Towards $2D$ random Kähler geometry

Conference in honor of Steve Zelditch

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Based on joint works with Hubert Lacoin (IMPA Rio)

Plan of the talk

Part I: 2*d* quantum gravity

- Which random geometries for 2*d* quantum gravity?
- Random geometries and classical functionals
- Liouville path integral

Part II: Mabuchi path integral

- Main result
- Construction
- Conjectures

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Euclidean 2d quantum gravity

Let M be a 2d manifold (without boundary) and $\mathcal{M}$ be the set of Riemannian metrics g on M (modulo diffeomorphisms).

Main question: give sense to the integral on $\mathcal{M}$

$$\int_{\mathcal{M}} \mathcal{Z}_m(g)\mathcal{D}g$$

where

- $\mathcal{D}g$ is the volume form of the L^2-metric on $\mathcal{M}$:

$$||\delta g||^2 = \int_M \mathrm{tr}(g^{-1}\delta g g^{-1}\delta g)\mathrm{d}v_g$$

  Other choices are possible!!! $\rightarrow$ Bilal-Ferrari-Klevtsov-Zelditch 11-14.

- $\mathcal{Z}_m(g)$ is the partition function of some model of statistical physics, called **matter field**, on the Riemannian manifold (M, g). Typically

$$\mathcal{Z}_m(g) = \Big(\frac{\det(-\triangle_g)}{v_g(M)}\Big)^{-\mathbf{c}_{\rm mat}/2}$$

  where $\triangle_g$ is the Laplacian and $\mathbf{c}_{\rm mat}$ is a constant called central charge.

Polyakov/DDK ansatz

Polyakov/DDK ansatz (80s): if the matter field is a CFT (take $\mathcal{Z}_m(g)$ as above) then the random metric g has law ruled by Liouville CFT (D-K-R-V 14').

Question: What if matter fields (slightly) move away from conformal symmetries?

Idea : Law of the random metric g determined by the way matter fields react to background changes of metrics (Weyl anomaly).

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Conformal matter: partition function $\mathcal{Z}_m$ satisfies **Polyakov's anomaly formula:** for $\omega : M \to \mathbb{R}$ smooth

$$\ln \frac{\mathcal{Z}_m(e^{\omega} g_0)}{\mathcal{Z}_m(g_0)} = \frac{\mathbf{c}_{\mathrm{mat}}}{96\pi} S_{\mathrm{L}}^{\mathrm{cl},0}(g_0, \omega)$$

where $S_{\mathrm{L}}^{\mathrm{cl},0}(g_0, \omega) := \int_M \left(|d\omega|_{g_0}^2 + 2R_{g_0}\omega\right) \mathrm{dv}_{g_0}$ is the classical Liouville functional and $\mathbf{c}_{\mathrm{mat}} \leq 1$ is called **central charge** of the matter field.

Polyakov/DDK ansatz

Polyakov/DDK ansatz (80s):

$$\int_{\mathcal{M}} \mathcal{Z}_m(g)\mathcal{D}g = \int \mathcal{Z}_m(g_\tau)\mathcal{Z}_{FP}(g_\tau)\mathcal{Z}_L(g_\tau)D\tau \tag{1}$$

where

- $D\tau$ is Weil-Petersson volume form and g_τ family of metrics.
- $\mathcal{Z}_{FP}(g_\tau)$ Fadeev-Popov ghosts, i.e. determinant of some Laplacian on forms (anomaly with constant -26)
- $\mathcal{Z}_L(g_\tau)$ Liouville partition function (anomaly with constant $\mathbf{c}_{\mathrm{L}}$)

Changing $g_\tau \to e^{\omega_\tau} g_\tau$ does not change r.h.s. of (1) if $\mathbf{c}_{\mathrm{mat}} - 26 + \mathbf{c}_{\mathrm{L}} = 0$.

