

Linear statistics of random zeros on complex manifolds
and
Bergman kernel asymptotics

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Distribution of *complex* zeros of $f_k(z) = \sum_{j=0}^k c_j z^j$
with independent random *complex* coefficients c_j :

- Hammersley 1956 studied the complex version of Kac's random polynomials:

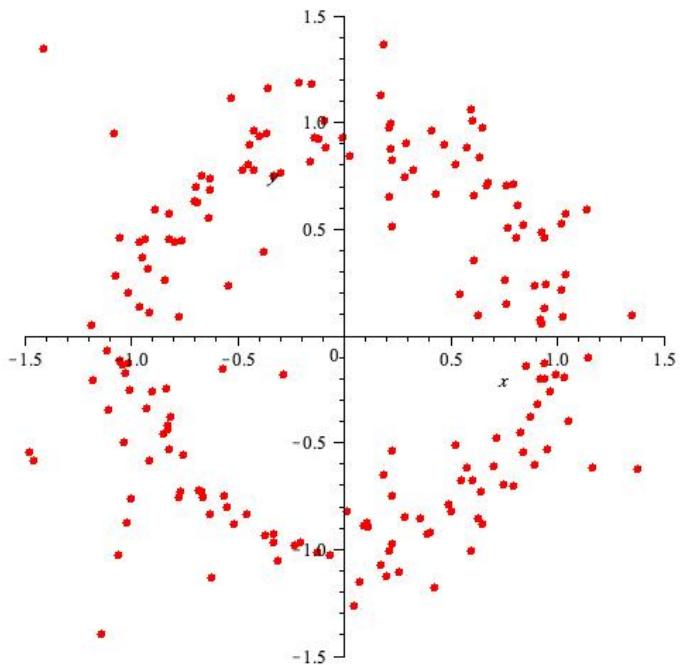
$$c_j = a_j + \sqrt{-1} b_j$$

where the a_j, b_j are independent real random variables, each with a Gaussian distribution of mean 0 and variance $1/2$:

$$\mathbf{E}(c_j) = 0, \quad \mathbf{E}(a_j^2) = \mathbf{E}(b_j^2) = \frac{1}{2}, \quad \mathbf{E}(|c_j|^2) = 1.$$

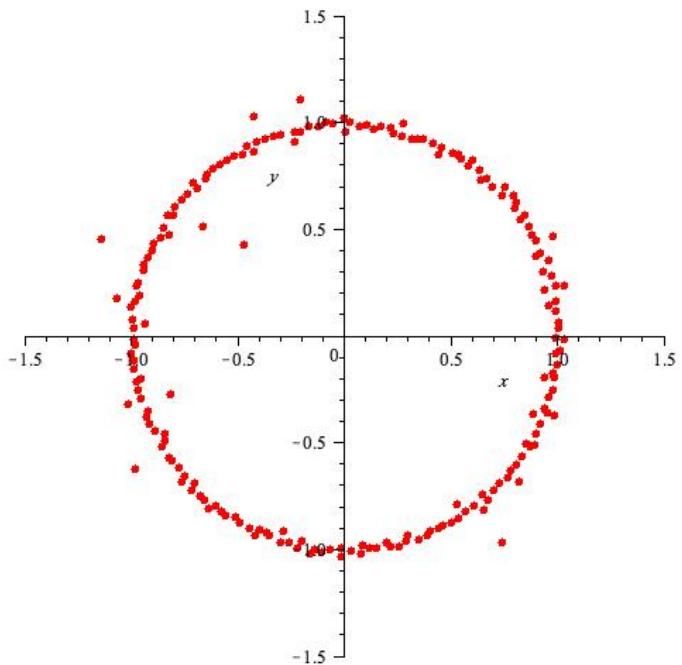
The zeros tend to be clustered around the unit circle.

