Weyl laws and closed geodesics on typical manifolds

Joint with Y. Canzani

Jeffrey Galkowski

Typicality?:

Typicality?: predominance

Typicality?: predominance

Definition

$\mathscr{G}$ (open subset of) Banach space. $G \subset \mathscr{G}$ is predominant if

Typicality?: predominance

Definition

$\mathscr{G}$ (open subset of) Banach space. $G \subset \mathscr{G}$ is predominant if

- for each g there is a submanifold $\mathcal{L}_g \subset \mathscr{G}$ with Borel measure μ_g:

$$G \cap \mathcal{L}_g \text{ has **full** measure.}$$

Typicality?: predominance

Definition

$\mathscr{G}$ (open subset of) Banach space. $G \subset \mathscr{G}$ is predominant if

- for each g there is a submanifold $\mathcal{L}_g \subset \mathscr{G}$ with Borel measure μ_g:

$$G \cap \mathcal{L}_g \text{ has **full** measure.}$$

- $g \in \mathcal{L}_g$ and $g \mapsto \mathcal{L}_g$ is C^1.

Typicality?: predominance

Definition

$\mathscr{G}$ (open subset of) Banach space. $G \subset \mathscr{G}$ is predominant if

- for each g there is a submanifold $\mathcal{L}_g \subset \mathscr{G}$ with Borel measure μ_g:

$$G \cap \mathcal{L}_g \text{ has } \textbf{full} \text{ measure.}$$

- $g \in \mathcal{L}_g$ and $g \mapsto \mathcal{L}_g$ is C^1.
- $\mu_g(U_g) > 0$ for every open neighborhood of g.

Typicality?: predominance

Definition

$\mathscr{G}$ (open subset of) Banach space. $G \subset \mathscr{G}$ is predominant if

- for each g there is a submanifold $\mathcal{L}_g \subset \mathscr{G}$ with Borel measure μ_g:

$$G \cap \mathcal{L}_g \text{ has } \textbf{full} \text{ measure.}$$

- $g \in \mathcal{L}_g$ and $g \mapsto \mathcal{L}_g$ is C^1.
- $\mu_g(U_g) > 0$ for every open neighborhood of g.

Properties.

Typicality?: predominance

Definition

$\mathscr{G}$ (open subset of) Banach space. $G \subset \mathscr{G}$ is predominant if

- for each g there is a submanifold $\mathcal{L}_g \subset \mathscr{G}$ with Borel measure μ_g:

$$G \cap \mathcal{L}_g \text{ has } \textbf{full} \text{ measure.}$$

- $g \in \mathcal{L}_g$ and $g \mapsto \mathcal{L}_g$ is C^1.
- $\mu_g(U_g) > 0$ for every open neighborhood of g.

Properties.

- predominant sets are dense

Typicality?: predominance

Definition

$\mathscr{G}$ (open subset of) Banach space. $G \subset \mathscr{G}$ is predominant if

- for each g there is a submanifold $\mathcal{L}_g \subset \mathscr{G}$ with Borel measure μ_g:

$$G \cap \mathcal{L}_g \text{ has } \textbf{full} \text{ measure.}$$

- $g \in \mathcal{L}_g$ and $g \mapsto \mathcal{L}_g$ is C^1.
- $\mu_g(U_g) > 0$ for every open neighborhood of g.

Properties.

- predominant sets are dense
- intersection of predominant sets are predominant

Typicality?: predominance

Definition

$\mathscr{G}$ (open subset of) Banach space. $G \subset \mathscr{G}$ is predominant if

- for each g there is a submanifold $\mathcal{L}_g \subset \mathscr{G}$ with Borel measure μ_g:

$$G \cap \mathcal{L}_g \text{ has } \textbf{full} \text{ measure.}$$

- $g \in \mathcal{L}_g$ and $g \mapsto \mathcal{L}_g$ is C^1.
- $\mu_g(U_g) > 0$ for every open neighborhood of g.

Properties.

- predominant sets are dense
- intersection of predominant sets are predominant
- in finite dimensions, predominant sets have full measure

Typicality?: predominance

Definition

$\mathscr{G}$ (open subset of) Banach space. $G \subset \mathscr{G}$ is predominant if

- for each g there is a submanifold $\mathcal{L}_g \subset \mathscr{G}$ with Borel measure μ_g:

$$G \cap \mathcal{L}_g \text{ has } \textbf{full} \text{ measure.}$$

- $g \in \mathcal{L}_g$ and $g \mapsto \mathcal{L}_g$ is C^1.
- $\mu_g(U_g) > 0$ for every open neighborhood of g.

Properties.

- predominant sets are dense
- intersection of predominant sets are predominant
- in finite dimensions, predominant sets have full measure

Note. We will work with $\mathscr{G}$ being the space of Riemannian metrics over a manifold M.

Counting Geodesics

(M^n, g) compact, no boundary.

Counting Geodesics

(M^n, g) compact, no boundary.

$$\mathfrak{c}(T, M, g) := \#\{\gamma \text{ a primitive, periodic, unit speed geodesic of length } \leq T\}.$$

Counting Geodesics

(M^n, g) compact, no boundary.

$$\mathfrak{c}(T, M, g) := \#\{\gamma \text{ a primitive, periodic, unit speed geodesic of length } \leq T\}.$$

- $\mathfrak{c}(T, M, g) < \infty$ for a Baire generic g [Abraham '70, Anosov '82]

Counting Geodesics

(M^n, g) compact, no boundary.

$$\mathfrak{c}(T, M, g) := \#\{\gamma \text{ a primitive, periodic, unit speed geodesic of length } \leq T\}.$$

- $\mathfrak{c}(T, M, g) < \infty$ for a Baire generic g [Abraham '70, Anosov '82]
- $\mathfrak{c}(T, M, g) \to \infty$ for a Baire generic g [Hingston '84]

Counting Geodesics

(M^n, g) compact, no boundary.

$$\mathfrak{c}(T, M, g) := \#\{\gamma \text{ a primitive, periodic, unit speed geodesic of length } \leq T\}.$$

- $\mathfrak{c}(T, M, g) < \infty$ for a Baire generic g [Abraham '70, Anosov '82]
- $\mathfrak{c}(T, M, g) \to \infty$ for a Baire generic g [Hingston '84]
- $\mathfrak{c}(T, M, g) \geq ce^{cT}$ for an open dense set of g [Contreras '10]