Simplest model of massive matter

- **Massive GFF:** partition function ($q \in \mathbb{R}$ and mass $m > 0$)

$$Z(g,q,m) = \int \exp\Big(- \frac{1}{4\pi} \int_M \big(|dX|_g^2 + iq\, R_g X + m^2 X^2 \big)\, \mathrm{dv}_g \Big) DX$$

problem: hard to understand the coupling of quantum gravity with this model

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- $m = 0$ model: remove divergencies to compute the $m \to 0$ partition function

$$Z_0(g, q) := \lim_{m \to 0} Z(g, q, m)$$

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Weyl anomaly: if $\omega : M \to \mathbb{R}$ smooth and $\mathbf{h}$ genus of M

$$\ln \frac{Z_0(e^{\omega} g_0, q)}{Z_0(g_0,q)} = \frac{1-6q^2}{96\pi} S_{\mathrm{L}}^{\mathrm{cl},0}(g_0,\omega) + \frac{q^2(1-\mathbf{h})}{4\pi} S_M(g_0,\omega)$$

where S_M the **Mabuchi K-energy**.

$\to$ Bilal-Ferrari-Klevtsov-Zelditch 11-14.

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- Upshot: Polyakov/DDK ansatz tells us that coupling $2d$ quantum gravity to this $m = 0$ GFF produces a random geometry governed by a path integral involving Liouville+K-energy.

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What is a (natural) random geometry?

- In geometry, basic objects in view of classification are manifolds with uniformized curvature
- such manifolds can generally be found by solving variational problems: one looks for the minimizer of some functional

$$\varphi \in \Sigma \mapsto S(\varphi).$$

- Corresponding random geometry is a functional measure (path integral) on Σ

$$e^{-S(\varphi)} D\varphi$$

where $D\varphi$ is the ”Lebesgue measure” on Σ.

Approach inherited from Feynmann's view on quantum mechanics.

Classical Liouville functional

Let $\gamma, \mu > 0$ be some parameters.

The map $\omega : M \to \mathbb{R}$ such that the metric $g = e^{\gamma\omega} g_0$ has uniformized curvature

$$R_g = -2\pi\mu\gamma^2$$

is a critical point of the Liouville functional

$$\omega \mapsto S_L(g_0, \omega) = \frac{1}{4\pi} \int_M \Big(|d\omega|^2_{g_0} + Q_c R_{g_0} \omega + 4\pi\mu e^{\gamma\omega} \Big) \, \mathrm{dv}_{g_0}$$

with

$$Q_c = \frac{2}{\gamma}$$

Notations: $\triangle_g =$ *Laplacian*, R_g=Ricci curvature, v_g=volume form

Kähler geometry

Consider a $2d$ manifold M equipped with a Riemannian metric g_0.

- Kähler potential ϕ of the metric $g = e^{\omega} g_0$ w.r.t. g_0 defined by

$$\frac{e^{\omega}}{\mathrm{v}_g(M)} - \frac{1}{\mathrm{v}_{g_0}(M)} = \tfrac{1}{2}\triangle_{g_0}\phi$$

  Another parametrization of the set of metrics that allows one to translate the search for constant curvature metrics in terms of complex Monge-Ampère equation.

- This has led to classification of Kähler manifolds (any dimension) with successive works by Aubin, Yau, Tian, Donaldson etc (1978-2015).

Notations: $\triangle_g = \mathit{Laplacian}$, R_g=Ricci curvature, v_g=volume form

Mabuchi K-energy

- Let ϕ be the Kähler potential of the metric $g = e^{\omega} g_0$ w.r.t. g_0

$$\frac{e^{\omega}}{\mathrm{v}_g(M)} - \frac{1}{\mathrm{v}_{g_0}(M)} = \tfrac{1}{2}\triangle_{g_0}\phi$$

- Definition of Mabuchi K-energy

$$S_{\mathrm{M}}(g_0, g) = \int_M \Big(2\pi(1-\mathbf{h})\phi\triangle_{g_0}\phi + \big(\frac{8\pi(1-\mathbf{h})}{\mathrm{v}_{g_0}(M)} - R_{g_0}\big)\phi + \frac{2}{\mathrm{v}_g(M)}\omega e^{\omega}\Big)\mathrm{d}\mathrm{v}_{g_0}$$

  with $\mathbf{h}$:= genus of M.

