash$-mutation (See Figure 2.14).

Remark 3.2.20. A configuration of k Δ-extra-decorated flags is equivalent to a Δ-extra-decorated local system (see Section 3.3 for the definition) on a disk with k external punctures and no internal puncture. Since on surfaces with no internal punctures twisted and non-twisted local systems are equivalent, we will show that the set of $\mathcal{A}$-coordinates determines the configuration as a corollary of the fact that the $\mathcal{A}$-coordinates of a local system can reconstruct it.

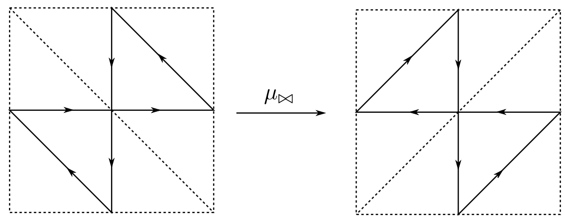


Figure 2.13: The flip mutation sequence for an A_1-type quiver.

3.2.5 Flip mutation sequence

Let P be a quadrilateral with a triangulation Δ, let $n \geq 2$ and let Q be the A_{n-1}-type quiver on P. We now describe a sequence of mutations that transform Q into the A_{n-1}-type quiver Q' associated with the other triangulation Δ' of P. The sequence of mutations is the same as the standard one described in [FG06] Section 10.3, except that at every step we need to apply all the possible $\boxslash$-mutations. We now describe the mutation sequence more precisely. Let (A, B, D, C) be the vertices of P, such that the internal edge of Δ is (B, C). The frozen vertices of Q lies on the edges of P, thus are uniquely determined by a quadruple (a, b, d, c) of integers such that only two of these integers are non-zero, and the remaining two add to n. The two non-zero coordinates indicate the edge of P the vertex is on, and their value indicate its location on the edge. For example, with $n = 4$, the frozen vertex of coordinates $(1, 3, 0, 0)$ is the vertex on the edge AB that is the closest to A. For all pair of positive integers (k, ℓ) with $k + \ell = n$, there are $k \times \ell$ mutable vertices of Q that lies in the interior the rectangle embedded in P whose vertices are $(k, \ell, 0, 0), (k, 0, 0, \ell), (0, \ell, k, 0)$ and $(0, 0, k, \ell)$. We denote this set of vertices of Q by $V_{k,\ell}$. For $1 \leq k \leq n - 1$, the k-th step is doing the $\bowtie$-mutation at every vertex of $V_{k,n-k}$, then doing once every possible $\boxslash$-mutation. The flip mutation sequence is the result of the $n - 1$ steps. See Figures 2.13, 2.14 and 2.15.

3.2.6 Pentagon relation

Let P be a pentagon and Δ a triangulation of P. Let Q be the A_{n-1}-type quiver associated to (P, Δ). The pentagon relation states that after having done the mutations corresponding to five flips (without doing twice the same flip in a row) in Q, the quiver Q and the triangulation Δ are back to their initial state. However, given a quintuplet (A, B, C, D, E) of flags in generic position and a Δ-extra-decoration the $\mathcal{A}$-coordinates associated to this configuration of flags is not back in their initial values after such a sequence of mutation. The reason for this is that the extra-decoration itself changes during the sequence of mutation that lead to a flip in the triangulation, namely during every $\boxslash$-mutation. In fact, after the five flips of the pentagon relation, the basis of a subspace $A_i \cap B_j \cap C_k$ with $i + j + k = 2n + 1$ is in general not mapped to a basis of the same subspace. This is easily seen for $n = 3$:

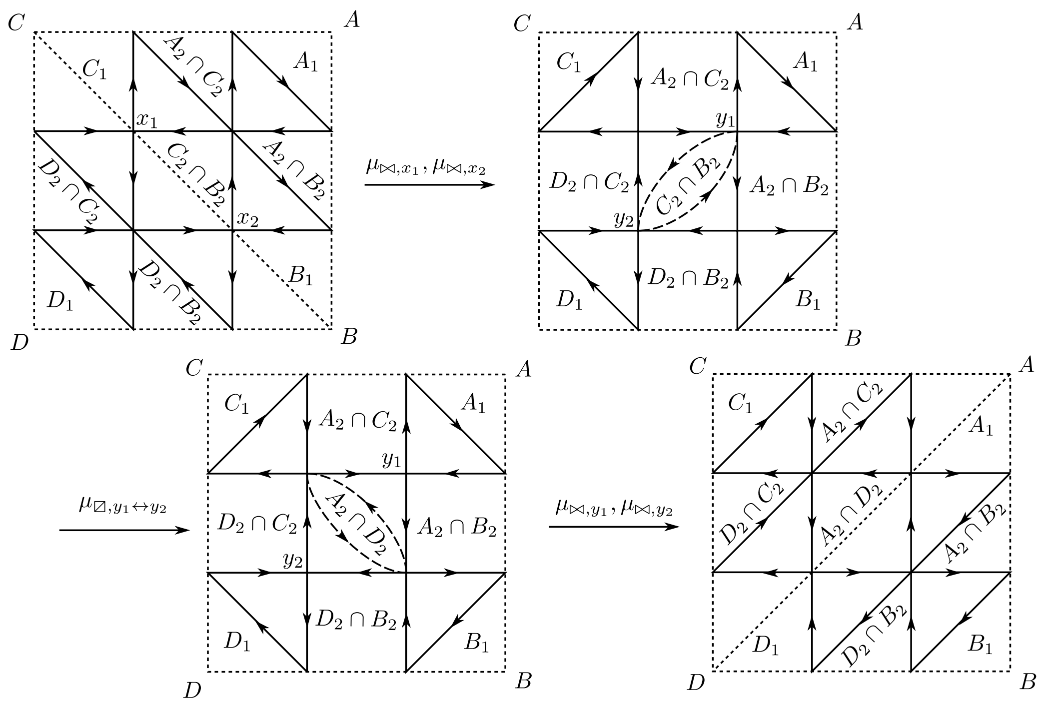


Figure 2.14: The flip mutation sequence for an A_2-type quiver. The anti-oriented cycles are labeled with the R-line associated to them.

