GEOMETRIC FLOWS IN SYMPLECTIC GEOMETRY

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Motivation from String Theory

- From time immemorial, the laws of nature at its most fundamental have been a source of inspiration for geometry and the theory of partial differential equations: electromagnetism, the weak and the strong interactions are governed by Yang-Mills equations, and general relativity by Einstein's equation.
- But the grand dream of theoretical physics is a unified theory of all interactions, of which the prime candidate is the five string theories, unified themselves into M Theory.
- Very early on, equations for the heterotic string had been proposed by Candelas, Horowitz, Strominger, and Witten, and they ushered in the need for canonical metrics, more specifically vector bundles equipped with a Hermitian-Einstein metric over a Calabi-Yau manifold.
- But over the years, motivated by mirror symmetry and developments in string theories themselves, it appeared that the equations from the other string theories would be important as well. Notably, one new feature of theirs is the implicit requirement of the existence of a covariantly constant spinor field, and the corresponding emergence of new geometric structures. Many such equations have been proposed in the physics literature, and the simplest ones have been formulated by L.S. Tseng and S.T. Yau.
- In this talk, we shall focus on the equations for the Type IIA string. One characteristic feature of these equations is that they take place on a 6-dimensional symplectic manifold. This is joint work with T. Fei, S. Picard, and X.W. Zhang.

Type IIA structures and Type IIA flow

- The Type IIA equations, as proposed by L.S. Tseng and S.T. Yau are as follows. Fix a 6-dimensional compact manifold X, with a symplectic form ω. We look for a primitive 3-form φ, a metric g_φ, and a current ρ_A satisfying

$$d\Lambda d(|\varphi|^2 \star \varphi) - \rho_A = 0, \quad d\varphi = 0,$$

  where Λ is the contraction with respect to ω, $\star$ and $|\cdot|$ are the Hodge star operator and the norm with respect to g_φ, and ρ_A is the Poincare dual of a linear combination of special Lagrangians.

- In this talk, we restrict ourselves to solutions with $\rho_A = 0$. In this case, we can try and find a solution to the Type IIA system of equations as the stationary points of a geometric flow.

- To do so, we recall that around 2000, Hitchin had shown that on any 6-dimensional manifold, one can associate, algebraically, to a generic 3-form φ an almost-complex structure J_φ. In our case, we have a symplectic form ω, and it makes sense to impose 3 more conditions: (a) the form φ is primitive, i.e. $\Lambda\varphi = 0$, which makes the form $g_\varphi(U, V) = \omega(U, J_\varphi V)$ Hermitian; (b) the form φ is positive, in the sense that g_φ is positive, and is hence a metric; and (c) the form φ is closed.

  Under such conditions, we say that the pair (ω, φ) is a Type IIA structure. Note that a Type IIA structure is in particular an almost-Kähler 3-fold.

The Type IIA flow (T. Fei, P., S. Picard, and X.W. Zhang)

We look for solutions of the above equation as stationary points of the following flow of 3-forms φ,

$$\partial_t \varphi = d\Lambda d(|\varphi|^2 \star \varphi)$$

for any initial data φ_0 which is positive, closed, and primitive. The underlying metric is the metric g_φ described previously which makes the triple $(J_\varphi, \omega, g_\varphi)$ into an almost-Kähler manifold.

Formally, this flow preserves the closedness of the form φ (the right hand side is a closed form), as well as the primitiveness of φ. Thus, as long as the flow exists and φ stays positive, it is a flow of Type IIA structures.

We shall

- Establish all these properties of the Type IIA flow
- Derive the corresponding flow of the metrics g_φ
- Derive Shi-type estimates
- Illustrate the properties of the Type IIA flow in examples

Connections on an almost-complex manifold

- Recall that associated to an almost-complex structure J is its Nijenhuis tensor $N \in \Lambda^2(M) \otimes T(M)$, defined by

$$N(X,Y) = \frac{1}{4}([JX,JY] - J[JX,Y] - J[X,JY] - [X,Y])$$

M is a genuine complex manifold if and only if $N = 0$.