Counting Geodesics

(M^n, g) compact, no boundary.

$$\mathfrak{c}(T, M, g) := \#\{\gamma \text{ a primitive, periodic, unit speed geodesic of length } \leq T\}.$$

- $\mathfrak{c}(T, M, g) < \infty$ for a Baire generic g [Abraham '70, Anosov '82]
- $\mathfrak{c}(T, M, g) \to \infty$ for a Baire generic g [Hingston '84]
- $\mathfrak{c}(T, M, g) \geq ce^{cT}$ for an open dense set of g [Contreras '10]
- $\mathfrak{c}(T, M, g) \sim ce^{hT}$ for g with negative curvature [Bowen '72]

Counting Geodesics

(M^n, g) compact, no boundary.

$$\mathfrak{c}(T, M, g) := \#\{\gamma \text{ a primitive, periodic, unit speed geodesic of length } \leq T\}.$$

- $\mathfrak{c}(T, M, g) < \infty$ for a Baire generic g [Abraham '70, Anosov '82]
- $\mathfrak{c}(T, M, g) \to \infty$ for a Baire generic g [Hingston '84]
- $\mathfrak{c}(T, M, g) \geq ce^{cT}$ for an open dense set of g [Contreras '10]
- $\mathfrak{c}(T, M, g) \sim ce^{hT}$ for g with negative curvature [Bowen '72]

What about *quantitative* upper bounds for 'typical' g?

Counting Geodesics

(M^n, g) compact, no boundary.

$$\mathfrak{c}(T, M, g) := \#\{\gamma \text{ a primitive, periodic, unit speed geodesic of length } \leq T\}.$$

- $\mathfrak{c}(T, M, g) < \infty$ for a Baire generic g [Abraham '70, Anosov '82]
- $\mathfrak{c}(T, M, g) \to \infty$ for a Baire generic g [Hingston '84]
- $\mathfrak{c}(T, M, g) \geq ce^{cT}$ for an open dense set of g [Contreras '10]
- $\mathfrak{c}(T, M, g) \sim ce^{hT}$ for g with negative curvature [Bowen '72]

What about *quantitative* upper bounds for 'typical' g?

Theorem (Canzani–G ('22))

Let M be a smooth manifold of dimension n. Then for all $\nu \geq 5$ the set of metrics, $g \in \mathcal{C}^\nu$, such that there is $C > 0$

$$\mathfrak{c}(T, M, g) \leq Ce^{CT^{\alpha_\nu}}$$

is predominant in the space of $\mathcal{C}^\nu$ metrics on M, where $\alpha_\nu = C_n + \log_2 \nu$.

Counting Eigenvalues

(M^n, g) compact, no boundary.

Counting Eigenvalues

(M^n, g) compact, no boundary. Eigenvalues of $-\Delta_g$: $0 = \lambda_0^2 < \lambda_1^2 \leq \lambda_2^2 \leq \ldots$

Counting Eigenvalues

(M^n, g) compact, no boundary. Eigenvalues of $-\Delta_g$: $0 = \lambda_0^2 < \lambda_1^2 \le \lambda_2^2 \le \dots$

$$\#\{\lambda_j \le \lambda\} = \frac{\mathrm{vol}_{\mathbb{R}^n}(B_1)\mathrm{vol}_g(M)}{(2\pi)^n}\lambda^n + E_\lambda$$

Counting Eigenvalues

(M^n, g) compact, no boundary. Eigenvalues of $-\Delta_g$: $0 = \lambda_0^2 < \lambda_1^2 \leq \lambda_2^2 \leq \dots$

$$\#\{\lambda_j \leq \lambda\} = \frac{\mathrm{vol}_{\mathbb{R}^n}(B_1)\mathrm{vol}_g(M)}{(2\pi)^n}\lambda^n + E_\lambda$$

- (M, g) general: $E_\lambda = O(\lambda^{n-1})$ [Levitan '52, Avakumoic '56, Hörmander '68]

Counting Eigenvalues

(M^n, g) compact, no boundary. Eigenvalues of $-\Delta_g$: $0 = \lambda_0^2 < \lambda_1^2 \leq \lambda_2^2 \leq \dots$

$$\#\{\lambda_j \leq \lambda\} = \frac{\mathrm{vol}_{\mathbb{R}^n}(B_1)\mathrm{vol}_g(M)}{(2\pi)^n}\lambda^n + E_\lambda$$

- (M, g) general: $E_\lambda = O(\lambda^{n-1})$ [Levitan '52, Avakumoic '56, Hörmander '68]
- (M, g) Zoll: $E_\lambda \neq o(\lambda^{n-1})$ [Duistermaat-Guillemin '75, Weinstein '74]

Counting Eigenvalues

(M^n, g) compact, no boundary. Eigenvalues of $-\Delta_g$: $0 = \lambda_0^2 < \lambda_1^2 \le \lambda_2^2 \le \dots$

$$\#\{\lambda_j \le \lambda\} = \frac{\mathrm{vol}_{\mathbb{R}^n}(B_1)\mathrm{vol}_g(M)}{(2\pi)^n}\lambda^n + E_\lambda$$

- (M, g) general: $E_\lambda = O(\lambda^{n-1})$ [Levitan '52, Avakumoic '56, Hörmander '68]
- (M, g) Zoll: $E_\lambda \neq o(\lambda^{n-1})$ [Duistermaat-Guillemin '75, Weinstein '74]
- (M, g) aperiodic: $E_\lambda = o(\lambda^{n-1})$ [Duistermaat-Guillemin '75, Ivrii '80]

Counting Eigenvalues

(M^n, g) compact, no boundary. Eigenvalues of $-\Delta_g$: $0 = \lambda_0^2 < \lambda_1^2 \leq \lambda_2^2 \leq \dots$

$$\#\{\lambda_j \leq \lambda\} = \frac{\mathrm{vol}_{\mathbb{R}^n}(B_1)\mathrm{vol}_g(M)}{(2\pi)^n}\lambda^n + E_\lambda$$