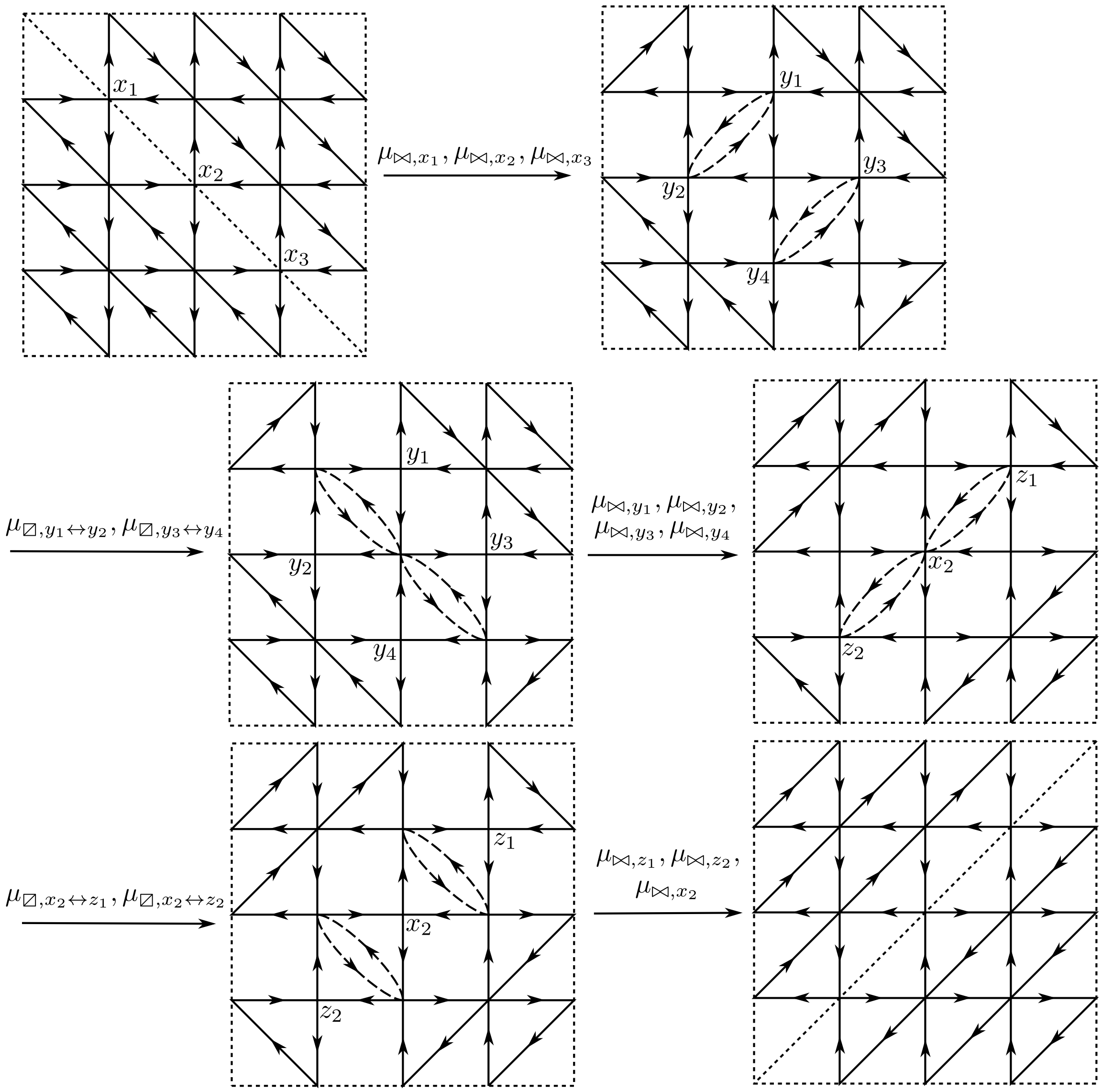


Figure 2.15: The flip mutation sequence for an A_3-type quiver. The first two groups of mutations correspond to the first step, the third and fourth groups of mutations correspond to the second step and the last group of mutations correspond to the third step.

after doing the flips corresponding to the pentagon relation as in Figure 2.16, the vector spanning $A_2 \cap C_2$ is mapped to a vector spanning $A_2 \cap D_2$ and conversely the vector spanning $A_2 \cap D_2$ is mapped to a vector spanning $A_2 \cap C_2$. In general, the pentagon flip sequence acts non-trivially on the extra-decoration of a fixed configuration of flags in generic position.

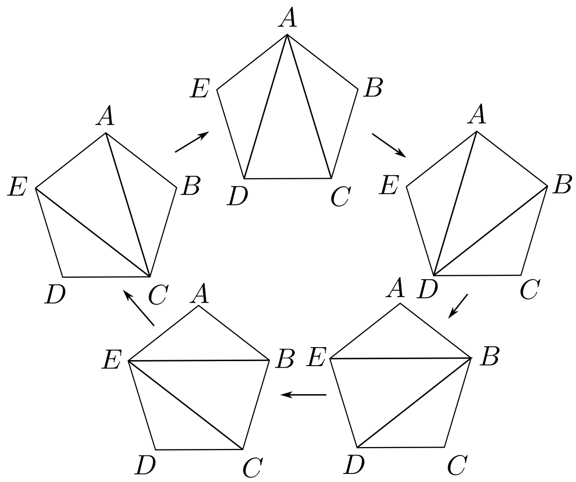


Figure 2.16: The sequence of flips corresponding to the pentagon relation.