- Critical points give metrics $g := e^{\omega} g_0$ with uniformized curvature

Notations: $\triangle_g = \mathit{Laplacian}$, R_g=Ricci curvature, v_g=volume form

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Random Riemannian geometry (or Liouville CFT)

Consider a Riemann surface M equipped with a metric g_0, and parameters $\mu > 0$, $\gamma \in (0, 2)$.

Quantum Liouville theory is a measure formally defined by

$$\langle F \rangle_{L,g_0} := \int F(\varphi) e^{-\mathcal{S}_L(g_0,\varphi)} D\varphi$$

where

- $\mathcal{S}_L$ is the quantum Liouville functional

$$\mathcal{S}_L(g_0, \varphi) = \frac{1}{4\pi} \int_M \Big(|d\varphi|^2_{g_0} + QR_{g_0}\varphi + 4\pi\mu e^{\gamma\varphi} \Big) d\mathrm{v}_{g_0}$$

- $D\varphi$ is the "Lebesgue measure" on the space of maps $\varphi : M \to \mathbb{R}$.
- Q is a parameter tuned at its quantum value

$$Q = \frac{2}{\gamma} + \frac{\gamma}{2}$$

David, Guillarmou, Kupiainen, R., V. (2014-2016):
Construction on compact Riemann surfaces

Random Riemannian geometry (or Liouville CFT)

- Random geometry is then understood as associated to the random metric tensor

$$e^{\gamma\varphi}g_0$$

where the random "function" φ has probability law characterized by functional expectations

$$\mathbb{E}[F(\varphi)] = \frac{1}{Z}\langle F\rangle_{L,g_0}$$

with

$$\langle F\rangle_{L,g_0} := \int F(\varphi)e^{-\mathcal{S}_L(g_0,\varphi)}D\varphi$$

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- As it turns out, φ is not a fairly defined function a.s. $\Rightarrow$ rich multifractal geometry
  - **Volume form**: uses Gaussian multiplicative chaos (GMC) theory for $\gamma \in (0, 2)$

    KAHANE (1985)
  - **Distance**: understood for $\gamma \in (0, 2)$

    DING, DUBÉDAT, DUPLANTIER, FALCONET, GWYNNE, MILLER, SHEFFIELD,... (2014-2022)

Symmetries of CFTs are encoded in the way they react to changes of background metrics

Conformal anomaly (David-Kupiainen-Rhodes-Vargas 14')

Consider a conformal metric $g = e^{\omega} g_0$ then

$$\langle F \rangle_{\mathrm{L},g} = \langle F(\cdot - \tfrac{Q}{2}\omega) \rangle_{\mathrm{L},g_0} \exp\Big(\frac{\mathbf{c}_{\mathrm{L}}}{96\pi} S_{\mathrm{L}}^{\mathrm{cl},0}(g_0, \omega) \Big) \tag{2}$$

where $S_{\mathrm{L}}^{\mathrm{cl},0}$ is the classical Liouville functional (with $\mu = 0$)

$$S_{\mathrm{L}}^{\mathrm{cl},0}(g_0, \omega) := \int_M \big(|d\omega|_{g_0}^2 + 2 R_{g_0} \omega \big) \mathrm{dv}_{g_0}, \tag{3}$$

and $\mathbf{c}_{\mathrm{L}} = 1 + 6Q^2$ is the *central charge* of the Liouville theory.

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Contains a great deal of information about the theory:

- connection with quantum gravity models (Polyakov, David-Distler-Kawai,...)
- Question: can we come up with a path integral producing a further Mabuchi term?

