Semiclassical scarring in KAM systems

Andrew Hassell

Conference in honour of Steve Zelditch, September 8-11 2022

Semiclassical scarring on KAM tori

- Background: my work showing that the stadium billiard is non-QUE for almost every value of the aspect ratio; in fact, that there is scarring onto the codimension-one set of ‘bouncing ball’ trajectories. (2008, published in 2010).
- I talked to Steve about this in 2008 at MSRI. He asked whether this method could be used to say something about non-equidistribution of KAM systems.
- Specifically he asked me if it could be used to show concentration onto KAM tori, using Popov’s construction of exponentially accurate quasimodes.

Comparison of the two problems (stadium/KAM)

- Both have a 1-parameter family of variations
- Both have quasimodes concentrating onto a codimension 1 subset of the energy shell
- The stadium is known to be QE (Schnirelman, Zelditch, Colin de Verdière, Gérard-Leichtnam, Zelditch-Zworski), KAM expected not to be (Gomes)
- KAM quasimodes are exponentially accurate, stadium is only $O(h^2)$ quasimode
- KAM tori are individually measure zero but together form a large measure set

The KAM problem has been studied in work of Sean Gomes in his thesis and then in collaboration with me:

- S. Gomes, *Generic KAM Hamiltonians are not Quantum Ergodic*, A&PDE, to appear; arXiv:1811.07718.
- S. Gomes, A. Hassell, *Semiclassical scarring on tori in KAM Hamiltonian systems*, J. Eur. Math. Soc. 2022.

Would not have been tackled without Steve's original question from 2008!

Spectral concentration

The key point in both problems is to bound the spectral concentration that can occur near the quasi-eigenvalues. Recall two key principles about eigenvalue/eigenfunction approximation: let T be a self-adjoint operator, and ϕ, $\|\phi\| = 1$ a normalized approximate eigenfunction, with $\|(T - \lambda)\phi\| \leq \epsilon$. Then:

• For $M > 1$ the spectral projector $\Pi_{[\lambda - M\epsilon, \lambda + M\epsilon]}$ on the larger interval $[\lambda - M\epsilon, \lambda + M\epsilon]$ satisfies

$$\|\Pi_{[\lambda - M\epsilon, \lambda + M\epsilon]}\phi - \phi\| \leq \frac{1}{M},$$

i.e. for large M ϕ is essentially composed of a sum of eigenfunctions with eigenvalues in the range $[\lambda - M\epsilon, \lambda + M\epsilon]$.

• if the spectral projector $\Pi_{[\lambda - M\epsilon, \lambda + M\epsilon]}$ has rank $\leq R$, then there is a normalized eigenfunction u with eigenvalue in the range $[\lambda - M\epsilon, \lambda + M\epsilon]$ such that

$$|\langle u, \phi \rangle| \geq R^{-1/2}(1 + O(M^{-1})).$$

Now, given a quasimode, i.e. a sequence ϕ_j with $\|(H_{h_j} - \lambda_j)\phi_j\| \leq \epsilon_j$ for some sequence $h_j \to 0$, the goal is to find a uniform M and R so that there are at most R eigenvalues (counted with multiplicity) in $[\lambda_j - M\epsilon_j, \lambda_j + M\epsilon_j]$, as $j \to \infty$. From this we can show that if the ϕ_j scar onto a set E of measure zero, then there is **scarring** of a sequence of true eigenfunctions u_j selected as above.

Key challenge: bound the multiplicity R uniformly for a sequence of quasimodes as $h \to 0$.

To do this we use the 'time' parameter t, and a formula for the flow of eigenvalues in t, to show that we avoid spectral concentration (i.e. $R \to \infty$) for most values of t.