- A Hermitian metric $g(X,Y)$ is a metric compatible with J in the sense $g(JX,JY) = g(X,Y)$. This is equivalent to the form ω defined by $\omega(X,Y) = g(X,JY)$ being antisymmetric, $\omega(X,Y) = -\omega(Y,X)$. Once a metric is fixed, important unitary connections are the following.
- The Levi-Civita connection ∇, characterized by zero torsion. However, it may not respect the almost-complex structure J.
- The Chern connection on almost-complex manifolds, characterized by

$$\nabla^C_{\bar{U}} V = [\bar{U}, V]^{1,0}, \quad U, V \in T^{1,0}(X)$$

- The Kähler case: this is the ideal situation, when the complex structure J is integrable, so that $N = 0$, and the form ω is symplectic, that is, $d\omega = 0$. In this case, the Chern connection coincides with the Levi-Civita connection.
- The Gauduchon line: on a general almost-Hermitian manifold, we still have the Chern connection, but we have to introduce a projected Levi-Civita connection, which now preserves the complex structure (i.e. $\mathcal{D}J = 0$). The Gauduchon line is a line of unitary connections $\mathcal{D}^{(t)}$ preserving the almost-complex structure J, which passes through these two connections.

Explicit formulas for connections on the Gauduchon line

$$\mathcal{D}_i^{(t)} X^m = \nabla_i X^m + g^{mk}(-N_{ijk} - V_{ijk} + tU_{ijk})X^j$$

where we have set $N_{ijk} = g_{im} N^m{}_{jk}$, $d\omega^c = -Jd\omega$, and defined the expressions U_{ijk} and V_{ijk} by

$$U^m{}_{bc} = \frac{1}{4}((d^c\omega)^m{}_{bc} + (d^c\omega)^m{}_{jk} J^j{}_b J^k{}_c)$$

$$V^m{}_{bc} = \frac{1}{4}((d^c\omega)^m{}_{bc} - (d^c\omega)^m{}_{jk} J^j{}_b J^k{}_c)$$

In this parametrization, $\mathcal{D}^{(1)}$ is the Chern connection, $\mathcal{D}^{(0)}$ is the projected Levi-Civita connection, and the connection $\mathcal{D}^{(-1)}$ is the Yano-Bismut connection, which is the unitary connection characterized by its torsion tensor being anti-symmetric in all of its indices.

- The almost-Kähler case: this is the case when $d\omega = 0$. Then $d^c\omega = -Jd\omega = 0$, and the Gauduchon line collapses to a point, and the Chern and the projected Levi-Civita connections coincide.
- However, if we maintain the almost-complex structure J fixed, and perform a Weyl transformation on the metric $\tilde{g} = f^2 g$, then we need to perform a corresponding Weyl transformation $\tilde{\omega} = |f|^2\omega$ on the symplectic form, in order for $(J, \tilde{\omega}, \tilde{g})$ to remain a compatible almost-Hermitian triplet. But under such transformation, even if ω is closed, the form $\tilde{\omega}$ is no longer closed. Thus, the Gauduchon line corresponding to $(J, \tilde{\omega}, \tilde{g})$ is then a whole line. We shall encounter this situation in the study of Type IIA structures.

Theorem 1 Let (ω, φ) be any Type IIA structure. Then the following holds:

(a) The Nijenhuis tensor has only 6 independent components.

(b) Set $\Omega_\varphi = \varphi + iJ_\varphi\varphi$. Then

$$\mathcal{D}^{0,1}\Omega_\varphi = 0$$

where $\mathcal{D}$ is the Chern connection (recall that $d\omega = 0$, so there is a single connection $\mathcal{D}$ on the Gauduchon line.) Thus Ω_φ is, formally, holomorphic with respect to J_φ.

(c) Set $\tilde{g}_\varphi = |\varphi|^2 g_\varphi$. Then

$$\tilde{\mathfrak{D}}\left(\frac{\Omega_\varphi}{|\Omega_\varphi|}\right) = 0$$

where $\tilde{\mathfrak{D}}$ is the projected Levi-Civita connection of the metric $\tilde{g}_\varphi$, with respect to the almost-complex structure J_φ. Thus Type IIA geometry has SU(3) holonomy, but with respect to the connection $\tilde{\mathfrak{D}}$.

Recall that, since $d(|\varphi|^2\omega) \neq 0$, there is a whole line of inequivalent connections on the Gauduchon line, and it is particularly important for (c) in the above theorem that the connection be the projected Levi-Civita connection with respect to J_φ and $\tilde{\omega} = |\varphi|^2\omega$.

Note also that a naive counting of tensors in $\Lambda^2 \otimes T(M)$ in dimension 6 would give 126 components. Thus the fact that the Nijenhuis tensor has only 6 independent components, is indicative of a very rich structure. This in fact results in many special identities which will be crucial later to getting estimates for the flow.