- (M, g) general: $E_\lambda = O(\lambda^{n-1})$ [Levitan '52, Avakumoic '56, Hörmander '68]
- (M, g) Zoll: $E_\lambda \neq o(\lambda^{n-1})$ [Duistermaat-Guillemin '75, Weinstein '74]
- (M, g) aperiodic: $E_\lambda = o(\lambda^{n-1})$ [Duistermaat-Guillemin '75, Ivrii '80]
- (M, g) no conjugate points: $E_\lambda = O(\frac{\lambda^{n-1}}{\log \lambda})$ [Berard '77 + Bonthoneau '17]

Counting Eigenvalues

(M^n, g) compact, no boundary. Eigenvalues of $-\Delta_g$: $0 = \lambda_0^2 < \lambda_1^2 \leq \lambda_2^2 \leq \dots$

$$\#\{\lambda_j \leq \lambda\} = \frac{\mathrm{vol}_{\mathbb{R}^n}(B_1)\mathrm{vol}_g(M)}{(2\pi)^n}\lambda^n + E_\lambda$$

- (M, g) general: $E_\lambda = O(\lambda^{n-1})$ [Levitan '52, Avakumoic '56, Hörmander '68]
- (M, g) Zoll: $E_\lambda \neq o(\lambda^{n-1})$ [Duistermaat-Guillemin '75, Weinstein '74]
- (M, g) aperiodic: $E_\lambda = o(\lambda^{n-1})$ [Duistermaat-Guillemin '75, Ivrii '80]
- (M, g) no conjugate points: $E_\lambda = O(\frac{\lambda^{n-1}}{\log \lambda})$ [Berard '77 + Bonthoneau '17]
- (M, g) Baire generic: $E_\lambda = o(\lambda^{n-1})$ [Duistermaat-Guillemin '75, Anosov '82]

Counting Eigenvalues

(M^n, g) compact, no boundary. Eigenvalues of $-\Delta_g$: $0 = \lambda_0^2 < \lambda_1^2 \leq \lambda_2^2 \leq \dots$

$$\#\{\lambda_j \leq \lambda\} = \frac{\mathrm{vol}_{\mathbb{R}^n}(B_1)\mathrm{vol}_g(M)}{(2\pi)^n}\lambda^n + E_\lambda$$

- (M, g) general: $E_\lambda = O(\lambda^{n-1})$ [Levitan '52, Avakumoic '56, Hörmander '68]
- (M, g) Zoll: $E_\lambda \neq o(\lambda^{n-1})$ [Duistermaat-Guillemin '75, Weinstein '74]
- (M, g) aperiodic: $E_\lambda = o(\lambda^{n-1})$ [Duistermaat-Guillemin '75, Ivrii '80]
- (M, g) no conjugate points: $E_\lambda = O(\frac{\lambda^{n-1}}{\log \lambda})$ [Berard '77 + Bonthoneau '17]
- (M, g) Baire generic: $E_\lambda = o(\lambda^{n-1})$ [Duistermaat-Guillemin '75, Anosov '82]

Theorem (Canzani–G '22)

Let M be a smooth manifold of dimension n. Then there is $\nu_0 > 0$ such that for all $\nu > \nu_0$ the set of metrics, $g \in \mathcal{C}^\nu$ such that

$$E_\lambda = O\Big(\frac{\lambda^{n-1}}{(\log \lambda)^{1/\alpha_\nu}}\Big)$$

is predominant in the space of $\mathcal{C}^\nu$ metrics on M, where $\alpha_\nu = C_n + \log_2 \nu$.

Weyl laws on Baire generic manifolds

Theorem (Duistermaat–Guillemin ('75) + Anosov ('82))

Let M be a smooth manifold of dimension n. The property

$$\#\{\lambda \leq \lambda_j\} = \frac{\mathrm{vol}_{\mathbb{R}^n}(B_1)\mathrm{vol}_g(M)}{(2\pi)^n}\lambda^n + E_\lambda, \qquad E_\lambda = o(\lambda^{n-1})$$

is Baire generic in the space of smooth metrics.

Proof.

$\square$

Weyl laws on Baire generic manifolds

Theorem (Duistermaat–Guillemin ('75) + Anosov ('82))

Let M be a smooth manifold of dimension n. The property

$$\#\{\lambda \leq \lambda_j\} = \frac{\mathrm{vol}_{\mathbb{R}^n}(B_1)\mathrm{vol}_g(M)}{(2\pi)^n}\lambda^n + E_\lambda, \qquad E_\lambda = o(\lambda^{n-1})$$

is Baire generic in the space of smooth metrics.

Proof.

- [Duistermaat–Guillemin ('75)] If the set of closed geodesics has measure zero in S^*M, then $E_\lambda = o(\lambda^{n-1})$.

$\square$

Weyl laws on Baire generic manifolds

Theorem (Duistermaat–Guillemin (‘75) + Anosov (’82))

Let M be a smooth manifold of dimension n. The property

$$\#\{\lambda \leq \lambda_j\} = \frac{\mathrm{vol}_{\mathbb{R}^n}(B_1)\mathrm{vol}_g(M)}{(2\pi)^n}\lambda^n + E_\lambda, \qquad E_\lambda = o(\lambda^{n-1})$$

is Baire generic in the space of smooth metrics.

Proof.

- [Duistermaat–Guillemin (’75)] If the set of closed geodesics has measure zero in S^*M, then $E_\lambda = o(\lambda^{n-1})$.
- [Anosov (’82)] The set of metrics such that for all $T > 0$, there are finitely many closed geodesics with length $\leq T$ is Baire generic.

$\square$

Weyl laws for predominant metrics

Theorem (Canzani–G ('22))

Let M be a smooth manifold of dimension n. Then there is $\nu_0 > 0$ such that for all $\nu > \nu_0$ the set of metrics, $g \in \mathcal{C}^\nu$ such that

$$\#\{\lambda_j \leq \lambda \,:\, \lambda_j^2 \in \sigma(-\Delta_g)\} = \frac{\mathrm{vol}_{\mathbb{R}^n}(B_1)\mathrm{vol}_g(M)}{(2\pi)^n}\lambda^n + E_\lambda, \qquad E_\lambda = O\Big(\frac{\lambda^{n-1}}{(\log \lambda)^{1/\alpha_\nu}}\Big)$$

is predominant in the space of $\mathcal{C}^\nu$ metrics on M, where $\alpha_\nu = C_n + \log_2 \nu$.