3.2.7 Quasideterminants

In this section we show that the $\mathcal{A}$-coordinates of an extra-decorated configuration of flags in R^n are quasideterminants. A general introduction to quasideterminants can be found in [GGRW05]. To get this representation of $\mathcal{A}$-coordinates as quasideterminants, it is necessary to fix a basis of the ambient space. In this section, R^n is endowed with its canonical basis $(e_1, \ldots, e_n)$.

Definition 3.2.21. Let $M \in \mathrm{GL}_n(R)$ and let $1 \leq i, j \leq n$. If the (j,i)-th coefficient of M^{-1} is invertible, we call its inverse the (i,j)-*th quasideterminant* of M. Otherwise, we say the quasideterminant does not exists. Given a matrix M of size $n \times n$ with coefficients in R and $1 \leq i, j \leq n$, we denote by $|M|_{i,j}$ the (i,j)-quasideterminant of M if it exists.

Let P be a k-gon, $k \geq 2$. Let Δ be a triangulation of P and let $(A^{(1)}, \ldots, A^{(k)}) \in \mathrm{DecConf}_k^{\Delta}(n)$. To realize $\mathcal{A}$-coordinates as quasideterminants we need to add a technical hypothesis on the configuration $(A^{(1)}, \ldots, A^{(k)})$: we suppose $(A^{(1)}, \ldots, A^{(k)}, F^{opp})$ is in generic position, where F^{opp} is the opposite standard flag, i.e. $F_i^{opp} = Re_n \oplus \cdots \oplus Re_{i+1}$.

Let $1 \leq i \leq k$. Let $(a_1^{(i)}, \ldots, a_n^{(i)})$ be the decoration of the flag $A^{(i)}$, i.e. $a_j^{(i)}$ freely spans $A_j^{(i)}/A_{j-1}^{(i)}$ for all $1 \leq j \leq n$. For all $1 \leq j \leq n$, choose a lift $\overline{a}_j^{(i)}$ of $a_j^{(i)}$ to $A_j \subset R^n$ (note that

$\overline{a}_1^{(i)} = a_1^{(i)}$). Denote by $M^{(i)}$ the matrix whose columns are $(\overline{a}_1^{(i)}|\dots|\overline{a}_n^{(i)})$. Let $1 \leq r \leq n$ and denote by $M_r^{(i)}$ the $r \times r$ submatrix of $M^{(i)}$ with the first r rows and the first r columns. Since the (p,q)-quasideterminant is invariant under columns operations except for adding a multiple of the q-th column to another, the quasideterminant $m_r^{(i)} = |M_r^{(i)}|_{r,r}$ does not depend on the choice of the lifts $\overline{a}_j^{(i)}$. Note that the fact that $(A^{(i)}, F^{opp})$ is generic ensure the submatrix $M_r^{(i)}$ is invertible, however its (p,r)-quasideterminants may not exist. Still, generically $m_r^{(i)}$ exists for all $1 \leq r \leq n$ and is invertible. Thus, we can consider a modified basis $(\tilde{a}_1^{(i)}, \dots, \tilde{a}_n^{(i)})$ of the flag $A^{(i)}$ given by $\tilde{a}_j^{(i)} = a_j^{(i)}(m_j^{(i)})^{-1}$. Let $\tilde{M}^{(i)}$ be the matrix whose columns are $(\tilde{a}_1^{(i)}|\dots|\tilde{a}_n^{(i)})$. This matrix satisfies $|\tilde{M}_r^{(i)}|_{r,r} = 1$.

Let (i,j,k) be a triangle of Δ, and let p,q,r such that $p+q+r = 2n+1$. Let $b_{p,q,r}$ be the basis of $A_p^{(i)} \cap A_q^{(j)} \cap A_r^{(k)}$ given by the extra-decoration. Then the matrix of $a_{p,q,r}^{A^{(i)}}$ in the bases $b_{p,q,r}$ and $\tilde{a}_p^{(i)}$ is

$$a_{p,q,r}^{A^{(i)}} = \left|\left(\tilde{a}_1^{(i)}|\dots|\tilde{a}_{p-1}^{(i)}|b_{p,q,r}\right)\right|_{p,p}.$$

3.3 Coordinates on extra-decorated local systems: the general case

In this section we extend the construction of $\mathcal{A}$-coordinates defined in Section 3.2 to twisted local systems on ciliated surfaces. Everything works similarly, except for one thing: the data of an extra-decoration on a twisted local system induce a technical sign difficulty. Indeed, because of the definition of the partial abelianization procedure, it is natural to assign the additional bases needed to an extra-decoration to the white vertices of the graph Γ_n. This is the purpose of the decorating curves introduced in Section 2.3.2. However, since the local system is twisted, the monodromy of a loop going once around a white vertex of Γ_n is $-\operatorname{Id}$ so a section of a line bundle over such a loop is only defined up to sign. We deal with this difficulty by instead taking a pair of sections summing to zero. Most $\mathcal{A}$-coordinates become then a pair of coordinates with opposite signs. Then being careful with the cyclic ordering around the white vertices, we can still compute the mutations for those coordinates.

In all this section, all triangulations of an hyperbolic ciliated surface S will be without self-folded triangles, i.e. triangles with two edges identified together in S.