Short-time existence of the flow

- A now well-known weakly parabolic system which admits nevertheless at least a short-time solution for any initial data is the Ricci flow. In his original 1982 paper, Hamilton established this by a highly non-trivial adaptation of the Nash-Moser scheme. The degeneracy in the Ricci flow can be traced back to the fact that the flow is invariant under diffeomorphisms. De Turck subsequently introduced a different way of lifting this degeneracy by a reparametrization with a time-dependent vector field.
- We can try the procedure of De Turck, and reparametrize the Type IIA flow by a time-dependent vector field V. The new difficulty is that the symplectic form ω is also reparametrized, and is now no longer time-independent. Thus the flow of φ has been transformed into a flow of pairs (φ, ω), given by

$$\partial_t \varphi = d\Lambda d(|\varphi|^2 \star \varphi) + d(\iota_V \varphi), \quad \partial_t \omega = d(\iota_V \omega)$$

  Unfortunately, this new flow is still only weakly parabolic, and the short-time existence can still not be established in this manner.
- To address this new difficulty, we introduce the following regularized flow,

$$\partial_t \varphi = d\Lambda d(|\varphi|^2 J_\varphi \varphi) - BdJ_\varphi d(|\varphi|^2 \Lambda J_\varphi \varphi) + d(\iota_V \varphi), \quad \partial_t \omega = d(\iota_V \omega)$$

  for a fixed positive constant B. This flow can be shown to admit short-time existence and preserve primitiveness. Thus φ remains primitive, so does $J_\varphi \varphi$, and $\Lambda J_\varphi \varphi = 0$. Thus the regularization term is 0, and the regularized flow reduces to the (reparametrized) original flow.

The Type IIA flow as a perturbation of the Ricci flow

Further analysis of the Type IIA flow requires working out the corresponding flow of metrics. The Type IIA flow is equivalent to the following flow of the pair (g_{ij}, u) where $u = \log |\varphi|^2$ is a scalar field,

$$\partial_t g_{ij} = e^u[-2R_{ij} + 2\nabla_i\nabla_j u - 4N^{kp}{}_i N_{pkj} + u_i u_j - u_p u_q J^p{}_i J^q{}_j + 4u_p(N_i{}^p{}_j + N_j{}^p{}_i)]$$

$$\partial_t u = e^u[\Delta u + 2(|\nabla u|^2 + |N|^2)]$$

The evolution of the Nijenhuis tensor

A distinctly new issue which arises with this flow is the flow of the Nijenhuis tensor, which has no analogue in e.g. Ricci flows, and which does not appear predictable from the flow of the metric. Nevertheless, we find that the norm of the Nijenhuis tensor also obeys a parabolic flow,

$$(\partial_t - e^u\Delta)|N|^2 = e^u[-2|\nabla N|^2 + (\nabla^2 u) \star N^2 + Rm \star N^2 + N \star \nabla N \star (N + \nabla u)] + \cdots]$$

Shi-type estimates

For a flow to be of practical value, it is essential that its singularities can be traced only to a finite number of geometric quantities blowing up. This turns out to be indeed the case for the Type IIA flow. In fact, we can show that, if on some time interval $[0, T)$, we have

$$|\log \varphi| + |Rm| \leq A$$

then for any α, we have

$$|\nabla^\alpha \varphi| \leq C(A, \alpha, T, \varphi(0))$$

In particular, if $[0, T)$ is the maximum time internal of existence, we must have

Examples of long-time behavior

- The integrable case

  If the almost-complex structure J_{φ_0} is integrable, then the Type IIA flow preserves the integrability, it exists for all times, and it converges as $t \to +\infty$ to a Kähler Ricci-flat metric.

  In fact, the flow turns out then to be equivalent to the dual flow to the Type IIB flow, as introduced earlier by Fei and Picard.

- An explicit nilmanifold example

  We consider the Type IIA flow on the nilmanifolds constructed by de Bartholomeis and Tomassini. There the basic structure is a nilpotent Lie group, and the natural ansatze for φ are preserved, and reduce the Type IIA flow as a system of ODE's which can be solved explicitly. We find in this way examples where the flow exists for all times, but the limit $\lim_{t\to\infty} J_t$ does not exist, but

$$\lim_{t\to\infty} |N|^2 = 0$$

- **An explicit solvmanifold example**

  Another very instructive example is provided by the symplectic half-flat structures on the solvmanifold introduced by Tomassini and Vezzoni. Some natural ansatze for φ are again preserved by the Type IIA flow, which reduces then to a system of ODE's which can be solved exactly. We find then the following interesting phenomena:

  • For any initial data, the flow for φ develops singularities in finite time. However, the limits of J_φ and g_φ continue to exist.

  • For certain initial data, φ is a self-expander, while J is stationary, and in fact a critical point of the energy functional of Blair-Ianus and Le-Wang.

  • For other initial data, as t approaches the maximum time of existence T, the limit $\lim_{t\to T} J$ exists, and is a harmonic almost-complex structure, and in particular a minimizer for $|N|^2$.

