"Microlocal Kakeya-Nikodym estimates and improved eigenfunction estimates"

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Based on earlier work with M. Blair & S. Zelditch

And ongoing work with
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Global Harmonic Analysis

Conference in honor of Steve Zelditch

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Happy Birthday Steve!

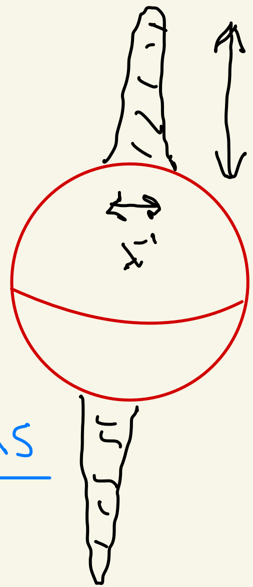
Setting: (M,g) compact n-dimensional manifold. Eigenfunctions:

$$\boxed{-\Delta_g e_\lambda = \lambda^2 e_\lambda} \; ; \; \lambda = \lambda_j \; ; \; 0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \dots$$

- Interested in "size" or concentration properties
  - How large can for $q > 2$

    $\|e_\lambda\|_{L^q(M)}$ be if $\|e_\lambda\|_{L^2(M)} = 1$?
- Do certain "geometries" ensure relatively small L^q-norms?
- For which exponents q ... ?

Worst-case : S^n

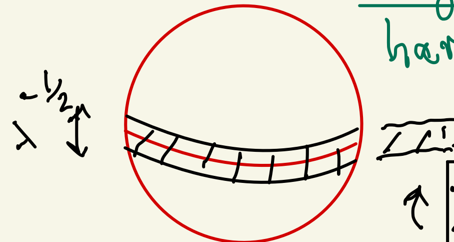


$$\boxed{\lambda^{\frac{n-1}{2}} = \text{size}}$$

Z_λ = zonal functions

$$\boxed{\|Z_\lambda\|_{L^q(S^n)} \approx \lambda^{n(\frac{1}{2}-\frac{1}{q})-\frac{1}{2}} \qquad q \gtrsim \frac{2n}{n-1}}$$

G_λ = Highest weight spherical harmonics



$$\boxed{\lambda^{\frac{n-1}{4}} = \text{size}}$$

$$\boxed{\|G_\lambda\|_{L^q(S^n)} \approx \lambda^{\frac{n-1}{2}(\frac{1}{2}-\frac{1}{q})} \qquad q \gtrsim 2,}$$

Summary of "enemies":

$$\|G_\lambda\|_q \approx \lambda^{\frac{n-1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)} \leq \lambda^{n\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{1}{2}} \approx \|Z_\lambda\|_q \iff q \geq q_c$$

$$\|G_\lambda\|_q \approx \lambda^{\frac{n-1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)} \geq \lambda^{n\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{1}{2}} \approx \|Z_\lambda\|_q \iff q \leq q_c$$

$$\boxed{q_c = \frac{2(n+1)}{n-1}}$$

Both powers of λ equal when $q = q_c$

$\frac{1}{q_c}$

$q \in [2, \; q_c, \; \infty]$

Highest weight Spherical Harmonic Regime

Zonal Regime

BOTH!!

"Transitional Height" $\alpha = \lambda^{\frac{n-1}{4}}$

Local estimates: Universal bounds

Theorem Any (M,g)

(i)
$$\|e_\lambda\|_{L^q(M)} \lesssim \begin{cases} \lambda^{n(\frac12-\frac1q)-\frac12}, & q\ge q_c=\frac{2(n+1)}{n-1} \\ \lambda^{\frac{n-1}{2}(\frac12-\frac1q)}, & 2<q\le q_c. \end{cases}$$

(ii) More generally, if $\mathbb{1}_{[\lambda-1,\lambda+1]}(P)$,

$P=\sqrt{-\Delta_g}$, denotes projection onto $[\lambda-1,\lambda+1]$ part of spectrum

(ii) $\|\mathbb{1}_{[\lambda-1,\lambda+1]}(P)\|_{L^2\to L^q} \lesssim$ RHS of (i)

Remark (ii) always sharp; expect improvements for (i) in many cases.