3.3.1 Decoration and extra-decoration

Let $k \geq 2$ and let S be a ciliated surface. Let $\mathcal{L}$ be a framed twisted $\mathrm{GL}_n(R)$-local system on S, and let Δ be a triangulation of S. Recall the decorating paths defined in Section 2.3.2. Let v be a white vertex of Γ_n and let i such that $v \in \gamma_p^{(i)}$. We use parallel transport along the path $\pi(T'\gamma_{p,v})$ to transport the i-dimensional subbundle $F_i^{(p)}$ along $T'\beta_v$. Since the monodromy around the loop going once around v is $-\mathrm{Id}$, this defines a subbundle of $\mathcal{L}|_{T'\beta_v}$. If v and v' are grouped together, i.e. $\beta_v = \beta_{v'}$, then the paths $T'\gamma_{p,v}$ and $T'\gamma_{p,v'}$ are homotopic so the respective parallel transport of $F_p^{(i)}$ induce the same subbundle over $T'\beta_v$.

Definition 3.3.1. Let $\mathcal{L}$ be a decorated twisted $\mathrm{GL}_n(R)$-local system on S and let Δ be a triangulation of S. Let v be a white vertex of Γ_n. Let p, q, r be the punctures at the vertex of the triangle t of Δ containing v. Let $1 < i, j, k < n$ such that $v \in \gamma_p^{(i)} \cap \gamma_q^{(j)} \cap \gamma_r^{(k)}$ (i.e. the coordinates of v in the triangle t are (i, j, k)). The three subbundles $F_i^{(p)}, F_j^{(q)}, F_k^{(r)}$ are well defined along $\pi(T'\beta_v)$, so we can define the 1-dimensional subbundle $F_i^{(p)} \cap F_j^{(q)} \cap F_k^{(r)} \subset \mathcal{L}|_{\pi(T'\beta_v)}$ which we denote by F_v or $F_{i,j,k}$. A Δ-*extra-decoration* of $\mathcal{L}$ with respect to the graph Γ_n is the data for each white vertex v of Γ_n of a flat section b_v of F_v that freely spans F_v. The data of a decorated twisted $\mathrm{GL}_n(R)$-local system on S together with a Δ-extra-decoration is called a Δ-*extra-decorated twisted* $\mathrm{GL}_n(R)$-*local system* on S.

A Δ-extra-decoration of a twisted $\mathrm{GL}_n(R)$-local system depends on the graph Γ_n, which depends on the triangulation Δ.

3.3.2 $\mathcal{A}$-coordinates

Let S be a ciliated surface, Δ a triangulation of S, $n \geq 2$ and Γ_n the associated bipartite graph. Let $\mathcal{L}$ be a Δ-extra-decorated twisted $\mathrm{GL}_n(R)$-local system on S. Let $\mathcal{E}$ be the abelianized $R^\times$-local system on Σ_n constructed in Section 2.5. The decoration and Δ-extra-decoration of $\mathcal{L}$ induce along all the decorating curves of Σ_n a flat section of $\mathcal{E}$. Indeed by construction, $\mathcal{E}$ is isomorphic to F_v in the neighborhood of a non-peripheral white vertex v in Σ_n, and isomorphic to $F_i^{(p)}/F_{i-1}^{(p)}$ in the neighborhood of the i-th lift p_i of a puncture p.

Let v be a white vertex of Γ_n, p, q, r the punctures at the vertices of the triangle containing v and (i, j, k) the coordinates of v in the triangle (p, q, r). Without loss of generality suppose $i > 1$. Let $\gamma_{p,v}$ be a joining path from $T'\beta_{p_i}$ to $T'\beta_v$. The fiber of $\mathcal{E}$ above the end point $T'\gamma_{p,v}(1)$ has a single basis given by the decoration of $\mathcal{L}$, namely $b_i^{(p)}$.

If the vertex v is peripheral, for instance if v is on $\gamma_q^{(1)}$, then the fiber of $\mathcal{E}$ above $T'\beta_v$ also have a single basis given by the decoration, namely $b_1^{(q)}$. Let $a_{i,j,k}^{(p)}$ be the holonomy in $\mathcal{E}$ of the path $T'\gamma_{p,v}$ in the bases $b_1^{(q)}$ and $b_i^{(p)}$. The $\mathcal{A}$-coordinate associated to the couple (p, v) is the singleton

$$\mathbf{a}_{i,j,k}^{(p)} = \left\{ a_{i,j,k}^{(p)} \right\}.$$

If the vertex v is not peripheral, then the fiber of $\mathcal{E}$ above the starting point $T'\gamma_{p,v}(0)$ have *two* bases given by $b_v(x_1)$ and $b_v(x_2)$, where $x_1, x_2 \in E(v)$ are such that $T'\gamma_{p,v}(0) =$

$T'\beta_v(x_1) = T'\beta_v(x_2)$. They satisfy $b_v(x_1) + b_v(x_2) = 0$. Let $a^{(p)}_{i,j,k}(x_r)$ be the holonomy in $\mathcal{E}$ of the path $T'\gamma_{p,v}$ in the bases $b_v(x_r)$ and $b^{(p)}_i$ for $r = 1, 2$. The $\mathcal{A}$-coordinate associated to the couple (p, v) is the pair

$$\mathbf{a}^{(p)}_{i,j,k} = \left\{ a^{(p)}_{i,j,k}(x_1), a^{(p)}_{i,j,k}(x_2) \right\}.$$

This pair satisfies $a^{(p)}_{i,j,k}(x_1) + a^{(p)}_{i,j,k}(x_2) = 0$ but there is no canonical choice of a ”positive” coordinate.

Note that on a white vertex v of coordinates (i, j, k) in the triangle (p, q, r) the map $\sigma : E(v) \to E(v)$ defined in Section 2.3.2 induce a cyclic order on the $\mathcal{A}$-coordinates associated to $(p, v), (q, v)$ and (r, v). Again, we see this cyclic order on the coordinates as a map σ such that $\sigma^{\# E(v)} = \mathrm{Id}$. For instance if $x_1 \in E(v)$ is such that $T'\gamma_{p,v}(1) = x_1$ and $\sigma(x_1) = x_2 = T'\gamma_{q,v}(1)$ then $\sigma(a^{(p)}_{i,j,k}(x_1)) = a^{(q)}_{i,j,k}(x_2)$.