- **Other properties verifiable in the half-flat case**

  These include the construction of ancient and immortal solutions of the flow by A. Raffero, and the explicit construction of the expected duality between the Type IIA and the Type IIB flows.

- Dynamical stability of the Type IIA flow
  The Type IIA flow is dynamically stable, in the following sense. Let $(\bar{\varphi}, \bar{\omega})$ be a Ricci flat Type IIA structure. Then there exists a constant $\epsilon > 0$ with the following property. For any (φ, ω) Type IIA structure satisfying

  $$|\varphi - \bar{\varphi}|_{W^{10,2}} + |\omega - \bar{\omega}|_{W^{10,2}} < \epsilon$$

  the Type IIA flow with intial data (φ, ω) converges to a Ricci-flat Type IIA structure (φ_∞, ω).

  The proof requires a classification of steady Type IIA solitons, and an adaptation of the methods of N. Sesum for the proof of the dynamical stability of the Kähler-Ricci flow.
- An application to symplectic geometry
  From the dynamic stability of the Type IIA flow, we can deduce the following theorem: assume that $c_1(M, \bar{\omega}) = 0$. If M is a compact symplectic 6d manifold, and $(M, \bar{\omega})$ admits a compatible complex structure, then there exists $\epsilon > 0$ with the property that, for any symplectic structure ω with $|\omega - \bar{\omega}|_{W^{10,2}} < \epsilon$, the manifold (M, ω) admits a compatible complex structure.
- More recently, Streets and Tian have announced a generalization of this theorem to all dimensions, using similar arguments with the stability of their Hermitian curvature flow.

Further questions

Singularities of Type IIA structures

The Type IIA structure suggests that, even an almost-complex structure which is not integrable may be quite rich. Some very speculative questions in this direction may be worth investigating:

- Just as natural singularities of complex manifolds are analytic varieties, we need to understand the singularities of Type IIA structures. Particularly interesting cases would be singularities supported on Lagrangian submanifolds and calibrated submanifolds.
- This would be directly relevant to the general case with $\rho_A \neq 0$. This case is really novel in geometric flows, as it is reminiscent of a free boundary problem.
- Can almost-integrable structures with richer function theory be identified by conditions of their Nijenhuis tensor more general than just vanishing identically ?

Other flows in symplectic geometry

For most of the geometric flows in geometric analysis which are weakly parabolic but not strictly parabolic, e.g. the Ricci flow, the Yang-Mills flow, spinor flows, the Type IIA flows, etc. we can still establish their short-time existence for arbitrary data. However there are some flows which still pose a challenge in this regard. Some notable examples are the following.

- The Hitchin functional and the corresponding gradient flow: on any compact 6-d manifold M, Hitchin introduced the functional on 3-forms

$$H(\varphi) = \frac{1}{2}\int_M \varphi \wedge (J_\varphi \varphi)$$

and showed that $\delta H = 0$ is equivalent to $d(J_\varphi \varphi) = 0$, and to J_φ being integrable. If M is equipped with a symplectic form and φ is primitive, then we get a compatible metric, and we can consider the gradient flow of $H(\varphi)$. It turns out that it is given by

$$\partial_t \varphi = dd^\dagger \varphi$$

Recall that Bryant's Laplacian flow on 7-manifolds is given exactly by this formula, so the Hitchin gradient flow can be viewed as the 6-manifold version of the 7-manifold Bryant's Laplacian flow. Remarkably, the Hitchin gradient flow can be cast in turn as a limiting version of the Type IIA flow

$$\partial_t = d\Lambda d(\star\varphi)$$

- The dual Ricci flow: This flow was introduced by T. Fei and P., as the symplectic dual of the Ricci flow on a complex manifold. It is a flow of positive and primitive 3-forms on a 6-dimensional symplectic manifold M, which can be worked out to be given by

$$\partial_t \varphi = d\Lambda d(\log|\varphi| \star \varphi)$$

which is similar to the Hitchin gradient flow, but this time with a $\log|\varphi|$ inserted.

- The regularized Type IIA flow: on the other hand, it is not hard to see that, for any $\epsilon > 0$, the modified Type IIA flow defined by

$$\partial_t \varphi = d\Lambda d(|\varphi|^\epsilon \star \varphi)$$

does admit short-time existence, and it is tempting that, from the possibly renormalized solutions φ_ϵ to these flows, we can extract a finite regularized limit, which can serve as solution to either the gradient Hitchin flow or the dual Ricci flow.

- All this raises the possibility of either identifying the specific conditions for the short-time existence of a given flow, or for finding global (stability ?) conditions on the underlying manifold under which a given flow would admit short-time existence for arbitrary initial data.