Idea of Proof:

Weyl laws for predominant metrics

Theorem (Canzani–G ('22))

Let M be a smooth manifold of dimension n. Then there is $\nu_0 > 0$ such that for all $\nu > \nu_0$ the set of metrics, $g \in \mathcal{C}^\nu$ such that

$$\#\{\lambda_j \leq \lambda \,:\, \lambda_j^2 \in \sigma(-\Delta_g)\} = \frac{\mathrm{vol}_{\mathbb{R}^n}(B_1)\mathrm{vol}_g(M)}{(2\pi)^n}\lambda^n + E_\lambda, \qquad E_\lambda = O\Big(\frac{\lambda^{n-1}}{(\log\lambda)^{1/\alpha_\nu}}\Big)$$

is predominant in the space of $\mathcal{C}^\nu$ metrics on M, where $\alpha_\nu = C_n + \log_2 \nu$.

Idea of Proof:

Definition

(M, g) is said to be $\mathbf{T}(R)$ non-periodic if

$$\mathrm{vol}\Big(\rho \,:\, \exists t \in [t_0, \mathbf{T}(R)] \text{ s.t. } d(\rho, \varphi_t^g(B(\rho, R))) \leq R\Big) \leq \frac{C}{\mathbf{T}(R)}, \qquad R \to 0^+.$$

Weyl laws for predominant metrics

Theorem (Canzani–G ('22))

Let M be a smooth manifold of dimension n. Then there is $\nu_0 > 0$ such that for all $\nu > \nu_0$ the set of metrics, $g \in \mathcal{C}^\nu$ such that

$$\#\{\lambda_j \leq \lambda \,:\, \lambda_j^2 \in \sigma(-\Delta_g)\} = \frac{\mathrm{vol}_{\mathbb{R}^n}(B_1)\mathrm{vol}_g(M)}{(2\pi)^n}\lambda^n + E_\lambda, \qquad E_\lambda = O\Big(\frac{\lambda^{n-1}}{(\log\lambda)^{1/\alpha_\nu}}\Big)$$

is predominant in the space of $\mathcal{C}^\nu$ metrics on M, where $\alpha_\nu = C_n + \log_2 \nu$.

Idea of Proof:

Definition

(M, g) is said to be $\mathbf{T}(R)$ non-periodic if

$$\mathrm{vol}\Big(\rho \,:\, \exists t \in [t_0, \mathbf{T}(R)] \text{ s.t. } d(\rho, \varphi_t^g(B(\rho, R))) \leq R\Big) \leq \frac{C}{\mathbf{T}(R)}, \qquad R \to 0^+.$$

Theorem (Canzani– G '20)

If (M, g) is $\mathbf{T}(R)$ non-periodic, then $E_\lambda = O(\lambda^{n-1}/\mathbf{T}(\lambda^{-1}))$.

Weyl laws for predominant metrics

Theorem (Canzani–G ('22))

Let M be a smooth manifold of dimension n. Then there is $\nu_0 > 0$ such that for all $\nu > \nu_0$ the set of metrics, $g \in \mathcal{C}^\nu$ such that

$$\#\{\lambda_j \leq \lambda \,:\, \lambda_j^2 \in \sigma(-\Delta_g)\} = \frac{\mathrm{vol}_{\mathbb{R}^n}(B_1)\mathrm{vol}_g(M)}{(2\pi)^n}\lambda^n + E_\lambda, \qquad E_\lambda = O\Big(\frac{\lambda^{n-1}}{(\log \lambda)^{1/\alpha_\nu}}\Big)$$

is predominant in the space of $\mathcal{C}^\nu$ metrics on M, where $\alpha_\nu = C_n + \log_2 \nu$.

Idea of Proof:

Definition

(M, g) is said to be $\mathbf{T}(R)$ non-periodic if

$$\mathrm{vol}\Big(\rho \,:\, \exists t \in [t_0, \mathbf{T}(R)] \text{ s.t. } d(\rho, \varphi_t^g(B(\rho, R))) \leq R\Big) \leq \frac{C}{\mathbf{T}(R)}, \qquad R \to 0^+.$$

Theorem (Canzani– G '20)

If (M, g) is $\mathbf{T}(R)$ non-periodic, then $E_\lambda = O(\lambda^{n-1}/\mathbf{T}(\lambda^{-1}))$.

- We need $\mathbf{T}(R) = (\log R^{-1})^{1/\alpha_\nu}$ non-periodicity for a predominant set of metrics.

Examples of non-periodic manifolds

Examples of **T** non-periodic manifolds with $\mathbf{T}(R) = \log(R^{-1})$

Examples of non-periodic manifolds

Examples of $\mathbf{T}$ non-periodic manifolds with $\mathbf{T}(R) = \log(R^{-1})$

- product manifolds

Examples of non-periodic manifolds

Examples of **T** non-periodic manifolds with $\mathbf{T}(R) = \log(R^{-1})$

- product manifolds
- manifolds with no conjugate points

Examples of non-periodic manifolds

Examples of $\mathbf{T}$ non-periodic manifolds with $\mathbf{T}(R) = \log(R^{-1})$

- product manifolds
- manifolds with no conjugate points (in fact, no 'maximal self conjugate' points)

Examples of non-periodic manifolds

Examples of $\mathbf{T}$ non-periodic manifolds with $\mathbf{T}(R) = \log(R^{-1})$

- product manifolds
- manifolds with no conjugate points (in fact, no 'maximal self conjugate' points)
- non-Zoll convex analytic surfaces of revolution [Volovoy '90]

Examples of non-periodic manifolds

Examples of **T** non-periodic manifolds with $\mathbf{T}(R) = \log(R^{-1})$

- product manifolds
- manifolds with no conjugate points (in fact, no 'maximal self conjugate' points)
- non-Zoll convex analytic surfaces of revolution [Volovoy '90]
- compact Lie group of rank > 1 with a bi-invariant metric [Volovoy '90]

Examples of non-periodic manifolds

Examples of **T** non-periodic manifolds with $\mathbf{T}(R) = \log(R^{-1})$

- product manifolds
- manifolds with no conjugate points (in fact, no 'maximal self conjugate' points)
- non-Zoll convex analytic surfaces of revolution [Volovoy '90]
- compact Lie group of rank > 1 with a bi-invariant metric [Volovoy '90]
- But is it predominant?