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Random Kähler geometry

-Riemann surface M with genus **h** and metric g_0

-Bilal-Ferrari-Klevtsov-Zelditch's proposal: construct the path integral

$$F \mapsto \int_{\{\phi:M\to\mathbb{R}\}} F(\phi) e^{-\beta\mathcal{S}_{\mathrm{M}}(g_0,g)-\mathcal{S}_L(g_0,\varphi)} D\phi$$

where $g = e^{\gamma\varphi} g_0$ and

- $\mathcal{S}_L$ is the quantum Liouville functional and $\mathcal{S}_M$ is the Mabuchi K-energy
- $g = e^{\gamma\varphi} g_0$ and ϕ is the Kähler potential of the metric g w.r.t. g_0

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-Our approach: change the integration variable $\phi \to \varphi$

- Jacobian of the form $e^{-\mathcal{S}_L(g_0, \varphi)}$
- leads to the study of the path integral

$$F \mapsto \int_{\{\varphi: M \to \mathbb{R}\}} F(\varphi) e^{-\beta \mathcal{S}_{\mathrm{M}}(g_0, g) - \mathcal{S}_L(g_0, \varphi)} D\varphi$$

Random Kähler geometry

Riemann surface M with genus $\mathbf{h}$ and metric g_0

Path integral (measure on a Sobolev type space)

$$\langle F\rangle_{\mathrm{ML},g_0} := \int_{\{\varphi:M\to\mathbb{R}\}} F(\varphi)e^{-\beta\mathcal{S}_{\mathrm{M}}(g_0,e^{\gamma\varphi}g_0)-\mathcal{S}_L(g_0,\varphi)}D\varphi$$

where $\beta > 0$

- $\mathcal{S}_L$ is the quantum Liouville functional with $\mu > 0$, $\gamma \in (0,2)$

$$\mathcal{S}_L(g_0,\varphi) = \frac{1}{4\pi}\int_M \Big(|d\varphi|_g^2 + QR_g\varphi + 4\pi\mu e^{\gamma\varphi}\Big)d\mathrm{v}_g, \qquad Q = \frac{2}{\gamma} + \frac{\gamma}{2}$$

- $\mathcal{S}_M$ is the quantum Mabuchi K-energy: if $g = e^{\gamma\varphi}g_0$

$$\mathcal{S}_{\mathrm{M}}(g_0,g) = \int_M \Big(2\pi(1-\mathbf{h})\phi\triangle_{g_0}\phi + (\frac{8\pi(1-\mathbf{h})}{\mathrm{v}_{g_0}(M)} - R_{g_0})\phi + \frac{2}{1-\frac{\gamma^2}{4}}\frac{1}{\mathrm{v}_g(M)}\gamma\varphi e^{\gamma\varphi}\Big)\mathrm{d}\mathrm{v}_{g_0}$$

and ϕ is the Kähler potential of the metric $e^{\gamma\varphi}g_0$ w.r.t. g_0

Existence: main results

Assume M has genus $\mathbf{h} \geq 2$ and $\gamma \in (0,1)$ and $\beta > 0$.

Theorem (LRV '18)

Probabilistic construction of the path integral

$$\langle F \rangle_{\mathrm{ML},g_0} := \int F(\varphi) e^{-\beta \mathcal{S}_{\mathrm{M}}(g_0, e^{\gamma\varphi} g_0) - \mathcal{S}_{\mathrm{L}}(g_0,\varphi)} D\varphi$$

This path integral has finite mass, i.e. $\langle 1 \rangle_{\mathrm{ML},g_0} < +\infty$ provided that the Mabuchi coupling constant is small enough

$$\beta \in \Big(0, \frac{\mathbf{h}-1}{2}\big(\tfrac{4}{\gamma^2} - \tfrac{\gamma^2}{4}\big)\Big)$$

Remark:

- the constraint on β is not a technical restriction, it is a **topological obstruction**!
- QFT with global conformal invariance but no locality...not a CFT!

