Quantitative unique continuation for restrictions of Laplace eigenfunctions

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Quantitative unique continuation for eigenfunctions

- Let (M,g) compact, closed manifold with Laplace-Beltrami operator $\Delta_g : C^\infty(M) \to C^\infty(M)$.
- L^2-orthonormal basis of eigenfunctions $u_{\lambda_j} : j = 1, 2, 3, ...,$

$$-\Delta_g u_{\lambda_j} = \lambda_j^2 u_{\lambda_j},\ 0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots.$$

- Set $h = \lambda_j^{-1}$ and $P(h) = -h^2\Delta_g - 1$, so that $P(h)u_h = 0$.
- Suppose $U \subset M$ is an open subset. Then, it is well-known that for some $C_U > 0$,

$$\int_U |u_h|^2 \geq e^{-C_U/h}.$$

This is **quantitative unique continuation** and is proved using Carleman estimates.

Goodness of eigenfunction sequences

- **QUESTION:** Suppose $H \subset M$ is a C^∞ hypersurface and $\{u_{h_{j_k}}\}, \ j_k \in \mathcal{S}$ is a sequence of eigenfunctions (possibly all of them). Under what conditions on H does there exist $C = C(H, M, g) > 0$ such that

$$\int_H |u_{h_{j_k}}|^2 d\sigma \geq e^{-C/h_{j_k}} \ , j_k \in \mathcal{S}? \tag{1}$$

- In the terminology of T-Zelditch, when (1) is satisfied, the eigenfunction sequence is said to be $\mathcal{S}$-GOOD relative to hypersurface H. When $\mathcal{S}$ is the entire sequence, H is said to be COMPLETELY GOOD relative to H.
- Goodness estimates *do not* hold in general: For example, when H is the fixed point set of $\mathbb{Z}_2$-isometric involution, odd eigenfunctions satisfy $u_h|_H = 0$.

Goodness of eigenfunction sequences

- The goodness bound in (1) has applications to eigenfunction nodal intersection bounds.
- We denote the nodal set of u_h by $Z_{u_h} := \{x \in M; u_h(x) = 0\}$.
- **Theorem 1** [T-Zelditch (2021)] Let (M^n, g) be a compact, real-analytic manifold and H be a connected, irreducible $\mathcal{S}$-good, real-analytic hypersurface. Then, there exists a constant $C = C(M, g, H) > 0$ such that

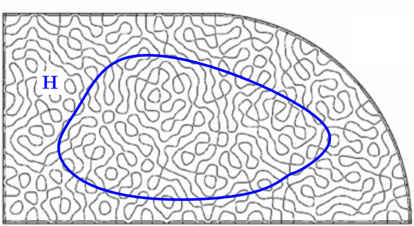
$$\mathcal{H}^{n-2}(Z_{u_{h_{j_k}}} \cap H) \leq C h_{j_k}^{-1}, \quad h_{j_k} \in \mathcal{S},$$

  where $\mathcal{H}^k$ denotes k-dimensional Hausdorff measure.

Goodness of eigenfunction sequences

- When M is a compact C^ω surface (with or without boundary) and H is a C^ω curve, Theorem 1 reduces to the nodal intersection bound [T-Zelditch (2008)]

$$\#\{Z_{u_{h_{j_k}}} \cap H\} = O(h_{j_k}^{-1}).$$



- Goodness estimates do not hold in general: For example, when H is the fixed point set of $\mathbb{Z}_2$-isometric involution, odd eigenfunctions satisfy $u_h|_H = 0$.

Goodness results

- When $M = \mathbb{H}/\Gamma$ is a compact hyperbolic surface and γ is a geodesic circle, or when M is non-compact finite volume and γ is a closed horocycle, J. Jung showed that for all eigenfunctions $\|u_h\|_{L^2(\gamma)} \geq e^{-C/h}$
- **Definition** A sequence of L^2-normalized eigenfunctions is Quantum Ergodic (QE) if for any $a \in S^0_{phg}(T^*M)$,

$$\langle Op(a)u_h, u_h \rangle \sim_{h \to 0^+} \int_{S^*M} a \, d\mu_L,$$

  where $d\mu_L$ is Liouville measure.
- **Theorem** (QE) [Shnirelman, Zelditch, Colin de Verdière]. Suppose (M^n, g) compact, closed Riemannian manifold with **ergodic** geodesic flow $G^t : S^*M \to S^*M$ with respect to Liouville measure $d\mu$ on S^*M. Then, there exists a set of density-one sequence $\mathcal{S}$ such that for $j_k \in \mathcal{S}$, the sequence $u_{u_{j_k}}, \ j_k \in \mathcal{S}$ is QE.