History: zeros of Hammersley polynomials



Zeros of 10 Hammersley polynomials of degree 20

History: zeros of Hammersley polynomials



Zeros of 1 Hammersley polynomial of degree 200

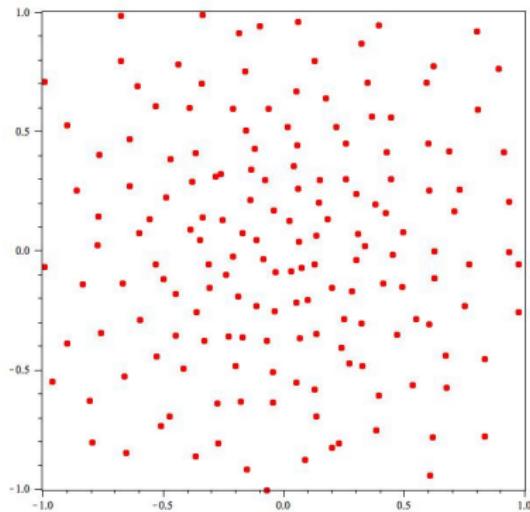
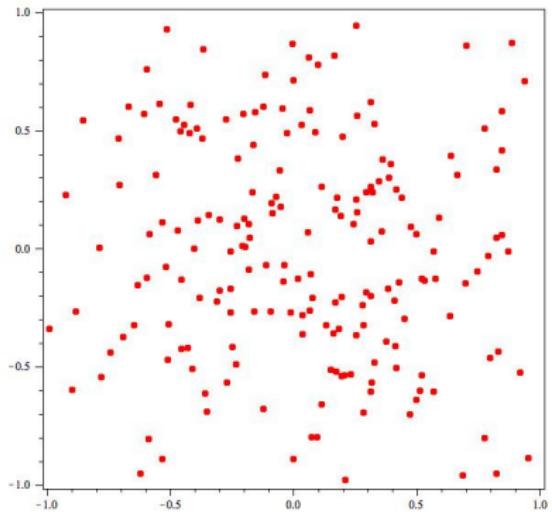
History: zeros of SU(2)-polynomials

Hannay (1996) studied the zeros of Gaussian SU(2)-polynomials:

$$f_k(z) = \sum_{j=0}^k c_j \binom{k}{j} z^j, \quad \{c_j\} \text{ i.i.d. Gaussians}$$

In contrast with the Hammersley polynomials, there is no clustering around the unit circle. In fact, the ensemble is invariant under the SU(2)-action on \mathbb{CP}^1 , and therefore the expected distribution is uniform on the Riemann sphere.

Here are plots of two different random point processes:



One of these depicts the zeros of a random SU(2)-polynomial.
The other is of independent random zeros (Bernoulli process).

The zeros of SU(2)-polynomials “repel” since their pair correlation

$$\text{Corr}_k(z, w) \approx \frac{k}{2} \text{dist}(z, w)^2, \quad \text{dist}(z, w) < 1/\sqrt{k}.$$

- Hannay (1996) determined the *scaling limit* pair correlation for the zeros: As $k \rightarrow \infty$,

$$\text{Corr}_k(0, \frac{z}{\sqrt{k}}) \rightarrow \frac{1}{2}|z|^2 - \frac{1}{36}|z|^6 + \frac{1}{720}|z|^{10} - \dots$$

- Nonnenmacher–Voros (1998) found that the correlations between zeros of holomorphic sections of line bundles* on Riemann surfaces of genus 1 (torus) have the same *scaling limit* as in the genus 0 case (\mathbb{CP}^1), as the degree increases.

**Holomorphic sections of a line bundle correspond to meromorphic functions with fixed poles.*

- This was generalized to Riemann surfaces of any genus in Bleher–S–Zelditch 2000,
- *Simultaneous zeros in higher dimension tend to cluster* (BSZ 2001)

Random holomorphic sections

We let $(L, h) \rightarrow M$ be a positive Hermitian holomorphic line bundle and we give M the Kähler form $\omega_h := -\frac{i}{2}\partial\bar{\partial}\log h$.