Proposition 3.3.2. *Let S be a ciliated surface, Δ a triangulation of S, $n \geq 2$ and $t = (p, q, r)$ a triangle of Δ. Let $\mathcal{L}$ be a Δ-extra-decorated twisted $\mathrm{GL}_n(R)$-local system on S. Let $i, j, k \in \leq n + 1$ s.t. $i + j + k = 2n + 2$ and let $\underline{a}^{(q)}_{i-1,j,k} \in \mathbf{a}^{(q)}_{i-1,j,k}$, $\underline{a}^{(p)}_{i,j,k-1} \in \mathbf{a}^{(p)}_{i,j,k-1}$ and $\underline{a}^{(r)}_{i,j-1,k} \in \mathbf{a}^{(r)}_{i,j-1,k}$. The $\mathcal{A}$-coordinates satisfy the following relations, called* triangle relations*:*

$$\sigma(\underline{a}^{(q)}_{i-1,j,k}).\left(\underline{a}^{(q)}_{i-1,j,k}\right)^{-1}.\sigma(\underline{a}^{(p)}_{i,j,k-1}).\left(\underline{a}^{(p)}_{i,j,k-1}\right)^{-1}.\sigma(\underline{a}^{(r)}_{i,j-1,k}).\left(\underline{a}^{(r)}_{i,j-1,k}\right)^{-1} = 1$$

Proof. These coordinates correspond to holonomies of paths in the $R^\times$-twisted local system $\mathcal{E}$ on Σ_n. The left-hand term correspond to the holonomy along a contractible loop in $T'\Sigma_n$, thus is trivial. □

3.3.3 Quiver and twisted affectations

Let Q be a triangulated quiver on S. To account for the $\mathcal{A}$-coordinates being defined up to sign, we slightly alter the definition of an affectation of Q. For this, the cycles in the quiver that are oriented with the opposite orientation to S will carry additional variables. Unfortunately, dealing with this sign indeterminacy greatly complexify the notations, even though the situation is very similar to the case of configurations of flags exposed in Section 3.2.3. All the following definitions are adaptations of the definitions in Section 3.2.3 to the case where many of the coordinates are defined ”up to sign”.

Definition 3.3.3. Let S be a marked surface. An *triangulated quiver* on S is a quiver Q embedded in S such that:

- Q has no trivial loops

- The (possibly empty) subset of vertices of Q lying on the boundary of S are called frozen.
- Each vertex of Q has as many arrows going in and going out of it. We define the *valency* of a vertex v, denoted by $\mathrm{val}(v)$, to be the number of arrows going to v if v is not frozen and this number plus one if v is frozen
- There are two types of arrows in Q, they are called either plain or dashed
- At each vertex of Q, there exists a pairing between arrows going in and arrows going out, such that each pair of arrows is part of an oriented triangle for which the orientation agrees with the orientation on S (in the following, an *oriented cycle* in an embedded quiver will always suppose matching orientation with the surface), and there are no edges inside such oriented triangles
- If an arrow $v \to v'$ is dashed, then there is another dashed arrow $v' \to v$ and those are the only arrows between v and v'
- Each oriented cycle of plain arrows of Q has an internal oriented triangulation of dashed arrows, and every dashed arrow belongs to one such triangulation
- If there is a plain arrow $v \to v'$, then there is no arrow $v' \to v$
- There is at most one arrow from a vertex v to another vertex v'

Definition 3.3.4. Let Q be a triangulated quiver on S. The complement of Q in S is a disjoint union of open disks, open punctured half-disks and open punctured disks, all with an oriented boundary. We call a cycle in Q with opposite orientation to the one on S which is the boundary of a disk without puncture an *anti-cycle* of Q.

Remark 3.3.5. Let Q be a triangulated quiver on S. Similarly to Remark 3.2.19, the complement of Q in S has three types of connected components:

- interior of oriented triangles, which carry triangle relations
- interior of cycles with opposite orientation to S (including the regions that contain portions of the boundary of S without punctures), which will carry coordinates with sign indeterminacy
- Punctured disks or half-disks on the boundary of S, which will carry coordinates with well-defined signs

A *twisted affectation* of Q is an affectation of Q, except the variables $a_1(v), \dots, a_r(v) \in R^\times$ affected to a vertex v of Q are replaced by sets $\mathbf{a}_1, \dots, \mathbf{a}_r$ of cardinal either 1 or 2. Moreover, we add the data of the sign indeterminacy: for each anti-cycle $c = (v_1, \dots, v_r)$ of Q and for each vertex v_k of c there is exactly one coordinate $\mathbf{a}_{i_k}(v_k)$ in the interior of c, i.e. such that $t_k < \mathbf{a}_{i_k}(v_k) < t_{k+1}$ where t_k (resp. t_{k+1}) is the triangle of Q containing (v_{k-1}, v_k) (resp. (v_k, v_{k+1})). We set for each vertex v_k of c a bijection $\sigma_k : \mathbf{a}_{i_k}(v_k) \to \mathbf{a}_{i_{k-1}}(v_{k-1})$ such that $\sigma^r = \tau_k$, where τ_k is either $\mathrm{Id} : \mathbf{a}_{i_k}(v_k) \to \mathbf{a}_{i_k}(v_k)$ if $\#\mathbf{a}_{i_k}(v_k) = 1$ or the transposition if $\#\mathbf{a}_{i_k}(v_k) = 2$, and where σ^r is a slight abuse of notation made by omitting the subscript k of the corresponding variable.