Stadium case

Using semiclassical notation, we look at eigenvalues of $h^2\Delta_t$ where Δ_t is the Dirichlet Laplacian for the stadium with aspect ratio t, i.e. the central rectangle is $[0, t] \times [0, 1]$.
Quasimodes are $\phi_j(x, y) = \chi(x)\sin(j\pi y)$ (supported in the central rectangle). Then for $h = 1/j$ we have

$$(h^2\Delta_t - \pi^2)\phi_j = O_{L^2}(h^2).$$

Assume, for a contradiction, that the stadium S_t is QUE for each t. The assumption of QUE implies a very uniform flow of eigenvalues, in fact

$$\dot{E}_j(t) = -c(t)E_j(t)(1 + o(1)).$$

Consideration of this flow together with Weyl's Law shows that, as the aspect ratio of the stadium ranges over an interval, most of the time we have a bounded number of eigenvalues in intervals $[\pi^2 - Mh^2, \pi^2 + Mh^2]$.

Why? There are $O(h^{-2})$ eigenvalues that cross the value π^2 in a given time interval, but each only spends time $O(h^2)$ in the interval $[\pi^2 - Mh^2, \pi^2 + Mh^2]$.

KAM system

Here we look at a one parameter variation $H + tP$ of a completely integrable Hamiltonian H, written in action-angle coordinates $(\theta_1, \theta_2, I_1, I_2)$ as $H(I_1, I_2)$, where $(\theta_1, \theta_2) \in \mathbb{T}^2$. The Hamiltonian flow is

$$\dot{\theta} = \omega(I) := \frac{\partial H}{\partial I}, \quad \dot{I} = 0.$$

We assume nondegeneracy, i.e. the frequency $\omega(I)$ is a locally invertible function of I. KAM theory tells us that the Lagrangian tori $\mathbb{T}(I)$ corresponding to 'sufficiently irrational' frequencies persist for some small time t, where sufficiently irrational corresponds to a Diophantine, or nonresonance, condition

$$|k \cdot \omega| \geq \frac{\kappa}{|k|^2}, \quad k \in \mathbb{Z}^2 \setminus \{0\}, \quad \kappa > 0 \text{ fixed, small.}$$

Assume that H and P have Gevrey regularity. We quantize $H + tP$ to a semiclassical pseudodifferential operator, $\hat{H} + t\hat{P}$. Using a quantum Birkhoff normal form, Popov constructed exponentially accurate quasimodes for small t. These are simply standard eigenfunctions $e^{im\cdot\theta}$ on the torus, mapped onto a nonresonant torus for $H + tP$ via the QBNF. For each nonresonant torus there is a sequence with semiclassical measure concentrated completely on that torus. Two key properties of Popov's construction:

- The quasimode error is $O(e^{-1/h^{\alpha}})$; actually, we only need $O(h^{\gamma})$ accuracy for fixed (but fairly large, e.g. $\gamma = 6$ will do) exponent γ.

- The nonresonant tori fill up a positive fraction of phase space. (So the density of quasimodes is comparable to that given by Weyl's Law.)

The variation of quasi-eigenvalue $H(I)$ corresponding to the torus with frequency $\omega = \omega(I)$ is, at time $t = 0$,

$$f(I) := \int_{\mathbb{T}_\omega} \dot{H}(\theta, I)\, d\theta.$$

We impose the generic condition that dH and df are linearly independent functions of I. This means that (H, f) furnish local coordinates in place of (I_1, I_2).

Using this condition, we are able to prove the following key property: if two quasi-eigenvalues $H(I_1; t)$ and $H(I_2; t)$, corresponding to two nonresonant frequencies ω_1, ω_2, are very close, then their time-derivatives are not close. That allows us to conclude that for most values of time t, all the quasi-eigenvalues are separated by some amount Mh^6.

Now recall that there are around $\sim h^{-2}$ quasi-eigenvalues, all distinct up to Mh^6 for the typical t. By the pigeonhole principle, there can't be more than a fixed number R of eigenvalues in each interval $[\lambda - Mh^6, \lambda + Mh^6]$. Thus we have obtained bounded multiplicity and the argument to obtain scarring is as described above.