Weyl anomaly (LRV 18')

Consider a conformal metric $g = e^{\omega} g_0$ then

$$\langle F \rangle_{\mathrm{ML},g} = \langle F(\cdot - \tfrac{Q}{2}\omega) \rangle_{\mathrm{ML},g_0} \exp\Big(\frac{\mathbf{c}_{\mathrm{L}}}{96\pi} S_{\mathrm{L}}^{\mathrm{cl},0}(g_0, \omega) + \beta S_{\mathrm{M}}(g_0, g) \Big) \tag{4}$$

where $S_{\mathrm{L}}^{\mathrm{cl},0}, S_{\mathrm{M}}$ are respectively classical Liouville functional (with $\mu = 0$) and classical Mabuchi K-energy.

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where $S^{\mathrm{cl},0}_{\mathrm{L}}$, S_{M} are respectively classical Liouville functional (with $\mu = 0$) and classical Mabuchi K-energy.

String susceptibility (LRV 18')

Under $\langle \cdot \rangle_{\mathrm{ML},g_0}$, the "volume of the manifold" $\int_M e^{\gamma\varphi} \mathrm{dv}_{g_0}$ has Gamma law $\Gamma(s,\mu)$.
The area scaling exponent s, called string susceptibility, has the expression

$$s := \frac{2Q}{\gamma}(\mathbf{h} - 1) - \frac{2\beta}{1 - \frac{\gamma^2}{4}}$$

Remark: agrees with the asymptotic expansion $\gamma \to 0$ given in physics by Bilal-Ferrari-Klevtsov-Zelditch.

Plan of the talk

Path integral for Liouville CFT

Riemann surface M, metric g_0

$$F \mapsto \int F(\Phi) e^{-\mathcal{S}_{\mathrm{L}}(g_0,\varphi)} D\varphi$$

Liouville action

$$\mathcal{S}_{\mathrm{L}}(g_0,\varphi) = \frac{1}{4\pi} \int_M \Big(|d\varphi|^2_{g_0} + QR_{g_0}\varphi + 4\pi\mu e^{\gamma\varphi} \Big) d\mathrm{v}_{g_0}$$

Parameters

$$\underline{\gamma \in (0,2)}, \quad Q = \frac{2}{\gamma} + \frac{\gamma}{2} \quad , \mu > 0$$

Gaussian Free Field on (M, g_0):

$$X_{g_0}(x) = \sqrt{2\pi} \sum_{n \geq 0} \frac{\alpha_n}{\sqrt{\lambda_n}} e_n(x)$$

with

- $(\alpha_n)_n$ iid standard Gaussians
- $(e_n)_n$ o.n.b. of eigenfunctions of Laplacian Δ_{g_0}

$$\Delta_{g_0} e_n = \lambda_n e_n, \qquad \int_M e_n \, \mathrm{dv}_{g_0} = 0$$

- Covariance $\mathbb{E}[X_{g_0}(x) X_{g_0}(x')] = G_{g_0}(x, x')$ Green function of the Laplacian.

Gaussian integral:

$$\boxed{\int_{\{\varphi : M \to \mathbb{R}\}} F(\varphi) e^{-\frac{1}{4\pi} \int_M |d\varphi|^2_{g_0} \mathrm{dv}_{g_0}} D\varphi = (\det{}'(\Delta_{g_0})/\mathrm{v}_{g_0}(M))^{-1/2} \int_{\mathbb{R}} \mathbb{E}\Big[F(c + X_{g_0})\Big] \, dc}$$

Gaussian Multiplicative chaos (GMC)

- Goal: construct a random measure formally given by
$$e^{\gamma X_{g_0}(x)}\,\mathrm{dv}_{g_0}(x).$$
Ill-defined as X_{g_0} is not a fairly defined function. At short scale
$$\mathbb{E}[X_{g_0}(x)X_{g_0}(x')] \approx \ln \frac{1}{d_{g_0}(x,x')}$$
- Call X_ϵ a regularization of the field X_{g_0} at scale ϵ
$$\mathbb{E}[X_\epsilon(z)X_\epsilon(z')] \approx \ln \frac{1}{d_{g_0}(z,z')+\epsilon}.$$

Theorem (Kahane 1985)

For $\gamma \in (0,2)$ there exists a non trivial random measure $\mathcal{G}_{g_0}^\gamma$ such that, almost surely, the limit
$$\lim_{\epsilon\to 0} \epsilon^{\frac{\gamma^2}{2}} e^{\gamma X_\epsilon(z)}\,\mathrm{v}_{g_0}(dz) = \mathcal{G}_{g_0}^\gamma(dz)$$
holds in the space of Radon measure. $\mathcal{G}_{g_0}^\gamma$ does not depend on the regularization.