Closed geodesics on predominant manifolds.

Theorem (Canzani–G '22)

The set of metrics, $g \in \mathcal{C}^{\nu}$, such that there is $C > 0$

$$\#\{\gamma : \gamma \text{ is a closed geodesic for } g \text{ with length } \leq T\} \leq Ce^{CT^{\alpha_\nu}}$$

is predominant in the space of $\mathcal{C}^{\nu}$ metrics on M, where $\alpha_\nu = C_n + \log_2 \nu$.

Closed geodesics on predominant manifolds.

Theorem (Canzani–G '22)

The set of metrics, $g \in \mathcal{C}^{\nu}$, such that there is $C > 0$

$$\#\{\gamma : \gamma \text{ is a closed geodesic for } g \text{ with length } \leq T\} \leq Ce^{CT^{\alpha_\nu}}$$

is predominant in the space of $\mathcal{C}^{\nu}$ metrics on M, where $\alpha_\nu = C_n + \log_2 \nu$.

Theorem (Canzani–G '22)

The set of metrics, $g \in \mathcal{C}^{\nu}$, such that there is $B > 0$ such that

$$\mathrm{vol}\Big(\rho : \exists t \in [t_0, T] \text{ s.t. } d(\rho, \varphi_t^g(\rho) \leq \epsilon\Big) \leq \epsilon^{2n-2} e^{BT^{\alpha_\nu}}.$$

is predominant in the space of $\mathcal{C}^{\nu}$ metrics on M, where $\alpha_\nu = C_n + \log_2 \nu$.

Closed geodesics on predominant manifolds.

Theorem (Canzani–G '22)

The set of metrics, $g \in \mathcal{C}^\nu$, such that there is $C > 0$

$$\#\{\gamma : \gamma \text{ is a closed geodesic for } g \text{ with length } \leq T\} \leq Ce^{CT^{\alpha_\nu}}$$

is predominant in the space of $\mathcal{C}^\nu$ metrics on M, where $\alpha_\nu = C_n + \log_2 \nu$.

Theorem (Canzani–G '22)

The set of metrics, $g \in \mathcal{C}^\nu$, such that there is $B > 0$ such that

$$\mathrm{vol}\Big(\rho : \exists t \in [t_0, T] \text{ s.t. } d(\rho, \varphi_t^g(\rho) \leq \epsilon\Big) \leq \epsilon^{2n-2} e^{BT^{\alpha_\nu}}.$$

is predominant in the space of $\mathcal{C}^\nu$ metrics on M, where $\alpha_\nu = C_n + \log_2 \nu$.

Corollary (Canzani–G '22)

The set of metrics, $g \in \mathcal{C}^\nu$ such that

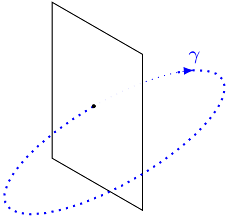
$$\#\{\lambda_j \leq \lambda : \lambda_j^2 \in \sigma(-\Delta_g)\} = \frac{\mathrm{vol}_{\mathbb{R}^n}(B_1)\mathrm{vol}_g(M)}{(2\pi)^n}\lambda^n + E_\lambda, \qquad E_\lambda = O\Big(\frac{\lambda^{n-1}}{(\log \lambda)^{1/\alpha_\nu}}\Big)$$

is predominant in the space of $\mathcal{C}^\nu$ metrics on M, where $\alpha_\nu = C_n + \log_2 \nu$.

Reduction to the Poincare map

$$\mathcal{C}(T, g) = \{\gamma \,:\, \text{periodic geodesic for } g,\ T \leq \text{length}(\gamma) \leq 2T\}.$$

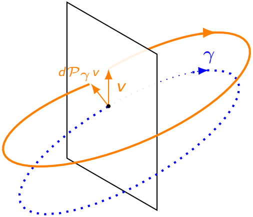
The Poincare map, $\mathcal{P}_\gamma$ associated to a closed geodesic γ



Reduction to the Poincare map

$$\mathcal{C}(T,g) = \{\gamma : \text{periodic geodesic for } g,\ T \leq \text{length}(\gamma) \leq 2T\}.$$

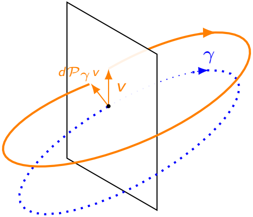
The Poincare map, $\mathcal{P}_\gamma$ associated to a closed geodesic γ



Reduction to the Poincare map

$$\mathcal{C}(T,g) = \{\gamma \,:\, \text{periodic geodesic for } g,\, T \leq \text{length}(\gamma) \leq 2T\}.$$

The Poincare map, $\mathcal{P}_\gamma$ associated to a closed geodesic γ



Theorem (Canzani–G '22)

The set of metrics $g \in \mathcal{C}^\nu$, such that there is $C > 0$ satisfying

$$\|(I - d\mathcal{P}_\gamma)^{-1}\| \leq Ce^{CT^{\alpha\nu}}, \qquad \gamma \in \mathcal{C}(T,g),$$

is predominant.

Goal: (G) $\|(I - d\mathcal{P}_\gamma)^{-1}\| \leq Ce^{CT^{\alpha\nu}}$ for $\gamma \in \mathcal{C}(T, g)$

- Hypothesis: Suppose $\mathcal{C}(2^j, g_\ell)$ satisfies (G) for $j \leq \ell$

Goal: (G) $\|(I - d\mathcal{P}_{\gamma})^{-1}\| \leq Ce^{CT^{\alpha\nu}}$ for $\gamma \in \mathcal{C}(T, g)$

- Hypothesis: Suppose $\mathcal{C}(2^j, g_\ell)$ satisfies (G) for $j \leq \ell$
- Want to find: large family of $g_{\ell+1}$ near g_ℓ such that $\mathcal{C}(2^j, g_{\ell+1})$ satisfies (G) for $j \leq \ell + 1$.

Goal: (G) $\|(I - d\mathcal{P}_\gamma)^{-1}\| \leq Ce^{CT^{\alpha\nu}}$ **for** $\gamma \in \mathcal{C}(T, g)$

- Hypothesis: Suppose $\mathcal{C}(2^j, g_\ell)$ satisfies (G) for $j \leq \ell$
- Want to find: large family of $g_{\ell+1}$ near g_ℓ such that $\mathcal{C}(2^j, g_{\ell+1})$ satisfies (G) for $j \leq \ell + 1$.