- Estimates for $q=q_c \Rightarrow$ all others. So improving critical L^{q_c}-bounds for e.f.'s $\Rightarrow$ improvements $\forall\, q \in (2,\infty]$

- Bérard ('77) $(\log\lambda)^{-1/2}$ improvements for $q=\infty$:

$$\boxed{\left\| \mathbb{1}_{[\lambda-\varepsilon(\lambda),\lambda+\varepsilon(\lambda)]}(P) f \right\|_{L^\infty} \lesssim \lambda^{\frac{n-1}{2}} \sqrt{\varepsilon(\lambda)}\, \|f\|_2, \quad \varepsilon(\lambda)=\frac{1}{\log\lambda}}$$

if (M,g) non-positive sectional curvatures

- By interpolation w/ universal L^{q_c}-bd get log-power improvements for $q \in (q_c,\infty)$ too.

- Better: Hassell-Tacy (2015): $(\log\lambda)^{-1/2}$ improvements for $q \in (q_c,\infty]$

- C.S.-Zelditch (2002): o-improvements on generic manifolds for $q \in (q_c,\infty]$.

Critical $q=q_c$ or subcritical $2<q<q_c$?

C.Z.-Zelditch (2014): For 2-d manifolds of non-positive curvature have o-improvements $\forall\, q\in(2,q_c)$, $q_c=6$.

M. Blair - C.S. (2015): Same for $n\geq 3$ & (2018) log-gains $q\in(2,q_c)$

C.S. (2017) $(\log\log\lambda)^{-1/2}$ gains $q=q_c$

M. Blair - C.S. (2019): $(\log\lambda)^{-\sigma_n}$ gains for $q=q_c$

Fix $\rho \in \mathcal{S}(\mathbb{R})$ w/ $\rho(0)=1$ and $\operatorname{supp} \hat{\rho} \subset [-1/2, 1/2]$. Then $\boxed{\rho(T(\lambda - P))e_\lambda = e_\lambda}$, $\boxed{T = c_0 \log \lambda.}$ and so obtain improved e.f. and $[\lambda - \varepsilon(\lambda), \lambda + \varepsilon(\lambda)]$, $\varepsilon = (\log \lambda)^{-1}$ spectral projection bounds via:

Theorem (M. Blair - X. Huang - C.S.) assume (M,g) has negative sectional curvatures then if $\sigma_n = \frac{1}{n+1}$, $n \gtrsim 3$ and $\sigma_2 = 1/6$

$$\boxed{\|\rho(T(\lambda - P))f\|_{q_c} \lesssim \lambda^{1/q_c} (\log \lambda)^{-\sigma_n} \|f\|_2.}$$

- Also have slightly weaker results for nonpositive curv. case.

- If $M = S^1 \times M^{n-1}$ product above would be sharp $n=2,3$, i.e., $O((\lambda/\log\lambda)^{1/q_c})$ bounds.

Strategy

Recall: $\boxed{\|G_\lambda\|_\infty \approx \lambda^{\frac{n-1}{4}}}$ highest wt sph. harmonics saturate L^{q_c}

- Basically cut at this height if $\|f\|_{L^2}=1$, $\delta = 1/8$:

$\boxed{A_+ = \{x\in M : |\rho(T(\lambda-P))f(x)| \geq \lambda^{\frac{n-1}{4}+\delta}\}}$; $\boxed{A_- = \{x\in M : |\rho(T(\lambda-P))f(x)| < \lambda^{\frac{n-1}{4}+\delta}\}}$

- Using very different methods (necessary) show

$$\boxed{\|\rho(T(\lambda-P))f\|_{L^{q_c}(A_+)} \lesssim \lambda^{1/q_c}(\log\lambda)^{-1/2}\|f\|_2} \quad \text{and} \quad \boxed{\|\rho(T(\lambda-P))f\|_{L^{q_c}(A_-)} \lesssim \lambda^{1/q_c}(\log\lambda)^{-\sigma_n}\|f\|_2}$$

"Directionless" size estimates

$\boxed{\text{N.B. LHS would} = 0 \text{ if } f = G_\lambda \text{ on } S^n\ !!!}$

Use harmonic analysis:
Bilinear methods/cancellations
Need "sense of direction"