Liouville path integral

Path integral defined by (assuming g_0 is uniformized)

$$\langle F\rangle_{L,g_0} := \int_{\mathbb{R}} e^{-2Q(1-\mathbf{h})c}\mathbb{E}\Big[F(c+X_{g_0})\exp\Big(-\mu e^{\gamma c}\mathcal{G}^{\gamma}_{g_0}(M)\Big)\Big]\,dc$$

where

- $\mathbf{h}$ is the genus of M
- X_{g_0} is a Gaussian Free Field under $\mathbb{E}$
- $\mathcal{G}^{\gamma}_{g}(M)$ is a Gaussian multiplicative chaos (GMC) formally understood as

$$\mathcal{G}^{\gamma}_{g_0}(M) = \int_M e^{\gamma X_{g_0}}\,\mathrm{dv}_{g_0}$$

Liouville-Mabuchi path integral: construction

Assuming g_0 is uniformized, defined by

$$\langle F\rangle_{\mathrm{ML},\mathrm{g}_0} := \int_{\mathbb{R}} e^{-2Q(1-\mathbf{h})c}\,\mathbb{E}\Big[e^{-\beta\mathcal{S}_{\mathrm{M}}(g_0,g)}F(c+X_{g_0})\exp\Big(-\mu e^{\gamma c}\mathcal{G}^{\gamma}_{g_0}(M)\Big)\Big]\,dc$$

with $g = e^{\gamma(c+X_{g_0})}g_0$ and

$$\mathcal{S}_{\mathrm{M}}(g_0,g) = \int_M \Big(2\pi(1-\mathbf{h})\Phi\triangle_{g_0}\Phi + \big(\frac{8\pi(1-\mathbf{h})}{\mathrm{v}_{g_0}(M)} - R_{g_0}\big)\Phi\Big)\mathrm{dv}_{g_0} + \frac{2}{1-\frac{\gamma^2}{4}}\frac{\mathcal{D}^{\gamma}_{g_0}(M)}{\mathcal{G}^{\gamma}_{g_0}(M)}$$

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Involves:

- Kähler potential of the "Liouville random metric"

$$\Phi(z) := -\frac{2}{\mathcal{G}^{\gamma}_{g}(M)}\int G_{g_0}(z,w)\mathcal{G}^{\gamma}_{g_0}(dw)$$

- $\mathcal{D}^{\gamma}_{g_0}(M)$ (formally $\int_M \gamma\varphi e^{\gamma\varphi}\mathrm{dv}_{g_0}$) is a variant of GMC that we call **derivative GMC**

$$\mathcal{D}^{\gamma}_{g_0}(M) := \lim_{\epsilon\to 0}\int_M (\gamma X_\epsilon(z) + \gamma^2\ln\epsilon)\epsilon^{\frac{\gamma^2}{2}}e^{\gamma X_\epsilon(z)}\,\mathrm{v}_{g_0}(dz)$$

Technical backbone

Establish negative exponential moments for the entropy

$$\forall \beta > 0, \quad \mathbb{E}\Big[\exp\Big(-\beta \frac{\mathcal{D}^{\gamma}_{g_0}(M)}{\mathcal{G}^{\gamma}_{g_0}(M)}\Big)\Big] < +\infty$$

Simple consequence of

- left **Gaussian concentration** for derivative GMC

$$\forall x > 0 \text{ large}, \quad \mathbb{P}(\mathcal{D}^{\gamma}_{g_0}(M) < -x) \leq C \exp\big(-C^{-1} x^2\big)$$