What to do:

Goal: (G) $\|(I - d\mathcal{P}_\gamma)^{-1}\| \leq Ce^{CT^{\alpha\nu}}$ for $\gamma \in \mathcal{C}(T, g)$

- Hypothesis: Suppose $\mathcal{C}(2^j, g_\ell)$ satisfies (G) for $j \leq \ell$
- Want to find: large family of $g_{\ell+1}$ near g_ℓ such that $\mathcal{C}(2^j, g_{\ell+1})$ satisfies (G) for $j \leq \ell + 1$.

What to do:

- $g_{\ell+1}$ such that $\mathcal{C}(2^j, g_{\ell+1})$ satisfies (G) for $j \leq \ell$

Goal: (G) $\|(I - d\mathcal{P}_\gamma)^{-1}\| \leq Ce^{CT^{\alpha\nu}}$ for $\gamma \in \mathcal{C}(T, g)$

- Hypothesis: Suppose $\mathcal{C}(2^j, g_\ell)$ satisfies (G) for $j \leq \ell$
- Want to find: large family of $g_{\ell+1}$ near g_ℓ such that $\mathcal{C}(2^j, g_{\ell+1})$ satisfies (G) for $j \leq \ell + 1$.

What to do:

- $g_{\ell+1}$ such that $\mathcal{C}(2^j, g_{\ell+1})$ satisfies (G) for $j \leq \ell$ (Small enough perturbation is good enough)

Goal: (G) $\|(I - d\mathcal{P}_\gamma)^{-1}\| \le Ce^{CT^{\alpha\nu}}$ **for** $\gamma \in \mathcal{C}(T, g)$

- Hypothesis: Suppose $\mathcal{C}(2^j, g_\ell)$ satisfies (G) for $j \le \ell$
- Want to find: large family of $g_{\ell+1}$ near g_ℓ such that $\mathcal{C}(2^j, g_{\ell+1})$ satisfies (G) for $j \le \ell + 1$.

What to do:

- $g_{\ell+1}$ such that $\mathcal{C}(2^j, g_{\ell+1})$ satisfies (G) for $j \le \ell$ (Small enough perturbation is good enough)
- $g_{\ell+1}$ such that $\mathcal{C}(2^{\ell+1}, g_{\ell+1}) \cap \{\text{primitive}\}$ satisfies (G)

Goal: (G) $\|(I - d\mathcal{P}_{\gamma})^{-1}\| \leq Ce^{CT^{\alpha\nu}}$ for $\gamma \in \mathcal{C}(T, g)$

- Hypothesis: Suppose $\mathcal{C}(2^j, g_\ell)$ satisfies (G) for $j \leq \ell$
- Want to find: large family of $g_{\ell+1}$ near g_ℓ such that $\mathcal{C}(2^j, g_{\ell+1})$ satisfies (G) for $j \leq \ell + 1$.

What to do:

- $g_{\ell+1}$ such that $\mathcal{C}(2^j, g_{\ell+1})$ satisfies (G) for $j \leq \ell$ (Small enough perturbation is good enough)
- $g_{\ell+1}$ such that $\mathcal{C}(2^{\ell+1}, g_{\ell+1}) \cap \{\text{primitive}\}$ satisfies (G)

- But!!! There are multiple geodesics in $\mathcal{C}(2^{\ell+1}, g_{\ell+1})$ so this is not enough

Goal: (G) $\|(I - d\mathcal{P}_\gamma)^{-1}\| \leq Ce^{CT^{\alpha\nu}}$ for $\gamma \in \mathcal{C}(T, g)$

- Hypothesis: Suppose $\mathcal{C}(2^j, g_\ell)$ satisfies (G) for $j \leq \ell$
- Want to find: large family of $g_{\ell+1}$ near g_ℓ such that $\mathcal{C}(2^j, g_{\ell+1})$ satisfies (G) for $j \leq \ell + 1$.

What to do:

- $g_{\ell+1}$ such that $\mathcal{C}(2^j, g_{\ell+1})$ satisfies (G) for $j \leq \ell$ (Small enough perturbation is good enough)
- $g_{\ell+1}$ such that $\mathcal{C}(2^{\ell+1}, g_{\ell+1}) \cap \{\text{primitive}\}$ satisfies (G)

- But!!! There are multiple geodesics in $\mathcal{C}(2^{\ell+1}, g_{\ell+1})$ so this is not enough
- It is difficult to control the effect of a perturbation on a multiple geodesic.

Goal: (G) $\|(I - d\mathcal{P}_\gamma)^{-1}\| \le Ce^{CT^{\alpha\nu}}$ for $\gamma \in \mathcal{C}(T, g)$

- Hypothesis: Suppose $\mathcal{C}(2^j, g_\ell)$ satisfies (G) for $j \le \ell$
- Want to find: large family of $g_{\ell+1}$ near g_ℓ such that $\mathcal{C}(2^j, g_{\ell+1})$ satisfies (G) for $j \le \ell + 1$.

What to do:

- $g_{\ell+1}$ such that $\mathcal{C}(2^j, g_{\ell+1})$ satisfies (G) for $j \le \ell$ (Small enough perturbation is good enough)
- $g_{\ell+1}$ such that $\mathcal{C}(2^{\ell+1}, g_{\ell+1}) \cap \{\text{primitive}\}$ satisfies (G)

- But!!! There are multiple geodesics in $\mathcal{C}(2^{\ell+1}, g_{\ell+1})$ so this is not enough
- It is difficult to control the effect of a perturbation on a multiple geodesic. To fix this, we need a condition that is 'inheritable'.

Goal: (G) $\|(I - d\mathcal{P}_\gamma)^{-1}\| \le Ce^{CT^{\alpha\nu}}$ for $\gamma \in \mathcal{C}(T, g)$

- Hypothesis: Suppose $\mathcal{C}(2^j, g_\ell)$ satisfies (G) for $j \le \ell$
- Want to find: large family of $g_{\ell+1}$ near g_ℓ such that $\mathcal{C}(2^j, g_{\ell+1})$ satisfies (G) for $j \le \ell + 1$.