Let $\{S_1^k, \dots, S_{d_k}^k\}$ be an orthonormal basis for $H^0(M, L^k)$ with respect to the inner product

$$\langle s_k, s'_k \rangle = \int_M h^k(s_k, \overline{s'_k}) \frac{1}{m!} \omega_h^m.$$

A (Gaussian) random holomorphic section of L^k is:

$$s_k(z) = \sum_j c_j S_j^k(z),$$

where the c_j are independent centered complex Gaussian random variables with $\mathbf{E}(|c_j|^2) = 1$.

Bergman kernel

The Bergman kernel for $H^0(M, L^k)$ is

$$B_k(z, w) = \sum_j S_j^k(z) \otimes \overline{S_j^k(w)}.$$

The Bergman kernel is the orthogonal projector
 $\Pi_k : \mathcal{L}^2(M, L^k) \rightarrow H^0(M, L^k)$:

$$(\Pi_k f)(z) = \int_M B_k(z, w) \cdot f(w) d\text{Vol}_M(w).$$

Let $s_k = \sum_j c_j S_j^k(z) \in H^0(M, L^k)$ denote a random section.

The Bergman kernel is the covariance of $s_k(z), s_k(w)$:

$$\mathbf{E} \left(s_k(z) \otimes \overline{s_k(w)} \right) = \sum_j S_j^k(z) \otimes \overline{S_j^k(w)} = B_k(z, w).$$

Variance of random zeros on Riemann surfaces

For an $SU(2)$ polynomial f_k of degree k on the Riemann sphere, we have $\mathbf{E}[\#(Z_{f_k} \cap U)] = \frac{k}{\pi} \text{Area}(U)$.

In fact, for a positive line bundle $(L, h) \rightarrow (M, \omega_h)$, Zelditch and I showed in our first joint paper in 1999 that

$$\mathbf{E}[\#(Z_{s_k} \cap U)] = \frac{k}{\pi} \int_U \omega_h + O(1).$$

What about the number variance for random zeros in U ? For independent random points thrown onto a Riemann surface, the variance of the number of points in U is exactly proportional to the number k of points, but

$$\text{Var}[\#(Z_{s_k} \cap U)] \approx \frac{\zeta(3/2)}{8\pi^{3/2}} k^{-1/2} \text{Length}(\partial U).$$

(S-Zelditch 2008)



We let $Z_{s_k} = \{z \in M : s_k(z) = 0\}$ denote the zero divisor of a random section $s_k \in H^0(M, L^k)$. Then for all domains $U \subset M$,

$$\mathbf{E} \left(k^{-1} \text{Vol}_{2m-2}(Z_{s_k} \cap U) \right) = \frac{m}{\pi} \text{Vol}_{2m}(U) + O(\tfrac{1}{k})$$

(S-Zelditch 1999)

Variance (S-Zelditch 2008):

- Let $U \subset M$ be a domain with piecewise smooth boundary.

$$\text{Var} \left(k^{-1} \text{Vol}_{2m-2}(Z_{s_k} \cap U) \right) \approx \nu_m \text{Vol}_{2m-1}(\partial U) k^{-m-1/2}$$

$$\nu_m = \frac{1}{8} \pi^{m-5/2} \zeta(m + \tfrac{1}{2})$$

- For a smooth test form $\varphi \in \mathcal{D}^{2m-2}(M)$:

$$\text{Var} \left(k^{-1} \int_{Z_{s_k}} \varphi \right) \approx \lambda_m \|\partial \bar{\partial} \varphi\|_{L^2}^2 k^{-m-2}$$

$$\lambda_m = \frac{1}{4} \pi^{m-2} \zeta(m+2)$$

$m = 1$: Sodin-Tsirelson 2004

The variance current

Let $X : \Omega \rightarrow \mathcal{D}'^k(M)$ be a random variable with values in the space $\mathcal{D}'_{\mathbb{R}}^k(M)$ of real currents of degree k on a manifold M . The variance of X is the current

$$\mathbf{Var}(X) := \mathbf{E}(X \boxtimes X) - \mathbf{E}(X) \boxtimes \mathbf{E}(X) \in \mathcal{D}'^{2k}(M \times M),$$

where we use the notation

$$S \boxtimes T = \pi_1^* S \wedge \pi_2^* T \in \mathcal{D}'^{2k}(M \times M), \quad \text{for } S, T \in \mathcal{D}'^p(M).$$