- sharp **small deviation** result for GMC (for some $s > 0$)

$$\forall x > 0 \text{ small}, \quad \mathbb{P}(\mathcal{G}^{\gamma}_{g_0}(M) < x) \leq C \exp\big(-C^{-1} |\ln x|^{\kappa} x^{-4/\gamma^2}\big)$$

Technical backbone

Establish negative exponential moments for the entropy

$$\forall \beta > 0, \quad \mathbb{E}\Big[\exp\Big(-\beta \frac{\mathcal{D}^{\gamma}_{g_0}(M)}{\mathcal{G}^{\gamma}_{g_0}(M)}\Big)\Big] < +\infty$$

Simple consequence of

- left **Gaussian concentration** for derivative GMC

$$\forall x > 0 \text{ large}, \quad \mathbb{P}(\mathcal{D}^{\gamma}_{g_0}(M) < -x) \leq C \exp\big(-C^{-1}\, x^2\big)$$

the field $X_{g_0} e^{\gamma X_{g_0}}$ is not bounded from below

- sharp **small deviation** result for GMC (for some $s > 0$)

$$\forall x > 0 \text{ small}, \quad \mathbb{P}(\mathcal{G}^{\gamma}_{g_0}(M) < x) \leq C \exp\big(-C^{-1} |\ln x|^{\kappa} x^{-4/\gamma^2}\big)$$

Plan of the talk

Part I: $2d$ quantum gravity

- Which random geometries for $2d$ quantum gravity?
- Random geometries and classical functionals
- Liouville path integral

Part II: Mabuchi path integral

- Main result
- Construction
- Conjectures

Conjecture I

Recall that the Liouville path integral is the invariant measure of the stochastic Ricci flow (see Dubedat/Shen)

$$\partial_t g = -2\mathrm{Ric}(g) - 2\lambda g + 2\kappa \xi_g$$

with $\kappa > 0$ and ξ_g L^2-white noise on the metrics on M, i.e.

$$\Big(f : \mathbb{R} \to C^\infty(M, S^2 T^* M)\Big) \mapsto \xi_g(f) \text{ Gaussian} \quad \text{and} \quad \mathbb{E}[\xi_g(f)^2] = \int_M \langle f_t, f_t \rangle_g \mathrm{dv}_g \, \mathrm{d}t.$$

Notations: $\triangle_g = \textit{Laplacian}$, R_g=Ricci curvature, $\bar{R}_g$=mean curvature, v_g=volume form

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- **Conjecture:** Liouville+Mabuchi path integral= invariant measure of the flow

$$\partial_t g = -2\mathrm{Ric}(g) - 2\lambda g - \beta\kappa^2 \frac{1}{\mathrm{v}_g(M)}(\psi_g + 1)g + 2\kappa\xi_g g$$

  with ψ_g the Ricci potential

$$-\triangle_g \psi_g = R_g - \bar{R}_g \qquad \text{and} \qquad \int_{\mathbb{R}}\int_M \psi_g\, d\mathrm{v}_g = 0.$$

Notations: $\triangle_g = \textit{Laplacian}$, R_g=Ricci curvature, $\bar{R}_g$=mean curvature, v_g=volume form

Conjecture II

Recall that

$$Z(g,q,m) = \int \exp\Big(-\frac{1}{4\pi}\int_M \big(|dX|_g^2 + iq\,R_g X + m^2 X^2\big)\,\mathrm{dv}_g\Big) DX$$

and

$$Z_0(g,q) := \lim_{m\to 0} Z(g,q,m)$$

- **Random planar maps:** put the flat metric on the faces of a triangulation T with N faces conformally embedded onto the manifold M to get a metric g_T on M. Pick such a T at random with law

$$\mathbb{P}_N(T) = cZ_0(g_T,q)$$

In the scaling limit $N\to\infty$, the law of g_T is described by the Liouville+Mabuchi path integral with

$$Q^2 = 4 + q^2 \quad \text{and} \quad \beta = \frac{q^2(\mathbf{h}-1)}{4\pi}$$




Happy birthday Steve!