Remark 3.3.6. For any consecutive coordinates $\mathbf{a} \to \mathbf{b}$ inside an anti-cycle the product $\sigma(\underline{a})\underline{a}^{-1}$ does not depend on the choice of $\underline{a} \in \mathbf{a}$. Indeed, the choice of $\underline{a} \in \mathbf{a}$ will change the sign of both $\underline{a}$ and $\sigma(\underline{a}) \in \mathbf{b}$. To simplify the notation in the formulas, we will write $\mathbf{ba}^{-1}$ instead of $\sigma(\underline{a})\underline{a}^{-1}$. Similarly, we will write $\mathbf{ab}^{-1}$ instead of $\underline{a}\sigma(\underline{a})^{-1}$. These products are equal to Goncharov-Kontsevich coordinates defined in [GK22].

Remark 3.3.7. When defining the map σ, it is sufficient to define the image of one element inside every coordinate. Indeed, either in every coordinate there is at most two elements, and knowing the image of one element is enough to determine a bijection between sets of cardinal 1 or 2.

Definition 3.3.8. Let Q be a triangulated quiver on S and $\mathcal{V}$ a twisted affectation of Q. We say $\mathcal{V}$ satisfies the triangle relations if for every oriented triangle t of Q, we have

$$(\mathbf{c}_1\mathbf{b}_2^{-1})(\mathbf{a}_1\mathbf{c}_2^{-1})(\mathbf{b}_1\mathbf{a}_2^{-1}) = 1$$

where the variable are defined as in Figure 3.17.

A *twisted seed* is the data of a couple $(Q, \mathcal{V})$ where Q is a triangulated quiver on S and $\mathcal{V}$ is a twisted affectation on Q satisfying the triangle relations.

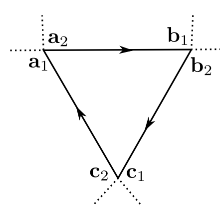


Figure 3.17: The variable around an oriented triangle of Q.

Definition 3.3.9. Let $(Q, \mathcal{V})$ be a twisted seed on S. Let x be a non-frozen vertex of Q of valency 2 and let a, b, c, d be the surrounding vertices as in Figure 3.18. Let $\underline{b}_1 \in \mathbf{b}_1$ and $\underline{c}_2 \in \mathbf{c}_2$. We say that x is mutable (or that the $\bowtie$-mutation at x is *admissible*) if the elements

$$\underline{x}'_1(\underline{b}_1) = \mathbf{c}_1\mathbf{x}_2^{-1}\underline{b}_1 + \mathbf{d}_2\mathbf{x}_1^{-1}\sigma(\underline{b}_1)$$
$$\underline{x}'_2(\underline{c}_2) = \mathbf{b}_2\mathbf{x}_1^{-1}\underline{c}_2 + \mathbf{a}_1\mathbf{x}_2^{-1}\sigma(\underline{c}_2)$$

are invertible in R, were the variables $\mathbf{a}_i, \mathbf{b}_i, \mathbf{c}_i, \mathbf{d}_i, \mathbf{x}_i$ are defined as in Figure 3.18. If x is mutable, we set

$$\mathbf{x}'_1 = \left\{ \underline{x}'_1(\underline{b}_1) \mid \underline{b}_1 \in \mathbf{b}_1 \right\}$$
$$\mathbf{x}'_2 = \left\{ \underline{x}'_2(\underline{c}_2) \mid \underline{c}_2 \in \mathbf{c}_2 \right\}$$

Note that if there are two elements in $\mathbf{b}_1$ (resp. $\mathbf{c}_2$), then there are also two elements in $\mathbf{x}'_1$ (resp. $\mathbf{x}'_2$).

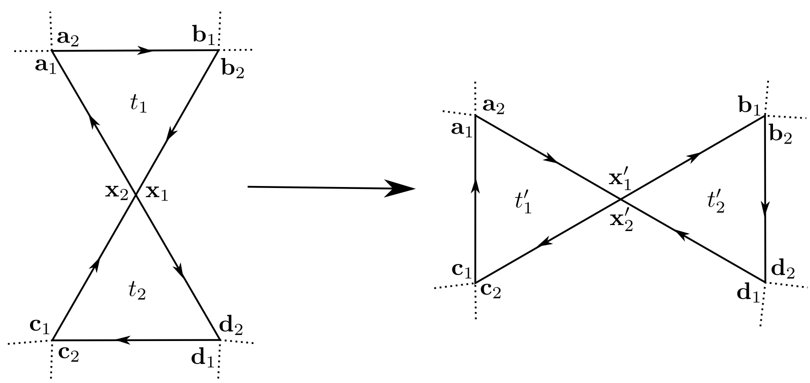


Figure 3.18: The situation before and after the $\bowtie$-mutation at the vertex x.

Remark 3.3.10. Each of the element in $\mathbf{x}_1$ (resp. $\mathbf{x}_2$) depends on the choice of an element in $\mathbf{b}_1$ (resp. $\mathbf{c}_2$). However the coordinate $\mathbf{x}'_1$ (resp. $\mathbf{x}'_2$) which is either a singleton or a pair depends only on $\mathbf{b}_1$ (resp. $\mathbf{c}_2$).

Definition 3.3.11. Let $(Q, \mathcal{V})$ be a twisted seed on S. Let x be a non-frozen vertex of Q of valency 2. If x is mutable, the $\bowtie$-*mutation* of $(Q, \mathcal{V})$ at x is a new seed $(Q', \mathcal{V}') = \mu_{\bowtie,x}(Q, \mathcal{V})$ defined as follows:

- The new quiver Q' is obtained by reversing the direction of the arrows incident to x and adding the four plain arrows needed to complete the triangles created. If this process creates a pair of plain arrows between two vertex with opposite directions, then if this pair lies inside an oriented plain cycle of the quiver we change the pair of arrows to be dashed, and if not we remove the pair of arrows.