Here, $\pi_1, \pi_2 : M \times M \rightarrow M$ are the projections. I.e.,

$$(S \boxtimes T)(z, w) = S(z) \wedge T(w), \quad (z, w) \in M \times M.$$

For a test form φ ,

$$\mathrm{Var}(X, \varphi) = (\mathrm{Var}(X), \varphi \boxtimes \varphi)$$

The variance current

The zero current: $(Z_{s^k}, \varphi) = \int_{Z_{s^k}} \varphi, \quad \varphi \in \mathcal{D}^{m-1, m-1}(M).$

Variance of the zero current:

$$\mathbf{Var}(Z_{s^k}) := \mathbf{E}(Z_{s^k} \boxtimes Z_{s^k}) - \mathbf{E}(Z_{s^k}) \boxtimes \mathbf{E}(Z_{s^k}) \in \mathcal{D}'^{2,2}(M \times M)$$

$$\mathbf{Var}\left(\int_{Z_{s^k}} \varphi\right) = \mathbf{Var}(Z_{s^k}, \varphi) = (\mathbf{Var}(Z_{s^k}), \varphi \boxtimes \varphi).$$

The variance current

Ingredients of the formula for $\text{Var}(Z_{s^k})$:

- The correlation (squared). Let $S_j^k = F_j^k e^{\otimes k}$.

$$P_k(z, w) = \frac{|B_k(z, w)|^2}{|B_k(z, z)| |B_k(w, w)|} = \frac{\left| \sum_j F_j^k(z) \overline{F_j^k(w)} \right|^2}{\sum_j |F_j^k(z)|^2 \sum_j |F_j^k(w)|^2}$$

- The (universal) function

$$G(t) := -\frac{1}{4\pi^2} \int_0^t \frac{\log(1-x)}{x} dx = \frac{1}{4\pi^2} \sum_{n=1}^{\infty} \frac{t^n}{n^2}.$$

The variance current

Theorem

The variance of the zero current for Gaussian random sections of $H^0(M, L^k)$ is given by

$$\mathbf{Var}(Z_{s^k}) = -\partial_z \bar{\partial}_z \partial_w \bar{\partial}_w Q_k(z, w) \in \mathcal{D}'^{2,2}(M \times M),$$

where the *pluri-bipotential of the variance current* Q_k is given by

$$Q_k = G \circ P_k = -\frac{1}{4\pi^2} \int_0^{P_k} \frac{\log(1-x)}{x} dx = \frac{1}{4\pi^2} \sum_{n=1}^{\infty} \frac{(P_k)^n}{n^2}.$$

$$P_k(z, w) = \frac{|B_k(z, w)|^2}{|B_k(z, z)| |B_k(w, w)|}.$$

Asymptotics:

- Tian-Yau-Zelditch expansion (Catlin, Zelditch 1998):

$$B_k(z, z) \sim k^m \left(\frac{1}{\pi^m} + a_1 k^{-1} + a_2 k^{-2} + \dots \right)$$

- Near-diagonal (Bleher–S–Zelditch 2000, S–Zelditch 2002):

For $|u| < b\sqrt{\log k}$, we have

$$\|B_k(z_0 + \frac{u}{\sqrt{k}}, z_0)\| \sim \frac{k^m}{\pi^m} e^{-\frac{1}{2}|u|^2} (1 + k^{-1/2} p_1 + k^{-1} p_2 + \dots)$$

(Lu–S 2015): Using K-coordinates about z_0 ,

$$p_1 = 0, \quad p_2 = \frac{1}{2}\rho + \frac{1}{8}R(u, \bar{u}, u, \bar{u}).$$

- Off-diagonal (Lindholm, Berndtsson 2003):

$$|B_k(z, w)| = O\left(k^m e^{-C\sqrt{k} \operatorname{dist}(z, w)}\right),$$

uniformly for $(z, w) \in M \times M$.

