QUANTUM ERGODICITY AND BEYOND

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CONFERENCE IN HONOR OF STEVE ZELDITCH

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MY FIRST ENCOUNTERS WITH STEVE WERE IN NEW YORK CITY IN THE EARLY 80'S. I RECALL LIVELY DISCUSSIONS ABOUT HELGASON'S THEOREMS FOR EIGENFUNCTIONS ON THE HYPERBOLIC PLANE AND HOW HE WAS USING THEM TO FORMULATE AND PROVE VERSIONS OF A REMARKABLE ANNOUNCEMENT OF SHNIRELMAN FROM 1974.

THIS LED TO THE STRIKING

QUANTUM ERGODICITY THEOREM

OF

COLIN DE VERDIERE - SHNIRELMAN - ZELDITCH.

- THE SETTING IS THE HAMILTONIAN WHICH IS THE GEODESIC MOTION g_t ON THE UNIT (CO)TANGENT BUNDLE $T_1^*(X)$ OF A COMPACT RIEMANNIAN X.

- THE SPECTRUM IS THAT OF THE LAPLACIAN

$$\Delta \phi_j + t_j^2 \phi_j = 0$$

$$0 = t_1 < t_2 \le t_3 \cdots \quad , \quad \phi_j \text{ O.N.B OF } L^2(X)$$

- THE PROBABILITY MEASURES

$$\mu_j = |\phi_j(x)|^2 d\mathrm{VOL}$$

HAVE MICRO-LOCAL LIFTS ν_j TO $T_1^*(X)$

$$\nu_j(a) = \langle \mathrm{Op}(a)\phi_j, \phi_j \rangle$$

FOR $a \in C^\infty(T^*X)$ HOMOGENEOUS OF DEGREE 0 IN THE COTANGENT VECTOR.

$\mathrm{Op}(a)$ IS A Ψ.D.O. WITH SYMBOL a.

QUANTUM ERGODICITY (CDV-S-Z):

IF THE GEODESIC FLOW g_t IS ERGODIC WRT THE VOLUME FORM μ, THEN FOR ALMOST ALL t_j'S IN THE SENSE OF DENSITY

$$\nu_j \to \mu .$$

- THIS "QE" IS A PRECISE QUANTUM ANALOGUE OF THE CLASSICAL BIRKHOFF ERGODIC THEOREM.
- FROM ITS INCEPTION STEVE HAS BEEN INVOLVED WITH ITS

EXTENSIONS:

- TO X WITH BOUNDARY GERARD-LEIGHTMAN BURQ, HASSELL-ZELDITCH
- SEMI-CLASSICAL VERSIONS FOR THE QUANTIZATIONS OF CLASSICAL HAMILTONIANS HELFFER-ROBERT-MARTINEZ; ZELDITCH-ZWORSKI

GENERALIZATIONS:

"QER" QUANTUM ERGODIC RESTRICTION CHRISTIANSON-TOTH-ZELDITCH.

QER: $H \subset X$ A SMOOTH CODIMENSION ONE EMBEDDED ORIENTED SEPARATING HYPERSURFACE; IF THE GEODESIC FLOW ON $T_1^*(X)$ IS ERGODIC THEN THERE IS A DENSITY ONE SUBSEQUENCE OF EIGEN FUNCTIONS WHOSE NORMALIZED MICRO-LOCAL CAUCHY DATA $(\phi|_H, \partial_\nu \phi|_H)$ RESTRICTED TO H IS EQUIDISTRIBUTED AS $t_j \to \infty$.

- IN HIS TREATMENT OF QE ZELDITCH INTRODUCED THE "QUANTUM VARIANCE" SUMS

$$V(a,T) := \sum_{t_j \le T} |\nu_{t_j}(a)|^2 \quad ; \text{ FOR } a \text{ WITH } \int a \, d\mu = 0$$

HE SHOWS THAT IF g_t IS ERGODIC THEN

$$V(a,T) = o(N(T)) \quad ; \quad N(T) = \sum_{t_j \le T} 1, \text{ AS } T \to \infty$$

AND IF X IS OF NEGATIVE CURVATURE THEN

$$V(a,T) \ll N(T)/\log T$$

ONE CAN ASK FOR THE SIZE OF V AND DOES IT HAVE AN ASYMPTOTICS?

IN THE LATE 80'S STEVE WAS VISITING STANFORD AND I REMEMBER MANY MORE LIVELY DISCUSSIONS ABOUT QE AND IN PARTICULAR IT BECAME CLEAR TO US THAT

IF ϕ_j'S ARE EISENSTEIN SERIES AND X IS AN ARITHMETIC SURFACE THEN FOR ZELDITCH'S CANONICAL QUANTIZATIONS ($T_1^*(X) \cong \Gamma \backslash SL_2(\mathbb{R})$)

THEN $\langle Op(a)\phi_j, \phi_j \rangle$ —— (*)

CAN BE EXPRESSED AS A SPECIAL VALUE ON THE CRITICAL LINE OF A RANKIN-SELBERG L-FUNCTION!