Goodness results

- The QE theorem was extended by Zelditch and Zworski to compact manifolds M with piecewise smooth boundary under the assumption that the associated billiard flow is ergodic and the test operators $A \in \Psi^0_{phg}(M)$ with supp $\;K_A \in \mathring{M} \times \mathring{M}$.
- **Theorem 2** [El-Hajj - T](2012) Let $\Omega \subset \mathbb{R}^2$ be a convex, piecewise C^∞ planar domain and let $u_{h_j}; j = 1, 2, ...$ be a sequence of Dirichlet (resp. Neumann) eigenfunctions that are QE. Let $H \subset \Omega$ be a strictly-convex, C^ω interior curve. Then,

$$\int_H |u_h|^2 d\sigma \geq e^{-C/h}, \qquad C = C(H, \Omega) > 0.$$

- As an example, Theorem 2 applies to QE sequences of eigenfunctions (which are of full-density) for the Bunimovich stadium and other ergodic billiards.

Special cases: polynomial bounds

- There are improvements in the case of flat tori due to Bourgain and Rudnick. When $n = 2, 3$, they prove that for any real-analytic hypersurface H with nonzero curvature,

$$\int_H |u_h|^2 \approx 1.$$

- When $M = SL_2(\mathbb{Z})/\mathbb{H}$ is a modular surface, u_h is an even Maas cusp form and H is a closed horocycle, it was proved by Ghosh, Reznikov and Sarnak that for any $\epsilon > 0$,

$$h^{-\epsilon} \gtrapprox_\epsilon \int_H |u_h|^2 \gtrapprox_\epsilon h^\epsilon.$$

Special cases: Quantum Ergodic Restriction (QER)

- **Quantum Ergodic Restriction (QER)**: When (M^n, g) is compact with ergodic geodesic flow and H is asymmetric with respect to the flow, it was proved by Zelditch-T that for Laplace eigenfunctions there is a density-one subsequence $\mathcal{S}$ such that

$$\int_H |u_{h_{j_k}}|^2 d\sigma_H \sim c_n \, vol(H), \quad j_k \in \mathcal{S}, \; h_{j_k} \to 0^+.$$

  The analogous result for general Schrödinger eigenfunctions was proved by Dyatlov and Zworski.

- Given semiclassical Cauchy data $CD(u_h) := \{(u_h|_H, \; hD_\nu u_h|_H)\}$, there are results by many authors for the corresponding restrictions $\int_H \|CD(u_h)\|^2 = \int_H (|u_h|^2 + |hD_\nu u_h|^2)$ under various conditions ([Burq], [Christianson-T-Zelditch], [Galkowski-Zelditch], [Ghosh-Reznikov-Sarnak], [Hassell-Tao], [Hassell-Zelditch],...

Defect measures

- **Definition:** Suppose $(u_{h_j})_{j=1}^{\infty}$ is a sequence of normalized eigenfunctions with the property that for any $a \in S^0$, the limit $\lim_{j\to\infty}\langle Op_h(a)u_{h_j}, u_{h_j}\rangle_{L^2(M)}$ exists. Then, the defect measure $d\mu$ is defined by
$$\int_{S^*M} a\, d\mu = \lim_{j\to\infty}\langle Op_{h_j}(a)u_{h_j}, u_{h_j}\rangle_{L^2(M)}.$$
- The QE case where $d\mu = d\mu_L$ is a special case.
- Let $\pi : T^*M \to M$ be canonical projection and assume that μ is *localized* in the sense that $K := \mathit{supp}\, \pi_*\mu \neq M$. In the following we assume that $H \in M \setminus K$ is a smooth hypersurface in the *microlocally forbidden* region.
- Given $H \subset (M \setminus K)$, let U_H be a Fermi neighbourhood of H.

Defect measures

- **Definition:** We say that $Q(h) \in \Psi_h^k(M)$ is a **lacunary** operator for the eigenfunction sequence u_h over U_H provided $Q(h)$ is h-elliptic on T^*U_H and for any $\chi_j \in C_0^\infty(U_H)$ with $\chi_1 \Subset \chi_2$,

$$\chi_1 Q(h) \chi_2 u_h = O(e^{-C/h}),$$

where $C > 0$ is a constant independent of H.