- see also Berman–Berndtsson–Sjöstrand 2008,
Ma–Marinescu 2008, 2013, Hezari–Lu–Xu 2017.

Variance of random zeros: smooth statistics

Write $R_u := R(u, \bar{u}, u, \bar{u})$. (Lu-S 2015):

$$P_k(z_0 + \frac{u}{\sqrt{k}}, z_0) = e^{-|u|^2} \left[1 + \frac{1}{4} R_u k^{-1} + O(k^{-3/2}) \right].$$

Let $\varphi \in \mathcal{D}^{2m-2}(M)$ be a smooth test form. Then

$$\begin{aligned} \text{Var}\left(\int_{Z_{s^k}} \varphi\right) &= (\mathbf{Var}(Z_{s^k}), \varphi \boxtimes \varphi) \\ &= \int_{M \times M} Q_k(z, w) i\partial\bar{\partial}\varphi(z) i\partial\bar{\partial}\varphi(w). \end{aligned}$$

$$Q_k = \frac{1}{4\pi^2} \sum_{n=1}^{\infty} \frac{(P_k)^n}{n^2} = \frac{1}{4\pi^2} \sum_{n=1}^{\infty} e^{-n|u|^2} \left[\frac{1}{n^2} + \frac{R_u}{4nk} \right] + O(k^{-\frac{3}{2}})$$

Variance of random zeros: smooth statistics

(S, 2021) Asymptotic formula:

On a Riemann surface: for $f \in \mathcal{C}^\infty$,

$$\text{Var} \left(\sum_{\{z: s_k(z)=0\}} f(z) \right) =$$

$$\frac{\zeta(3)}{16\pi} \|\Delta f\|^2 k^{-1} - \frac{\pi^3}{2880} \left\{ \int_M \rho |\Delta f|^2 \omega + \|d\Delta f\|_2^2 \right\} k^{-2} + O(k^{-3})$$

On a Kähler manifold:

$$\text{Var} \left(\int_{Z_{s^k}} \varphi \right) = \frac{1}{4} \pi^{m-2} k^{-m}$$

$$\times \left[\zeta(m+2) \|\partial\bar{\partial}\varphi\|_2^2 + \zeta(m+3) \left\{ \int \frac{\rho}{2} |\partial\bar{\partial}\varphi|^2 + \|\bar{\partial}^* \partial\bar{\partial}\varphi\|_2^2 \right\} k^{-1} \right.$$

$$+ \dots]$$



Number variance

So far, we have discussed the hypersurface case. For the point case (codimension m), let f_1^k, \dots, f_m^k be independent Gaussian sections of L^k . Consider the random variable

$$\mathcal{N}_k^U(f_1^k, \dots, f_m^k) := \#\{z \in U : f_1^k(z) = \dots = f_m^k(z) = 0\}.$$

$$\mathbf{E}(\mathcal{N}_k^U) = k^m \int_U \omega^m = \frac{m!}{\pi^m} \text{Vol}_{\mathbb{CP}^m}(U) k^m.$$

Theorem

(S-Zelditch 2008)

$$\mathbf{Var}(k^{-m} \mathcal{N}_k^U) = k^{-m-1/2} [\nu_m \text{Vol}(\partial U) + O(k^{-1/2+\varepsilon})],$$

where ν_m is a (universal) constant.

It follows from the variance estimates that $\mathbf{E}(Z_{s^k}, \varphi)$ almost surely converges to its limiting average value. In fact, Sodin–Tsirelson (2004) showed that the distributions (Z_{s^k}, φ) (suitably normalized) converge to a normal distribution, in dimension 1. This was generalized in S–Zelditch (2008) to higher dimension, but only for the codimension 1 case. A generalization for non-smooth φ was given by Nazarov–Sodin (2011).

Open problems

- Asymptotic expansion for the number variance (and intermediate codimensions) in dimension ≥ 2 .
- Central limit theorem for the number variance in dimension ≥ 2 .