$\boxed{\text{LHS would be} \ll \lambda^{1/q_c} \text{ if } f = Z_\lambda}$

(Semi-) Global Harmonic Analysis: Tool Chest:

$$\rho_\lambda = \rho(T(\lambda - P))$$

"Global"

$$\sigma_\lambda = \rho(\lambda - P)$$

"Local"

$$\tilde{\rho}_\lambda = \sigma_\lambda \circ \rho_\lambda$$

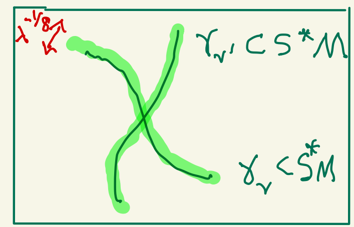
"Semi-local"

Also need ΨDO operators Q_ν

$Q_\nu(x,\xi)$ supp'd in $\lambda^{-1/8}$ ngbhd γ_ν

$$I \approx \sum_\nu Q_\nu(x,D)$$

$$\|g\|_2^2 = \sum_\nu \|Q_\nu g\|_2^2 .$$



Useful Facts: $\|\tilde{\rho}_\lambda - \rho_\lambda\|_{L^2 \to L^{q_c}}$ small $\Longrightarrow$ can replace ρ_λ by $\tilde{\rho}_\lambda$ in previous slide

and $\|\sigma_\lambda Q_\nu - Q_\nu \sigma\|_2$ small.

Estimates Required

To handle $\boxed{L^{q_c}(A_+) \text{ bounds}}$ use:

$\tilde{\rho}_\lambda = L_\lambda + G_\lambda$ ("local" + "global"):

$$\boxed{\|L_\lambda\|_{L^{q_c'}(M) \to L^{q_c}(M)} \lesssim \lambda^{2/q_c} (\log \lambda)^{-1},}$$

and

$$\boxed{|G_\lambda(x,y)| = O(\lambda^{\frac{n-1}{2}} \exp(C_0 T))}$$

Use these and adapt earlier arguments of Bourgain / Hassell - Tacy to prove $L^{q_c}(A_+)$-bounds: for region where $\boxed{|\tilde{\rho}_\lambda f| \geq \lambda^{\frac{n-1}{4}+} \text{ (large).}}$ $\boxed{\text{FLOOR NEEDED!!}}$

$\boxed{\text{For } L^{q_c}(A_-)}$ ($\tilde{\rho}_\lambda f$ can be of "size" of highest wt. sph. harmonic)

Need "microlocal L^{q_c} Kakeya-Nikodym estimates"

$$\boxed{\sup_\nu \|Q_\nu \rho_\lambda\|_{L^2(M) \to L^{q_c}(M)} \lesssim (\lambda / \log \lambda)^{1/q_c}}$$

- Similar to what Zelditch suggested!!

Semiglobal Harmonic Analysis: Step 1 (non-diagonal power gains via "cancellation")

$I \approx \sum Q_\nu$ (loc near $\gamma_\nu \subset S^*M$), $\sum_\nu \|Q_\nu h\|_2^2 \lesssim \|h\|_2^2$; $\tilde{\rho}_\lambda = \sigma_\lambda \circ \rho_\lambda$, $\sigma_\lambda = \rho(\lambda - P)$, $\rho_\lambda = \rho(T(\lambda - P))$

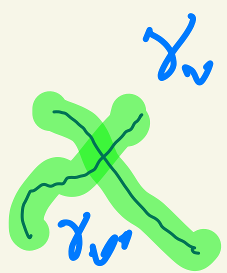
$$(*) \quad \|\tilde{\rho}_\lambda f\|^2_{L^{q_c}(A_-)} = \|(\tilde{\rho}_\lambda f)^2\|_{L^{q_c/2}(A_-)} \approx \Big\|\sum_{\nu,\nu'} Q_\nu \sigma_\lambda h \cdot Q_{\nu'} \sigma_\lambda h\Big\|_{L^{q_c/2}(A_-)}, \quad \boxed{h = \rho_\lambda f}.$$