- The two $\mathcal{A}$-coordinates $\mathbf{x}_1, \mathbf{x}_2$ at the vertex x are replaced by the new variables $\mathbf{x}'_1$ and $\mathbf{x}'_2$ defined above. The new cyclic ordering at the vertex x is described on Figure 3.18.

- The new map σ' is defined as follows:

$$\sigma'(\underline{a}_1) = \sigma^2(\underline{a}_1) \in \mathbf{c}_1, \underline{a}_1 \in \mathbf{a}_1$$

$$\sigma'(\underline{d}_2) = \sigma^2(\underline{d}_2) \in \mathbf{b}_2, \underline{d}_2 \in \mathbf{d}_2$$

$$\sigma'(\underline{b}_1) = \underline{x}'_1(\underline{b}_1), \sigma'(\underline{x}'_1(\underline{b}_1)) = \sigma(\underline{b}_1)$$

$$\sigma'(\underline{c}_2) = \underline{x}'_2(\underline{c}_2), \sigma'(\underline{x}'_2(\underline{c}_2)) = \sigma(\underline{c}_2)$$

and σ' coincides with σ for any coordinate $\mathbf{y} \notin \{\mathbf{a}_1, \mathbf{d}_2, \mathbf{b}_1, \mathbf{c}_2, \mathbf{x}'_1, \mathbf{x}'_1\}$.

Definition 3.3.12. Let $(Q, \mathcal{V})$ be a seed on S. Let a and d two vertex of Q with a pair of dashed arrows between them. Let $b, c \in Q_0$ such that the oriented triangle containing $d \to a$ (resp. $a \to d$) is $t_1 = d \to a \to b \to d$ (resp. $t_2 = a \to d \to c \to a$), see Figure 3.19. The $\boxslash$-*mutation* of $(Q, \mathcal{V})$ at the pair of arrows $a \leftrightarrow c$ is a new seed $(Q', \mathcal{V}') = \mu_{\boxslash, a \leftrightarrow d}(Q, \mathcal{V})$ defined as follows:

- The pair of dashed arrow is part of an internal triangulation of a plain oriented cycle of length 4. The new quiver Q' is obtained by doing a flip of this edge in the internal triangulation:

$$Q'_1 = (Q_1 \setminus \{a \to d, d \to a\}) \cup \{b \to c, c \to b\}.$$

  The new arrows are also dashed.

- The valency of a and c is decreased by 1 so the variables $\mathbf{a}_3$ and $\mathbf{d}_3$ in between the removed dashed arrows are also removed. We then need to add two new variables in between the new pair of dashed arrows, see Figure 3.19. The new added variable are:

$$c_3(\underline{a}_3) = \mathbf{d}_1 \mathbf{b}_2^{-1} \underline{a}_3 = \mathbf{a}_2 \mathbf{b}_1^{-1} \sigma(\underline{a}_3), \underline{a}_3 \in \mathbf{a}_3$$

$$b_3(\underline{d}_3) = \mathbf{a}_1 \mathbf{c}_2^{-1} \underline{d}_3 = \mathbf{d}_2 \mathbf{c}_1^{-1} \sigma(\underline{d}_3), \underline{d}_3 \in \mathbf{d}_3$$

- The new map σ' is defined as follows:

$$\sigma'(\underline{c}_3(\underline{a}_3)) = \underline{b}_3(\sigma(\underline{a}_3)) \in \mathbf{b}_3$$

$$\sigma'(\underline{b}_3(\underline{d}_3)) = \underline{c}_3(\sigma(\underline{d}_3)) \in \mathbf{c}_3$$

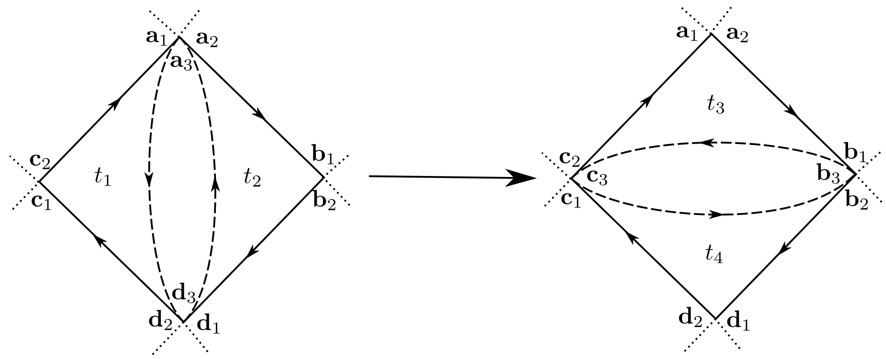


Figure 3.19: The situation before and after the ⊠-mutation in an oriented quadrilateral of Q.

Proposition 3.3.13. *If $(Q, \mathcal{V})$ is a twisted seed on S, then the pair $(Q', \mathcal{V}')$ obtained after any admissible $\bowtie$-mutation or any ⊠-mutation is a twisted seed on S.*

3.3.4 Twisted affectation associated to an extra-decorated twisted local system

Let S be a ciliated surface and let Δ be a triangulation of S. Let $n \geq 2$ and let Q be the A_{n-1}-type quiver on (S, Δ). Let t be a triangle of Δ and let (p, q, r) be the punctures at the vertices of t. Let $\mathcal{L}$ be a Δ-extra-decorated twisted local system on S.