What to do:

- $g_{\ell+1}$ such that $\mathcal{C}(2^j, g_{\ell+1})$ satisfies (G) for $j \le \ell$ (Small enough perturbation is good enough)
- $g_{\ell+1}$ such that $\mathcal{C}(2^{\ell+1}, g_{\ell+1}) \cap \{\text{primitive}\}$ satisfies (G)

- But!!! There are multiple geodesics in $\mathcal{C}(2^{\ell+1}, g_{\ell+1})$ so this is not enough
- It is difficult to control the effect of a perturbation on a multiple geodesic. To fix this, we need a condition that is 'inheritable'.

Observe:

$$d\mathcal{P}_{k\gamma} = (d\mathcal{P}_\gamma)^k \qquad .$$

Goal: (G) $\|(I - d\mathcal{P}_\gamma)^{-1}\| \le Ce^{CT^{\alpha\nu}}$ for $\gamma \in \mathcal{C}(T, g)$

- Hypothesis: Suppose $\mathcal{C}(2^j, g_\ell)$ satisfies (G) for $j \le \ell$
- Want to find: large family of $g_{\ell+1}$ near g_ℓ such that $\mathcal{C}(2^j, g_{\ell+1})$ satisfies (G) for $j \le \ell + 1$.

What to do:

- $g_{\ell+1}$ such that $\mathcal{C}(2^j, g_{\ell+1})$ satisfies (G) for $j \le \ell$ (Small enough perturbation is good enough)
- $g_{\ell+1}$ such that $\mathcal{C}(2^{\ell+1}, g_{\ell+1}) \cap \{\text{primitive}\}$ satisfies (G)

- But!!! There are multiple geodesics in $\mathcal{C}(2^{\ell+1}, g_{\ell+1})$ so this is not enough
- It is difficult to control the effect of a perturbation on a multiple geodesic. To fix this, we need a condition that is 'inheritable'.

Observe:

$$d\mathcal{P}_{k\gamma} = (d\mathcal{P}_\gamma)^k \qquad \longrightarrow \qquad (I - d\mathcal{P}_{k\gamma})^{-1} \text{ exists } \Leftrightarrow e^{2\pi i p/k} \notin \operatorname{Spec}(d\mathcal{P}_\gamma).$$

Goal: (G) $\|(I - d\mathcal{P}_\gamma)^{-1}\| \leq Ce^{CT^{\alpha\nu}}$ for $\gamma \in \mathcal{C}(T, g)$

- Hypothesis: Suppose $\mathcal{C}(2^j, g_\ell)$ satisfies (G) for $j \leq \ell$
- Want to find: large family of $g_{\ell+1}$ near g_ℓ such that $\mathcal{C}(2^j, g_{\ell+1})$ satisfies (G) for $j \leq \ell + 1$.

What to do:

- $g_{\ell+1}$ such that $\mathcal{C}(2^j, g_{\ell+1})$ satisfies (G) for $j \leq \ell$ (Small enough perturbation is good enough)
- $g_{\ell+1}$ such that $\mathcal{C}(2^{\ell+1}, g_{\ell+1}) \cap \{\text{primitive}\}$ satisfies (G)

- But!!! There are multiple geodesics in $\mathcal{C}(2^{\ell+1}, g_{\ell+1})$ so this is not enough
- It is difficult to control the effect of a perturbation on a multiple geodesic. To fix this, we need a condition that is 'inheritable'.

Observe:

$$d\mathcal{P}_{k\gamma} = (d\mathcal{P}_\gamma)^k \qquad \longrightarrow \qquad (I - d\mathcal{P}_{k\gamma})^{-1} \text{ exists } \Leftrightarrow e^{2\pi i p/k} \notin \mathrm{Spec}(d\mathcal{P}_\gamma).$$

Hope 1: $\gamma \in \mathcal{C}(2^j, g_{\ell+1})$, then $d\mathcal{P}_\gamma$ has eigenvalues 'far' from the unit circle.

Goal: (G) $\|(I - d\mathcal{P}_\gamma)^{-1}\| \leq Ce^{CT^{\alpha\nu}}$ for $\gamma \in \mathcal{C}(T, g)$

- Hypothesis: Suppose $\mathcal{C}(2^j, g_\ell)$ satisfies (G) for $j \leq \ell$
- Want to find: large family of $g_{\ell+1}$ near g_ℓ such that $\mathcal{C}(2^j, g_{\ell+1})$ satisfies (G) for $j \leq \ell + 1$.

What to do:

- $g_{\ell+1}$ such that $\mathcal{C}(2^j, g_{\ell+1})$ satisfies (G) for $j \leq \ell$ (Small enough perturbation is good enough)
- $g_{\ell+1}$ such that $\mathcal{C}(2^{\ell+1}, g_{\ell+1}) \cap \{\text{primitive}\}$ satisfies (G)

- But!!! There are multiple geodesics in $\mathcal{C}(2^{\ell+1}, g_{\ell+1})$ so this is not enough
- It is difficult to control the effect of a perturbation on a multiple geodesic. To fix this, we need a condition that is 'inheritable'.

Observe:

$$d\mathcal{P}_{k\gamma} = (d\mathcal{P}_\gamma)^k \qquad \longrightarrow \qquad (I - d\mathcal{P}_{k\gamma})^{-1} \text{ exists } \Leftrightarrow e^{2\pi i p/k} \notin \operatorname{Spec}(d\mathcal{P}_\gamma).$$

Hope 1: $\gamma \in \mathcal{C}(2^j, g_{\ell+1})$, then $d\mathcal{P}_\gamma$ has eigenvalues 'far' from the unit circle. (This doesn't work since $d\mathcal{P}_\gamma$ is symplectic)

Goal: (G) $\|(I - d\mathcal{P}_\gamma)^{-1}\| \leq Ce^{CT^{\alpha\nu}}$ for $\gamma \in \mathcal{C}(T, g)$

- Hypothesis: Suppose $\mathcal{C}(2^j, g_\ell)$ satisfies (G) for $j \leq \ell$
- Want to find: large family of $g_{\ell+1}$ near g_ℓ such that $\mathcal{C}(2^j, g_{\ell+1})$ satisfies (G) for $j \leq \ell + 1$.