"Expect" terms $\boxed{\Big\|\sum_{\text{non-diag}} Q_\nu \sigma_\lambda h \, Q_{\nu'} \sigma_\lambda h\Big\|_{q_c/2}}$ small if non-dig means γ_ν not close to $\gamma_{\nu'}$:

$$(\mathrm{ND}) \quad \boxed{\Big\|\sum_{\text{non-diag}} Q_\nu \sigma_\lambda h \cdot Q_{\nu'} \sigma_\lambda h\Big\|_{L^{q_c/2}(M)} \lesssim \lambda^{2/q_c - \delta_n} \|h\|_2^2} \lesssim \lambda^{2/q_c - \delta_n} \|f\|_2^2, \quad (\text{POWER Improve!!})$$

(ND) by bilinear (local) harmonic analysis $\boxed{\text{(Tao–Vargas–Vega)}}$.

non-diag:



Step 2: Diagonal Terms $\quad \gamma_\nu \approx \gamma_{\nu'}$

Using "microlocal KN" estimates and another of $\boxed{T-V-V}$ can estimate other terms !! $(n \gtrsim 3)$

$$\text{(D)} \quad \boxed{\Big\| \sum_{\text{diag}} Q_\nu \sigma_\lambda h \, Q_{\nu'} \sigma_\lambda h \Big\|_{L^{q_c/2}(M)}^{q_c/2} \lesssim \sum_\nu \| Q_\nu \sigma_\lambda h \|_{q_c}^{q_c}} = \sum_\nu \| Q_\nu \sigma_\lambda h \|_{q_c}^2 \cdot \| Q_\nu \tilde{\rho}_\lambda f \|_{q_c}^{q_c - 2}$$

$$\lesssim \sum_\nu \| \sigma_\lambda Q_\nu h \|_{q_c}^2 \cdot \sup_\nu \| Q_\lambda \rho_\lambda f \|_{q_c}^{q_c-2} \lesssim \lambda^{2/q_c} \sum_\nu \| Q_\nu h \|_2^2 \cdot \left(\lambda^{1/q_c} (\log \lambda)^{-\sigma} \right)^{q_c - 2} \| f \|_2^{q_c - 2}$$

$$\lesssim \lambda (\log \lambda)^{-\tilde{\sigma}} \| \rho_\lambda f \|_2^2 \cdot \| f \|_2^{q_c - 2} \lesssim \lambda (\log \lambda)^{-\tilde{\sigma}} \| f \|_2^{q_c}$$

loc ests σ_λ $\qquad$ MKN

Combining 2 Steps would get ($\Downarrow$ theorem):

$$(\dagger)\qquad \boxed{\|\tilde{\rho}_\lambda f\|_{L^{q_c}(A_-)} \lesssim \left(\lambda^{1/q_c-\delta_n} + \lambda^{1/q_c}(\log\lambda)^{-\sigma_n}\right)\|f\|_2}$$

Lied though: Only have

$$(ND_q)\qquad \boxed{\Big\|\sum_{\text{non-diag}} Q_\gamma\sigma_\lambda h\cdot Q_{\gamma'}\sigma_\lambda h\Big\|_{L^{q/2}(M)} \lesssim \lambda^{(n-1)(\frac{1}{2}-\frac{1}{q})-\delta(q,n)}\|h\|_2^2,\ \text{if } q\in\left(\frac{2(n+2)}{n}, q_c\right)}$$

(Regime of G_λ)

Use ceiling to fix: If $q<q_c$ close to q_c:

$$\|\tilde{\rho}_\lambda f\|_{L^{q_c}(A_-)} \lesssim \|\tilde{\rho}_\lambda f\|_{L^\infty(A_-)}^{\varepsilon(q)}\cdot\|\tilde{\rho}_\lambda f\|_{L^q(M)}^{1-\varepsilon(q)} \lesssim \left(\lambda^{\frac{n-1}{4}+}\right)^{\varepsilon(q)}\|\tilde{\rho}_\lambda f\|_{L^q(M)}^{1-\varepsilon(q)}$$

Miraculously, can use $\delta(q,n)$ power gains in (ND_q) and this ceiling to get $(\dagger)$!!

$\boxtimes$

“Steve is a unicorn. Unique on the planet,”

Bill Minicozzi

Thanks,

Steve!