Inside t, every vertex of Q is determined by a triple of positive integers (i, j, k) such that $i + j + k = 2n$ and $i, j, k \leq n$, where i (resp. j, k) is the distance of the vertex to the puncture p (resp. q, r). We denote the vertex with coordinates (i, j, k) by $v_{i,j,k}$. The two vertices closest to p are then $v_{1,n-1,n}$ and $v_{1,n,n-1}$.

Every non-frozen vertex $v_{i,j,k}$ of Q that is not on an edge of t has valency three, and the three oriented triangles containing $v_{i,j,k}$ are:

$$t^p_{i,j,k} = v_{i,j,k} \to v_{i-1,j+1,k} \to v_{i-1,j,k+1} \to v_{i,j,k}$$

$$t^q_{i,j,k} = v_{i,j,k} \to v_{i,j-1,k+1} \to v_{i+1,j-1,k} \to v_{i,j,k}$$

$$t^r_{i,j,k} = v_{i,j,k} \to v_{i+1,j,k-1} \to v_{i,j+1,k-1} \to v_{i,j,k}$$

We assign to $v_{i,j,k}$the coordinates $\mathbf{a}^{(p)}_{i+1,j,k}, \mathbf{a}^{(q)}_{i,j+1,k}, \mathbf{a}^{(r)}_{i,j,k+1}$ with the following cyclic order:

$$\mathbf{a}^{(p)}_{i+1,j,k} < t^r_{i,j,k} < \mathbf{a}^{(q)}_{i,j+1,k} < t^p_{i,j,k} < \mathbf{a}^{(r)}_{i,j,k+1} < t^q_{i,j,k} < \mathbf{a}^{(p)}_{i+1,j,k}$$

To remember this, $t^p_{i,j,k}$ is the triangle the closest to p and $\mathbf{a}^{(p)}_{i,j,k+1}$ is the coordinate the farthest from p, with a similar statement for the other triangles/coordinates.

If one of the edges of t is a boundary component then the vertices of Q that lies on this edge are frozen. Suppose $v_{i,j,k}$ is on the boundary between p and q (which means that $k = n$ and $i + j = n$), the other cases being symmetric. Then $v_{i,j,n}$ is part of exactly one oriented triangle $t^r_{i,j,n}$ of Q. Let $b_{p,q}$ be the boundary between p and q. We assign to $v_{i,j,n}$ the variables $\mathbf{a}^{(p)}_{i+1,j,n}$ and $\mathbf{a}^{(q)}_{i,j+1,n}$ with the following cyclic ordering:

$$\mathbf{a}^{(p)}_{i+1,j,n} < t^r_{i,j,n} < \mathbf{a}^{(q)}_{i,j+1,n} < b_{A,B} < \mathbf{a}^{(p)}_{i+1,j,n}.$$

It remains to assign variables to non-frozen vertices that lie on an edge of t, which is then an edge shared by two triangles $t = (p, q, r)$ and $t' = (q, p, s)$, so the cyclic order on the punctures is (p, s, q, r). Such a vertex v has coordinates (i, j, n) in t with $i + j = n$ and coordinates (j, i, n) in t'. The two oriented triangles of Q containing v are $t^r_{i,j,n}$ in t and $t'^s_{j,i,n}$ in t'. We assign to v the variables $\mathbf{a}^{(p)}_{i+1,j,n}(t) = \mathbf{a}^{(p)}_{j,i+1,n}(t')$ and $\mathbf{a}^{(q)}_{i,j+1,n}(t) = \mathbf{a}^{(q)}_{j+1,i,n}(t')$ with the following cyclic ordering:

$$\mathbf{a}^{(p)}_{i+1,j,n}(t) < t^r_{i,j,n} < \mathbf{a}^{(q)}_{i,j+1,n} < t'^s_{j,i,n} < \mathbf{a}^{(p)}_{i+1,j,n}(t).$$

As an immediate corollary from Proposition 3.3.2, we have:

Proposition 3.3.14. *The set of variables described above is an affectation of Q and satisfy the triangle relations.*

Remark 3.3.15. The coordinates defined by Goncharov and Kontsevich in [GK22] are products of two of the $\mathcal{A}$-coordinates described here, as in Remark 3.3.6. As such, they do not require an extra-decoration but only a decoration, and the sign indeterminacy doesn't appear. However, the mutations formulas described in [GK22] are closer to the mutations formulas for $\mathcal{Q}$-coordinates described in Section 3.1.5.

Further considerations

Here is a non-exhaustive list of directions in which expand the work presented in this manuscript, as well as question arising naturally:

- Is it possible to have a geometric interpretation of the $\mathcal{A}$-coordinates obtained by mutations of the quiver that are not part of the flip sequence? Without such interpretation, we would need another approach to define a generalization of Berenstein-Retakh algebra for coordinates on $\mathrm{GL}_n(R)$-local systems.

- Is there similar non-commutative coordinates on local systems for other Lie groups and/or other parabolic subgroups? A particularly interesting case of this would be for $G = SO(p, q)$, $p \neq q$ with a framing which takes values in the flag variety G/P_Θ, where P_Θ is the parabolic subgroup such that (G, P_Θ) admits a Θ-positive structure. Such coordinates would mix both commutative and non-commutative behaviors.

- Is there an additional structure on the space of $\mathcal{A}$-coordinates, such as the symplectic structure on the space of Fock-Goncharov $\mathcal{A}$-coordinates or the non-commutative 2-form on the space of Goncharov-Kontsevich coordinates?

- Is it possible to describe the elementary mutations as "mutations" of the spectral surface and of the spectral network, such that the intermediary $\mathcal{A}$-coordinates arising in the middle of the flip mutation sequence would be also holonomies of certain paths in the abelianized local system?

- Does the partial abelianization procedure also hold for local systems whose group is not of type A? Generalizations of spectral networks and abelianization already exists for such groups, see [IM21].

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