What to do:

- $g_{\ell+1}$ such that $\mathcal{C}(2^j, g_{\ell+1})$ satisfies (G) for $j \leq \ell$ (Small enough perturbation is good enough)
- $g_{\ell+1}$ such that $\mathcal{C}(2^{\ell+1}, g_{\ell+1}) \cap \{\text{primitive}\}$ satisfies (G)

- But!!! There are multiple geodesics in $\mathcal{C}(2^{\ell+1}, g_{\ell+1})$ so this is not enough
- It is difficult to control the effect of a perturbation on a multiple geodesic. To fix this, we need a condition that is 'inheritable'.

Observe:

$$d\mathcal{P}_{k\gamma} = (d\mathcal{P}_\gamma)^k \qquad \longrightarrow \qquad (I - d\mathcal{P}_{k\gamma})^{-1} \text{ exists } \Leftrightarrow e^{2\pi i p/k} \notin \operatorname{Spec}(d\mathcal{P}_\gamma).$$

Hope 1: $\gamma \in \mathcal{C}(2^j, g_{\ell+1})$, then $d\mathcal{P}_\gamma$ has eigenvalues 'far' from the unit circle. (This doesn't work since $d\mathcal{P}_\gamma$ is symplectic)

Hope 2: $\gamma \in \mathcal{C}(2^j, g_{\ell+1})$, then $d\mathcal{P}_\gamma$ has eigenvalues 'far' from roots of unity.

Goal: (G) $\|(I - d\mathcal{P}_\gamma)^{-1}\| \leq Ce^{CT^{\alpha\nu}}$ for $\gamma \in \mathcal{C}(T, g)$

- Hypothesis: Suppose $\mathcal{C}(2^j, g_\ell)$ satisfies (G) for $j \leq \ell$
- Want to find: large family of $g_{\ell+1}$ near g_ℓ such that $\mathcal{C}(2^j, g_{\ell+1})$ satisfies (G) for $j \leq \ell + 1$.

What to do:

- $g_{\ell+1}$ such that $\mathcal{C}(2^j, g_{\ell+1})$ satisfies (G) for $j \leq \ell$ (Small enough perturbation is good enough)
- $g_{\ell+1}$ such that $\mathcal{C}(2^{\ell+1}, g_{\ell+1}) \cap \{\text{primitive}\}$ satisfies (G)

- But!!! There are multiple geodesics in $\mathcal{C}(2^{\ell+1}, g_{\ell+1})$ so this is not enough
- It is difficult to control the effect of a perturbation on a multiple geodesic. To fix this, we need a condition that is 'inheritable'.

Observe:

$$d\mathcal{P}_{k\gamma} = (d\mathcal{P}_\gamma)^k \qquad \longrightarrow \qquad (I - d\mathcal{P}_{k\gamma})^{-1} \text{ exists } \Leftrightarrow e^{2\pi i p/k} \notin \operatorname{Spec}(d\mathcal{P}_\gamma).$$

Hope 1: $\gamma \in \mathcal{C}(2^j, g_{\ell+1})$, then $d\mathcal{P}_\gamma$ has eigenvalues 'far' from the unit circle. (This doesn't work since $d\mathcal{P}_\gamma$ is symplectic)

Hope 2: $\gamma \in \mathcal{C}(2^j, g_{\ell+1})$, then $d\mathcal{P}_\gamma$ has eigenvalues 'far' from roots of unity. This works, but requires delicate adjustments at every step of the induction

How to perturb g_ℓ?

Let $\gamma_{g_\ell} \in \mathcal{C}(2^{\ell+1}, g_\ell)$ is primitive. How can we perturb?

How to perturb g_ℓ?

Let $\gamma_{g_\ell} \in \mathcal{C}(2^{\ell+1}, g_\ell)$ is primitive. How can we perturb?

- Use primitivity: find a (physical!) ball, B over which γ_{g_ℓ} passes only once.

How to perturb g_ℓ?

Let $\gamma_{g_\ell} \in \mathcal{C}(2^{\ell+1}, g_\ell)$ is primitive. How can we perturb?

- Use primitivity: find a (physical!) ball, B over which γ_{g_ℓ} passes only once.

- Make a family of perturbations $\mathbb{R}^N \ni \sigma \to g_\sigma$ in B so that $\sigma \mapsto (\mathcal{P}_\sigma, d\mathcal{P}_\sigma)$ is a submersion.

How to perturb g_ℓ?

Let $\gamma_{g_\ell} \in \mathcal{C}(2^{\ell+1}, g_\ell)$ is primitive. How can we perturb?

- Use primitivity: find a (physical!) ball, B over which γ_{g_ℓ} passes only once.
- Make a family of perturbations $\mathbb{R}^N \ni \sigma \to g_\sigma$ in B so that $\sigma \mapsto (\mathcal{P}_\sigma, d\mathcal{P}_\sigma)$ is a submersion.
- Use a quantitative Sard theorem due to Yomdin to guarantee that

$$\mathrm{m}\big(\{\sigma : (\mathcal{P}_\sigma, d\mathcal{P}_\sigma) \text{ is near } (Id, \mathcal{M}_K)\}\big) \ll 1$$

$$\mathcal{M}_K := \{\text{matrices with eigenvalue } e^{2\pi i p/k} \text{ for some } 1 \leq k \leq K\}$$

How to perturb g_ℓ?

Let $\gamma_{g_\ell} \in \mathcal{C}(2^{\ell+1}, g_\ell)$ is primitive. How can we perturb?

- Use primitivity: find a (physical!) ball, B over which γ_{g_ℓ} passes only once.
- Make a family of perturbations $\mathbb{R}^N \ni \sigma \to g_\sigma$ in B so that $\sigma \mapsto (\mathcal{P}_\sigma, d\mathcal{P}_\sigma)$ is a submersion.
- Use a quantitative Sard theorem due to Yomdin to guarantee that

$$\mathrm{m}\big(\{\sigma : (\mathcal{P}_\sigma, d\mathcal{P}_\sigma) \text{ is near } (Id, \mathcal{M}_K)\}\big) \ll 1$$
$$\mathcal{M}_K := \{\text{matrices with eigenvalue } e^{2\pi i p/k} \text{ for some } 1 \leq k \leq K\}$$

- Note! This only allows to inherit up to iterates of length K, have to update these later.

Happy birthday!