THIS CERTAINLY SPARKED MY INTEREST IN THESE QUESTIONS AS THE FUNDAMENTAL QUESTIONS ABOUT L-FUNCTIONS (EG THE RIEMANN HYPOTHESIS) YIELDS DECISIVE INFORMATION ABOUT THE SIZE OF (*).

- IN PART THIS LED RUDNICK AND MYSELF TO THE QUANTUM UNIQUE ERGODICITY CONJECTURE; "QUE".

T. WATSON'S THESIS GIVES THE EXPLICIT RELATION BETWEEN

$$|\langle Op(a)\phi_j, \phi_j\rangle|^2$$

AND SPECIAL VALUES OF L-FUNCTIONS, WHEN X IS ARITHMETIC AND ϕ_j AND THE OBSERVABLE a ARE ALL HECKE DIAGONALIZED.

(JAKOBSON'S THESIS CLARIFIED THE EISENSTEIN SERIES)

IT HAS SERVED AS A BASIC TOOL IN FURTHER ADVANCES.

<u>ZELDITCH QUANTUM VARIANCE FOR ARITHMETIC X:</u>

<u>THEOREM</u> (LUO-SARNAK-ZHAO ; NELSON)

FOR $\int a\, d\mu = 0$ AND ϕ_j HECKE-EIGENBASIS.

$$VAR(a,T) = \sum_{t_j \leq T}' |\langle Op(a)\phi_j, \phi_j\rangle|^2 \sim V_{QUAN}(a)\, N(T)^{1/2},$$

AS $T \to \infty$

THE QUADRATIC FORM $V_{QUAN}(a)$ IS DIAGONALIZED BY THE DECOMPOSITION OF $L^2(\Gamma\backslash G)$ UNDER THE G-ACTION $G = SL_2(\mathbb{R})$

AND $V_{QUAN}(a) = * V_{CLASSICAL}(a)$ ON EACH IRREDUCIBLE WITH $*$ A SUBTLE ARITHMETIC FACTOR

$$V_{CLASSICAL}(a) = \int_{-\infty}^{\infty} \left(\int_{T^*(X)} a(g_t v)\, a(v)\, d\mu(v) \right) dt .$$

APPLICATIONS OF QE, QUE AND QER

WHEN COUPLED WITH L^p-ESTIMATES FOR EIGENFUNCTION ON X AND THEIR RESTRICTIONS TO H, THE QUANTUM ERGODIC THEOREMS HAVE MANY APPLICATIONS.

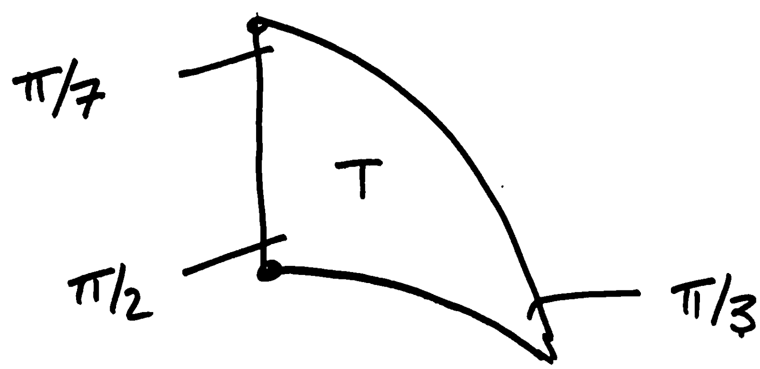
- ONE OF THE MOST INTERESTING IS TO NODAL DOMAINS IN DIMENSION 2.

COURANT'S UPPER BOUND FOR THE NUMBER OF NODAL DOMAINS IS VERY GENERAL. IT IS NOTORIOUSLY (OR EMBARASSINGLY) DIFFICULT TO PRODUCE MANY NODAL DOMAINS FOR A GIVEN X, FOR WHICH THE EIGENFUNCTIONS ARE NOT GIVEN EXPLICITLY (EG BY SEPARATION OF VARIABLES)

FOR EXAMPLE FOR A NEGATIVELY CURVED SURFACE X.

- IF X HAS AN INVOLUTIVE ISOMETRY σ WITH A FIXED SET H WHICH IS A UNION OF CLOSED GEODESICS THEN ONE CAN PRODUCE (GLOBAL) NODAL DOMAINS BY LOOKING FOR SIGN CHANGES IN $\phi_j|_H$ AND $\partial_n \phi_j|_H$.