- **Theorem [Canzani-T](2022)** Let u_h be a sequence of L^2-normalized Laplace eigenfunctions with localized defect measure, $H \subset (M \setminus K)$ a smooth hypersurface sufficiently close to K and $Q(h)$ a lacunary operator associated with the eigenfunction sequence as above. Then, for any $\epsilon > 0$ there exist $C(\epsilon) > 0$ and $h(\epsilon) > 0$ such that for $h \in (0, h_0(\epsilon)]$,

$$\int_H |u_h|^2 \geq C(\epsilon) e^{-[d(H,K)+\epsilon]/h},$$

where $d(H,K) > 0$ is the distance between H and $K = \text{supp}\pi_* \mu$.

Examples

- There is a companion result that holds for sequences of eigenfunctions of Schrödinger operators $-h^2\Delta_g + V - E$ under the localization assumption on the corresponding defect measure.

- **Examples:**

- (i) (M^n, g) any smooth compact manifold with $Q(h) = -h^2\Delta + V(x) - E(h),\;\; E(h) \to E$ a regular value of $V \in C^\infty(M)$. Result applies to all eigenfunctions with eigenvalues $E(h)$ and hypersurfaces $H \subset \{V > E\}$.

Examples

- (ii) (M^n, g) smooth compact manifold and $P_1(h), ..., P_n(h)$ a quantum completely integrable (QCI) system with $P_1(h) = -h^2\Delta_g$ (or $P_1(h) = -h^2\Delta_g + V$) and $[P_i, P_j] = 0$ for all $i, j \in \{1, ..., n\}$. Let $E = (E_1, ...E_n)$ be a regular value of the moment map $\mathcal{P} := (p_1, ..., p_n) : T^*M \to \mathbb{R}^n$ and set

$$Q(h) = \sum_{j=1}^{n} (P_j^* - E_j(h))(P_j - E_j(h))$$

  where $E(h) \to E$ as $h \to 0^+$. Result applies to all (joint) eigenfunctions with joint eigenvalues $E(h)$ and hypersurfaces $H \subset (M \setminus \mathcal{P}^{-1}(E))$.
- (iii) Result also applies to appropriate eigenfunction subsequences on general twisted products $M \otimes_f N$ with metrics $g = g_M \oplus f^2 g_N$, where $f \in C^\infty(M; \mathbb{R} - 0)$.

Sketch of proof

- The proof of the theorem consists of two main steps.
- (i) An argument involving Carleman estimates shows that in an ϵ-Fermi neighbourhood $U_H(\epsilon)$ of H, for any $\epsilon > 0$,

$$\int_{U_H(\epsilon)} |u_h|^2 \geq C(\epsilon) e^{-[d(H,K)+\epsilon]/h}, \quad h \in (0, h_0(\epsilon)]. \ (*)$$

- (ii) Let $(x', x_n) : U_H(\epsilon) \to \mathbb{R}^n$ be Fermi coordinates. Since the lacunary operator $Q(h)$ is by assumption h-elliptic over $U_H(\epsilon)$, one can carry out a factorization

$$Q(h) =_{U_H(\epsilon)} A(h)\big(hD_{x_n} - iB_0\big) + R(h), \quad (**)$$

where $B_0 > 0$ is constant, $A(h)$ is h-elliptic over $U_H(\epsilon)$ and $\|R(h)\|_{L^2 \to L^2} = O(h^\infty)$.

Defect measures

- Using the lacunary assumption $Qu_h =_{U_H(\epsilon)} O(e^{-C/h})$, the factorization in $(**)$ and a local parametrix construction for $A(h)$ one shows that
- $$\begin{aligned}\big(hD_{x_n} - iB_0\big) v_h &= O(e^{-C/h}),\\ v_n &= (Id + E(h))u_h,\end{aligned} \tag{2}$$

  where $\|E(h)\|_{L^2(U_H(\epsilon)) \to L^2(U_H^+(\epsilon))} = O(h^\infty)$ with $U_H^+(\epsilon) := \{\tau \in [-\epsilon/2, 0]\}$ and $v_h|_H = u_h|_H$. Setting $H_\tau = \{x_n = \tau\}$ with $H_0 = H$ and integrating $h\partial_\tau \|v_h\|^2_{L^2(H_\tau)}$ over $U_H^+(\epsilon)$ gives
- $$h\|v_h\|_H^2 - h\|v_h\|^2_{H_{-\epsilon/2}} \geq B_0 h \|v_h\|^2_{L^2(U_H^+(\epsilon))} + O(e^{-C/h}). \tag{3}$$
- Apply the Carleman bound in $(*)$ to the RHS in (3) to finish the proof. $\square$