THIS WAS OBSERVED AND EXPLOITED BY GHOSH-REZNIKOV-S IN THE CASE THAT X IS ARITHMETIC.



$\mathbb{H}^2$

T IS THE ARITHMETIC HYPERBOLIC TRIANGLE WITH ANGLES INDICATED.
(FUN FACT: (MAZAC ET AL) $\lambda_1(T) = 41. \ldots$ AND IS MAXIMAL AMONG ALL HYPERBOLIC ORBIFOLDS)

FOR THE EIGENFUNCTIONS ϕ_t WE ASSUME (AS WE MAY) THAT ϕ IS EITHER A DIRICHLET OR NEUMANN EIGENFUCTION AS WELL AS OF THE HECKE ALGEBRA (COMMUTATIVE) OF OPERATORS COMING FROM CORRESPONDENCES.

THEOREM (A. GHOSH – A. REZNIKOV – S ; S. JANG – J. JUNG)

THE NUMBER OF NODAL DOMAINS OF AN EIGENFUNCTION ϕ_t ON T GOES TO INFINITY AS $t \to \infty$.

COMMENTS :

(1) G-R-S PROVE THIS (IN FACT A BIT MORE) CONDITIONAL ON THE RIEMANN HYPOTHESIS FOR CERTAIN L-FUNCTIONS.

(2) JANG–JUNG AVOID THE L-FUNCTION ESTIMATE BY AN INGENIOUS RENORMALISING ARGUMENT COUPLED WITH A RELLICH IDENTITY AND ARGUMENTS IN (CHRISTIANSON - TOTH - ZELDITCH'S) QER PAPER.

(3). AMONG VARIOUS OTHER INPUTS IN THE ABOVE THEOREM IS LINDENSTRAUSS' QUE FOR T.

• FOR X OF NEGATIVE CURVATURE (SURFACE) WE DON'T KNOW QUE, BUT QE AND QER CAN BE USED TO GIVE A SIMILAR STATEMENT FOR ALMOST ALL ϕ_j's (J. JUNG – ZELDITCH).

THE PLANAR DOMAIN VERSION OF THE ABOVE IS AS ALWAYS THE MOST APPEALING

THEOREM (H. HEZARI 2018)
LET Ω BE A PLANAR DOMAIN WITH PIECEWISE SMOOTH BOUNDARY AND ERGODIC BILLIARD BALL, THEN THE NUMBER OF NODAL DOMAINS GOES TO INFINITY FOR ALMOST ALL ϕ_j's.

NOTE: THE NUMBER OF NODAL DOMAINS THAT ARE PRODUCED IN THE ABOVE (ARITHMETIC X's) AND SINAI BILLIARDS IS MUCH SMALLER THAN WHAT IS FOUND NUMERICALLY (~~HE~~JHAL / BARNETT ET AL).

IN FACT THE NO' OF NODAL DOMAINS IS CONSISTENT WITH WHAT IS PREDICTED FOR RANDOM MONOCHROMATIC WAVES (NAZAROV-SODIN) AND IS A FIXED FRACTION (ABOUT 6%) OF ~~THE~~ COURANT'S UPPER BOUND.

DIMENSION 3 AND HIGHER :

FOR X NEGATIVELY CURVED AND OF DIMENSION 3, I AM NOT AWARE OF ANY MEANS TO PRODUCE NODAL DOMAINS, AND PERHAPS FOR GOOD REASON. IF MANY SUCH EXIST THEY ARE APPARENTLY INVISIBLE.

IF WE ASSUME THAT THE EIGENFUNCTIONS BEHAVE LIKE RANDOM MONOCHROMATIC WAVES (BERRY) THE ACCORDING TO NAZAROV-SODIN THERE SHOULD BE A POSITIVE FRACTION OF COURANT'S BOUND OF NODAL DOMAINS. HOWEVER WE ARE APPARENTLY IN A SUPERCRITICAL PERCOLATION REGIME AND THERE IS (NUMERICALLY BARNETT) A UNIQUE GIANT NODAL HYPERSURFACE LEAVING ALMOST NO ROOM FOR OTHER NODAL SURFACES. THAT IS THE NAZAROV-SODIN FRACTION IS TINY.

EVEN FOR SEMISEPARABLE X'S OF DIMENSION 3 THE NODAL COUNT STORY IS DRAMATICALLY DIFFENT:

- J. JUNG-ZELDITCH SHOW THAT FOR $T^3 = \mathbb{R}^3/L$ (AS WELL AS OTHER KALUZA-KLEIN SPACES) THERE IS AN O.N.B OF ϕ_j'S WHICH HAVE ONLY TWO NODAL DOMAINS.
- THIS SHOULD BE CONTRASTED WITH THE TWO DIMENSIONAL RESULTS OF A. STERN AND H. LEWY.